



# *Geometry & Topology*

Volume 27 (2023)

**Classifying sufficiently connected PSC manifolds  
in 4 and 5 dimensions**

OTIS CHODOSH

CHAO LI

YEVGENY LIOKUMOVICH



# Classifying sufficiently connected PSC manifolds in 4 and 5 dimensions

OTIS CHODOSH

CHAO LI

YEVGENY LIOKUMOVICH

We show that if  $N$  is a closed manifold of dimension  $n = 4$  (resp.  $n = 5$ ) with  $\pi_2(N) = 0$  (resp.  $\pi_2(N) = \pi_3(N) = 0$ ) that admits a metric of positive scalar curvature, then a finite cover  $\widehat{N}$  of  $N$  is homotopy equivalent to  $S^n$  or connected sums of  $S^{n-1} \times S^1$ . Our approach combines recent advances in the study of positive scalar curvature with a novel argument of Alpert, Balitskiy and Guth.

Additionally, we prove a more general mapping version of this result. In particular, this implies that if  $N$  is a closed manifold of dimensions 4 or 5, and  $N$  admits a map of nonzero degree to a closed aspherical manifold, then  $N$  does not admit any Riemannian metric with positive scalar curvature.

53C21

## Introduction

We are concerned here with the problem of classification of manifolds admitting positive scalar curvature (PSC). For closed (compact, no boundary) 2- and 3-manifolds, this problem is completely resolved; namely, the sphere and projective plane are the only closed surfaces admitting positive scalar curvature and a 3-manifold admits positive scalar curvature if and only if it has no aspherical factors in its prime decomposition. In particular, a 3-manifold admitting positive scalar curvature has a finite cover diffeomorphic to  $S^3$  or to a connected sum of finitely many  $S^2 \times S^1$ .

The main result of this paper is the following partial generalization of this statement to dimensions  $n = 4, 5$ :

**Theorem 1** *Suppose that  $N$  is a closed smooth  $n$ -manifold admitting a metric of positive scalar curvature and*

- $n = 4$  and  $\pi_2(N) = 0$ , or
- $n = 5$  and  $\pi_2(N) = \pi_3(N) = 0$ .

*Then a finite cover  $\widehat{N}$  of  $N$  is homotopy equivalent to  $S^n$  or connected sums of  $S^{n-1} \times S^1$ .*

Chodosh and Li [7] and Gromov [14] showed that, if a closed  $N^n$  is aspherical (ie  $\pi_k(N) = 0$  for all  $k \geq 2$ ) and  $n = 4, 5$ , then there is no Riemannian metric of positive scalar curvature on  $N$ . [Theorem 1](#) can thus be seen as a refinement of this into a positive result.

**Remark** By Theorem 1.3 of Gadgil and Seshadri [12] (see also Freedman [11], Milnor [26] and Kreck and Lück [22]), we have that if  $n = 4$  and  $\widehat{N}$  is homotopy equivalent to  $S^4$  or  $S^3 \times S^1$ , or if  $n = 5$  (with no further restriction on the homotopy type), then homotopy equivalence in the conclusion to [Theorem 1](#) can be upgraded to homeomorphism.

We also prove a more general “mapping” version of [Theorem 1](#).

**Theorem 2** *Suppose that  $N$  is a closed smooth  $n$ -manifold with a metric of positive scalar curvature and there exists a nonzero degree map  $f: N \rightarrow X$ , to a manifold  $X$  satisfying*

- $n = 4$  and  $\pi_2(X) = 0$ , or
- $n = 5$  and  $\pi_2(X) = \pi_3(X) = 0$ .

*Then a finite cover  $\widehat{X}$  of  $X$  is homotopy equivalent to  $S^n$  or connected sums of  $S^{n-1} \times S^1$ .*

We note that the following result immediately follows from [Theorem 2](#):

**Corollary 3** *Let  $n \in \{4, 5\}$ ,  $X$  and  $N$  be closed oriented manifolds of dimension  $n$ , and  $X$  be aspherical. Suppose there exists a map  $f: N \rightarrow X$  with  $\deg f \neq 0$ . Then  $N$  does not admit any Riemannian metric of positive scalar curvature.*

Recall that it was previously shown in [7; 14] that closed aspherical (ie  $\pi_k(N) = 0$  for all  $k \geq 2$ )  $n$ -manifolds do not admit PSC for  $n = 4, 5$ . In [14] a related statement

was proven for manifolds admitting proper, distance-decreasing maps to uniformly contractible manifolds. In fact, [Corollary 3](#) seems to have been asserted by Gromov [[15](#), page 144–145], but the (relatively simple) lifting argument does not appear there.

## 0.1 Urysohn width bounds

Recall that a metric space  $(X, d)$  has *Urysohn  $q$ -width*  $\leq \Lambda$  if there is a  $q$ -dimensional simplicial complex  $K$  and a continuous map  $X \rightarrow K$  such that  $\text{diam } f^{-1}(s) \leq \Lambda$  for all  $s \in K$ . As such, having finite Urysohn  $q$ -width implies that a manifold looks  $\leq q$ -dimensional in some macroscopic sense.

A well-known conjecture (see [[15](#), page 63]) of Gromov posits that an  $n$ -manifold with scalar curvature  $\geq 1$  has finite Urysohn  $(n-2)$ -width. Various forms of this conjecture are proven for  $n = 3$ —see Gromov and Lawson [[16](#)], Katz [[21](#)], Marques and Neves [[24](#)] and Liokumovich and Maximo [[23](#)]—while the conjecture is largely open for  $n \geq 4$  (some progress has been achieved by Bolotov and Dranishnikov [[2](#); [3](#)]).

A key component in the proof of [Theorem 1](#) is the following result:

**Theorem 4** *For  $(N^n, g)$  satisfying the hypothesis of [Theorem 1](#), the universal cover  $(\tilde{N}, \tilde{g})$  has finite Urysohn 1-width.*

This follows by combining [Corollary 7](#) and [Proposition 8](#) below. A simple example where [Theorem 1](#) applies is the product metric on  $S^1 \times S^3$ , whose universal cover is  $\mathbb{R} \times S^3$ , clearly of finite Urysohn 1-width. On the other hand, we note that the higher connectivity hypothesis in [Theorem 4](#) is necessary: compare with  $T^2 \times S^2$ .

**Remark** Consider a metric  $g_R$  on  $S^3$  formed by capping off a cylinder  $[-R, R] \times S^2(1)$  with hemispheres and smoothing out the resulting metric, so that the scalar curvature is  $\geq 1$ . The product metric  $(S^1(1), g_S) \times (S^3, g_R)$  has scalar curvature  $\geq 1$  but the universal cover has Urysohn 1-width  $\sim R$ . As such, the estimate in [Theorem 4](#) cannot be made quantitative (essentially, the issue is that the universal cover converges to  $\mathbb{R}^2 \times S^2(1)$ , which has nontrivial  $\pi_2$ ).

As we were finishing this paper, we discovered that, recently, Gromov has indicated a proof of the classification of PSC 3-manifolds [[15](#), page 135] by using finiteness of the 1-Urysohn width of the universal cover. Our proof of [Theorem 1](#) follows a similar strategy once [Theorem 4](#) is proven.

## 0.2 Remarks on positive isotropic curvature

**Theorem 1** has an interesting relationship to well-known conjectures of Gromov [13, Section 3(b)] and Schoen [29] concerning the topology of closed  $n$ -manifolds admitting a metric with positive isotropic curvature (PIC). Namely, they (respectively) conjecture that if a closed manifold has a PIC metric then the fundamental group is virtually free and a finite cover is diffeomorphic to either a sphere or connected sums of finitely many  $S^1 \times S^{n-1}$ .

There have been distinct approaches to such a question, relying on either minimal surface theory or Ricci flow. Using minimal surface theory, Micallef and Moore have shown that, if  $M^n$  is a closed PIC manifold then  $\pi_k(M) = 0$  for  $k = 2, \dots, \lfloor \frac{1}{2}n \rfloor$  [25]. In particular, if  $M$  is simply connected, then it is homeomorphic to a sphere. In related work, Fraser has proven that an  $n$ -manifold ( $n \geq 5$ ) with PIC does not contain a subgroup isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$  [10].

On the other hand, using Ricci flow, Hamilton has classified 4-manifolds admitting PIC that do not contain nontrivial incompressible  $(n-1)$ -dimensional space forms [18]. This was extended to prove the Gromov–Schoen conjectures for  $n = 4$  by Chen, Tang and Zhu [6]. In higher dimensions, Brendle and Schoen [5] and Nguyen [27] proved the PIC condition is preserved under the Ricci flow; this is an important ingredient in Brendle and Schoen’s proof of the differentiable sphere theorem. Recently, Brendle has achieved a breakthrough in the study of the Ricci flow of PIC manifolds and has extended Hamilton’s result to dimensions  $n \geq 12$  [4]; as above, this result has been used to prove the Gromov–Schoen conjectures for  $n \geq 12$  by Huang [20].

We note that, since PIC implies PSC, combining [25] with **Theorem 1** yields an alternative proof of Gromov’s conjecture (the fundamental group is virtually free) for  $n = 4$  and proves a weak version of Schoen’s conjecture for  $n = 4$  (ie with homotopy equivalence replacing diffeomorphism). Furthermore, **Theorem 1** implies that a PIC 5-manifold with  $\pi_3(M) = 0$  satisfies Gromov’s conjecture and the same weak version of Schoen’s conjecture. It is an interesting question if a 5-manifold with PIC has  $\pi_3(M) = 0$  (note that  $\pi_2(M) = 0$  by [25]).

