Cloaking via change of variables for the Helmholtz equation

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Abstract

The transformation optics approach to cloaking uses a singular change of coordinates, which blows up a point to the region being cloaked. This paper examines a natural regularization, obtained by (i) blowing up a ball of radius ρ rather than a point, and (ii) including a well-chosen lossy layer at the inner edge of the cloak. We assess the performance of the resulting near-cloak as the regularization parameter ρ tends to 0, in the context of (Dirichlet and Neumann) boundary measurements for the time-harmonic Helmholtz equation. Since the goal is to achieve cloaking regardless of the content of the cloaked region, we focus on estimates that are uniform with respect to the physical properties of this region. In three space dimensions our regularized construction performs relatively well: the deviation from perfect cloaking is of order ρ. In two space dimensions it does much worse: the deviation is of order $1/|\log \rho|$. In addition to proving these estimates, we give numerical examples demonstrating their sharpness. Some authors have argued that perfect cloaking can be achieved without losses by using the singular change-of-variable-based construction. In our regularized setting the analogous statement is false: without the lossy layer, there are certain resonant inclusions (depending in general on ρ) that have a huge effect on the boundary measurements.

1 Introduction

We say a region of space is cloaked for a particular class of measurements if its contents – and even the existence of the cloak – are invisible using such measurements.

A change-of-variable-based scheme for cloaking was proposed by Pendry, Schurig, and Smith in [21] for measurements that can be modelled using the time-harmonic

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Maxwell equations. Essentially the same scheme was discussed earlier by Greenleaf, Lassas, and Uhlmann in [7] for electric impedance tomography. Recent reviews with many references to the rapidly growing literature on cloaking and other applications of "transformation optics" can be found in [12, 13, 23, 29]; see also [28] for an enlightening treatment, [14] for information about earlier work along similar lines, and [3, 5] for an application to scalar wave propagation (the focus of the present paper). For discussion of the literature most related to the present work, see Section 2.7.

The change-of-variable-based scheme proposed in [7, 21] is rather singular. This makes it difficult to analyze; in particular, multiple proposals have emerged about the appropriate notion of a "weak solution" of Maxwell's equations in such a singular setting [8, 25, 26, 28]. The proposals could all be correct, if they represent the limiting behavior of different regularizations. However there has been relatively little work on the limiting behavior of any regularization. Such work has mainly been restricted to uniform inclusions (whose properties remain fixed as the regularization varies), analyzed via separation of variables [5, 9, 22, 25, 29, 30].

This paper develops a different viewpoint, which avoids singular structures and weak solutions. We shall study change-of-variable-based "near-cloaks," defined using a natural regularization of the singular scheme. Briefly: the framework of [7, 21] uses a singular change of variable, which blows up a point to a finite-size region. Our near-cloaks replace this with a regular change of variable, which blows up a small ball to a finite-size region.

The key issues from our perspective are (a) specifying the precise structure of the near-cloak, and (b) assessing its performance. We shall address these issues for the scalar Helmholtz equation

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \omega^2 q(x)u = 0 \quad \text{in } \Omega \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N = 2$ or 3. This PDE describes time-harmonic solutions $U = ue^{-i\omega t}$ of the scalar wave equation $q(x)U_{tt} - \nabla\cdot(A(x)\nabla U) = 0$.

Any analysis of cloaking must specify the class of measurements being considered. We shall focus on "boundary measurements," i.e. the correspondence between Dirichlet and Neumann data (u and $(A\nabla u)\cdot \nu$) at $\partial\Omega$.

Our main results are summarized in Section 2. They encompass the following key points:

(i) If there are no constraints on the material properties of the objects to be cloaked, then change-of-variable-based cloaking from boundary measurements requires the use of lossy materials.

(ii) The change-of-variable-based scheme works much better in 3D than in 2D. In fact, our near-cloaks come within ρ of perfect cloaking in 3D, but only within $1/|\log \rho|$ of perfect cloaking in 2D. Here ρ is our regularization parameter – the radius of the small ball that is blown up to a finite-size region – and the deviation from perfect cloaking is measured by the difference between the Neumann-to-Dirichlet map and that of a uniform body.

Our viewpoint was introduced in [15], which focused on electric impedance tomography. This viewpoint was recently adopted by Liu [17], who studied near-cloaking achieved by change of variables when a homogeneous Dirichlet boundary condition is imposed at the inner edge of the cloak; his performance estimates are similar to ours (see point (ii) above). Other regularizations – of a more direct "truncation" nature, and sometimes involving other boundary conditions – are considered in [5, 9, 10, 11, 22, 25, 29, 30]. The recent articles [10, 11] note the possibility of resonance, which is directly related to point (i) above.

2 Main Ideas

2.1 Cloaking with respect to boundary measurements

As stated in the Introduction, we shall focus on "boundary measurements," i.e. the correspondence between Dirichlet and Neumann data. In the context of Helmholtz's equation (1.1), this means we consider the map

$$\Lambda_{A,q} : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega) \ ,$$

defined by

$$\Lambda_{A,q}(\psi) = u\big|_{\partial\Omega} \text{ where } u \in H^1(\Omega) \text{ solves (1.1) with } \sum A_{ij}\frac{\partial u}{\partial x_j}\nu_i = \psi \ . \tag{2.1}$$

This map is well-defined and invertible provided $A_{ij}(x)$ is a uniformly elliptic symmetric-matrix-valued function and ω^2 avoids a discrete set of eigenvalues. Throughout this paper we shall impose this restriction on ω^2 relative to the homogeneous medium, $A = I$, $q = 1$. The Sobolev space $H^{1/2}(\partial\Omega)$ consists functions with "$\frac{1}{2}$ derivative in L^2" and $H^{-1/2}(\partial\Omega)$ is its dual. These are the natural spaces for Dirichlet and Neumann data of finite-energy solutions, since $\phi \in H^{1/2}(\partial\Omega)$ if and only if ϕ is the restriction to $\partial\Omega$ of some function in $H^1(\Omega)$.

Fixing Ω, we shall say that $A(x)$ and $q(x)$ "look uniform" if the associated boundary measurements are identical to those obtained when $A = I$, $q = 1$, in other words if $\Lambda_{A,q} = \Lambda_{I,1}$.

Rather than define "cloaks of arbitrary geometry", let us explain what it means for a specific structure $A_c(x), q_c(x)$ defined in the shell $1 < |x| < 2$ to cloak the unit ball $B_1 = \{|x| < 1\}$. Given a domain Ω containing B_2, we say that A_c, q_c cloaks B_1 if whenever

$$A(x), q(x) = \begin{cases} I, 1 & \text{for } x \in \Omega \setminus B_2 \\ A_c, q_c & \text{in } B_2 \setminus B_1 \\ \text{arbitrary} & \text{in } B_1 \end{cases} \tag{2.2}$$

then $\Lambda_{A,q} = \Lambda_{I,1}$. In other words, Ω looks uniform regardless of the content of the "cloaked region" B_1. To make the definition complete one must specify the meaning of "arbitrary" in (2.2): for example one might ask that A and q be real-valued in B_1, with $A(x)$ uniformly elliptic. It is easy to see that the above definition depends only on the "cloak" A_c, q_c, not on the choice of Ω. In particular, if cloaking is achieved for $\Omega = B_2$ then it is also achieved for any larger domain.

