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Clean Semiprime *f*-Rings with Bounded Inversion

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ABSTRACT

An element in a ring is called *clean* if it may be written as a sum of a unit and idempotent. The ring itself is called *clean* if every element is clean. Recently, Anderson and Camillo (Anderson, D. D., Camillo, V. (2002). Commutative rings whose elements are a sum of a unit and an idempotent. *Comm. Algebra* 30(7):3327–3336) has shown that for commutative rings every von-Neumann regular ring as well as zero-dimensional rings are clean. Moreover, every clean ring is a pm-ring, that is every prime ideal is contained in a unique maximal ideal. In the same article, the authors give an example of a commutative ring which is a pm-ring yet not clean, e.g., $C(\mathbb{R})$. It is this example which interests us. Our discussion shall take place in a more general setting. We assume that all rings are commutative with 1.

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One of the main results is the following:

Theorem. *For a Tychonoff space X, the ring $C(X)$ of continuous real-valued functions on X is a clean ring if and only if X is strongly zero-dimensional.*

The more general setting hinted at above is that of semiprime *f*-rings with bounded inversion. In general, most results concerning this class of rings involve a concept that involves the lattice structure. As we see above the idea of a clean ring is completely independent of this extra structure. We shall lay the groundwork here but for additional information we suggest the reader check Darnel (1995) or the classic reference Bigard et al. (1977).

1. SEMIPRIME *f*-RINGS WITH BOUNDED INVERSION

Definition 1. *A* lattice-ordered ring *is a ring A equipped with a lattice order $\leq$ such that translation preserves the order and such that if $a, b \geq 0$, then $ab \geq 0$. If A satisfies the additional property that if $a \wedge b = 0$ and $c \geq 0$ then $ca \wedge b = 0$ we say A is an* f-ring*. It is known that if one assumes the Axiom of Choice or more generally the Boolean Prime Ideal Theorem, then the condition of being an f-ring is equivalent to being ℓ-isomorphic to an ℓ-subring of a direct product of totally-ordered rings (with coordinate-wise operations.) It is customary to assume this condition as the definition of an f-ring. We shall simply assume the Axiom of Choice.*

It is known that in an f-ring, squares, and hence 1*, are positive, and orthogonal elements annihilate each other. If A has the property that for any positive elements a, b if $na \leq b$ for all $n \in \mathbb{N}$ then it follows that $a = 0$, we call A an* archimedean *f*-ring*. It is known that archimedean f-rings are semiprime (that is, they have no nonzero nilpotent elements) and hence may be embedded into a direct product of totally-ordered integral domains.*

Max (A) *is the set of maximal ideals of A equipped with the hull-kernel topology. This means that sets of the form $U(a) = \{M : a \notin M\}$ for some $a \in A$ form a base for the open sets of the topology. Recall that a topological* *space X is called* zero-dimensional *if it has a base of clopen sets. If the space X in question is compact then X is zero-dimensional if and only if it is totally disconnected.*

Just for the record

Definition 2. *An f-ring A is said to have the* bounded inversion property *if u is a unit whenever* $u \geq 1$.

The next lemma shall be useful.

Lemma 3 (Henriksen et al., 1961**).** *Let A be a semi-prime f-ring. Then A has bounded inversion if and only if each maximal ideal is a convex* ℓ*-subgroup. In this case,* $\mathrm{Max}(A)$ *is a compact, Hausdorff space.*

What we can immediately derive from the lemma is that for any $a \in A$, $U(a) = U(|a|)$. Thus, we need only consider positive elements when considering basic open sets of $\mathrm{Max}(A)$. Also, observe that if $f, a \in A$ then we may define a subset of $\mathrm{Max}(A)$ by setting

$$f^{-1}(a) = \{M \in \mathrm{Max}(A) : f + M = a + M\}.$$

When $a = 0$, we instead denote this set by $V(f)$. These are the complements of $U(f)$ and so form a base for the topology of closed sets on $\mathrm{Max}(A)$. Since we will dealing with zero-dimensional spaces it is helpful to know when a subset of $\mathrm{Max}(A)$ is clopen. We recite without proof.