## Organization of the paper

In **Section 1** we revisit the filling radius estimates from [7; 14]. In **Section 2** we show that such estimates imply **Theorem 4**. Then, we complete the proof of **Theorem 1** in **Section 3**. Finally, in **Section 4** we prove **Theorem 2**.

## Acknowledgements

Chodosh was supported by a Sloan Fellowship, a Terman Fellowship and NSF grant DMS-2016403. Li was supported by NSF grant DMS-2005287. Liokumovich was supported by a NSERC Discovery grant and an NSERC Accelerator award. We are grateful to Hannah Alpert, Larry Guth, Ciprian Manolescu, Kasra Rafi and Boyu Zhang for various discussions related to this article. We would like to thank Georg Frenck for suggesting the statement of [Theorem 2](#) to us as well as the referee for several helpful suggestions.

## 1 Filling estimates

In [7; 14], it was shown that a closed aspherical  $n$ -manifold does not admit positive scalar curvature for  $n = 4, 5$  by combining a linking argument with a filling radius inequality in the presence of positive scalar curvature. In this section we observe that this filling radius inequality carries over to the setting considered here.

We begin by summarizing the results contained in [7] that will be needed in this paper.

**Theorem 5** *Consider  $(N^n, g)$  a closed Riemannian  $n$ -manifold with scalar curvature  $R \geq 1$ . Fix a Riemannian cover  $(\hat{N}, \hat{g})$ .*

(1) *Suppose that  $n = 4$ . There is a universal constant  $L_0 > 0$  with the following property. Consider a closed embedded 2-dimensional submanifold  $\hat{\Sigma}_2 \subset \hat{N}$  with  $[\hat{\Sigma}_2] = 0 \in H_2(\hat{N}; \mathbb{Z})$ . Then there is a 3-chain  $\hat{\Sigma}'_3 \subset B_{L_0}(\hat{\Sigma}_2)$  and a closed embedded 2-dimensional submanifold  $\hat{\Sigma}'_2$  with*

$$\partial \hat{\Sigma}'_3 = \hat{\Sigma}_2 - \hat{\Sigma}'_2$$

*as chains such that, for every connected component  $S$  of  $\hat{\Sigma}'_2$ , the extrinsic diameter of  $S$  satisfies  $\text{diam}(S) \leq L_0$ .*

(2) *Suppose that  $n = 5$ . There is a universal constant  $L_0 > 0$  with the following property. Consider a closed embedded 3-dimensional submanifold  $\hat{\Sigma}_3 \subset \hat{N}$  with  $[\hat{\Sigma}_3] = 0 \in H_3(\hat{N}; \mathbb{Z})$ . Then there is a 4-chain  $\hat{\Sigma}'_4 \subset B_{L_0}(\hat{\Sigma}_3)$  and a closed embedded 3-dimensional submanifold  $\hat{\Sigma}'_3$  with*

$$\partial \hat{\Sigma}'_4 = \hat{\Sigma}_3 - \hat{\Sigma}'_3$$

*as chains as well as 3-chains  $\hat{U}_1, \dots, \hat{U}_m$  with  $\text{diam}(\hat{U}_j) \leq L_0$  and 2-cycles*

$$\{\hat{\Gamma}_j^l : j = 1, \dots, m, l = 1, \dots, k(j)\}$$

with  $\text{diam}(\widehat{\Gamma}_j^l) \leq L_0$  such that

$$\widehat{\Sigma}_3 = \sum_{j=1}^m \widehat{U}_j \quad \text{and} \quad \partial \widehat{U}_j = \sum_{l=1}^{k(j)} \widehat{\Gamma}_j^l \quad \text{for each } j = 1, \dots, m,$$

where both equalities hold as chains (not just in homology). Finally, there is an integer  $q$  and a function

$$u: \{(j, l) : j = 1, \dots, m, l = 1, \dots, k(j)\} \rightarrow \{1, \dots, q\}$$

such that, for  $r \in \{1, \dots, q\}$ , we have

$$\text{diam}\left(\bigcup_{(j,l) \in u^{-1}(r)} \widehat{\Gamma}_j^l\right) \leq L_0$$

and, moreover,

$$\sum_{(j,l) \in u^{-1}(r)} \widehat{\Gamma}_j^l = 0$$

as 2-chains for  $r \in \{1, \dots, q\}$ .

**Proof** When  $n = 4$ , one can solve Plateau’s problem to find  $\widehat{\Sigma}_3$  minimizing area with  $\partial \widehat{\Sigma}_3 = \widehat{\Sigma}_2$ . Applying the “ $\mu$ -bubble technique” (see [7, Section 3]), we can find  $\widehat{\Sigma}'_2 \subset \widehat{\Sigma}_3$  with  $d_{\widehat{\Sigma}_3}(\widehat{\Sigma}'_2, \widehat{\Sigma}_2) \leq L_0$  and such that  $\widehat{\Sigma}'_2 \subset \widehat{\Sigma}_3$  is a “stable  $\mu$ -bubble” in the sense of [7, Lemma 14]. By [7, Lemma 16], the intrinsic diameter of each component is  $\leq L_0$  (taking  $L_0$  larger if necessary). This proves the assertion (since extrinsic distances are bounded by the intrinsic distances).

Similarly, when  $n = 5$ , we can solve Plateau’s problem to find  $\widehat{\Sigma}_4$  minimizing area with  $\partial \widehat{\Sigma}_4 = \widehat{\Sigma}_3$ . As before, we can find a “stable  $\mu$ -bubble”  $\widehat{\Sigma}'_3$  with  $d_{\widehat{\Sigma}_4}(\widehat{\Sigma}'_3, \widehat{\Sigma}_3) \leq L_0$ . Finally, the construction of the  $\widehat{U}_j$  and  $\widehat{\Gamma}_j^k$  follows from the “slice-and-dice” procedure from [7, Sections 6.3–6.4].

Note that the last conclusion (ie that  $\sum_{(j,l) \in u^{-1}(r)} \widehat{\Gamma}_j^l = 0$ ) was stated slightly differently in [7]. To be precise, it was proven that the cycles  $\sum_{(j,l) \in u^{-1}(r)} \widehat{\Gamma}_j^l$  are disjoint for distinct  $r$  (see [7, Section 6.4]). Now, by using  $\sum_{r=1}^q \sum_{(j,l) \in u^{-1}(r)} \widehat{\Gamma}_j^l = \partial(\sum_{j=1}^m U_j) = 0$ , we find that each term in the sum must vanish.  $\square$

**Example 1** We illustrate the “slice-and-dice” procedure and its relevance to the statements of Theorem 5 with Figure 1, where  $\widehat{\Sigma}'_3$  is diffeomorphic to  $S^2 \times S^1$ . We first cut (slice)  $\widehat{\Sigma}'_3$  by an embedded  $S^2$  and view the result as a 3-manifold with boundary, which we further cut (dice) into seven 3-chains  $\widehat{U}_1, \dots, \widehat{U}_7$  such that each

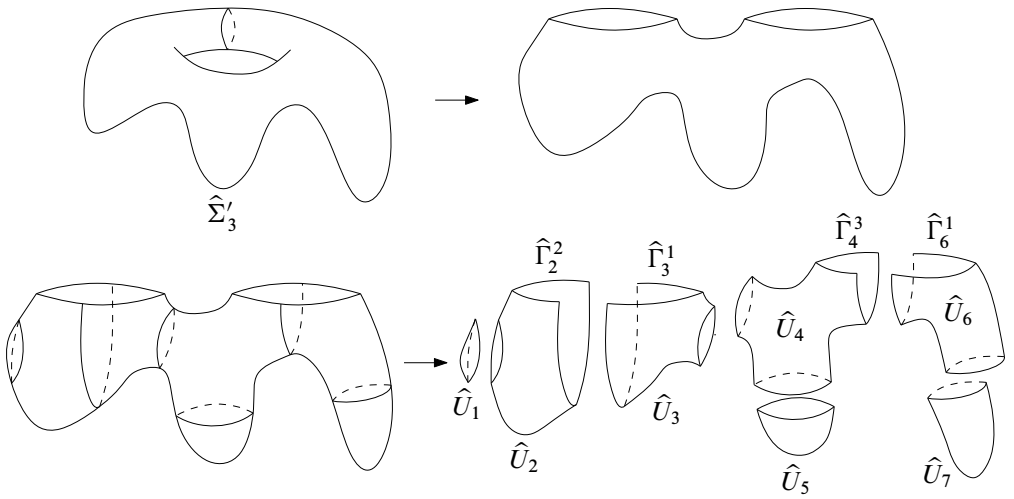


Figure 1: Cutting the 3-cycle  $\hat{\Sigma}'_3$  into small pieces.