2.2 The "pushforwards" $F_*(A)$ and $F_*(q)$

The change-of-variable-based cloaking scheme relies on the following basic fact.

Let $F : \Omega \to \Omega$ be a differentiable, orientation-preserving, surjective and invertible map such that $F(x) = x$ at $\partial\Omega$. Then $u(x)$ solves $\nabla_x \cdot (A(x)\nabla_x u) + \omega^2 q(x) u = 0$ if and only if $w(y) = u(F^{-1}(y))$ solves $\nabla_y \cdot (F_*A(y)\nabla_y w) + \omega^2 F_*q(y) w = 0$ with

$$F_*A(y) = \frac{DF(x)A(x)DF^T(x)}{\det DF(x)}, \quad F_*q(y) = \frac{q(x)}{\det DF(x)}, \quad x = F^{-1}(y) \ . \tag{2.3}$$

Moreover A, q and F_*A, F_*q give the same boundary measurements:

$$\Lambda_{A,q} = \Lambda_{F_*A, F_*q} \ . \tag{2.4}$$

In (2.3) DF is the matrix whose (i,j)th element is $\partial F_i/\partial x_j$. Note that A and F_*A are symmetric-matrix-valued functions, while q and F_*q are scalar-valued functions; our use of the same symbol F_* for both cases is a convenient abuse of notation.

The proof of the preceding statement is elementary. The weak form of the PDE $\nabla_x \cdot (A(x)\nabla_x u) + \omega^2 q(x) u = 0$ is the assertion that

$$\int_\Omega \left[\sum_{i,j} A_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial \phi}{\partial x_i} - \omega^2 q(x) u(x) \phi(x) \right] dx = 0$$

for all ϕ that vanish at $\partial\Omega$. Changing variables to $y = F(x)$, this becomes the statement that

$$\int_\Omega \left[\sum_{i,j} (F_*A)_{ij}(y) \frac{\partial w}{\partial y_j} \frac{\partial \psi}{\partial y_i} - \omega^2 F_*q(y) w(y) \psi(y) \right] dy = 0$$

with $\psi(y) = \phi(x)$. As ϕ varies over test functions vanishing at $\partial\Omega$ so does ψ, so we conclude that $\nabla_y \cdot (F_*A(y)\nabla_y w) + \omega^2 F_*q(y) w = 0$. In fact the two PDE's are equivalent, since the argument is reversible. To see that A, q and F_*A, F_*q give the same boundary measurements, it suffices to note that the above two integrals agree for any smooth function ϕ (and the associated $\psi(y) = \phi(x)$) whether it vanishes or not on $\partial\Omega$. Integration by parts now gives that $\sum (F_*A)_{ij} \frac{\partial w}{\partial y_j} \nu_i(y) = \sum A_{ij} \frac{\partial u}{\partial x_j} \nu_i(x)$. Since $y = F(x) = x$ on $\partial\Omega$ (and therefore $w = u$ on $\partial\Omega$) it follows that $\Lambda_{A,q} = \Lambda_{F_*A, F_*q}$.

2.3 A lossless regularization of the singular cloaking scheme

Suppose Ω contains the ball B_2. For any (small) $\rho > 0$, consider the change of variables F_ρ defined by

$$F_\rho(x) = \begin{cases} x & \text{for } x \in \Omega \setminus B_2 \\ \left(\frac{2-2\rho}{2-\rho} + \frac{1}{2-\rho}|x|\right)\frac{x}{|x|} & \text{for } \rho \le |x| \le 2 \\ \frac{x}{\rho} & \text{for } |x| \le \rho \end{cases} . \tag{2.5}$$

Its key properties are that

- F_ρ is continuous and piecewise smooth,
- F_ρ expands B_ρ to B_1, while mapping B_2 to itself; and
- $F(x) = x$ outside B_2.

The arguments in [7, 21] applied to Helmholtz suggest that B_1 should be cloaked by $A_c = (F_0)_* I, q_c = (F_0)_* 1$, where $F_0 = \lim_{\rho \to 0} F_\rho$ is the singular transformation that blows up the origin to the ball B_1. We might therefore think that if ρ is small then $(F_\rho)_* I, (F_\rho)_* 1$ should nearly cloak B_1, in the sense that if

$$A(y), q(y) = \begin{cases} I, 1 & \text{for } y \in \Omega \setminus B_2 \\ (F_\rho)_* I, (F_\rho)_* 1 & \text{in } B_2 \setminus B_1 \\ \text{arbitrary} & \text{in } B_1 \end{cases} \tag{2.6}$$

then $\Lambda_{A,q} \approx \Lambda_{1,1}$.

Such a statement is true at frequency 0; this is the main result of [15]. It is however not true when $\omega \neq 0$; we shall explain why not in Section 2.5.

2.4 Reduction to the study of small inclusions

To assess the whether $A_c = (F_\rho)_* I, q_c = (F_\rho)_* 1$ achieves approximate cloaking, we must study the boundary operator associated with (2.6). By the change of variable principle, this is the same as the boundary operator associated with

$$(F_\rho^{-1})_* A(x), (F_\rho^{-1})_* q(x) = \begin{cases} I, 1 & \text{for } x \in \Omega \setminus B_\rho \\ \text{arbitrary} & \text{in } B_\rho. \end{cases} \tag{2.7}$$

Here we have used the fact that $(F_\rho^{-1})_* \circ (F_\rho)_* = \text{id}$, and so if A, q are arbitrary in B_1, then their transforms $(F_\rho^{-1})_* A$ and $(F_\rho^{-1})_* q$ are similarly arbitrary in B_ρ. Thus:

(2.8)
$(F_\rho)_* I, (F_\rho)_* 1$ approximately cloak B_1 if and only if
an inclusion of radius ρ with arbitrary content has little
effect on the boundary map of an otherwise uniform domain.

2.5 Failure of the lossless regularization

The lossless regularized scheme discussed in Sections 2.3–2.4 does *not* achieve approximate cloaking. To explain why not, it suffices by (2.8) to show that a small inclusion in an otherwise uniform domain can have a large effect on the boundary operator.

We use separation of variables, focusing on the 2D case for simplicity. Let $\Omega = B_2$, and consider

$$A_\rho, q_\rho = \begin{cases} I, 1 & \text{in } B_2 \setminus B_\rho \\ \tilde{A}_\rho, \tilde{q}_\rho & \text{in } B_\rho \end{cases}$$

where $\tilde{A}_\rho > 0$ and $\tilde{q}_\rho$ are real-valued constants. The general solution of the associated Helmholtz equation can be expressed in polar coordinates as

$$u = \sum_{k=-\infty}^{\infty} \alpha_k J_k\left(\omega r \sqrt{\tilde{q}_\rho / \tilde{A}_\rho}\right) e^{ik\theta} \quad \text{for } r \leq \rho \ ,$$
$$u = \sum_{k=-\infty}^{\infty} \left[\beta_k J_k(\omega r) + \gamma_k H_k^{(1)}(\omega r)\right] e^{ik\theta} \quad \text{for } \rho < r \leq 2 \ ,$$

for appropriate choices of α_k, β_k and γ_k. When we solve a Neumann problem, the three unknowns at mode k $(\alpha_k, \beta_k, \gamma_k)$ are determined by three linear equations: agreement with the Neumann data at $r = 2$ and satisfaction of the two transmission conditions at $r = \rho$. However, for any $\omega \neq 0$ and any k, *this linear system has determinant zero at selected values of* $\tilde{A}_\rho$ *and* $\tilde{q}_\rho$. (We shall show this in Section 4, where we also study the asymptotics of such special values of $\tilde{A}_\rho, \tilde{q}_\rho$ as $\rho \to 0$ for $k = 0$ and $k = 1$.) When the linear system is degenerate (for some k), the homogeneous Neumann problem has a nonzero solution, and the boundary map Λ_{A_ρ, q_ρ} is not even well-defined. In brief: no matter how small the value of ρ, for any $\omega \neq 0$ there are *cloak-busting* choices of $\tilde{A}_\rho$ and $\tilde{q}_\rho$ for which the ball with such an inclusion is resonant at frequency ω.