Lemma 4 (Woodward, 1993**).** *Let A be a semi-prime f-ring with bounded inversion. A set* $K \subseteq \mathrm{Max}(A)$ *is clopen if and only if* $K = U(e)$ *for some idempotent* $e \in A$.

Definition 5. *To each ring A recall that*

$$A^* = \{a \in A : |a| \leq n \quad \textit{for some } n \in \mathbb{N}\}$$

is the bounded *subring of A.* A^* *is precisely the convex subring generated by* 1. *If A has bounded inversion then so does* A^*. *Furthermore, in* Woodward (1993) *Corollary* 2.1*, it is shown that whenever A is a semiprime f-ring with bounded inversion then* $\mathrm{Max}(A) \cong \mathrm{Max}(A^*)$.

Theorem 6. *Let A be a semi-prime f-ring with bounded inversion. The following are equivalent*:

(i) $\mathrm{Max}(A)$ *is zero-dimensional.*
(ii) *A is clean.*

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(iii) A^* *is clean.*
(iv) *Every element of* A *may be written as a sum of unit and a square root of* 1, (*that is,* $t^2 = 1$).
(v) *For every pair of relatively prime positive elements* $a, b \in A$ *there is a* $t \in A$ *such that* $a - bt^2$ *is a unit.*

Proof. First observe that once we show that (i) and (ii) are equivalent since $\mathrm{Max}(A) \cong \mathrm{Max}(A^*)$ it follows that the first three are equivalent. That (ii) and (iv) are equivalent follows from the fact that bounded inversion implies the hypothesis of Proposition 10 (Camillo and Yu, 1994).

(i) implies (ii). Let $f \in A$ and set $Z_1 = V(f-1)$. If $Z_1 = \emptyset$, then $f-1$ is contained in no maximal ideal, and hence is a unit. Therefore $f = (f-1) + 1$ is clean. So without loss of generality we assume that Z_1 is nonempty. Set $Z_2 = V(f)$. These are disjoint closed sets in $\mathrm{Max}(A)$. Our hypothesis implies that we may find a clopen subset $K \subseteq \mathrm{Max}(A)$ such that $Z_2 \subseteq K$ and $K \cap Z_1 = \emptyset$. By Lemma 4, we may choose an idempotent e such that $U(e) = K$. Then define

$$g = e(f-1) \qquad h = (1-e)f$$

and set $u = g + h$. Observe that

$$u + e = g + h + e = ef - e + f - ef + e = f;$$

thus it suffices to show that u is a unit. But observe that $Z(u) = \emptyset$. To see this let $M \in \mathrm{Max}(A)$. If $M \in K$, then $1 - e \in M$. Now, $h(M) = 0$ and $g(M) = f(M) - 1 \neq 0$ as $M \notin Z_1$. So $u(M) = g(M) \neq 0$. Similarly, if $M \notin K$, then $e \in M$ and so $g(M) = 0$ and $h(M) = f(M) \neq 0$ as $M \notin Z_2$. Again, $u(M) = h(M) \neq 0$. Thus, u belongs to no maximal ideal of A, Whence u is a unit.

(ii) implies (i). Let $M \in U(a)$. Since $U(a) = U(|\mathrm{a}|) = U(|a| \wedge 1)$ without loss generality we may assume that $0 \leq a \leq 1$. Since A/M is a totally ordered field and $a \notin M$, there is an $0 \leq f \in A$ such that $af + M = 1 + M$. Observe that $M \in U(fa) = U(f) \cap U(a) \subseteq U(a)$. Thus, we may further impose on a that $a + M = 1 + M$. Now, by (ii) there is a unit $u \in A$ and an idempotent $e \in A$ such that $a = u + e$. Notice that if $e \notin M$ then

$$\begin{aligned} 1 + Ma + M &= (u + e) + M \\ &= (u + M) + e + M = (u + M) + (1 + M) \end{aligned}$$

and so $u \in M$ contradicting the fact that u is a unit. Thus, $e \in M$. Since e is idempotent it follows that $1 - e \notin M$, i.e., $M \in U(1-e)$ where $U(1-e)$ is a clopen set. Furthermore, if $N \in U(1-e)$, then $e \in N$ and so $a \notin N$. Thus, $M \in U(1-e) \subseteq U(a)$. It follows that $\mathrm{Max}(A)$ is zero-dimensional.

