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Clique Coverings of Glued Graphs at Complete Clones

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Abstract

A clique covering of a graph G is a set of cliques of G in which each edge of G is contained in at least one clique. The smallest cardinality of clique coverings of G is called the clique covering number of G. A glued graph results from combining two nontrivial vertex-disjoint graphs by identifying nontrivial connected isomorphic subgraphs of both graphs. Such subgraphs are referred to as the clones. The two nontrivial vertex-disjoint graphs are referred to the original graphs.

In this paper, we investigate bounds of clique covering numbers of glued graphs at clone which is isomorphic to K_n in terms of clique covering numbers of their original graphs, and give a characterization of a glued graph with the clique covering number of each possible value.

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Keywords: clique coverings, glued graphs

1 Introduction

Let G_1 and G_2 be two nontrivial vertex-disjoint graphs. Let H_1 and H_2 be nontrivial connected subgraphs of G_1 and G_2, respectively, such that $H_1 \cong H_2$ with an isomorphism f. The *glued graph between G_1 and G_2 at H_1 and H_2 with respect to f*, denoted by $\underset{H_1 \cong_f H_2}{G_1 \triangleleft\triangleright G_2}$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f between H_1 and H_2. Let H be the copy of H_1 and H_2 in the glued graph. We refer to H, H_1 and H_2 as the *clones* of the glued graph,

G_1 and G_2, respectively, and refer to G_1 and G_2 as the *original graphs*. We use $u \equiv v$ to denote the vertex in $\underset{H_1 \cong_f H_2}{G_1 \lhd\!\rhd G_2}$ where $u \in V(G_1)$ and $v \in V(G_2)$. The *glued graph between G_1 and G_2 at the clone H*, written $G_1 \underset{H}{\lhd\!\rhd} G_2$, means that there exist subgraph H_1 of G_1 and subgraph H_2 of G_2 and isomorphism f between H_1 and H_2 such that $\underset{H_1 \cong_f H_2}{G_1 \lhd\!\rhd G_2}$ and H is the copy of H_1 and H_2 in the resulting graph. We denote $G_1 \lhd\!\rhd G_2$ an arbitrary graph resulting from gluing graphs G_1 and G_2 at any isomorphic subgraph $H_1 \cong H_2$ with respect to any of their isomorphism. More details regarding glued graphs can be explored in Promsakon's Thesis [6].

Cliques are complete subgraphs of a graph that are not necessarily maximal. A *clique covering* of a graph G is a set of cliques of G in which each edge of G is contained in at least one clique. The smallest cardinality of clique coverings of G is called the *clique covering number* of G, and is denoted by $cc(G)$. A *minimum clique covering of* G is a clique covering of G with cardinality $cc(G)$. A problem of representing of a graph by set intersections by Erdős, et al.[4] became the subject of clique coverings of graphs. Many authors have studied clique covering numbers of some class of graphs(e.g. [1], [2], [3], [7]). A *glued graph at complete clone* is a glued graph at clone which is a clique of both original graphs. An *n-clique* of a graph G is a clique of G with n vertices. We use symbol $K_n(v_1, ..., v_n)$ for the complete graph K_n on the set of vertices $\{v_1, ..., v_n\}$. Other standard notations we follow West [8].

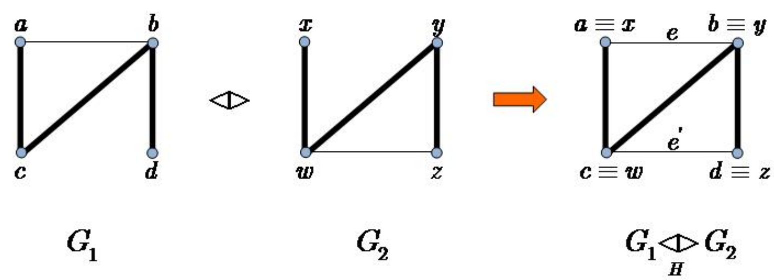

Figure 1: A glued graph.