2.6 Our near-cloaks

The standard way to deal with resonance is to introduce a mechanism for damping or loss. There are many alternatives, most of which amount to considering an open rather than a closed system (for example, use of a scattering boundary condition permits energy to be lost at infinity).

In this paper we choose a particular damping mechanism, which permits us to remain focused on boundary measurements for the Helmholtz equation (1.1). Specifically: we take q to be complex, choosing the geometry in such a way that it maintains the equivalence between near-cloaking and insensitivity to small inclusions.

Our construction (nearly) cloaks $B_{1/2}$ by surrounding it with two concentric shells: an isotropic but lossy one of thickness $1/2$, coated by an anisotropic but lossless shell similar to the one in Section 2.3. Besides the regularization parameter ρ, it also has a damping parameter $\beta > 0$. The analogue of (2.6) is

$$A(y), q(y) = \begin{cases} I, 1 & \text{for } y \in \Omega \setminus B_2 \\ (F_{2\rho})_* I, (F_{2\rho})_* 1 & \text{in } B_2 \setminus B_1 \\ (F_{2\rho})_* I, (F_{2\rho})_* (1 + i\beta) & \text{in } B_1 \setminus B_{1/2} \\ \text{arbitrary real, elliptic} & \text{in } B_{1/2}. \end{cases} \tag{2.9}$$

To be clear: in $B_{1/2}$ we permit $q(y)$ to be any L^∞ real-valued function, and we permit $A(y)$ to be any real symmetric-matrix-valued function that is uniformly bounded and uniformly positive definite. (See Section 2.7 for comments on the hypothesis that $A > 0$ in the cloaked region.) When A, q are arbitrary in this sense in $B_{1/2}$, their pullbacks $(F_{2\rho}^{-1})_* A, (F_{2\rho}^{-1})_* q$ are similarly arbitrary in B_ρ. So the boundary operator associated with $A(y), q(y)$ is the same as that of

$$A_\rho, q_\rho = (F_{2\rho}^{-1})_* A(x), (F_{2\rho}^{-1})_* q(x) = \begin{cases} I, 1 & \text{for } x \in \Omega \setminus B_{2\rho} \\ I, 1 + i\beta & \text{in } B_{2\rho} \setminus B_\rho \\ \text{arbitrary real, elliptic} & \text{in } B_\rho \end{cases} \tag{2.10}$$

(this is the analogue of (2.7)). We shall show in Section 3 that when β is chosen properly – specifically, when $\beta \sim \rho^{-2}$ – this construction approximately cloaks $B_{1/2}$ in the sense that

$$\|\Lambda_{A,q} - \Lambda_{I,1}\| = \|\Lambda_{A_\rho, q_\rho} - \Lambda_{I,1}\| \le C e(\rho) \tag{2.11}$$

where the left hand side uses the operator norm[1] on maps from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$ and

$$e(\rho) = \begin{cases} 1/|\log \rho| & \text{in space dimension 2} \\ \rho & \text{in space dimension 3} \end{cases}. \tag{2.12}$$

We emphasize that this near-cloaking is achieved *regardless of the content of the cloaked region*, i.e. the constant C in (2.11) is entirely independent of the values of $A(y)$ and $q(y)$ in $B_{1/2}$ (provided they are real, with A symmetric and positive definite).

The estimate (2.11) is essentially optimal. In fact, we shall show in Section 4 that there exist (constant) values of $\tilde{A}_\rho > 0$ and $\tilde{q}_\rho$ and Neumann data ψ such that when

$$A_\rho(x), q_\rho(x) = \begin{cases} I, 1 & \text{for } x \in \Omega \setminus B_{2\rho} \\ I, 1 + i\beta & \text{in } B_{2\rho} \setminus B_\rho \\ \tilde{A}_\rho, \tilde{q}_\rho & \text{in } B_\rho \end{cases}$$

then

$$\frac{\left\|(\Lambda_{A_\rho, q_\rho} - \Lambda_{I,1})\psi\right\|_{H^{1/2}}}{\|\psi\|_{H^{-1/2}}} \sim e(\rho).$$

[1] To be completely explicit: $\|\Lambda_{A,q} - \Lambda_{I,1}\| = \sup_{\|\psi\|_{H^{-1/2}} \le 1} \|\Lambda_{A,q}\psi - \Lambda_{I,1}\psi\|_{H^{1/2}}$; thus, it measures the worst-case difference between the Dirichlet data associated with coefficients A, q and $I, 1$ when the associated PDE's are solved using the same (normalized) Neumann data.

Note that our near-cloak is not very successful in space dimension 2, since $1/|\log \rho|$ decays very slowly as $\rho \to 0$. It is much more successful in space dimension 3. The reason for such dimension-dependent behavior lies in the different decay of the fundamental solution of the Laplacian in dimensions 2 and 3. (In space dimension $N > 3$, arguments similar to the ones presented here would give a corresponding estimate with $e(\rho) = \rho^{N-2}$.)

2.7 Discussion

Our presentation used the radial transformation $F_{2\rho}$ defined by (2.5), but our analysis of the scheme involves only the study of the inclusion problem (2.10). By replacing $F_{2\rho}$ by a more general change of variable, one easily gets a similar scheme for cloaking a non-spherical cavity.

We explained in Section 2.5 that the lossless version of our regularization must fail, if the goal is to achieve cloaking without regard to the physical properties of the region being cloaked. The papers [5, 9, 10, 22, 25, 29, 30] take a different viewpoint: translated into our terminology they assume that the properties of the cloaked region remain fixed as $\rho \to 0$. It appears that perfect cloaking is achieved without losses for 3D Maxwell and 3D Helmholtz; however the results we present in Section 4 indicate that this should not be the case for 2D Helmholtz (see the discussion associated with Figure 4.2).

Our near-cloaks use loss parameter $\beta \sim \rho^{-2}$. Numerically we can say a little more: the *optimal* choice of β is about $c\rho^{-2}$ with $c \approx 2.5$ in 2D and $c \approx 4$ in 3D (see the discussion of Figures 4.4 and 4.5 in Section 4). When β is significantly smaller near-cloaking is not achieved, because the loss is not sufficient to hide certain "cloak-busting" inclusions. When β is larger the performance of the near-cloak is slightly worse, however near-cloaking is apparently achieved even in the limit $\beta \to \infty$. This limit corresponds, at least heuristically, to the imposition of a Dirichlet boundary condition at the inner edge of the cloak, the case considered in [17]. Thus our results are closely related to those of [17], however we achieve near-cloaking using a finite value of the loss parameter.