$\hat{U}_j$  satisfies  $\text{diam } \hat{U}_j \leq L_0$  (of course, the number of chains may vary in different examples). We label the boundary components of  $\hat{U}_j$ , from left to right in the figure, by  $\hat{\Gamma}_j^l$  for  $l = 1, \dots, k(j)$ . Note that, in this case, there are four such boundary components that are nonsmooth, namely  $\hat{\Gamma}_2^2, \hat{\Gamma}_3^1, \hat{\Gamma}_4^3$  and  $\hat{\Gamma}_6^1$ . The function  $u$  groups different  $\hat{\Gamma}_j^l$  that glue together into a 2-cycle. For example, we have

$$u(2, 2) = u(3, 1) = u(4, 3) = u(6, 1),$$

and

$$u(1, 1) = u(2, 1), \quad u(3, 2) = u(4, 1), \quad u(4, 2) = u(5, 1), \quad u(6, 2) = u(7, 1).$$

Moreover, the values of  $u$  on different groups of  $\hat{\Gamma}_j^l$  are different (eg  $u(2, 2) \neq u(1, 1)$ ). Note here that  $\sum_{(j,l) \in u^{-1}(r)} \hat{\Gamma}_j^l = 0$  for each  $r$ .

The following proposition will be used to replace [7, Proposition 10] in the more general setting considered here:

**Proposition 6** Consider  $\pi : (\hat{N}, \hat{g}) \rightarrow (N, g)$  a regular<sup>1</sup> Riemannian covering map of  $n$ -dimensional manifolds, with  $(N, g)$  compact. Assume that  $H_l(\hat{N}, \mathbb{Z}) = 0$ . Then, for  $r > 0$ , there is  $R = R(r) < \infty$  such that  $H_l(B_r(x), \mathbb{Z}) \rightarrow H_l(B_R(x), \mathbb{Z})$  is the zero map for any  $x \in \hat{N}$ .

<sup>1</sup>Recall that a cover is regular if the group of deck transformations acts transitively on the fibers. In particular, the universal cover is a regular cover.



**Proof** We first fix  $x = x_0$ . For any  $r > 0$ , there is  $r_1 \in [r, 2r]$  with  $\overline{B_{r_1}(x_0)}$  a compact manifold (with boundary). By Corollaries A.8 and A.9 in [19], the homology groups of  $\overline{B_{r_1}(x_0)}$  are finitely generated. Assume that  $\alpha_1, \dots, \alpha_J$  generates  $H_l(\overline{B_{r_1}(x_0)}, \mathbb{Z})$ . By assumption, for each  $i$ ,  $\alpha_i = \partial\beta_i$  for some  $(l+1)$ -chains  $\beta_1, \dots, \beta_J$ . Choose  $R_1 = R_1(r)$  so that  $\beta_i \in B_{R_1}(x_0)$  for  $i = 1, \dots, J$ . Then we see that  $H_l(\overline{B_{r_1}(x_0)}, \mathbb{Z}) \rightarrow H_l(B_{R_1}(x_0), \mathbb{Z})$  is the zero map, so, in particular,

$$H_l(B_r(x_0), \mathbb{Z}) \rightarrow H_l(B_{R_1}(x_0), \mathbb{Z})$$

is the zero map.

Now, for any  $x \in \widehat{N}$ , we can assume (using a deck transformation) that  $d(x, x_0) \leq \text{diam } N$ . Thus,

$$B_r(x) \subset B_{r+\text{diam } N}(x_0) \quad \text{and} \quad B_{R_1(r+\text{diam } N)}(x_0) \subset B_{R_1(r+\text{diam } N)+\text{diam } N}(x).$$

Thus, we find that the assertion holds for  $R(r) = R_1(r + \text{diam } N) + \text{diam } N$ . □

Putting these facts together, we thus obtain the following generalization of the filling estimate obtained in [7; 14]:

**Corollary 7** *Suppose that, for  $n \in \{4, 5\}$ ,  $(N^n, g)$  is a closed Riemannian  $n$ -manifold with positive scalar curvature and  $\pi_2(N) = \dots = \pi_{n-2}(N) = 0$ . Then there is  $L = L(N, g) > 0$  with the following property. Consider  $\Sigma_{n-2}$  an closed embedded  $(n-2)$ -submanifold in  $\widetilde{N}$  the universal cover. Then  $\Sigma_{n-2}$  is nullhomologous in  $B_L(\Sigma_{n-2})$ .*

**Proof** Observe that the universal cover  $\widetilde{N}$  has  $\pi_1(\widetilde{N}) = \dots = \pi_{n-2}(\widetilde{N}) = 0$ . By the Hurewicz theorem,  $H_{n-3}(\widetilde{N}, \mathbb{Z}) = H_{n-2}(\widetilde{N}, \mathbb{Z}) = 0$ .

When  $n = 4$ , the assertion immediately follows from a combination of Theorem 5 with Proposition 6. Indeed, Theorem 5 implies that  $\Sigma_2$  is homologous to  $\Sigma'_2$  in  $B_{L_0}(\Sigma_2)$ , where  $\text{diam}(\Sigma'_2) \leq L_0$ . Proposition 6 implies that  $\Sigma'_2$  can be filled in an  $R(L_0)$ -neighborhood. Thus,  $\Sigma_2$  can be filled in an  $(L_0 + R(L_0))$ -neighborhood.

When  $n = 5$ , the proof is more complicated due to the nature of the “slice-and-dice” decomposition in Theorem 5. Fix  $\widehat{\Sigma}'_3 \subset B_{L_0}(\Sigma_3)$  homologous to  $\Sigma_3$  and  $\{\widehat{U}_j\}$  and  $\{\widehat{\Gamma}^l_j\}$  with the properties described in Theorem 5. We can now fill  $\widehat{\Sigma}'_3$  in a bounded neighborhood following [7, Section 6.4], which we explain here. Since  $\text{diam}(\widehat{\Gamma}^l_j) \leq L_0$ , Proposition 6 implies that  $\widehat{\Gamma}^l_j = \partial\widetilde{\Gamma}^l_j$  for a 3-chain with  $\text{diam}(\widetilde{\Gamma}^l_j) \leq R(L_0)$ . Then, because  $\text{diam}(\widehat{U}_j) \leq L_0$ ,

$$\widehat{U}_j - \sum_{l=1}^{k(j)} \widetilde{\Gamma}^l_j$$

is a 3–cycle of diameter  $\leq L_0 + 2R(L_0)$ . Thus, by Proposition 6, there is a 4–chain  $\tilde{U}_j$  with  $\text{diam}(\tilde{U}_j) \leq R(L_0 + 2R(L_0))$  and

$$\partial\tilde{U}_j = \hat{U}_j - \sum_{l=1}^{k(j)} \tilde{\Gamma}_j^l.$$

On the other hand, as was proven in Theorem 5, there is

$$u: \{(j, l) : j = 1, \dots, m, l = 1, \dots, k(j)\} \rightarrow \{1, \dots, q\}$$

such that

$$\text{diam}\left(\bigcup_{(j,l) \in u^{-1}(r)} \hat{\Gamma}_j^l\right) \leq L_0$$

and

$$\sum_{(j,l) \in u^{-1}(r)} \hat{\Gamma}_j^l = 0$$

as 2–chains.

As such, for  $r \in \{1, \dots, q\}$ ,  $\sum_{(j,l) \in u^{-1}(r)} \tilde{\Gamma}_j^l$  is a 3–cycle of diameter bounded by  $2R(L_0) + L_0$  and thus there is a 4–chain  $\tilde{\Xi}_r$  with  $\text{diam}(\tilde{\Xi}_r) \leq R(L_0 + 2R(L_0))$  and

$$\partial\tilde{\Xi}_r = \sum_{(j,l) \in u^{-1}(r)} \tilde{\Gamma}_j^l.$$

This yields

$$\hat{\Sigma}'_3 = \partial\left[\sum_{r=1}^q \tilde{\Xi}_r + \sum_{j=1}^m \tilde{U}_j\right]$$

with

$$\sum_{r=1}^q \tilde{\Xi}_r + \sum_{j=1}^m \tilde{U}_j \subset B_{R(L_0+2R(L_0))}(\hat{\Sigma}'_3).$$

Thus,  $\Sigma_3$  is nullhomologous in an  $(R(L_0+2R(L_0))+R(L_0))$ –neighborhood. □

**Example 2** Continuing Example 1, we illustrate in Figure 2 how Corollary 7 works for  $\hat{\Sigma}'_3$  in Figure 1. Consider all 2–cycles  $\hat{\Gamma}_j^i$  with  $u(j, i) = r$ . Fill in  $\hat{\Gamma}_j^i$  with a 3–chain  $\tilde{\Gamma}_j^i$ . By construction, the sum of these  $\tilde{\Gamma}_j^i$  forms a 3–cycle, which can then be filled in by a 4–chain  $\tilde{\Xi}_r$ . By Proposition 6 and Corollary 7, the diameter of all these fill-ins are bounded by  $R(L_0 + 2R(L_0))$ .

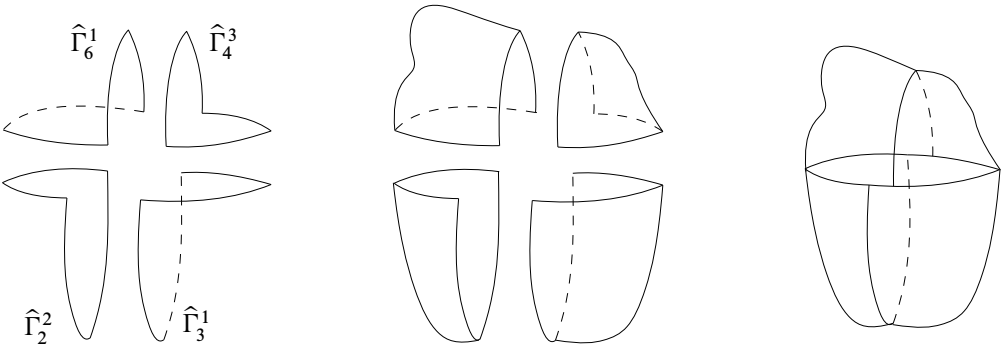


Figure 2: To fill in  $\{\hat{\Gamma}_j^l : u(j, l) = r\}$ , we first fill in each  $\hat{\Gamma}_j^l$  with  $\tilde{\Gamma}_j^l$  (center), then fill in by  $\Xi_r$ , to obtain a 3-cycle (right).

