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A NOTE ON THE MATRIX EXPONENTIAL*

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Abstract. This note is devoted to simplifying one method to calculate the exponential of a matrix presented by I.E. Leonard in [*SIAM Rev.*, 38 (1996), pp. 507–512].

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1. Introduction.

In [1] one method to calculate the exponential of a matrix is presented with the goal of minimizing the mathematical prerequisites. Such a method follows from Theorem 2 in [1], and it requires to solve, for an $n \times n$ matrix, n initial value problems for an nth order linear differential equation with constant coefficients.

In this note we present a result which is deduced easily from the above mentioned theorem and makes the practical method to calculate the exponential of a matrix simpler. We give an example to illustrate our result.

2. Main result.

For the sake of completeness, we enunciate here the key result of [1, Theorem 2, p. 509].

THEOREM 2.1. *Let A be a constant $n \times n$ matrix with characteristic polynomial*

$$p(\lambda) = \lambda^n + c_{n-1}\,\lambda^{n-1} + \cdots + c_1\,\lambda + c_0\,.$$

Then

$$e^{At} = x_1(t)I + x_2(t)A + \cdots + x_n(t)A^{n-1}\,,$$

where, for each $k = 1, 2, \ldots, n$, x_k is the solution to the nth order scalar differential equation

$$x^{(n)} + c_{n-1}\,x^{(n-1)} + \cdots + c_1\,x' + c_0\,x = 0\,, \tag{1}$$

satisfying the following initial conditions:

$$x_k^{(k-1)}(0) = 1,\ x_k^{(i)}(0) = 0 \ \text{ for } \ i \neq k-1,\ 0 \leq i \leq n-1\,. \tag{2}$$

Before enunciating our result, we recall that the homogeneous differential equation (1) always has a fundamental system of exactly n solutions

$$S = \{\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)\}\,;$$

that is, S is a basis for the linear space of all the solutions of (1).

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Moreover, the matrix

$$B_t = \begin{pmatrix} \varphi_1(t) & \varphi_1'(t) & \cdots & \varphi_1^{(n-1)}(t) \\ \varphi_2(t) & \varphi_2'(t) & \cdots & \varphi_2^{(n-1)}(t) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_n(t) & \varphi_n'(t) & \cdots & \varphi_n^{(n-1)}(t) \end{pmatrix} \tag{3}$$

is invertible for every $t \in \mathbf{R}$.

We also recall that the *characteristic equation* of (1) is the algebraic equation

$$\lambda^n + c_{n-1}\,\lambda^{n-1} + \cdots + c_1\,\lambda + c_0 = 0\,.$$

Taking into account these preliminaries, we may prove the following result.

THEOREM 2.2. *Let A be a constant $n \times n$ matrix with characteristic polynomial $p(\lambda)$. Then*

$$e^{At} = x_1(t)I + x_2(t)A + \cdots + x_n(t)A^{n-1}\,,$$

where

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = B_0^{-1} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \varphi_n(t) \end{pmatrix}, \tag{4}$$

B_0 being the matrix defined in (3) *(for $t = 0$) and $S = \{\varphi_1(t),\, \varphi_2(t), \ldots, \varphi_n(t)\}$ being a fundamental system of solutions for the homogeneous linear differential equation whose characteristic equation is the characteristic equation of A, $p(\lambda) = 0$.*

Proof. Let $p(\lambda) = \lambda^n + c_{n-1},\, \lambda^{n-1} + \cdots + c_1,\, \lambda + c_0$ be the characteristic polynomial of A, and let us consider the nth order differential equation (1), whose characteristic equation is $p(\lambda) = 0$.

In view of Theorem 2.1,

$$e^{At} = x_1(t)I + x_2(t)A + \cdots + x_n(t)A^{n-1}\,,$$

where $x_k(t)$ is the solution of (1) with initial conditions (2) for each $k = 1, 2, \ldots, n$.

Now, let us observe that the set $T = \{x_1(t), x_2(t), \ldots, x_n(t)\}$ is also a fundamental set of solutions for (1) since the Wronskian $W(x_1(t), \ldots x_n(t))$ takes the value 1 at $t = 0$. Moreover, by using the uniqueness of solution for the initial value problem (1)–(2), we obtain that, for each $k = 1, 2, \ldots, n$,

$$\varphi_k(t) = \varphi_k(0)\,x_1(t) + \varphi_k'(0)\,x_2(t) + \cdots + \varphi_k^{(n-1)}(0)\,x_n(t)\,;$$

that is,

$$\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \varphi_n(t) \end{pmatrix} = B_0 \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}.$$

Since B_0 is invertible, we obtain the equality (4) and the proof is complete. ∎

3. Example. Let us consider the 3×3 real matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of A is $p(\lambda) = (\lambda^2+1)(\lambda-1)$, and hence a fundamental system for the differential equation whose characteristic equation is $p(\lambda) = 0$ is given by

$$S = \{\cos t, \sin t, e^t\}.$$

The matrices B_0 and its inverse are

$$B_0 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad B_0^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = B_0^{-1}\begin{pmatrix} \cos t \\ \sin t \\ e^t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \cos t - \sin t + e^t \\ 2\sin t \\ -\cos t - \sin t + e^t \end{pmatrix}.$$

Finally, taking into account that

$$A^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain

$$\begin{aligned} e^{At} &= x_1(t)I + x_2(t)A + x_3(t)A^2 \\ &= \begin{pmatrix} x_1(t) - x_3(t) & x_2(t) & 0 \\ -x_2(t) & x_1(t) - x_3(t) & 0 \\ 0 & 0 & x_1(t) + x_2(t) + x_3(t) \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^t \end{pmatrix}. \end{aligned}$$

REFERENCE

[1] I. E. LEONARD, *The matrix exponential*, SIAM Rev., 38 (1996), pp. 507–512.