Much of the literature on cloaking focuses on scattering rather than boundary measurements. It would be interesting to know whether our near-cloaks work equally well in that setting, e.g. whether there is an estimate analogous to (2.11) for the scattering of plane waves from Ω (embedded in uniform space with $A = I$, $q = 1$). We conjecture that this is the case.[2] (The results in [17] provide such an estimate when $\beta = \infty$.)

In assessing the performance of our near-cloak, we focus on the worst-case behavior. In particular, our estimate (2.11) applies regardless of the material properties of the cloaked region, provided only that $A(y)$ is real-valued, positive-definite, and finite there, and $q(y)$ is real-valued function. The constant in the estimate does

[2] A treatment of the scattering problem in much the same spirit as the present paper has recently been completed by Nguyen [20].

not depend on the upper or lower bounds for A or q in the cloaked region. The recent paper [4] argues that by taking $A < 0$ in part of the cloaked region, one can defeat the effect of the (singular, lossless) change-of-variable-based cloak. We doubt that our lossy near-cloak would be defeated by such a scheme. But to discuss a situation where the real part of A changes sign it is necessary to include losses (A must be complex). As the losses tend to zero and ellipticity is lost, the local fields may become increasingly oscillatory (this is case, for example, in the "anomalous localized resonances" of [19]). Since our analysis assumes that A, q are real in the cloaked region, we assume $A > 0$ to know that the PDE has a well-defined solution.

Is our approach the best way to achieve near-cloaking without singular materials? Not necessarily. The papers [9, 30] suggest that a truncation-based regularization combined with a different choice of boundary condition at the inner edge of the cloak may do better. But these papers keep the material in the cloaked region fixed as the regularization parameter tends to zero. It would be interesting to examine whether their lossless near-cloaks can be defeated by special "cloak-busting" inclusions, as discussed in Section 2.5.

Is the change-of-variable-based approach optimal? Or might there be an entirely different approach to (approximate) cloaking – using materials less singular than $(F_{2\rho})_* I$, $(F_{2\rho})_* 1$, and achieving an error estimate much better than $e(\rho)$? This question remains open. The recent paper [27] used separation of variables and a genetic algorithm to optimize cloaking of a *fixed, constant* inclusion with respect to scattering measurements, obtaining a better result with less complexity than the change-of-variable-based scheme. But their cloak would probably not work as well for non-constant inclusions. Moreover, since it was obtained by numerical optimization, the example in [27] lacks the intuitiveness and universality of the change-of-variable-based scheme.

This paper focuses entirely on change-of-variable-based cloaking. But we note in passing the existence of other promising schemes for achieving similar goals, including one based on optical conformal mapping [16], another using anomalous localized resonance [19], and a third based on special (object-dependent) coatings [1].

3 The effect of a small inclusion

The goal of this section is to prove (2.11). We begin by giving the result a more formal statement. Throughout this section, Ω is a bounded domain in $\mathbb{R}^N$ ($N = 2$ or 3), whose boundary is C^2 (so we may use elliptic estimates), with $0 \in \Omega$ (our inclusions will be centered at 0). We are interested in Helmholtz's equation at frequency ω: given $\psi \in H^{-1/2}(\partial\Omega)$, let u_0 be the solution of

$$\begin{cases} \Delta u_0 + \omega^2 u_0 = 0 & \text{in } \Omega \\ \dfrac{\partial u_0}{\partial \nu} = \psi & \text{on } \partial\Omega \end{cases} . \tag{3.1}$$

We suppose that $-\omega^2$ is not an eigenvalue of the Neumann Laplacian. The boundary value problem (3.1) is therefore well-posed, and

$$\|u_0\|_{H^1(\Omega)} \leq C\|\psi\|_{H^{-1/2}(\partial\Omega)} \ .$$

Now consider the solution u_ρ of

$$\begin{cases} \operatorname{div}(A_\rho \nabla u_\rho) + \omega^2 q_\rho u_\rho = 0 & \text{in } \Omega \\ \dfrac{\partial u_\rho}{\partial \nu} = \psi & \text{on } \partial\Omega \end{cases} , \tag{3.2}$$

where A_ρ and q_ρ have the form:

$$\begin{cases} A_\rho = I,\ q_\rho = 1 & \text{in } \Omega \setminus B_{2\rho} \\ A_\rho = 1,\ q_\rho = 1 + i\beta & \text{in } B_{2\rho} \setminus B_\rho \\ A_\rho, q_\rho \text{ arbitrary real, elliptic} & \text{in } B_\rho \end{cases} .$$

Here β is a positive constant, and the "arbitrary real, elliptic" A_ρ and q_ρ in B_ρ are assumed to be positive definite, symmetric-matrix-valued and real-valued functions respectively, in $L^\infty(B_\rho)$ (q_ρ need not be of one sign). We assume that Ω contains a neighborhood of $B_{2\rho}$ (this is a smallness condition on ρ). The existence and uniqueness of u_ρ is easy to see using the positivity of β (see Section 3.1). We claim that if β is chosen appropriately then u_ρ is close to u_0:

Theorem 3.1. *Suppose $-\omega^2$ is not an eigenvalue of the Laplacian on Ω with Neumann boundary condition. Let u_0 and u_ρ be the solutions of (3.1) and (3.2) respectively, and suppose $\beta = d_0\rho^{-2}$ for some positive constant d_0. Then there exist constants ρ_0 and C (independent of ψ) such that for any $\rho < \rho_0$,*

$$\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \leq Ce(\rho)\|\psi\|_{H^{-1/2}(\partial\Omega)} \tag{3.3}$$

where $e(\rho)$ is defined by (2.12). In other words, the difference between the two boundary operators Λ_{A_ρ,q_ρ} and $\Lambda_{I,1}$ has norm at most $Ce(\rho)$, when viewed as an operator from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. The constants ρ_0 and C depend on ω and d_0, but they are completely independent of the values of A_ρ and q_ρ in B_ρ.

Our strategy for proving this theorem is as follows:

- In *Section 3.1* we use the energy identity and the positivity of β to control the L^2 norm of u_ρ in $B_{2\rho} \setminus B_\rho$. We also deduce, by a duality argument, an estimate for the restriction of u_ρ to $\partial B_{2\rho}$.
- In *Section 3.2* we prove a general result comparing the Helmholtz equation in Ω to the same equation in the punctured domain $\Omega \setminus B_{2\rho}$. It is obvious that if the latter problem is solved using Dirichlet data $u_0|_{\partial B_{2\rho}}$ at the edge of the "hole" , and normal flux data ψ on $\partial\Omega$, then the solution is u_0. The main estimate of Section 3.2 is an associated stability result: it asserts that if Dirichlet data at the edge of the hole are close to u_0, then the solution of Helmholtz in the punctured domain is close to u_0 at $\partial\Omega$.