## 2 Filling versus Urysohn width

The next result is inspired by work of Hannah Alpert, Alexey Balitskiy and Larry Guth [1], which we learnt about from a talk by Alpert. The strategy should be compared with [16, Corollary 10.11].

**Proposition 8** *Assume that  $(N^n, g)$  has the property that any closed embedded  $(n-2)$ -submanifold in the universal cover  $\Sigma_{n-2} \subset \tilde{N}$  can be filled in  $B_L(\Sigma_{n-2})$ . Then the universal cover  $(\tilde{N}, \tilde{g})$  satisfies:*

(\*) *For any point  $p \in \tilde{N}$ , each connected component of a level set of  $d(p, \cdot)$  has diameter  $\leq 20L$ .*

Note that Corollary 7 implies a manifold  $(N, g)$  in Theorem 1 satisfies the assumptions of Proposition 8. By the argument in [17, Corollary 10.11], this shows that the universal cover  $(\tilde{N}, \tilde{g})$  has Urysohn 1-width  $\leq 20L$ . In particular, the macroscopic dimension of  $\tilde{N}$  is 1.

**Proof** Let  $p \in \tilde{N}$  be a point and consider level sets of the distance function  $f(x) = d(p, x)$ .

For the sake of contradiction, suppose that there is a curve  $\gamma \subset f^{-1}(t)$  connecting points  $x$  and  $y$  with  $d(x, y) \geq 20L$ . Fix a minimizing geodesic  $\eta_x$  from  $p$  to  $x$  (and similarly for  $\eta_y$ ) and consider the triangle  $T = \eta_x * \gamma * -\eta_y$ . Fix  $0 < l < L$  such that  $\partial B_{4L+l}(x)$  and  $\partial B_{L+l}(\eta_x)$  are smooth hypersurfaces intersecting transversely. Set  $\Sigma_{n-2} := \partial B_{4L+l}(x) \cap \partial B_{L+l}(\eta_x)$ . Note that we have not ruled out  $\Sigma_{n-2} = \emptyset$ ; in this case we will take  $d(\Sigma_{n-2}, \cdot) = \infty$  below.

By construction,

$$d(\Sigma_{n-2}, \eta_x) > L.$$

Set  $\Sigma'_{n-1} := \partial B_{4L+l}(x) \cap \overline{B_{L+l}(\eta_x)}$  and note that  $\partial \Sigma'_{n-1} = \Sigma_{n-2}$ . Observe that, since  $\eta_x$  is a minimizing geodesic between  $p$  and  $x$  and  $\Sigma'_{n-1}$  is a subset of  $\partial B_{4L+l}(x)$ , it must hold that  $\eta_x$  intersects  $\Sigma'_{n-1}$  exactly once and does so orthogonally (and thus transversally). We will return to this observation below.

**Lemma 9**  $d(\Sigma_{n-2}, \gamma) \geq d(\Sigma'_{n-1}, \gamma) > L.$

**Proof** We first prove that  $d(\Sigma'_{n-1}, \gamma) > L$ . Choose  $s \in \Sigma'_{n-1}$  with  $d(s, \gamma) = d(\Sigma'_{n-1}, \gamma)$ . There is  $e \in \eta_x$  such that  $d(s, e) \leq L + l$ . We have

$$d(x, e) \geq d(x, s) - d(s, e) \geq 4L + l - (L + l) = 3L.$$

Since  $\eta_x$  is minimizing (and has length  $t$ ), we have  $d(p, e) \leq t - 3L$ . Thus,

$$d(p, s) \leq d(p, e) + d(e, s) \leq t - 3L + L + l = t - 2L + l.$$

Thus,

$$d(s, \gamma) \geq d(p, \gamma) - d(p, s) \geq t - (t - 2L + l) = 2L - l.$$

This completes the proof of  $d(\Sigma'_{n-1}, \gamma) > L$ . Since  $\Sigma_{n-2} \subset \Sigma'_{n-1}$ , it clearly holds that  $d(\Sigma_{n-2}, \gamma) \geq d(\Sigma'_{n-1}, \gamma)$ . □

**Lemma 10**  $\Sigma'_{n-1} \cap \eta_y = \emptyset.$

**Proof** Suppose the contrary. Consider  $s \in \Sigma'_{n-1} \cap \eta_y$ . Note that  $d(s, x) = 4L + l$  and there is  $e \in \eta_x$  with  $d(s, e) \leq L + l$ . We have

$$d(x, e) \leq d(x, s) + d(s, e) \leq 5L + 2l.$$

As such,

$$d(p, e) \geq t - 5L - 2l,$$

so

$$d(p, s) \geq d(p, e) - d(e, s) \geq t - 5L - 2l - L - l = t - 6L - 3l.$$

Thus,

$$d(s, y) \leq 6L + 3l.$$

However, this contradicts

$$20L \leq d(x, y) \leq d(x, s) + d(s, y) \leq 4L + l + 6L + 3l = 10L + 4l. \quad \square$$

**Lemma 11**  $d(\Sigma_{n-2}, \eta_y) > L.$

**Proof** The proof is similar to the previous argument. Suppose we have  $s \in \Sigma_{n-2}$  and  $e_y \in \eta_y$  with  $d(s, e_y) \leq L$ . There is  $e_x \in \eta_x$  with  $d(s, e_x) = L + l$ . Note that

$d(s, x) = 4L + l$ . Thus,

$$d(p, e_x) = t - d(x, e_x) \geq t - d(x, s) - d(s, e_x) \geq t - 5L - 2l.$$

Thus,

$$d(p, e_y) \geq d(p, e_x) - d(e_x, e_y) \geq d(p, e_x) - d(e_x, s) - d(s, e_y) = t - 7L - 3l.$$

This implies that

$$d(y, e_y) \leq 7L + 3l.$$

However, this contradicts

$$20L \leq d(x, y) \leq d(x, s) + d(s, e_y) + d(e_y, y) \leq 12L + 4l. \quad \square$$

We can now complete the proof of [Proposition 8](#). Perturb the triangle  $T$  to be a smooth embedded curve  $T'$  still intersecting  $\Sigma'_{n-1}$  transversely. As long as the perturbation is small,  $T' \cap \Sigma'_{n-1}$  will consist of a single point (thanks to [Lemmas 9](#) and [10](#), along with the observation that  $\eta_x$  intersects  $\Sigma'_{n-1}$  transversely in exactly one point). Assume first that  $\Sigma_{n-2} \neq \emptyset$ . By assumption, there is  $\Sigma_{n-1} \subset B_L(\Sigma_{n-2})$  with  $\partial\Sigma_{n-1} = \Sigma_{n-2}$ . Using [Lemmas 9](#) and [11](#) as well as  $d(\Sigma_{n-2}, \eta_x) = L + l$ , we find that  $\Sigma_{n-1} \cap T' = \emptyset$ . As such,  $T'$  has nontrivial algebraic intersection with the cycle  $\Sigma'_{n-1} - \Sigma_{n-1}$ . This is a contradiction since  $\tilde{N}$  is simply connected.

If  $\Sigma_{n-2} = \emptyset$ , then the argument is similar but simpler. In this case, we note that  $\Sigma'_{n-1}$  is a cycle and, combining [Lemmas 9](#) and [10](#) with the fact that  $\Sigma'_{n-1}$  intersects  $\eta_x$  transversely exactly once, we see that  $\Sigma'_{n-1}$  is a cycle with nontrivial algebraic intersection with  $T'$ , a contradiction as before. □

### 3 Fundamental group and homotopy type

In this section, we prove [Theorem 1](#). We first prove (see [Corollary 14](#) below) that a manifold  $(N^n, g)$  whose universal cover satisfies the conclusion of [Proposition 8](#) has virtually free fundamental group. (Recall that a group is virtually free if it processes a free subgroup of finite index.) This fact seems to be well known among certain experts (in particular, see [\[15, page 135\]](#)). We give a proof here, roughly following the strategy used in [\[28\]](#). The argument is based on notion of the number of *ends* of a group.

**Definition 12** Given a group  $G$ , its number of ends,  $e(G)$ , is defined as the number of topological ends of  $\tilde{K}$ , where  $\tilde{K} \rightarrow K$  is a regular covering of finite simplicial complexes  $K$  and  $\tilde{K}$ , and  $G$  is the group of deck transformations.