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That (i) and (v) are equivalent is shown in Theorem 4.1 (Woodward, 1993). □

We end this section with a remark. In Woodward (1993) it is shown that for semiprime *f*-rings with bounded inversion the property of being local-global implies that Max(A) is zero-dimensional and hence clean. When A is furthermore assumed to be archimedean then all 3 conditions are equivalent. To date there are no examples showing that they differ.

2. $C(X)$

Definition 7. *X denotes a topological space. Throughout, we assume that X is Tychonoff, that is, it is Hausdorff and completely regular. Recall that a space X is called completely regular if whenever $V \subseteq X$ is closed and $x \notin V$ then there exists an $f \in C(X)$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in V$. $C(X)$ is the set of continuous real-valued functions with domain X. $C(X)$ is an archimedean f-ring with bounded inversion. $C^*(X)$ denotes the subring of bounded continuous functions on X. To any given Tychonoff space X, we let βX denote its Stone-Čech compactification. It is known that* Max($C(X)$) *and βX are homeomorphic in a natural way. It follows that* Max($C(X)$) $\cong$ Max*($C(X)$).

In general, there are examples of spaces with the property that X is zero-dimensional yet βX is not (see the exercises of Chapter 16 (Gillman and Jerrison, 1960)). *If X has the property that βX is zero-dimensional, then we call X* strongly zero-dimensional. (*It is necessary that X be zero-dimensional for it to be strongly zero-dimensional.*) *Most zero-dimensional spaces are strongly zero-dimensional; e.g., Lindelöff and basically disconnected spaces (the Stone duals of σ-complete boolean algebras). It should be noted for the nonexpert that the definition of zero-dimensional in* Gillman and Jerrison (1960) *is precisely what we are now calling strongly zero-dimensional. Our main reference for rings of continuous functions is* Gillman and Jerrison (1960)*, while our main reference for zero-dimensional spaces is* Porter and Woods (1988).

We now recall some needed terminology.

Definition 8. *Subsets of X of the form*

$$Z(f) = \{x \in X : f(x) = 0\} \quad \textit{or} \quad coz(f) = \{x \in X : f(x) \neq 0\}$$

for some $f \in C(X)$ are called zerosets and cozerosets, respectively. Observe that each zeroset is closed and each cozeroset is open. The set on which f is nonnegative is symbolized by pos(f), and neg(f) is analogously

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defined. Obviously, $coz(f) = pos(f) \cup neg(f)$*. A characteristic function, i.e., a* $\{0,1\}$*-valued continuous function, defined on a clopen set* K *shall be denoted by* χ_K*.*

Lemma 9. *Given disjoint zero-sets* Z_1, Z_2 *there exists a* $g \in C(X)$ *such that* $Z_1 = g^{-1}\{1\}$ *and* $Z_2 = g^{-1}\{-1\}$*.*

Theorem 10 (4.7(g) Porter and Woods, 1988)**.** *Let X be a zero-dimensional space. Then X is strongly zero-dimensional if and only if for every pair of disjoint zerosets there is a clopen set separating them.*

In proving the main theorem we are able to gather several characterizations of cleanliness in the class of $C(X)$s.