- In *Section 3.3* we show how the estimates in Sections 3.1 and 3.2 combine to prove Theorem 3.1.
- The discussion of Section 3.2 uses the well-posedness of Helmholtz's equation in the punctured domain $\Omega \setminus B_{2\rho}$ (with Neumann data at $\partial\Omega$ and Dirichlet data at $\partial B_{2\rho}$). This well-posedness result is not surprising (if the hole is small its effect should be small) but we do not know a convenient reference. So we give a self-contained proof in *Section 3.4*.
- The arguments in Sections 3.2 and 3.4 use some estimates for solutions of Laplace's equation in the exterior of a small ball. Those estimates are not difficult, but we do not know a suitable reference. So we give a self-contained proof in *Section 3.5*.

3.1 Some estimates based on the positivity of β

We noted above that the well-posedness of (3.2) follows easily from the positivity of β. The proof, which is standard, uses the energy identity. The following Lemma uses a variant of that argument to bound the L^2 norm of u_ρ in the shell $\rho < |x| < 2\rho$ by $\|u_0 - u_\rho\|_{H^{1/2}(\partial\Omega)}$.

Lemma 3.2. *The solutions of (3.1) and (3.2) satisfy*

$$\omega^2\beta \int_{B_{2\rho}\setminus\overline{B_\rho}} |u_\rho|^2\, dx \le C\|\psi\|_{H^{-1/2}(\partial\Omega)}\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \;,$$

where C is an absolute constant (depending only on Ω).

Proof. Multiplying (3.2) by $\bar{u}_\rho$ (the complex conjugate of u_ρ) and integrating by parts gives

$$-\int_\Omega A_\rho \nabla u_\rho \nabla \bar{u}_\rho\, dx + \omega^2 \int_\Omega q_\rho u_\rho \bar{u}_\rho\, dx = -\int_{\partial\Omega} (A_\rho \nabla u_\rho)\cdot\nu\, \bar{u}_\rho\, d\sigma_x \;.$$

The first term on the left hand side is real. Therefore taking the imaginary part of each side (and remembering that $A_\rho = I$ near $\partial\Omega$) we get

$$\begin{aligned}\omega^2\beta \int_{B_{2\rho}\setminus\overline{B_\rho}} |u_\rho|^2\, dx &= -\mathrm{Im}\left(\int_{\partial\Omega} \frac{\partial u_\rho}{\partial\nu}\cdot \bar{u}_\rho\, d\sigma_x\right) \\ &= -\mathrm{Im}\left(\int_{\partial\Omega} \psi(\bar{u}_\rho - \bar{u}_0)\, d\sigma_x\right) \;. \end{aligned} \tag{3.4}$$

For the second equality we have used that $\partial u_\rho/\partial\nu = \psi$, and the fact that

$$\int_{\partial\Omega} \psi \bar{u}_0\, d\sigma_x = \int_\Omega |\nabla u_0|^2\, dx - \omega^2 \int_\Omega |u_0|^2\, dx$$

is real. The assertion of the lemma is an immediate consequence of (3.4). □

The functions u_0 and u_ρ solve the same PDE in $\Omega\setminus\overline{B_{2\rho}}$, with the same Neumann data at the outer boundary $\partial\Omega$. We will compare them in Sections 3.2 and 3.3 using elliptic estimates on this punctured domain. So it is crucial to control u_ρ at $\partial B_{2\rho}$.

We achieve such control (in the $H^{-1/2}$ norm) by combining the last result with a duality argument.

Lemma 3.3. *The solutions of (3.1) and (3.2) satisfy*

$$\|u_\rho(\rho\ \cdot)\|^2_{H^{-1/2}(\partial B_2)} \leq C\frac{[(1+\beta)\omega^2\rho^2+1]^2}{\omega^2\beta}\rho^{-N}\|\psi\|_{H^{-1/2}(\partial\Omega)}\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \ ,$$

where C is an absolute constant (depending only on Ω).

Proof. We use the fact that

$$\|u_\rho(\rho\ \cdot)\|_{H^{-1/2}(\partial B_2)} = \sup_{\|\phi\|_{H^{1/2}(\partial B_2)}\leq 1} \left|\int_{\partial B_2} u_\rho(\rho x)\phi(x)\, d\sigma_x\right| \ .$$

Now, for any $\phi \in H^{1/2}(\partial B_2)$ there exists $w \in H^2(B_2)$ such that

$$\begin{aligned} &(a) \quad w = 0 \text{ on } \partial B_2\ ,\ \frac{\partial w}{\partial\nu} = \phi \text{ on } \partial B_2\ ,\\ &(b) \quad \|w\|_{H^2(B_2)} \leq C\|\phi\|_{H^{1/2}(\partial B_2)}\ ,\\ &(c) \quad w \text{ vanishes inside } B_1\ \ . \end{aligned}$$

Using this w we have

$$\int_{\partial B_2} u_\rho(\rho x)\phi(x)\, d\sigma_x = \int_{\partial B_2} u_\rho(\rho x)\frac{\partial w}{\partial\nu}\, d\sigma_x \ ,$$

whence after integration by parts

$$\begin{aligned} \int_{\partial B_2} u_\rho(\rho x)\phi(x)\ d\sigma_x &= \rho\int_{B_2}\nabla u_\rho(\rho x)\nabla w\ dx + \int_{B_2} u_\rho(\rho x)\Delta w\ dx\\ &= -\rho^2\int_{B_2}\Delta u_\rho(\rho x) w\ dx + \int_{B_2} u_\rho(\rho x)\Delta w\ dx \ . \end{aligned}$$

Since w vanishes in B_1 and $\Delta u_\rho + (1+i\beta)\omega^2 u_\rho = 0$ in $B_{2\rho}\setminus\overline{B_\rho}$, we conclude that

$$\begin{aligned} \left|\int_{\partial B_2} u_\rho(\rho x)\phi(x) d\ \sigma_x\right| &\leq \omega^2(1+\beta)\rho^2\left(\int_{1<|x|<2}|u_\rho|^2(\rho x)\right)^{\frac{1}{2}}\|w\|_{L^2(B_2)}\\ &\qquad + \left(\int_{1<|x|<2}|u_\rho|^2(\rho x)\right)^{\frac{1}{2}}\|w\|_{H^2(B_2)}\\ &\leq C[\omega^2(1+\beta)\rho^2+1]\|u_\rho(\rho\ \cdot)\|_{L^2(1<|x|<2)}\|\phi\|_{H^{1/2}(\partial B_2)} \ . \end{aligned}$$

Maximizing over ϕ subject to $\|\phi\|_{H^{1/2}(\partial B_2)} \leq 1$ and using the relation

$$\|u_\rho(\rho\ \cdot)\|_{L^2(B_2\setminus\overline{B_1})} = \rho^{-N/2}\|u_\rho\|_{L^2(B_{2\rho}\setminus\overline{B_\rho})}$$

we conclude that

$$\|u_\rho(\rho\ \cdot)\|_{H^{-1/2}(\partial B_2)} \leq C[\omega^2(1+\beta)\rho^2+1]\rho^{-N/2}\|u_\rho\|_{L^2(B_{2\rho}\setminus\overline{B_\rho})} \ . \tag{3.5}$$

Squaring both sides and combining the result with Lemma 3.2 leads easily to the desired estimate. □

3.2 Estimates for Helmholtz on the punctured domain

As noted above, u_0 and u_ρ solve the same PDE in $\Omega \setminus \overline{B_{2\rho}}$, with the same Neumann data at the outer boundary $\partial\Omega$. If in addition their values are similar at the inner boundary $\partial B_{2\rho}$, then u_0 should be globally close to u_ρ. The following Lemma makes this rigorous. For notational simplicity we take the inclusion to be B_r rather than $B_{2\rho}$.