It follows from [9] that a finitely generated group can have 0, 1, 2 or infinitely many ends. Our main result here is as follows:

**Proposition 13** *Suppose  $(N, g)$  is a closed Riemannian manifold satisfying the conclusions of Proposition 8. Then any finitely generated subgroup  $G$  of  $\pi_1(N)$  cannot have one end.*

We will prove this below, but first we note that it yields the desired statement:

**Corollary 14** *Suppose  $(N, g)$  is a closed Riemannian manifold satisfying the conclusions of Proposition 8. Then  $\pi_1(N)$  is virtually free.*

**Proof** We follow the proof of [28, Theorem 2.5]. Indeed, by combining the main result of [8] (see [30, Section 7]) with Proposition 13,  $\pi_1(N)$  is the fundamental group of a finite graph of groups with finite edge and vertex groups. The assertion now follows from Proposition 11 in Chapter II, Section 2.6 of [31] (or eg [30, Theorem 7.3]).  $\square$

Moreover, we observe that given these results, we can finish the proof of Theorem 1.

**Proof of Theorem 1** By Corollary 7, Proposition 8, and Corollary 14,  $\pi_1(N)$  is virtually free. Let  $G \subset \pi_1(N)$  be a finite-index subgroup which is a free group. Consider the finite covering  $\hat{N} \xrightarrow{\hat{p}} N$  such that the image of  $\hat{p}_\#$  is  $G$ . Then  $\pi_1(\hat{N})$  is a finitely generated free group. Since  $\pi_2(\hat{N}) = \cdots = \pi_{n-2}(\hat{N}) = 0$ , Sections 2 and 3 of [12] imply that  $\hat{N}$  is homotopy equivalent to  $S^n$  or connected sums of  $S^{n-1} \times S^1$ .  $\square$

We now give the proof of Proposition 13:

**Proof of Proposition 13** Suppose there is a finitely generated subgroup  $G$  of  $\pi_1(N)$  with one end. We will show that this leads to a contradiction.

We divide the proof into several steps. Take a cover  $N_0 \xrightarrow{p} N$  such that  $p_\#(\pi_1(N_0)) = G$ . Because  $p_\#: \pi_1(N_0) \rightarrow \pi_1(N)$  is injective, this ensures that  $\pi_1(N_0) = G$ . If  $G$  is finite then  $e(G) = 0$ , so we can assume  $G$  is infinite.

Since  $G$  is finitely generated, we can find  $K \subset N_0$  a compact submanifold with boundary containing representatives of all of the generators of  $G$ . Write  $i: K \rightarrow N_0$  for the inclusion map and note that  $i_\#: \pi_1(K) \rightarrow \pi_1(N_0) = G$  is surjective. Let  $H = \ker i_\#$ , so  $G = \pi_1(K)/H$ . Choose  $j: \tilde{K} \rightarrow K$  the cover (with  $\tilde{K}$  path connected) of  $K$  so that

$j_{\#}(\pi_1(\tilde{K})) = H$ . Since  $H$  is a normal subgroup of  $\pi_1(K)$ , the covering  $j: \tilde{K} \rightarrow K$  is regular, and the group of deck transformations of  $j$  is isomorphic to  $\pi_1(K)/H = G$ . Thus,  $\tilde{K}$  is noncompact. Note that  $j_{\#} \circ i_{\#}: \pi_1(\tilde{K}) \rightarrow \pi_1(N_0)$  is the zero map, so we can lift  $i$  to  $\tilde{i}: \tilde{K} \rightarrow \tilde{N}$ , where  $\tilde{N}$  is the universal cover of  $N$ . (We emphasize that  $\tilde{K}$  is not necessarily the universal cover of  $K$ .)

As such, we have the diagram of spaces

$$\begin{array}{ccc}
 \tilde{K} & \xrightarrow{\tilde{i}} & \tilde{N} \\
 \downarrow j & & \downarrow \tilde{p} \\
 K & \xrightarrow{i} & N_0 \\
 & & \downarrow p \\
 & & N
 \end{array}$$

The maps  $i$  and  $\tilde{i}$  are inclusions of codimension zero submanifolds with boundary; indeed:

**Lemma 15**  $\tilde{i}$  is a proper embedding.

**Proof** We first show that  $\tilde{i}$  is injective. Suppose that  $\tilde{i}(a) = \tilde{i}(b)$ . Connect  $a$  and  $b$  by a curve  $\tilde{\eta}$  in  $\tilde{K}$ . By assumption,  $\tilde{i}(\tilde{\eta})$  is a loop in  $\tilde{N}$ , so  $\eta = j(\tilde{\eta})$  has  $[\eta] = e \in \pi_1(N_0) = G$ . Thus,  $[\eta] \in H \subset \pi_1(K)$ . This implies that  $\tilde{\eta}$  is a loop, ie  $a = b$ . It is straightforward to check that  $\tilde{i}$  is a closed map, using the fact that  $K$  is compact and  $\tilde{p}$  is a covering map. Therefore,  $\tilde{i}$  is proper. □

Note that  $N$  is equipped with a Riemannian metric  $g$ , so that, for any  $p \in \tilde{N}$ , each connected component of a level set of  $f_p(x) = d_g(x, p)$  has diameter  $\leq C$ , where  $C = 20L$  as given in Proposition 8. The embeddings  $i, \tilde{i}$  induce metric structures on  $K$  and  $\tilde{K}$ , respectively.

**Lemma 16** For each  $r > 0$ , there exists  $R(r) > 0$  such that, for any  $a, b \in \tilde{K}$  with  $d_{\tilde{N}}(a, b) \leq r$ , we have  $d_{\tilde{K}}(a, b) \leq R(r)$ .

**Proof** Fix  $x \in \tilde{K}$ . By applying a deck transformation, we can assume that  $d_{\tilde{N}}(a, x) \leq c_0$  (here  $c_0$  only depends on  $K$ ), so  $d_{\tilde{N}}(b, x) \leq r + c_0$ . Since  $\tilde{K}$  is connected (and  $\tilde{i}$  is proper), there exists  $R = R(r + c_0)$  so that  $d_{\tilde{K}}(a, x), d_{\tilde{K}}(b, x) \leq R$ . The assertion follows from the triangle inequality. □

In the following lemma, we will call a curve  $\tilde{\gamma} : \mathbb{R} \rightarrow \tilde{K}$  a *line* if it minimizes length on compact subintervals relative to competitors in  $\tilde{K}$ . Note that such a curve is a geodesic in the sense of metric geometry, but not necessarily in the sense of Riemannian geometry, since it could stick to  $\partial\tilde{K}$  in places. Similarly, we will call  $\sigma' : [0, \infty) \rightarrow \tilde{K}$  a *minimizing ray* if it minimizes length in the same sense.

**Lemma 17** *There exists a line  $\tilde{\gamma}$  in  $\tilde{K}$ .*

**Proof** Fix  $p \in \tilde{K}$ , and choose  $p_j \in \tilde{K}$  diverging. Let  $\sigma_j$  denote a curve that minimizes length in  $\tilde{K}$  between  $p$  and  $p_j$ . We assume that  $\sigma_j$  is parametrized by unit speed. In particular,  $\sigma_j$  is a 1-Lipschitz map from an interval to  $\tilde{K}$ . Consider an exhaustion of  $\tilde{K}$  by nested compact sets containing  $p$ . Applying Arzelà–Ascoli in each compact set and taking a diagonal sequence, we obtain that, after passing to a subsequence,  $\sigma_i$  converges to a minimizing ray  $\sigma' : [0, \infty) \rightarrow \tilde{K}$ . Since  $G$  is the group of deck transformations of  $\tilde{K} \rightarrow K$  acting transitively on  $\tilde{K}$  and  $K$  is compact, we can choose  $t_i \rightarrow \infty$  and deck transformations  $\Phi_i$  so that  $d_{\tilde{g}}(p, \Phi_i(\sigma'(t_i)))$  is uniformly bounded. Then  $\sigma'_i(t) = \Phi_i(\sigma'(t + t_i))$  subsequentially converges to a geodesic line  $\sigma$  (using Arzelà–Ascoli again).  $\square$

Parametrize the curve  $\tilde{\gamma}$  so that  $d_{\tilde{K}}(\tilde{\gamma}(a), \tilde{\gamma}(b)) = |a - b|$  (note that  $d_{\tilde{N}}(\tilde{\gamma}(a), \tilde{\gamma}(b))$  might be smaller than  $|a - b|$ ). Let  $\gamma = \tilde{\iota} \circ \tilde{\gamma}$ . Note that  $\tilde{\gamma}$  is automatically proper in  $\tilde{K}$ , and thus Lemma 15 implies that  $\gamma$  is proper in  $\tilde{N}$ .

For each  $R > 0$ , consider the open geodesic ball  $B_R(\gamma(0)) \subset \tilde{N}$ . Define parameters

$$t_-(R) = \max\{t : \gamma(-\infty, t) \cap B_R(\gamma(0)) = \emptyset\},$$

$$t_+(R) = \min\{t : \gamma(t, \infty) \cap B_R(\gamma(0)) = \emptyset\}.$$

Note that  $t_{\pm}(R) \rightarrow \pm\infty$  as  $R \rightarrow \infty$ .