Consider graphs G_1, G_2 and $G_1 \underset{H}{\lhd\!\rhd} G_2$ whose clone H is shown as bold edges in Figure 1 . Let $\mathrm{C}_1 = \{K_3(a, b, c), K_2(b, d)\}$, $\mathrm{C}_2 = \{K_2(x, w), K_3(w, y, z)\}$ and $\mathrm{C} = \{K_3(a \equiv x, b \equiv y, c \equiv w), K_3(d \equiv z, b \equiv y, c \equiv w)\}$. We can see that C_1, C_2 and C are clique coverings of G_1, G_2 and $G_1 \underset{H}{\lhd\!\rhd} G_2$, respectively. Apparently, $\mathrm{C}_1 \cup \mathrm{C}_2$ is not a clique covering of $G_1 \underset{H}{\lhd\!\rhd} G_2$, though the copies of cliques in C_1 and C_2 lie in C. For convenience, we here abuse the terminologies by considering subgraphs of original graphs as subgraphs of the glued graph, and subgraphs in the clone of the glued graph are subgraphs in the clones of both original graphs. For example, if Q is a clique in G_1, then Q is also a clique in $G_1 \Leftrightarrow G_2$; or if C_1 and C_2 are clique coverings of G_1 and G_2, respectively, then $\mathrm{C}_1 \cup \mathrm{C}_2$ is a clique covering of $G_1 \Leftrightarrow G_2$; also an edge in the clone of $G_1 \Leftrightarrow G_2$ is an edge in G_1 and G_2.

Throughout the paper, G_1 and G_2 are two nontrivial vertex-disjoint graphs and the clone H is a nontrivial connected clone.

2 Preliminaries

We first consider a trivial bound of the clique covering numbers of any glued graphs. Since any clone of $G_1 \Leftrightarrow G_2$ is nontrivial, $cc(G_1 \Leftrightarrow G_2) \geq 1$. Moreover, we can use minimum clique coverings of G_1 and G_2 to cover $G_1 \Leftrightarrow G_2$. Therefore,

$$1 \leq cc(G_1 \Leftrightarrow G_2) \leq cc(G_1) + cc(G_2).$$

If G_1 is a Hamiltonian graph on n vertices with a Hamiltonian path P_n and $G_2 = \overline{G}_1 \cup P_n$, then $G_1 \underset{P_n}{\Leftrightarrow} G_2$ is K_n. It is evident that the graph gluing of original graphs with any arbitrary large clique covering number could yield a resulting glued graph with the clique covering number 1. This circumstance occurs because of the existence of a new clique in the glued graph. We then investigate glued graphs involving the existence of new cliques.

An edge $e = ab$ in any glued graph $G_1 \Leftrightarrow G_2$ is *a new edge for the original graph* G_i, $i = 1$ or 2 if the corresponding vertices of a and b in G_i are not adjacent. An n-clique $Q = K_n(v_1, ..., v_n)$ in any glued graph $G_1 \Leftrightarrow G_2$ is *a new clique for the original graph* G_i, $i = 1$ or 2 if corresponding vertices of $v_1, ..., v_n$ in G_i do not form an n-clique in G_i.

As illustrated in Figure 1, the glued graph $G_1 \underset{H}{\Leftrightarrow} G_2$ contains the edge $e = xy$ while x and y are not adjacent in G_2. So, e is a new edge for G_2. Similarly, e' is a new edge for G_1. Besides, we have that $K_3(b \equiv y, c \equiv w, d \equiv z)$ and $K_3(a \equiv x, b \equiv y, c \equiv w)$ are new cliques for G_1 and G_2, respectively.

Remark 2.1. Followings are simple observations:

1. If a glued graph $G_1 \Leftrightarrow G_2$ has a new clique for G_i, $i = 1$ or 2, then $G_1 \Leftrightarrow G_2$ has a new edge for G_i.
2. Any new edge of a glued graph cannot be a new edge for both original graphs at the same time.

3. Both endpoints of a new edge of a glued graph must lie in the clone.

Any glued graph without new cliques has its clique covering number with a new lower bound as in the following theorem.

Theorem 2.2. [5] *If $G_1 \Diamond G_2$ does not have a new clique of size at least 3 for any original graphs, then $max\{cc(G_1), cc(G_2)\} \leq cc(G_1 \Diamond G_2)$.*

For $n \geq 2$, $G_1 \underset{K_n}{\Diamond} G_2$ denotes an arbitrary glued graph between graphs G_1 and G_2 containing a subgraph K_n at any clone which is isomorphic to K_n. $G_1 \underset{K_n}{\Diamond} G_2$ is called as *a glued graph at complete clone.*

Remark 2.3. By Remark 2.1(3) and the clone of $G_1 \underset{K_n}{\Diamond} G_2$ is a complete graph, we have that $G_1 \underset{K_n}{\Diamond} G_2$ does not have a new edge or, consequently a new clique, for any original graphs.

In [5], the authors have investigated bounds of clique coverings of glued graphs at clone which is isomorphic to K_2 and given a characterization of these glued graphs with the clique covering number of each possible value illustrated in Theorem 2.4, Theorem 2.5 and Corollary 2.6.