Lemma 3.4. *Suppose $-\omega^2$ is not an eigenvalue of the Laplacian on Ω with Neumann boundary condition. There are constants r_0 and C with the following property: suppose $r < r_0$, suppose u_0 solves (3.1) with boundary data $\psi \in H^{-1/2}(\partial\Omega)$, and suppose u_r solves*

$$\begin{cases} \Delta u_r + \omega^2 u_r = 0 & \text{in } \Omega \setminus \overline{B_r} \\ u_r = \varphi & \text{on } \partial B_r \\ \dfrac{\partial u_r}{\partial \nu} = \psi & \text{on } \partial\Omega \end{cases} \tag{3.6}$$

using the same Neumann data ψ as for u_0 on $\partial\Omega$, and Dirichlet data $\varphi \in H^{1/2}(\partial B_r)$, then

$$\|u_r - u_0\|_{H^{1/2}(\partial\Omega)} \le Ce(r)\|(\varphi - u_0)(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ , \tag{3.7}$$

where $e(r)$ is given by (2.12). The constants r_0 and C depend on ω and Ω, but they are entirely independent of ψ, φ, and r.

Proof. We shall show in Section 3.4 that if Helmholtz's equation is well-posed on Ω, then it is also well-posed on $\Omega \setminus \overline{B_r}$ when r is sufficiently small and ∂B_r carries a homogeneous Dirichlet condition. In particular, if w solves

$$(\Delta + \omega^2)w = F \text{ in } \Omega \setminus \overline{B_r}\ , \ \ \frac{\partial w}{\partial \nu} = f \text{ on } \partial\Omega\ , \ \ w = 0 \text{ on } \partial B_r\ , \tag{3.8}$$

then

$$\|w\|_{H^1(\Omega\setminus\overline{B_r})} \le C\left(\|F\|_{L^2(\Omega\setminus\overline{B_r})} + \|f\|_{H^{-1/2}(\partial\Omega)}\right) \ , \tag{3.9}$$

with C independent of r.

We want to estimate $u_r - u_0$ using (3.9). It isn't zero at ∂B_r, but we can fix this by subtracting a harmonic function. We shall show in Section 3.5 that there is a solution of $\Delta V = 0$ in $\Omega \setminus \overline{B_r}$ with $V = \varphi - u_0$ on ∂B_r satisfying

$$\begin{aligned} \|\frac{\partial}{\partial \nu} V\|_{L^2(\partial\Omega)} &\le Ce(r)\|(\varphi - u_0)(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \\ \|V\|_{H^{1/2}(\partial\Omega)} &\le Ce(r)\|(\varphi - u_0)(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \\ \|V\|_{L^2(\Omega\setminus\overline{B_r})} &\le Ce(r)\|(\varphi - u_0)(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \end{aligned} \tag{3.10}$$

(see Proposition 3.8). The function $w_r = u_r - u_0 - V$ satisfies (3.8) with $F = -\omega^2 V$ and $f = -\partial V/\partial \nu$. So the estimate (3.9) gives

$$\begin{aligned}
\|u_r - u_0\|_{H^{1/2}(\partial\Omega)} &\leq \|w_r\|_{H^{1/2}(\partial\Omega)} + \|V\|_{H^{1/2}(\partial\Omega)} \\
&\leq C\|w_r\|_{H^1(\Omega\setminus\overline{B_r})} + \|V\|_{H^{1/2}(\partial\Omega)} \\
&\leq C\left(\|\omega^2 V\|_{L^2(\Omega\setminus\overline{B_r})} + \|\frac{\partial}{\partial\nu}V\|_{H^{-1/2}(\partial\Omega)} + \|V\|_{H^{1/2}(\partial\Omega)}\right) \\
&\leq Ce(r)\|(\varphi - u_0)(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \quad ,
\end{aligned}$$

which is the desired estimate. □

3.3 Proof of Theorem 3.1

Theorem 3.1 follows by elementary manipulation from Lemmas 3.3 and 3.4:

Proof of Theorem 3.1. Lemma 3.4 with $r = 2\rho$ and $\varphi = u_\rho|_{\partial B_{2\rho}}$ gives

$$\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \leq Ce(\rho)\|(u_\rho - u_0)(\rho\,\cdot)\|_{H^{-1/2}(\partial B_2)} \quad .$$

Therefore by the triangle inequality

$$\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \leq Ce(\rho)\left(\|u_0(\rho\,\cdot)\|_{H^{-1/2}(\partial B_2)} + \|u_\rho(\rho\,\cdot)\|_{H^{-1/2}(\partial B_2)}\right) \quad .$$

The first term is easy to estimate, using the well-posedness of the PDE on Ω and elliptic regularity:

$$\|u_0(\rho\,\cdot)\|_{H^{-1/2}(\partial B_2)} \leq C\|u_0(\rho\,\cdot)\|_{L^\infty(\partial B_2)} \leq C\|\psi\|_{H^{-1/2}(\partial\Omega)}.$$

To estimate the second term we apply Lemma 3.3. Since $\beta = d_0\rho^{-2}$ by hypothesis, the conclusion of Lemma 3.3 is

$$\|u_\rho(\rho\,\cdot)\|_{H^{-1/2}(\partial B_2)} \leq C_2\rho^{(2-N)/2}\|\psi\|^{1/2}_{H^{-1/2}(\partial\Omega)}\|u_\rho - u_0\|^{1/2}_{H^{1/2}(\partial\Omega)} \tag{3.11}$$

where C_2 depends only on d_0, ω, and Ω. The right hand side is bounded, for $\varepsilon > 0$, by

$$\begin{aligned}
C_2\rho^{\frac{2-N}{2}}&\left(\frac{\rho^{(2-N)/2}e(\rho)}{4\varepsilon}\|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)} + \frac{\varepsilon}{\rho^{(2-N)/2}e(\rho)}\|u_\rho - u_0\|_{H^{\frac{1}{2}}(\partial\Omega)}\right) \\
&= C_2\frac{\rho^{2-N}e(\rho)}{4\varepsilon}\|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)} + C_2\frac{\varepsilon}{e(\rho)}\|u_\rho - u_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \quad .
\end{aligned}$$

Combining these results we get

$$\begin{aligned}
\|u_\rho - u_0\|_{H^{\frac{1}{2}}(\partial\Omega)} &\leq Ce(\rho)\|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)} \\
&\quad + C_2 e(\rho)\frac{\rho^{2-N}e(\rho)}{4\varepsilon}\|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)} + C_2\varepsilon\|u_\rho - u_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \quad .
\end{aligned}$$

We now choose ε so that $C_2\varepsilon < 1$. Then the last term on the right hand side can be absorbed by the left hand side, and we conclude that

$$\|u_\rho - u_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \le Ce(\rho)\|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)} + Ce(\rho)\rho^{2-N}e(\rho)\|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)}$$

with C independent of ρ, ψ, and the values of A_ρ and q_ρ in B_ρ. When $N=2$, $\rho^{2-N}e(\rho) = e(\rho) \to 0$ as $\rho \to 0$. When $N=3$, $\rho^{2-N}e(\rho) = 1$ is constant. In either case we get

$$\|u_\rho - u_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \le Ce(\rho)\|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)} \;,$$

which is the desired conclusion. □

3.4 Uniform well-posedness for the punctured domain

This section provides the proof of (3.9). Actually we shall prove a slightly stronger statement, in which $\|F\|_{L^2(\Omega\setminus\overline{B_r})}$ is replaced by a weaker norm (see equation (3.16)). A concise statement of our well-posedness result is given at the end of the section (see Proposition 3.5).