Since  $e(\tilde{K}) = 1$ ,  $\gamma(t_{\pm}(R))$  can be connected in  $\tilde{K} \setminus B_R(\gamma(0))$ . Because  $\tilde{N}$  is simply connected, this implies that  $\gamma(t_{\pm}(R))$  lie in the *same* component of  $\partial B_R(\gamma(0)) \subset \tilde{N}$  and thus

$$d_{\tilde{N}}(\gamma(t_-(R)), \gamma(t_+(R))) \leq C.$$

On the other hand, we have

$$d_{\tilde{K}}(\gamma(t_-(R)), \gamma(t_+(R))) = |t_-(R) - t_+(R)| \rightarrow \infty.$$

This contradicts Lemma 16. This completes the proof of Proposition 13.  $\square$



## 4 Generalization to the mapping problem

In this section we prove [Theorem 2](#). The proof here is partly motivated by [[14](#), Section 5], where nonexistence of PSC metrics on certain noncompact manifold admitting a proper, distance-decreasing map to a uniformly contractible manifold is established. We first observe that we may assume, without loss of generality, that  $\pi_1(X)$  is infinite. Indeed, if  $\pi_1(X)$  is finite, then the universal cover  $\tilde{X}$  is compact and satisfies that  $\pi_1(\tilde{X}) = \cdots = \pi_{n-2}(\tilde{X}) = 0$ . By the Hurewicz theorem, we have that  $H_1(\tilde{X}) = \cdots = H_{n-2}(\tilde{X}) = 0$ . Poincaré duality further implies that  $H_1(\tilde{X}) = \cdots = H_{n-1}(\tilde{X}) = 0$ , and hence  $\tilde{X}$  is homeomorphic to  $S^n$ .

We begin with the following general lemma. Note that it is tempting to try to lift a map of nonzero degree to the universal covers, but this map may not be proper (and hence the degree will not be well defined). We note that the construction of the appropriate cover is somewhat analogous to the construction of  $\tilde{K}$  in [Section 3](#).

**Lemma 18** *Suppose that  $X$  and  $N$  are closed oriented manifolds and  $f: N \rightarrow X$  has nonzero degree. Letting  $\tilde{X}$  denote the universal cover of  $X$ , there exists a connected cover  $\hat{N} \rightarrow N$  and a lift  $\hat{f}: \hat{N} \rightarrow \tilde{X}$  such that  $\hat{f}$  is proper and  $\deg \hat{f} = \deg f$ .*

**Proof** Choose a regular value  $x \in X$  and set  $f^{-1}(x) = \{z_1, \dots, z_k\}$ . Consider  $H := \ker f_{\#}: \pi_1(N, z_1) \rightarrow \pi_1(X, x)$ . Choose a covering space  $p: \hat{N} \rightarrow N$  so that image  $p_{\#}: \pi_1(\hat{N}, \hat{z}_1) \rightarrow \pi_1(N, z_1)$  is  $H$ . Below we will show that the map  $f$  lifts to  $\hat{f}: \hat{N} \rightarrow \tilde{X}$  and that  $\hat{f}$  satisfies the assertions made above.

**Noncompactness of  $\hat{N}$**  We claim that  $\hat{N}$  is noncompact. We first show that the image of  $f_{\#}$  is a subgroup of  $\pi_1(X, x)$  with finite index. Let  $G = f_{\#}(\pi_1(N, z_1))$  and  $\bar{\pi}: (\bar{X}, \bar{x}) \rightarrow (X, x)$  be a covering map such that image  $((\bar{\pi})_{\#}: \pi_1(\bar{X}, \bar{x}) \rightarrow \pi_1(X, x))$  is  $G$ . The map  $f$  lifts to a map  $\bar{f}: (N, z_1) \rightarrow (\bar{X}, \bar{x})$  such that  $f = \bar{\pi} \circ \bar{f}$ . Since  $N$  is compact and  $f$  is surjective, we see that  $\bar{X}$  is compact. Hence, we have  $\deg f = \deg \bar{\pi} \cdot \deg \bar{f}$ . It follows that  $\deg \bar{\pi}$  is an integer factor of  $\deg f$ , and thus  $G$  is a subgroup of  $\pi_1(X, x)$  of finite index.

The number of sheets of the covering map  $p$  is the index of  $H = p_{\#}(\pi_1(\hat{N}, \hat{z}_1))$  in  $\pi_1(N, z_1)$ . Since  $H$  is a normal subgroup, this is equal to the number of elements of the group  $\pi_1(N, z_1)/H$ , which is isomorphic to  $G$  and thus of infinite order. This implies that  $\hat{N}$  is noncompact, as claimed.

**Lifting the map  $f$**  Consider  $f \circ p: \widehat{N} \rightarrow X$ . Note that  $(f \circ p)_\#: \pi_1(\widehat{N}) \rightarrow \pi_1(X)$  is the zero map. Thus, we can lift  $f \circ p$  to the universal cover of  $X$ :

$$\begin{array}{ccc} (\widehat{N}, \widehat{z}_1) & \xrightarrow{\widehat{f}} & (\widetilde{X}, \widetilde{x}) \\ p \downarrow & & \downarrow \pi \\ (N, z_1) & \xrightarrow{f} & (X, x) \end{array}$$

Clearly, a loop in  $N$  lifts to a loop in  $\widehat{N}$  if and only if it is in  $H$  (recall that  $H$  is normal).

**Counting lifts of preimages** We now claim that  $\#(\widehat{f}^{-1}(\widetilde{x}) \cap p^{-1}(z_j)) = 1$ . To this end, suppose that  $a, b \in \widehat{f}^{-1}(\widetilde{x}) \cap p^{-1}(z_j)$ . Choose a path  $\widehat{\gamma}$  in  $\widehat{N}$  connecting the two points. Then  $\gamma = p \circ \widehat{\gamma}$  is a loop in  $N$  based at  $z_j$ . On the other hand,  $\widetilde{\gamma} := \widehat{f} \circ \widehat{\gamma}$  is a loop in  $\widetilde{X}$  based at  $\widetilde{x}$ . Since  $e = \pi_\#[\widetilde{\gamma}] = f_\#[\gamma]$ , we thus see that  $[\gamma] \in H$ . This is a contradiction since this would imply that  $\gamma$  lifts to a loop (as remarked above).

**Properness** We now show that  $\widehat{f}$  is proper. Assume that  $\widehat{r}_i \rightarrow \infty$  in  $\widehat{N}$  but  $\widehat{f}(\widehat{r}_i) \rightarrow q$  in  $\widetilde{X}$ . Since  $N$  is compact, we can pass to a subsequence such that  $p(\widehat{r}_i) \rightarrow r \in N$ . Then  $\pi(q) = f(r)$ .

Choose a contractible neighborhood  $U \subset N$  with  $r \in U$ . By shrinking  $U$ , we can assume that  $f(U)$  is contained in a contractible open set  $W \subset X$ . Then  $\pi^{-1}(W)$  consists of disjoint copies of  $W$ . We can assume that  $\widehat{f}(\widehat{r}_i)$  are all contained in the copy containing  $q$ .

Assume that  $p(\widehat{r}_i) \in U$  for all  $i$ . Fix paths  $\eta_i$  from  $p(\widehat{r}_i)$  to  $r$  in  $U$  and paths  $\widehat{\gamma}_i$  from  $\widehat{r}_1$  to  $\widehat{r}_i$  in  $\widehat{N}$ . Then

$$\alpha_i := (\eta_i) * (p \circ \widehat{\gamma}_i) * (-\eta_i)$$

is a loop from  $r$  to  $r$ . Lift  $\alpha_i$  to  $\widehat{\alpha}_i$  a path in  $\widehat{N}$  that agrees with  $\widehat{\gamma}_i$  on that portion of  $\widehat{\alpha}_i$ . Note that  $\widehat{\alpha}_i$  cannot be a loop for  $i$  large, since the  $\widehat{r}_i$  are diverging.

We now consider  $\widetilde{\alpha}_i := \widehat{f} \circ \widehat{\alpha}_i$  a path in  $\widetilde{X}$ . By construction,  $\widetilde{\alpha}_i$  is a loop in  $\widetilde{X}$ . This is a contradiction as before.

**Degree** Finally, we check that  $\deg \widehat{f} = \deg f$ . The lift  $\widetilde{x}$  is a regular point for  $\widehat{f}$  and we have seen that each element of  $f^{-1}(x)$  lifts to a unique element of  $\widehat{f}^{-1}(\widetilde{x})$ . But the local degree of  $\widehat{f}$  at each preimage  $\widehat{z}_i$  is the same as the degree of  $f$  at the corresponding point  $p(\widehat{z}_i)$  (since  $p$  is a covering map). □

**Remark** With some trivial modifications in the proof, a similar result holds for possibly nonorientable  $X$  and  $N$  with a map  $f : N \rightarrow X$  of nonzero mod 2 degree.

Using the lifted map  $\hat{f}$  we can now follow [14, Section 5] to show that the conclusion of Corollary 7 holds in the setting of Theorem 2.

**Lemma 19** *Let  $X$  and  $N$  be oriented Riemannian manifolds and  $f : (N, g) \rightarrow (X, g_X)$  with  $f$  distance-decreasing and  $\deg f \neq 0$ . Assume that  $N$  admits a metric of positive scalar curvature and that either  $n = 4$  and  $\pi_2(X) = 0$ , or  $n = 5$  and  $\pi_2(X) = \pi_3(X) = 0$ .*

*Then there exists  $L > 0$  with the following property: if  $\Sigma_{n-2}$  is an  $(n-2)$ -dimensional nullhomologous cycle in the universal cover  $\tilde{X}$  of  $X$ , then the cycle  $\deg(f)\Sigma_{n-2}$  can be filled inside  $B_L(\Sigma_{n-2})$ .*

**Proof** We consider  $n = 5$  since the  $n = 4$  case is similar (but simpler). By scaling, we can assume that  $(N, g)$  has scalar curvature  $R \geq 1$ . As in Corollary 7,  $H_2(\tilde{X}, \mathbb{Z}) = H_3(\tilde{X}, \mathbb{Z}) = 0$ .