Theorem 2.4. [5] *For any graphs G_1 and G_2,*

$$cc(G_1) + cc(G_2) - 1 \leq cc(G_1 \underset{K_2}{\Diamond} G_2) \leq cc(G_1) + cc(G_2).$$

Theorem 2.5. [5] *For any graphs G_1 and G_2, the following statements are equivalent:*

(i) $cc(G_1 \underset{K_2}{\Diamond} G_2) = cc(G_1) + cc(G_2) - 1$.

(ii) *There exists a minimum clique covering of G_1 or G_2 containing the clone K_2.*

(iii) *$cc(G_1 - e) = cc(G_1) - 1$ or $cc(G_2 - e) = cc(G_2) - 1$ where e is the edge of the clone K_2.*

Corollary 2.6. [5] *For any graphs G_1 and G_2, the following statements are equivalent:*

(i) $cc(G_1 \underset{K_2}{\Diamond} G_2) = cc(G_1) + cc(G_2)$.

(ii) *There is no minimum clique covering of G_1 and G_2 containing the clone K_2.*

(iii) $cc(G_1 - e) \geq cc(G_1)$ *and* $cc(G_2 - e) \geq cc(G_2)$ *where* e *is the edge of the clone* K_2.

In this paper, we extend such results to the glued graphs at clone K_n where $n \geq 3$.

In order to investigate lower bounds of $cc(G_1 \diamond G_2)$, we consider the set of all cliques in C which belong to each original graph when C is a minimum clique covering of a glued graph $G_1 \diamond G_2$. The next notations are defined for convenience.

For a glued graph $G_1 \underset{H}{\diamond} G_2$, let C be a minimum clique covering of $G_1 \underset{H}{\diamond} G_2$. Define $\mathrm{C}[G_1] = \{C \in \mathrm{C} \mid C \text{ is a clique of } G_1\}$, $\mathrm{C}[G_2] = \{C \in \mathrm{C} \mid C \text{ is a clique of } G_2\}$ and $\mathrm{C}[H] = \{C \in \mathrm{C} \mid C \text{ is a clique of } H\}$. In general, C may not be the same as $\mathrm{C}[G_1] \cup \mathrm{C}[G_2]$.

Proposition 2.7. *For a minimum clique covering* C *of* $G_1 \diamond G_2$ *which does not have a new clique for any original graphs,* $\mathrm{C} = \mathrm{C}[G_1] \cup \mathrm{C}[G_2]$*, and hence,* $\mathrm{C}[G_1] \cup \mathrm{C}[G_2]$ *is also a minimum clique covering of* $G_1 \diamond G_2$.

Proof. Let C be a minimum clique covering of $G_1 \diamond G_2$. Then $\mathrm{C}[G_1] \cup \mathrm{C}[G_2] \subseteq \mathrm{C}$ by definitions. Since $G_1 \diamond G_2$ does not have a new clique for any original graphs, every clique in a glued graph must be a copy of a clique in G_1 or G_2. Thus $\mathrm{C} \subseteq \mathrm{C}[G_1] \cup \mathrm{C}[G_2]$. □

When a glued graph does not have a new clique for any original graphs, an improved lower bound is obtained next.

Theorem 2.8. *For* $G_1 \underset{H}{\diamond} G_2$ *which does not have a new clique for any original graphs,*

$$cc(G_1) + cc(G_2) - 2cc(H) \leq cc(G_1 \underset{H}{\diamond} G_2) \leq cc(G_1) + cc(G_2).$$

Proof. As the upper bound has been already examined, here we present the lower bound for the clique covering number of a glued graph which does not have a new clique for any original graphs. Let C be a minimum clique covering of $G_1 \underset{H}{\diamond} G_2$. By Proposition 2.7, we have $cc(G_1 \underset{H}{\diamond} G_2) = |\mathrm{C}[G_1]| + |\mathrm{C}[G_2]| - |\mathrm{C}[G_1] \cap \mathrm{C}[G_2]| = |\mathrm{C}[G_1]| + |\mathrm{C}[G_2]| - |\mathrm{C}[H]|$.

Let D be a minimum clique covering of H. Note that, for $i = 1, 2$, $(\mathrm{C}[G_i] \setminus \mathrm{C}[H]) \cap \mathrm{D} = \emptyset$, so $|(\mathrm{C}[G_i] \setminus \mathrm{C}[H]) \cup \mathrm{D}| = |\mathrm{C}[G_i]| - |\mathrm{C}[H]| + cc(H)$. Since $(\mathrm{C}[G_i] \setminus \mathrm{C}[H]) \cup \mathrm{D}$ is a clique covering of G_i, $|\mathrm{C}[G_i]| - |\mathrm{C}[H]| + cc(H) \geq cc(G_i)$.