We are concerned with the PDE

$$\left\{\begin{array}{ll} \Delta w_0 + \omega^2 w_0 = F & \text{in } \Omega \\ \dfrac{\partial w_0}{\partial \nu} = f & \text{at } \partial\Omega \end{array}\right. \tag{3.12}$$

and its analogue (3.8) in the punctured domain $\Omega\setminus\overline{B_r}$. Since ω is real, it suffices to consider the case when F, f and w_0 are real-valued. (The corresponding estimates for complex-valued solutions are immediate, by considering the real and imaginary parts separately.)

We begin by reviewing the equivalence of well-posedness and the "inf-sup condition." For any domain Ω, it is well-known (and fairly easy to prove) that the condition

$$\inf_{\substack{w\in H^1(\Omega)\\ \|w\|_{H^1}=1}} \sup_{\substack{v\in H^1(\Omega)\\ \|v\|_{H^1}\le 1}} \left| \int_\Omega \nabla w\cdot\nabla v\,dx - \omega^2\int_\Omega wv\,dx \right| \ge c_0 > 0 \tag{3.13}$$

is necessary and sufficient for the wellposedness of the boundary value problem (3.12) (see for instance [2]). To be quite precise, (3.13) is necessary and sufficient for the existence of a bounded inverse $H^1(\Omega)' \to H^1(\Omega)$ to the linear operator associated with the bilinear form

$$B(w,v) = \int_\Omega \nabla w\cdot\nabla v\,dx - \omega^2\int_\Omega wv\,dx \;,$$

which in turn yields a (unique) weak solution of (3.12) satisfying

$$\|w_0\|_{H^1(\Omega)} \le C_0\left(\|F\|_{H^1(\Omega)'} + \|f\|_{H^{-1/2}(\partial\Omega)}\right) \;.$$

Here $H^1(\Omega)'$ is the dual of $H^1(\Omega)$. Elliptic regularity implies that w_0 is a strong solution of (3.12) provided F and f are sufficiently regular. The requirement that

$-\omega^2$ not be an eigenvalue for the Laplacian on Ω with Neumann boundary condition is equivalent to this notion of wellposedness.

The situation for a punctured domain $\Omega \setminus \overline{B_r}$ with $w_r = 0$ at ∂B_r is similar (and equally standard). If $H^1_*(\Omega \setminus \overline{B_r})$ denotes the space

$$H^1_*(\Omega \setminus \overline{B_r}) = H^1(\Omega \setminus \overline{B_r}) \cap \{ \ w|_{\partial B_r} = 0 \ \}$$

equipped with the H^1-norm, then the "inf-sup" condition

$$\inf_{\substack{w \in H^1_*(\Omega\setminus\overline{B_r}) \\ \|w\|_{H^1}=1}} \ \sup_{\substack{v \in H^1_*(\Omega\setminus\overline{B_r}) \\ \|v\|_{H^1}\le 1}} \left| \int_{\Omega\setminus\overline{B_r}} \nabla w \cdot \nabla v \, dx - \omega^2 \int_{\Omega\setminus\overline{B_r}} wv \, dx \right| \ge c_1 > 0 \tag{3.14}$$

is necessary and sufficient for the unique solvability of the boundary value problem

$$(\Delta + \omega^2) w_r = F \text{ in } \Omega \setminus \overline{B_r} \, , \ \ \frac{\partial w_r}{\partial \nu} = f \text{ on } \partial\Omega \, , \ \ w_r = 0 \text{ on } \partial B_r \, , \tag{3.15}$$

with the associated estimate

$$\|w_r\|_{H^1(\Omega\setminus\overline{B_r})} \le C_1 \left(\|F\|_{H^1_*(\Omega\setminus\overline{B_r})'} + \|f\|_{H^{-1/2}(\partial\Omega)} \right) . \tag{3.16}$$

Our task is now clear. To prove (3.16), we must show that if Ω satisfies the inf-sup condition (3.13) then $\Omega \setminus \overline{B_r}$ satisfies the inf-sup condition (3.14) when r is sufficiently small, with a constant c_1 that remains uniform as $r \to 0$.

So suppose (3.13) holds, and consider any $w_* \in H^1_*(\Omega \setminus \overline{B_r})$ such that $\|w_*\|_{H^1} = 1$. Extend w_* by 0 to all of Ω, and call the extension $\tilde{w}$. Then $\tilde{w} \in H^1(\Omega)$, with $\|\tilde{w}\|_{H^1(\Omega)} = 1$. So by (3.13) there exists $v \in H^1(\Omega)$ with

$$\left| \int_\Omega \nabla \tilde{w} \cdot \nabla v \, dx - \omega^2 \int_\Omega \tilde{w} v \, dx \right| \ge \frac{c_0}{2} \ \text{ and } \|v\|_{H^1(\Omega)} \le 1 \ \ .$$

Let P denote orthogonal projection onto $H^1(\Omega) \cap \{w = 0 \text{ on } B_r\}$, using the $H^1(\Omega)$ inner-product, and define $v_* \in H^1_*(\Omega \setminus \overline{B_r})$ by

$$v_* = P(v)|_{\Omega\setminus\overline{B_r}} \ \ .$$

Since v_* is (the restriction of) a projection

$$\|v_*\|_{H^1(\Omega\setminus\overline{B_r})} \le \|v\|_{H^1(\Omega)} \le 1 \ \ . \tag{3.17}$$

Decomposing $\int_{\Omega\setminus\overline{B_r}} \nabla w_* \cdot \nabla v_* \, dx - \omega^2 \int_{\Omega\setminus\overline{B_r}} w_* v_* \, dx$ as

$$\int_{\Omega\setminus\overline{B_r}} \nabla w_* \cdot \nabla v \, dx - \omega^2 \int_{\Omega\setminus\overline{B_r}} w_* v \ dx \\ + \int_{\Omega\setminus\overline{B_r}} \nabla w_* \cdot \nabla (v_* - v) \, dx - \omega^2 \int_{\Omega\setminus\overline{B_r}} w_* (v_* - v) \, dx \ \ ,$$

we have

$$\left| \int_{\Omega\setminus\overline{B_r}} \nabla w_* \cdot \nabla v \, dx - \omega^2 \int_{\Omega\setminus\overline{B_r}} w_* v \, dx \right| = \left| \int_\Omega \nabla \tilde{w} \cdot \nabla v \, dx - \omega^2 \int_\Omega \tilde{w} v \, dx \right| \ge \frac{c_0}{2} \ \ ,$$

from which it follows that

$$\left|\int_{\Omega\setminus\overline{B_r}} \nabla w_* \cdot \nabla v_*\, dx - \omega^2 \int_{\Omega\setminus\overline{B_r}} w_* v_*\, dx\right| \geq \frac{c_0}{2} - \left|\int_{\Omega\setminus\overline{B_r}} \nabla w_* \cdot \nabla(v_* - v)\ dx - \omega^2 \int_{\Omega\setminus\overline{B_r}} w_*(v_* - v)\, dx\right| \ . \tag{3.18}$$