By assumption,  $\Sigma_3 = \partial\Sigma_4$  in  $\tilde{X}$  for some chain  $\Sigma_4$ . Up to a small perturbation, we can assume that  $\hat{f}$  is transversal to  $\Sigma_3$  and  $\Sigma_4$ . Set  $\hat{\Sigma}_4 := \hat{f}^{-1}(\Sigma_4)$  and similarly  $\hat{\Sigma}_3 = \partial\hat{\Sigma}_4$ . Note that  $\hat{\Sigma}_3$  is nullhomologous in  $\hat{N}$  (by construction). Using Theorem 5, we can find  $\hat{\Sigma}'_3 \subset B_{L_0}(\hat{\Sigma}_3)$  homologous to  $\hat{\Sigma}_3$  as well as 3-chains  $\hat{U}_1, \dots, \hat{U}_m$  with  $\text{diam}(U_j) \leq L_0$  and 2-cycles  $\{\hat{\Gamma}^l_j : j = 1, \dots, m, l = 1, \dots, k(j)\}$  with  $\text{diam}(\hat{\Gamma}^l_j) \leq L_0$  and such that

$$\hat{\Sigma}'_3 = \sum_{j=1}^m \hat{U}_j \quad \text{and} \quad \partial\hat{U}_j = \sum_{l=1}^{k(j)} \hat{\Gamma}^l_j \quad \text{for each } j = 1, \dots, m,$$

where both equalities hold as chains (not just in homology). Finally, there is an integer  $q$  and a function

$$u : \{(j, l) : j = 1, \dots, m, l = 1, \dots, k(j)\} \rightarrow \{1, \dots, q\}$$

such that, for  $r \in \{1, \dots, q\}$ , we have

$$\text{diam}\left(\bigcup_{(j,l) \in u^{-1}(r)} \hat{\Gamma}^l_j\right) \leq L_0$$

and

$$\sum_{(j,l) \in u^{-1}(r)} \hat{\Gamma}^l_j = 0$$

as 2-cycles for  $r \in \{1, \dots, q\}$ .

Denote by  $\Sigma'_3$  the 3-cycle in  $\tilde{X}$  obtained by pushing  $\hat{\Sigma}'_3$  forward by the map  $\hat{f}$  and similarly for  $U_j$  and  $\Gamma_j^l$ .

Since  $\hat{f}$  is transversal to  $M_3$ , it is easy to check that  $\deg \hat{f}|_{M_3} = \deg \hat{f}$ . Hence,  $\hat{f}_\#([\hat{\Sigma}'_3]) = (\deg \hat{f})[\Sigma_3]$ . Moreover, since  $f$  (and thus  $\hat{f}$ ) was assumed to be distance-decreasing, we see that  $d_{(\tilde{X}, g_{\tilde{X}})}(\Sigma_3, \Sigma'_3) \leq L_0$ . As such, it suffices to bound  $\Sigma'_3$  in a controlled neighborhood.

To do so, we follow the argument used in [Corollary 7](#). Because  $\text{diam}(\hat{\Gamma}_j^l) \leq L_0$ , we can use [Proposition 6](#) to find a 3-chain  $\tilde{\Gamma}_j^l$  with  $\text{diam}(\tilde{\Gamma}_j^l) \leq R(L_0)$  and  $\partial \tilde{\Gamma}_j^l = \Gamma_j^l$  and then a 4-chain  $\tilde{U}_j$  with

$$\partial \tilde{U}_j = U_j - \sum_{l=1}^{k(j)} \tilde{\Gamma}_j^l$$

and  $\text{diam}(\tilde{U}_j) \leq R(L_0 + 2R(L_0))$ . Thus,

$$\Sigma'_3 = \sum_{j=1}^m \partial \tilde{U}_j + \sum_{r=1}^q \sum_{(j,l) \in u^{-1}(r)} \tilde{\Gamma}_j^l.$$

and

$$\text{diam}\left(\sum_{(j,l) \in u^{-1}(r)} \tilde{\Gamma}_j^l\right) \leq 2R(L_0) + L_0.$$

We can thus complete the proof as in [Corollary 7](#). □

Granted [Lemma 19](#), [Theorem 2](#) follows. Indeed, in order to prove the Urysohn width estimate of [Proposition 8](#), it is enough to assume that the filling radius estimate holds for a multiple  $\text{deg}(f)\Sigma_{n-2}$  of every cycle  $\Sigma_{n-2}$ . The rest of the proof of [Theorem 2](#) proceeds exactly as the proof of [Theorem 1](#).

## References

- [1] **H Alpert, A Balitskiy, L Guth**, *Macroscopic scalar curvature and codimension 2 width*, preprint (2021) [arXiv 2112.04594](#)
- [2] **D Bolotov**, *About the macroscopic dimension of certain PSC-manifolds*, *Algebr. Geom. Topol.* 9 (2009) 21–27 [MR](#) [Zbl](#)
- [3] **D Bolotov, A Dranishnikov**, *On Gromov’s scalar curvature conjecture*, *Proc. Amer. Math. Soc.* 138 (2010) 1517–1524 [MR](#) [Zbl](#)
- [4] **S Brendle**, *Ricci flow with surgery on manifolds with positive isotropic curvature*, *Ann. of Math.* 190 (2019) 465–559 [MR](#) [Zbl](#)

- [5] **S Brendle, R Schoen**, *Manifolds with  $1/4$ -pinched curvature are space forms*, J. Amer. Math. Soc. 22 (2009) 287–307 [MR](#) [Zbl](#)
- [6] **B-L Chen, S-H Tang, X-P Zhu**, *Complete classification of compact four-manifolds with positive isotropic curvature*, J. Differential Geom. 91 (2012) 41–80 [MR](#) [Zbl](#)
- [7] **O Chodosh, C Li**, *Generalized soap bubbles and the topology of manifolds with positive scalar curvature*, preprint (2020) [arXiv 2008.11888](#)
- [8] **MJ Dunwoody**, *The accessibility of finitely presented groups*, Invent. Math. 81 (1985) 449–457 [MR](#) [Zbl](#)
- [9] **DBA Epstein**, *Ends*, from “Topology of 3-manifolds and related topics” (MK Fort, Jr, editor), Prentice-Hall, Englewood Cliffs, NJ (1962) 110–117 [MR](#) [Zbl](#)
- [10] **AM Fraser**, *Fundamental groups of manifolds with positive isotropic curvature*, Ann. of Math. 158 (2003) 345–354 [MR](#) [Zbl](#)
- [11] **MH Freedman**, *The topology of four-dimensional manifolds*, J. Differential Geometry 17 (1982) 357–453 [MR](#) [Zbl](#)
- [12] **S Gadgil, H Seshadri**, *On the topology of manifolds with positive isotropic curvature*, Proc. Amer. Math. Soc. 137 (2009) 1807–1811 [MR](#) [Zbl](#)
- [13] **M Gromov**, *Positive curvature, macroscopic dimension, spectral gaps and higher signatures*, from “Functional analysis on the eve of the 21st century, II” (S Gindikin, J Lepowsky, R L Wilson, editors), Progr. Math. 132, Birkhäuser, Boston, MA (1996) 1–213 [MR](#) [Zbl](#)
- [14] **M Gromov**, *No metrics with positive scalar curvatures on aspherical 5-manifolds*, preprint (2020) [arXiv 2009.05332](#)
- [15] **M Gromov**, *Four lectures on scalar curvature*, preprint (2021) [arXiv 1908.10612v5](#)
- [16] **M Gromov, HB Lawson, Jr**, *Spin and scalar curvature in the presence of a fundamental group, I*, Ann. of Math. 111 (1980) 209–230 [MR](#) [Zbl](#)
- [17] **M Gromov, HB Lawson, Jr**, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. 58 (1983) 83–196 [MR](#) [Zbl](#)
- [18] **RS Hamilton**, *Four-manifolds with positive isotropic curvature*, Comm. Anal. Geom. 5 (1997) 1–92 [MR](#) [Zbl](#)
- [19] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) [MR](#) [Zbl](#)
- [20] **H Huang**, *Compact manifolds of dimension  $n \geq 12$  with positive isotropic curvature*, preprint (2019) [arXiv 1909.12265](#)
- [21] **M Katz**, *The first diameter of 3-manifolds of positive scalar curvature*, Proc. Amer. Math. Soc. 104 (1988) 591–595 [MR](#) [Zbl](#)
- [22] **M Kreck, W Lück**, *Topological rigidity for non-aspherical manifolds*, Pure Appl. Math. Q. 5 (2009) 873–914 [MR](#) [Zbl](#)