Definition 11. *Let A be a commutative ring with identity.* $U(A)$ *and* $Id(A)$ *denote the set of units and idempotents of A, respectively. An element* $x \in A$ *is said to be a* root of an idempotent *if* $x_n \in Id(R)$ *for some natural n. Clearly, in a semiprime f-ring a root of an* idempotent *is a square root of an idempotent. We call an element in a ring* almost clean *if it may be written as the sum of a regular element (an element which is not a zero divisor) and an idempotent. It is obvious that a clean ring is almost clean.*

The proof of the following is patent. There is also a separate formulation involving cozerosets.

Proposition 12. *Let X be a Tychonoff space and* $f \in C(X)$*. Then*

(1) *f is a unit if and only if* $Z(f) = \emptyset$*.*
(2) *f is a regular element if and only if* $Z(f)$ *is nowhere dense.*
(3) *f is an idempotent if and only if* $f = \chi_K$ *for some clopen* $K \subseteq X$*.*
(4) *f is a root of an idempotent if and only if f is* {-1, 0, 1}*-valued.*

Theorem 13. *Let X be a Tychonoff space. The following statements are equivalent:*

(i) $C(X)$ *is clean.*
(ii) $C^*(X)$ *is clean.*
(iii) *X is strongly zero-dimensional.*
(iv) *Every element of* $C(X)$ *may be written as the sum of a unit and a root of an idempotent.*
(v) $C(X)$ *is almost clean.*

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(vi) *$C^*(X)$ is almost clean.*
(vii) *For every pair of disjoint zerosets Z_1, Z_2 there exists a clopen set K such that $K \cap Z_1$ and $(X-K) \cap Z_2$ are nowhere dense.*

Proof. That (i), (ii), (iii), and (iv) are equivalent is Theorem 6.

We now show that (v), (vi), and (vii) are equivalent. Clearly, (v) implies (vi) and so we prove (vi) implies (vii). Let Z_1, Z_2 be disjoint zerosets and choose $f \in C^*(X)$ such that $f^{-1}\{1\} = Z_1$ and $Z(f) = Z_2$. Write $f = r + e$ where e is an idempotent and r is a regular element. Let $K = coz(e)$ a clopen set. Observe that on K, $f = r + 1$ and that $K \cap Z_1 \subseteq Z(r)$. Similarly, on $X-K, f = r$ and so $(X-K) \cap Z_2 \subseteq Z(r)$. Since r is regular it follows that these sets are nowhere dense. Thus, (vi) implies (vii).

(vii) implies (v). Let $f \in C(X)$ and choose a clopen set K such that $K \cap Z(f-1)$, $(X-K) \cap Z(f)$ are nowhere dense. Define r to be equal to $f-1$ on K and equal to f on $X-K$. Then $f = r + e$. Since K is clopen we get that $r \in C(X)$. Also, $Z(r) \subseteq (K \cap Z(f-1)) \cup (X-K) \cap Z(f)$ which is nowhere dense, whence r is regular and $C(X)$ is almost clean.

To finish off the proof we need only demonstrate that (vi) implies (iii). To this end suppose $C^*(X)$ is almost clean. Thus, $C(\beta X)$ is almost clean so that βX has property (vii). Without loss of generality we assume that X is compact. Recall that for compact spaces zero-dimensionality and total disconnectedness are the same property and so if X is not zero-dimensional, then there are distinct $x, y \in X$ which lie in the same connected component. Choose disjoint zerosets Z_1, Z_2 such that $x \in Z_1$, $y \in Z_2$. Moreover, we may assume that each point is in the interior of the chosen zeroset. By (vii) we may find a clopen set K for which $K \cap Z_1$ and $(X-K) \cap Z_2$ are nowhere dense. Since K is clopen either $x, y \in K$ or $x, y \notin K$. In the first case we obtain that y lies in the interior of $(X-K) \cap Z_2$ and in the second case we have that x is in the interior of $K \cap Z_1$. Either way we derive a contradiction. Thus, X is zero-dimensional. □

Remark 14. The almost clean property deals with writing an element as a sum of a regular element and an idempotent. Thus it is natural to question when $C(X)$ has the property that every element may be written as a product of a regular element and an idempotent. This happens precisely when every principal ideal is projective; in Endo (1960) one direction is proved. We supply the other direction below. Thus, every element of $C(X)$ would satisfy this property if and only if every principal ideal is projective; a so-called *p.p. ring*. In Theorem 8.4.4 of Glaz (1989), the spaces X for which $C(X)$ is a p.p. ring are characterized as basically disconnected spaces; a theorem due (independently) to Neville (1990), Brookshear (1978), and De Marco (1983).