Thus

$$\begin{aligned} cc(G_1 \underset{H}{\triangleleft\!\triangleright} G_2) &= |\mathrm{C}[G_1]| + |\mathrm{C}[G_2]| - |\mathrm{C}[H]| \\ &\geq cc(G_1) + |\mathrm{C}[H]| - cc(H) + cc(G_2) + |\mathrm{C}[H]| - cc(H) - |\mathrm{C}[H]| \\ &\geq cc(G_1) + cc(G_2) - 2cc(H). \end{aligned}$$

□

Since $G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2$ does not have a new clique for any original graphs, the next theorem is followed.

Theorem 2.9. *For any graphs G_1 and G_2 containing K_n as a subgraph,*

$$cc(G_1) + cc(G_2) - 2 \leq cc(G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2) \leq cc(G_1) + cc(G_2). \tag{2.1}$$

Example 2.10. *The sharpness of the lower bound in (2.1)*. Let Q and Q' be two graphs such that $Q \cong K_n$ and $Q' \cong K_n$, $V(Q) = \{v_1, ..., v_n\}$, $V(Q') = \{v'_1, ..., v'_n\}$, $E(Q) = \{e_1, ..., e_k\}$, $E(Q') = \{e'_1, ..., e'_k\}$ where $|E(K_n)| = k$. Thus $Q \cong Q'$ by isomorphism f such that $f(v_i) = v'_i$. We can assume that $e_i = e'_i$.

- Let G_1 be an $(n+1)$-vertex graph containing Q as a subgraph such that the remaining vertex a_1 joins with endpoints of e_1 in Q.
- Let G_2 be a graph with $V(G_2) = V(Q') \cup \{a_2 ..., a_k\}$ and contains Q' as a subgraph such that each vertex a_i joins with endpoints of each e_i in Q' where $i = 2, ..., k$.

It can be easily seen that $cc(G_1) = 2$ and $cc(G_2) = k$. Moreover, $cc(G_1 \underset{Q \cong_f Q'}{\triangleleft\!\triangleright} G_2) = k = cc(G_1) + cc(G_2) - 2$.

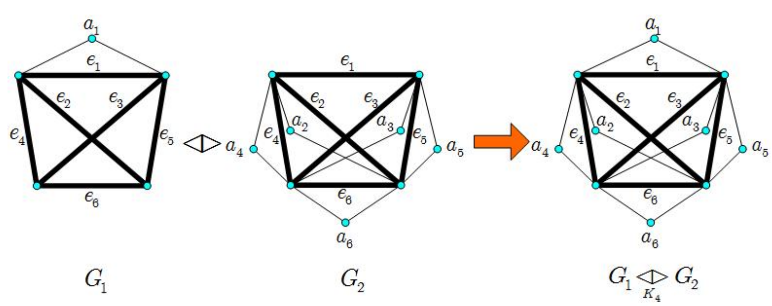

Figure 2: The sharpness of the lower bound in Equation (2.1) with $n = 4$

Our main result of the paper is to characterize each possible value of $cc(G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2)$ in Equation (2.1). As $\mathrm{C}[G_1]$ and $\mathrm{C}[G_2]$ play an important role to obtain such characterizations, we then consider properties of $\mathrm{C}[G_1]$ and $\mathrm{C}[G_2]$ for any minimum clique covering C in $G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2$.

Remark 2.11. Let C be a minimum clique covering of $G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2$. Then

1. There exists at most one nontrivial subgraph of K_n in C.

2. $\mathrm{C}[G_1] \cap \mathrm{C}[G_2] = \begin{cases} \{S\} & \text{, if there exists a nontrivial subgraph } S \text{ of } K_n \text{ in C;} \\ \emptyset & \text{, otherwise.} \end{cases}$

3. $|\mathrm{C}| = \begin{cases} |\mathrm{C}[G_1]| + |\mathrm{C}[G_2]| - 1 & \text{, if there exists a nontrivial subgraph } S \text{ of } K_n \text{ in C;} \\ |\mathrm{C}[G_1]| + |\mathrm{C}[G_2]| & \text{, otherwise.} \end{cases}$

For convenience, in our proofs, K_n in the the glued graph $G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2$ is referred to only the clone K_n, not an arbitrary copy of K_n in the glued graph.