- [23] **Y Liokumovich, D Maximo**, *Waist inequality for 3-manifolds with positive scalar curvature*, preprint (2020) [arXiv 2012.12478](#)
- [24] **F C Marques, A Neves**, *Rigidity of min-max minimal spheres in three-manifolds*, *Duke Math. J.* 161 (2012) 2725–2752 [MR](#) [Zbl](#)
- [25] **M J Micallef, J D Moore**, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, *Ann. of Math.* 127 (1988) 199–227 [MR](#) [Zbl](#)
- [26] **J Milnor**, *Lectures on the h-cobordism theorem*, Princeton Univ. Press (1965) [MR](#) [Zbl](#)
- [27] **H T Nguyen**, *Isotropic curvature and the Ricci flow*, *Int. Math. Res. Not.* 2010 (2010) 536–558 [MR](#) [Zbl](#)
- [28] **M Ramachandran, J Wolfson**, *Fill radius and the fundamental group*, *J. Topol. Anal.* 2 (2010) 99–107 [MR](#) [Zbl](#)
- [29] **R Schoen**, *Manifolds with pointwise 1/4-pinched curvature*, preprint (2007)
- [30] **P Scott, T Wall**, *Topological methods in group theory*, from “Homological group theory” (C T C Wall, editor), London Math. Soc. Lecture Note Ser. 36, Cambridge Univ. Press (1979) 137–203 [MR](#) [Zbl](#)
- [31] **J-P Serre**, *Trees*, Springer (2003) [MR](#) [Zbl](#)

*Department of Mathematics, Stanford University*  
*Stanford, CA, United States*

*Courant Institute of Mathematical Sciences, New York University*  
*New York, NY, United States*

*Department of Mathematics, University of Toronto*  
*Toronto ON, Canada*

[ochodosh@stanford.edu](mailto:ochodosh@stanford.edu), [chaoli@nyu.edu](mailto:chaoli@nyu.edu), [ylio@math.toronto.edu](mailto:ylio@math.toronto.edu)

Proposed: John Lott

Seconded: Urs Lang, Gang Tian

Received: 21 June 2021

Revised: 13 September 2021

# GEOMETRY & TOPOLOGY

[msp.org/gt](http://msp.org/gt)

## MANAGING EDITOR

András I. Stipsicz    Alfréd Rényi Institute of Mathematics  
[stipsicz@renyi.hu](mailto:stipsicz@renyi.hu)

## BOARD OF EDITORS

Dan Abramovich	Brown University <a href="mailto:dan_abramovich@brown.edu">dan_abramovich@brown.edu</a>	Mark Gross	University of Cambridge <a href="mailto:mgross@dpmms.cam.ac.uk">mgross@dpmms.cam.ac.uk</a>
Ian Agol	University of California, Berkeley <a href="mailto:ianagol@math.berkeley.edu">ianagol@math.berkeley.edu</a>	Rob Kirby	University of California, Berkeley <a href="mailto:kirby@math.berkeley.edu">kirby@math.berkeley.edu</a>
Mark Behrens	Massachusetts Institute of Technology <a href="mailto:mbehrens@math.mit.edu">mbehrens@math.mit.edu</a>	Frances Kirwan	University of Oxford <a href="mailto:frances.kirwan@balliol.oxford.ac.uk">frances.kirwan@balliol.oxford.ac.uk</a>
Mladen Bestvina	Imperial College, London <a href="mailto:bestvina@math.utah.edu">bestvina@math.utah.edu</a>	Bruce Kleiner	NYU, Courant Institute <a href="mailto:bkleiner@cims.nyu.edu">bkleiner@cims.nyu.edu</a>
Martin R. Bridson	Imperial College, London <a href="mailto:m.bridson@ic.ac.uk">m.bridson@ic.ac.uk</a>	Urs Lang	ETH Zürich <a href="mailto:urs.lang@math.ethz.ch">urs.lang@math.ethz.ch</a>
Jim Bryan	University of British Columbia <a href="mailto:jbryan@math.ubc.ca">jbryan@math.ubc.ca</a>	Marc Levine	Universität Duisburg-Essen <a href="mailto:marc.levine@uni-due.de">marc.levine@uni-due.de</a>
Dmitri Burago	Pennsylvania State University <a href="mailto:burago@math.psu.edu">burago@math.psu.edu</a>	John Lott	University of California, Berkeley <a href="mailto:lott@math.berkeley.edu">lott@math.berkeley.edu</a>
Ralph Cohen	Stanford University <a href="mailto:ralph@math.stanford.edu">ralph@math.stanford.edu</a>	Ciprian Manolescu	University of California, Los Angeles <a href="mailto:cm@math.ucla.edu">cm@math.ucla.edu</a>
Tobias H. Colding	Massachusetts Institute of Technology <a href="mailto:colding@math.mit.edu">colding@math.mit.edu</a>	Haynes Miller	Massachusetts Institute of Technology <a href="mailto:hrm@math.mit.edu">hrm@math.mit.edu</a>
Simon Donaldson	Imperial College, London <a href="mailto:s.donaldson@ic.ac.uk">s.donaldson@ic.ac.uk</a>	Tom Mrowka	Massachusetts Institute of Technology <a href="mailto:mrowka@math.mit.edu">mrowka@math.mit.edu</a>
Yasha Eliashberg	Stanford University <a href="mailto:eliash-gt@math.stanford.edu">eliash-gt@math.stanford.edu</a>	Walter Neumann	Columbia University <a href="mailto:neumann@math.columbia.edu">neumann@math.columbia.edu</a>
Benson Farb	University of Chicago <a href="mailto:farb@math.uchicago.edu">farb@math.uchicago.edu</a>	Jean-Pierre Otal	Université d'Orleans <a href="mailto:jean-pierre.otal@univ-orleans.fr">jean-pierre.otal@univ-orleans.fr</a>
Steve Ferry	Rutgers University <a href="mailto:sferry@math.rutgers.edu">sferry@math.rutgers.edu</a>	Peter Ozsváth	Columbia University <a href="mailto:ozsvath@math.columbia.edu">ozsvath@math.columbia.edu</a>
Ron Fintushel	Michigan State University <a href="mailto:ronfint@math.msu.edu">ronfint@math.msu.edu</a>	Leonid Polterovich	Tel Aviv University <a href="mailto:polterov@post.tau.ac.il">polterov@post.tau.ac.il</a>
David M. Fisher	Rice University <a href="mailto:davidfisher@rice.edu">davidfisher@rice.edu</a>	Colin Rourke	University of Warwick <a href="mailto:gt@maths.warwick.ac.uk">gt@maths.warwick.ac.uk</a>
Mike Freedman	Microsoft Research <a href="mailto:michaelf@microsoft.com">michaelf@microsoft.com</a>	Stefan Schwede	Universität Bonn <a href="mailto:schwede@math.uni-bonn.de">schwede@math.uni-bonn.de</a>
David Gabai	Princeton University <a href="mailto:gabai@princeton.edu">gabai@princeton.edu</a>	Peter Teichner	University of California, Berkeley <a href="mailto:teichner@math.berkeley.edu">teichner@math.berkeley.edu</a>
Stavros Garoufalidis	Southern U. of Sci. and Tech., China <a href="mailto:stavros@mpim-bonn.mpg.de">stavros@mpim-bonn.mpg.de</a>	Richard P. Thomas	Imperial College, London <a href="mailto:richard.thomas@imperial.ac.uk">richard.thomas@imperial.ac.uk</a>
Cameron Gordon	University of Texas <a href="mailto:gordon@math.utexas.edu">gordon@math.utexas.edu</a>	Gang Tian	Massachusetts Institute of Technology <a href="mailto:tian@math.mit.edu">tian@math.mit.edu</a>
Lothar Götsche	Abdus Salam Int. Centre for Th. Physics <a href="mailto:gotsche@ictp.trieste.it">gotsche@ictp.trieste.it</a>	Ulrike Tillmann	Oxford University <a href="mailto:tillmann@maths.ox.ac.uk">tillmann@maths.ox.ac.uk</a>
Jesper Grodal	University of Copenhagen <a href="mailto:jg@math.ku.dk">jg@math.ku.dk</a>	Nathalie Wahl	University of Copenhagen <a href="mailto:wahl@math.ku.dk">wahl@math.ku.dk</a>
Misha Gromov	IHÉS and NYU, Courant Institute <a href="mailto:gromov@ihes.fr">gromov@ihes.fr</a>	Anna Wienhard	Universität Heidelberg <a href="mailto:wienhard@mathi.uni-heidelberg.de">wienhard@mathi.uni-heidelberg.de</a>

See inside back cover or [msp.org/gt](http://msp.org/gt) for submission instructions.

The subscription price for 2023 is US \$740/year for the electronic version, and \$1030/year (+ \$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [Current Mathematical Publications](#) and the [Science Citation Index](#).

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW<sup>®</sup> from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing  
<http://msp.org/>

© 2023 Mathematical Sciences Publishers

# GEOMETRY & TOPOLOGY

Volume 27    Issue 4 (pages 1273–1655)    2023

---

- Floer theory and reduced cohomology on open manifolds 1273  
YOEL GROMAN
- Geodesic coordinates for the pressure metric at the Fuchsian locus 1391  
XIAN DAI
- Birational geometry of the intermediate Jacobian fibration of a cubic fourfold 1479  
GIULIA SACCÀ
- Cohomological  $\chi$ -independence for moduli of one-dimensional sheaves and moduli of Higgs bundles 1539  
DAVESH MAULIK and JUNLIANG SHEN
- Coarse injectivity, hierarchical hyperbolicity and semihyperbolicity 1587  
THOMAS HAETTEL, NIMA HODA and HARRY PETYT
- Classifying sufficiently connected PSC manifolds in 4 and 5 dimensions 1635  
OTIS CHODOSH, CHAO LI and YEVGENY LIOKUMOVICH