Proposition 15 (Endo, 1960)**.** *Every element of a commutative ring R may be written as a product of a regular element and an idempotent if and only if R is a p.p. ring.*

Proof. In Endo (1960), the author showed the sufficiency. Let $x \in R$. Write $x = re$ where r is regular and e is idempotent. Clearly, the ideal (e) is projective. Define a map $\varphi : (e) \to (x)$ by $\varphi(se) = sx$. Since r is regular this is well defined and φ is a surjective R-module homomorphism. The kernel of this map is simply $\{se : sx = 0\} = \{se : sre = 0\} = \{0\}$ again by regularity of r. Thus, (x) being isomorphic to a projective module is projective itself, and therefore R is a p.p. ring. □

Proposition 16. *Suppose R is a p.p. ring. Then R is almost clean.*

Proof. Let $x \in R$ and write $x = re$ where r is regular and e is idempotent. Let $v = re - (1 - e)$ and $f = 1 - e$ then $x = v + f$. As e is idempotent so is f. Thus, all we need demonstrate is that v is regular. Suppose $sv = 0$. Then we obtain that $sre = s(1 - e)$, whence $sre = 0 = s(1 - e)$. Since r is regular it follows that $se = 0$. Now, $s \in \mathrm{Ann}(1 - e) = (e)$ so that for some $t \in R$ we have $s = te$. So $0 = se = te^2 = te = s$, whence r is regular and x is almost clean. □

Corollary 17. *Every basically disconnected space is strongly zero-dimensional.*

In Proposition 1.9 (Nicholson, 1977) it is shown that every commutative clean ring is potent. Recall that a ring R is *potent* if idempotents may be lifted module $\mathfrak{J}(R)$ and that every ideal not contained in $\mathfrak{J}(R)$ contains a nonzero idempotent. ($\mathfrak{J}(R)$ denotes the Jacobson radical of R.) If $\mathfrak{J}(R) = 0$, then it follows R is potent if and only if every principal ideal contains an idempotent. We conclude with a characterization of when $C(X)$ is potent. This condition is weaker than cleanliness. Recall that a space X is said to have a *clopen π-base* if every open set contains a clopen subset.

Proposition 18. *For a Tychonoff space X, C(X) is potent if and only if X has a clopen π-base.*

Corollary 19. *$C(X)$ is potent if and only if $C^*(X)$ is potent.*

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Proof. We will show that X has a clopen π-base if and only if βX has a clopen π-base. This is known and proved in Martinez and Woodward (1996) but our proof avoids any mention of specker algebras.

Sufficiency: This follows from the fact that a dense subspace of a space with a clopen π-base has a clopen π-base.

Necessity: Let $f \in C^*(X)$ be nonzero with $f \geq 0$. Let $U = \{x \in X : f(x) > \frac{1}{\epsilon}\}$ for some $\epsilon > 0$ so that $U \neq \emptyset$. Now, U is a cozerset of X and so the hypothesis says there is a clopen set $K \subseteq U$. Define $g \in C(X)$ to be $\frac{1}{f}$ on K and 0 otherwise. Then $\chi_K = fg$. Finally observe that g is bounded and so $C^*(X)$ is potent. □

Example 20. The class of spaces for which $C(X)$ is potent (properly) includes zero-dimensional spaces and specker spaces (see Bella et al., 1996). Also, since there are zero-dimensional spaces X for which βX is not zero-dimensional it follows that there are examples of a potent $C(X)$ which is not clean. There is another f-ring characterization for when $C(X)$ is potent given in Martinez and Woodward (1996).

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