Proposition 2.12. *For a minimum clique covering* C *of* $G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2$,

$$cc(G_1) - 1 \leq |\mathrm{C}[G_1]| \leq cc(G_1) \text{ and } cc(G_2) - 1 \leq |\mathrm{C}[G_2]| \leq cc(G_2).$$

Proof. Note first that $\mathrm{C}[G_1] \cup \{K_n\}$ is a clique coverings of G_1. Thus $cc(G_1) \leq |\mathrm{C}[G_1] \cup \{K_n\}| = |\mathrm{C}[G_1]| + 1 - |\mathrm{C}[G_1] \cap \{K_n\}|$. So, $|\mathrm{C}[G_1]| \geq cc(G_1) - 1$.

Suppose that $|\mathrm{C}[G_1]| > cc(G_1)$. Let C_1 be a minimum clique covering of G_1. Thus $|\mathrm{C}_1| = cc(G_1)$ and $\mathrm{C}_1 \cup \mathrm{C}[G_2]$ is a clique covering of $G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2$. Therefore there exists at most one nontrivial subgraph of K_n in $\mathrm{C}_1 \cap \mathrm{C}[G_2]$. Hence, $|\mathrm{C}_1 \cup \mathrm{C}[G_2]| = cc(G_1) + |\mathrm{C}[G_2]| - |\mathrm{C}_1 \cap \mathrm{C}[G_2]|$ and $|\mathrm{C}[G_1] \cup \mathrm{C}[G_2]| > cc(G_1) + |\mathrm{C}[G_2]| - |\mathrm{C}[G_1] \cap \mathrm{C}[G_2]|$.

Case 1. $\mathrm{C}_1 \cap \mathrm{C}[G_2] \neq \emptyset$. Thus there is unique nontrivial subgraph of K_n, say C, in $\mathrm{C}_1 \cap \mathrm{C}[G_2]$. So, $C \in \mathrm{C}[G_1]$. Hence $|\mathrm{C}[G_1] \cup \mathrm{C}[G_2]| > cc(G_1) + |\mathrm{C}[G_2]| - 1$. We have that $|\mathrm{C}_1 \cup \mathrm{C}[G_2]| = cc(G_1) + |\mathrm{C}[G_2]| - 1$. This contradicts the fact that $\mathrm{C}[G_1] \cup \mathrm{C}[G_2]$ is a minimum clique covering of $G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2$.

Case 2. $\mathrm{C}_1 \cap \mathrm{C}[G_2] = \emptyset$. Thus $|\mathrm{C}_1 \cup \mathrm{C}[G_2]| = cc(G_1) + |\mathrm{C}[G_2]|$.

Subcase 2.1. $\mathrm{C}[G_2]$ does not contain a nontrivial subgraph of K_n, also is $\mathrm{C}[G_1]$. We have that $|\mathrm{C}[G_1] \cup \mathrm{C}[G_2]| > cc(G_1) + |\mathrm{C}[G_2]|$. This contradicts the fact that $\mathrm{C}[G_1] \cup \mathrm{C}[G_2]$ is a minimum clique covering of $G_1 \underset{K_n}{\triangleleft\!\triangleright} G_2$.

Subcase 2.2. $\mathrm{C}[G_2]$ contains a nontrivial subgraph of K_n, say C'. Thus $C' \in \mathrm{C}[G_1]$ and $C' \notin \mathrm{C}_1$. Hence $(\mathrm{C}_1 \cup \mathrm{C}[G_2]) \setminus C'$ is a clique covering of

$G_1\underset{K_n}{\diamond}G_2$. Consider $\left|(\mathrm{C}_1 \cup \mathrm{C}[G_2]) \setminus C'\right| = cc(G_1) + |\mathrm{C}[G_2]| - 1$. It is easy to see that $|\mathrm{C}[G_1] \cup \mathrm{C}[G_2]| > cc(G_1) + |\mathrm{C}[G_2]| - 1$. Again it is a contradiction.

We have a contradiction for all cases, and hence $|\mathrm{C}[G_1]| \leq cc(G_1)$. Similarly, $cc(G_2) - 1 \leq |\mathrm{C}[G_2]| \leq cc(G_2)$. □

From Remark 2.11(1), there exists at most one nontrivial subgraph of K_n in C, here we consider the case that there exists exactly one nontrivial subgraph of K_n in C.

Proposition 2.13. *Let* C *be a minimum clique covering of* $G_1\underset{K_n}{\diamond}G_2$. *If there exists a nontrivial subgraph* S *of* K_n *such that* $S \in \mathrm{C}$, *then*

(i) $|\mathrm{C}[G_1]| = cc(G_1)$ *and* $|\mathrm{C}[G_2]| = cc(G_2)$, *and*

(ii) $(\mathrm{C}[G_1] \cup \{K_n\}) \setminus \{S\}$ *and* $(\mathrm{C}[G_2] \cup \{K_n\}) \setminus \{S\}$ *are minimum clique coverings of* G_1 *and* G_2, *respectively.*

Proof. Assume that there exists a nontrivial subgraph S of K_n such that $S \in \mathrm{C}$. Hence $(\mathrm{C}[G_1] \cup \{K_n\}) \setminus \{S\}$ and $(\mathrm{C}[G_2] \cup \{K_n\}) \setminus \{S\}$ are clique coverings of G_1 and G_2, respectively. Consider $|(\mathrm{C}[G_1] \cup \{K_n\}) \setminus \{S\}| = |\mathrm{C}[G_1]|$ and $|(\mathrm{C}[G_2] \cup \{K_n\}) \setminus \{S\}| = |\mathrm{C}[G_2]|$. Hence $|\mathrm{C}[G_1]| \geq cc(G_1)$ and $|\mathrm{C}[G_2]| \geq cc(G_2)$. By Proposition 2.12, we have $|\mathrm{C}[G_1]| = cc(G_1)$ and $|\mathrm{C}[G_2]| = cc(G_2)$. Moreover, $(\mathrm{C}[G_1] \cup \{K_n\}) \setminus \{S\}$ and $(\mathrm{C}[G_2] \cup \{K_n\}) \setminus \{S\}$ are minimum clique coverings of G_1 and G_2, respectively. □

3 Characterizations

We have proved that $cc(G_1) + cc(G_2) - 2 \leq cc(G_1\underset{K_n}{\diamond}G_2) \leq cc(G_1) + cc(G_2)$, that is, there are three possible values of $cc(G_1\underset{K_n}{\diamond}G_2)$. To characterize glued graphs with the clique covering number of each possible value, we start with giving a condition to get $cc(G_1\underset{K_n}{\diamond}G_2) = cc(G_1) + cc(G_2) - 1$ as illustrated in the next lemma.

Lemma 3.1. *If there exists a minimum clique covering of* $G_1\underset{K_n}{\diamond}G_2$ *containing a nontrivial subgraph of the clone* K_n, *then* $cc(G_1\underset{K_n}{\diamond}G_2) = cc(G_1) + cc(G_2) - 1$.

Proof. Let C be a minimum clique covering of $G_1\underset{K_n}{\diamond}G_2$. Assume that there exists a nontrivial subgraph S of K_n such that $S \in \mathrm{C}$. By Proposition 2.13(i), we have $|\mathrm{C}[G_1]| = cc(G_1)$ and $|\mathrm{C}[G_2]| = cc(G_2)$. Hence, by Remark 2.11(3), we have $cc(G_1\underset{K_n}{\diamond}G_2) = |\mathrm{C}[G_1]| + |\mathrm{C}[G_2]| - 1 = cc(G_1) + cc(G_2) - 1$. □

Lemmas 3.2 and 3.3 help us to characterize $cc(G_1 \underset{K_n}{\diamond} G_2)$ of each possible value.

Lemma 3.2. *Let G_1 and G_2 be any graphs containing K_n as a subgraph. If $cc(G_1 \underset{K_n}{\diamond} G_2) = cc(G_1) + cc(G_2) - 1$, then there exists a minimum clique covering of G_1 or G_2 containing the clone K_n.*

Proof. Assume that $cc(G_1 \underset{K_n}{\diamond} G_2) = cc(G_1) + cc(G_2) - 1$. Let C be a minimum clique covering of $G_1 \underset{K_n}{\diamond} G_2$. Thus $|\mathrm{C}| = cc(G_1) + cc(G_2) - 1$. If there exists a nontrivial subgraph S of K_n such that $S \in \mathrm{C}$, then the statement is proved by Proposition 2.13(ii). We may assume that there is no nontrivial subgraph of K_n contained in C. By Remark 2.11(3), $|\mathrm{C}| = |\mathrm{C}[G_1]| + |\mathrm{C}[G_2]|$. Assume that all minimum clique coverings of G_1 do not contain K_n. Note that $\mathrm{C}[G_1] \cup \{K_n\}$ is a clique covering of G_1. If $|\mathrm{C}[G_1]| = cc(G_1) - 1$, then $\mathrm{C}[G_1] \cup \{K_n\}$ is a minimum clique covering of G_1 which contains K_n, a contradiction. By Proposition 2.12, $|\mathrm{C}[G_1]| = cc(G_1)$. Because $cc(G_1) + cc(G_2) - 1 = |\mathrm{C}| = |\mathrm{C}[G_1]| + |\mathrm{C}[G_2]|$, we have $|\mathrm{C}[G_2]| = cc(G_2) - 1$. Since $\mathrm{C}[G_2] \cup \{K_n\}$ is a clique covering of G_2 and $|\mathrm{C}[G_2] \cup \{K_n\}| = cc(G_2)$, $\mathrm{C}[G_2] \cup \{K_n\}$ is a minimum clique covering of G_2. □

Lemma 3.3. *Let G_1 and G_2 be any graphs containing K_n as a subgraph. If $cc(G_1 \underset{K_n}{\diamond} G_2) = cc(G_1) + cc(G_2) - 2$, then there exist minimum clique coverings of G_1 and G_2, both containing the clone K_n.*

Proof. Let C be a minimum clique covering of $G_1 \underset{K_n}{\diamond} G_2$. Thus $|\mathrm{C}| = cc(G_1) + cc(G_2) - 2$. By Lemma 3.1, $K_n \notin \mathrm{C}$. Note that $\mathrm{C}[G_1] \cup \{K_n\}$ and $\mathrm{C}[G_2] \cup \{K_n\}$ are clique coverings of G_1 and G_2, respectively. By Remark 2.11(3), we have $cc(G_1) + cc(G_2) - 2 = |\mathrm{C}[G_1]| + |\mathrm{C}[G_2]|$. By Proposition 2.12, $|\mathrm{C}[G_1]| = cc(G_1) - 1$ and $|\mathrm{C}[G_2]| = cc(G_2) - 1$. Hence $|\mathrm{C}[G_1] \cup \{K_n\}| = |\mathrm{C}[G_1]| + 1 = cc(G_1)$ and $|\mathrm{C}[G_2] \cup \{K_n\}| = |\mathrm{C}[G_2]| + 1 = cc(G_2)$. Therefore, $\mathrm{C}[G_1] \cup \{K_n\}$ and $\mathrm{C}[G_2] \cup \{K_n\}$ are minimum clique coverings of G_1 and G_2, respectively. □

Theorem 3.4. *Let G_1 and G_2 be any graphs containing K_n as a subgraph. Then, $cc(G_1) + cc(G_2) - 2 \leq cc(G_1 \underset{K_n}{\diamond} G_2) \leq cc(G_1) + cc(G_2) - 1$ if and only if there exists a minimum clique covering of G_1 or G_2 containing the clone K_n.*

Proof. Necessity follows immediately from Lemmas 3.2 and 3.3. For sufficiency, assume that K_n is contained in at least one minimum clique covering of G_1 or G_2. Without loss of generality, we choose a minimum clique covering of G_1 containing K_n, say C_1. Let C_2 be a minimum clique covering of G_2. Then $(\mathrm{C}_1 \setminus \{K_n\}) \cup \mathrm{C}_2$ is a clique covering of $G_1 \underset{K_n}{\diamond} G_2$. Since $(\mathrm{C}_1 \setminus \{K_n\}) \cap \mathrm{C}_2 = \emptyset$,

$|(\mathrm{C}_1 \setminus \{K_n\}) \cup \mathrm{C}_2| = |\mathrm{C}_1| - 1 + |\mathrm{C}_2| = cc(G_1) + cc(G_2) - 1$. Hence $cc(G_1 \underset{K_n}{\lhd\!\rhd} G_2) \leq cc(G_1) + cc(G_2) - 1$. By Theorem 2.9, $cc(G_1 \underset{K_n}{\lhd\!\rhd} G_2) \geq cc(G_1) + cc(G_2) - 2$. □

From Equation (2.1), the contrapositive of Theorem 3.4 gives a characterization of glued graphs with $cc(G_1 \underset{K_n}{\lhd\!\rhd} G_2) = cc(G_1) + cc(G_2)$.

Corollary 3.5. *Let G_1 and G_2 be any graphs containing K_n as a subgraph. Then,*
$cc(G_1 \underset{K_n}{\lhd\!\rhd} G_2) = cc(G_1) + cc(G_2)$ *if and only if there is no minimum clique covering of G_1 or G_2 containing the clone K_n.*

Now, we consider a characterization of $cc(G_1 \underset{K_n}{\lhd\!\rhd} G_2) = cc(G_1) + cc(G_2) - 2$.

Theorem 3.6. *Let G_1 and G_2 be any graphs containing K_n as a subgraph. Then,*
$cc(G_1 \underset{K_n}{\lhd\!\rhd} G_2) = cc(G_1) + cc(G_2) - 2$ *if and only if there exist minimum clique coverings of G_1 and G_2 where both contain the clone K_n and the union of them deleting the clone K_n is a clique covering of $G_1 \underset{K_n}{\lhd\!\rhd} G_2$.*

Proof. For sufficiency, we can choose a minimum clique covering of G_1 containing K_n, say C_1 and a minimum clique covering of G_2 containing K_n, say C_2 such that $(\mathrm{C}_1 \setminus \{K_n\}) \cup (\mathrm{C}_2 \setminus \{K_n\})$ is a clique covering of $G_1 \underset{K_n}{\lhd\!\rhd} G_2$. Hence $|(\mathrm{C}_1 \setminus \{K_n\}) \cup (\mathrm{C}_2 \setminus \{K_n\})| \geq cc(G_1 \underset{K_n}{\lhd\!\rhd} G_2)$. Suppose that $(\mathrm{C}_1 \setminus \{K_n\}) \cap (\mathrm{C}_2 \setminus \{K_n\}) \neq \emptyset$. Thus there is a clique Q' contained in $(\mathrm{C}_1 \setminus \{K_n\}) \cap (\mathrm{C}_2 \setminus \{K_n\})$. Since an edge of G_1 and G_2 in $G_1 \underset{K_n}{\lhd\!\rhd} G_2$ must be in K_n , we have that Q' is a subgraph of K_n. Hence, both K_n and Q' are in $\mathrm{C}_1 \cap \mathrm{C}_2$. This contradicts Remark 2.11(1). Therefore, $(\mathrm{C}_1 \setminus \{K_n\}) \cap (\mathrm{C}_2 \setminus \{K_n\}) = \emptyset$. Consider $|(\mathrm{C}_1 \setminus \{K_n\}) \cup (\mathrm{C}_2 \setminus \{K_n\})| = |\mathrm{C}_1| - 1 + |\mathrm{C}_2| - 1 = cc(G_1) + cc(G_2) - 2$. Hence $cc(G_1) + cc(G_2) - 2 \geq cc(G_1 \underset{K_n}{\lhd\!\rhd} G_2)$. By Theorem 2.9, $cc(G_1 \underset{K_n}{\lhd\!\rhd} G_2) \geq cc(G_1) + cc(G_2) - 2$. Therefore $cc(G_1 \underset{K_n}{\lhd\!\rhd} G_2) = cc(G_1) + cc(G_2) - 2$.

For necessity, assume that $cc(G_1 \underset{K_n}{\lhd\!\rhd} G_2) = cc(G_1) + cc(G_2) - 2$. Let C be a minimum clique covering of $G_1 \underset{K_n}{\lhd\!\rhd} G_2$. By Lemma 3.1, we have that $K_n \notin \mathrm{C}$. As the proof of Lemma 3.3, we can see that $\mathrm{C}[G_1] \cup \{K_n\}$ and $\mathrm{C}[G_2] \cup \{K_n\}$ are minimum clique coverings of G_1 and G_2, respectively. Note that $[(\mathrm{C}[G_1] \cup \{K_n\}) \cup (\mathrm{C}[G_2] \cup \{K_n\})] \setminus \{K_n\} = \mathrm{C}[G_1] \cup \mathrm{C}[G_2]$. By Proposition 2.7, $\mathrm{C}[G_1] \cup \mathrm{C}[G_2]$ is a minimum clique covering of $G_1 \underset{K_n}{\lhd\!\rhd} G_2$. □

Any glued graph at clone K_n, $G_1 \underset{K_n}{\lhd\!\rhd} G_2$, which does not satisfy the conditions in Corollary 3.5 and Theorem 3.6 has $cc(G_1 \underset{K_n}{\lhd\!\rhd} G_2) = cc(G_1) + cc(G_2) - 1$ by

Equation (2.1). However, the opposite of these two conditions is not simple. It would be interesting if one can find another simpler characterization of $G_1 \underset{K_n}{\Diamond} G_2$ with $cc(G_1 \underset{K_n}{\Diamond} G_2) = cc(G_1) + cc(G_2) - 1$.

When the clone H of $G_1 \underset{H}{\Diamond} G_2$ is an induced subgraph of both original graphs, $G_1 \underset{H}{\Diamond} G_2$ has no new clique for any original graphs. Therefore, $cc(G_1) + cc(G_2) - 2cc(H) \leq cc(G_1 \underset{H}{\Diamond} G_2) \leq cc(G_1) + cc(G_2)$. It is possible to further characterize glued graphs with each possible value in such a bound when the clone is an induced subgraph of both original graphs.

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