Our essential task is thus to show that the expression in absolute values on the right hand side of (3.18) is small. For any $\phi_* \in H^1_*(\Omega\setminus\overline{B_r})$ let $\tilde{\phi} \in H^1(\Omega) \cap \{\, w = 0 \text{ on } B_r \,\}$ denote its extension (by zero) to all of Ω. Then

$$\int_{\Omega\setminus\overline{B_r}} \nabla(v_* - v)\cdot\nabla\phi_*\, dx + \int_{\Omega\setminus\overline{B_r}} (v_* - v)\phi_*\, dx = \int_\Omega \nabla(P(v) - v)\nabla\tilde{\phi}\, dx + \int_\Omega (P(v) - v)\tilde{\phi}\, dx = 0 \ , \tag{3.19}$$

and as a consequence (using $\phi_* = w_*$)

$$\int_{\Omega\setminus\overline{B_r}} \nabla w_* \cdot \nabla(v_* - v)\, dx - \omega^2 \int_{\Omega\setminus\overline{B_r}} w_*(v_* - v)\, dx = -(\omega^2 + 1)\int_{\Omega\setminus\overline{B_r}} w_*(v_* - v)\, dx \ .$$

Inserting this into (3.18), we get

$$\left|\int_{\Omega\setminus\overline{B_r}} \nabla w_* \cdot \nabla v_*\, dx - \omega^2 \int_{\Omega\setminus\overline{B_r}} w_* v_*\, dx\right| \geq \frac{c_0}{2} - (\omega^2 + 1)\left|\int_{\Omega\setminus\overline{B_r}} w_*(v_* - v)\, dx\right| \ . \tag{3.20}$$

We shall show below (see Lemma 3.7) the existence of constants C and r_0 such that

$$\|v_* - v\|_{L^2(\Omega\setminus\overline{B_r})} \leq Ce(r)^{1/2}\|v\|_{H^1(\Omega\setminus\overline{B_r})} \text{ provided } 0 < r < r_0 \ . \tag{3.21}$$

Accepting this for a moment, the rest of the argument is easy. Combining (3.20) with (3.21), and recalling that $\|w_*\|_{H^1(\Omega\setminus\overline{B_r})} = 1$ and $\|v\|_{H^1(\Omega)} \leq 1$, we get

$$\begin{aligned}
&\left|\int_{\Omega\setminus\overline{B_r}} \nabla w_* \cdot \nabla v_*\, dx - \omega^2 \int_{\Omega\setminus\overline{B_r}} w_* v_*\, dx\right| \\
&\qquad \geq \frac{c_0}{2} - (\omega^2 + 1)\|w_*\|_{L^2(\Omega\setminus\overline{B_r})}\|v_* - v\|_{L^2(\Omega\setminus\overline{B_r})} \\
&\qquad \geq \frac{c_0}{2} - Ce(r)^{1/2}\|v\|_{H^1(\Omega\setminus\overline{B_r})} \\
&\qquad \geq \frac{c_0}{2} - Ce(r)^{1/2} \geq \frac{c_0}{4} > 0
\end{aligned}$$

provided r is sufficiently small (less than $e^{-(4C/c_0)^2}$ for $N = 2$, and less than $(c_0/4C)^2$ for $N = 3$). Thus the “inf-sup” condition (3.14) holds, with a positive constant

c_1 independent of r. In summary, once (3.21) has been established we will have proved

Proposition 3.5. *Suppose $-\omega^2$ is not an eigenfrequency for the Laplacian on Ω with Neumann boundary condition. Then there exists $r_0 > 0$ such that the problem (3.15) has a unique solution for all $0 < r < r_0$ and all $F \in H^1_*(\Omega \setminus \overline{B_r})'$, $f \in H^{-1/2}(\partial\Omega)$. Furthermore, the solution to (3.15) satisfies (3.16) with a constant C_1 that is independent of r.*

The rest of this subsection is devoted to proving (3.21). The proof, presented in Lemma 3.7, makes use of the following correctly-scaled trace estimate.

Lemma 3.6. *Suppose Ω contains B_{2r_0}, $r_0 < 1$. Assume the spatial dimension is $N = 2$ or 3, and let $e(r)$ be defined by (2.12). Then there is a constant C such that*

$$\|w\|_{L^2(\partial B_r)} \leq C\left(\frac{r^{N-1}}{e(r)}\right)^{1/2} \|w\|_{H^1(\Omega\setminus\overline{B_r})} \quad , \tag{3.22}$$

for any $0 < r < r_0$ and any $w \in H^1(\Omega \setminus \overline{B_r})$.

Proof. We may suppose that w vanishes outside B_{2r_0}. (The general case is easily reduced to this one, by replacing w with $w\chi$ where χ is a smooth function such that $\chi = 1$ on B_{r_0} and $\chi = 0$ off B_{2r_0}.) Our plan is to decompose w as

$$w = w - \frac{1}{|\partial B_r|}\int_{\partial B_r} w\, d\sigma + \frac{1}{|\partial B_r|}\int_{\partial B_r} w\, d\sigma \quad ,$$

and to prove that

$$\|w - \frac{1}{|\partial B_r|}\int_{\partial B_r} w\, d\sigma\|_{L^2(\partial B_r)} \leq Cr^{1/2}\|w\|_{H^1(\Omega\setminus\overline{B_r})} \quad \text{, and} \tag{3.23}$$

$$\|\frac{1}{|\partial B_r|}\int_{\partial B_r} w\, d\sigma\|_{L^2(\partial B_r)} \leq C\left(\frac{r^{N-1}}{e(r)}\right)^{1/2} \|w\|_{H^1(\Omega\setminus\overline{B_r})} \quad . \tag{3.24}$$

The desired result (3.22) is an immediate consequence of these inequalities.

To prove (3.23), consider the function

$$w_r(y) = w(ry) - \frac{1}{|\partial B_r|}\int_{\partial B_r} w\, d\sigma \quad .$$

It is defined on $\left(\frac{1}{r}\Omega\right) \setminus \overline{B_1}$, and it has mean value zero on the inner boundary ∂B_1. Therefore

$$\begin{aligned}
\frac{1}{r^{(N-1)/2}}\|w - \frac{1}{|\partial B_r|}\int_{\partial B_r} w\, d\sigma\|_{L^2(\partial B_r)} &= \|w_r\|_{L^2(\partial B_1)} \\
&\leq C\|\nabla w_r\|_{L^2(B_2\setminus\overline{B_1})} \\
&\leq C\|\nabla w_r\|_{L^2(\left(\frac{1}{r}\Omega\right)\setminus\overline{B_1})} \\
&= Cr^{(2-N)/2}\|\nabla w\|_{L^2(\Omega\setminus\overline{B_r})} \quad .
\end{aligned}$$

This gives (3.23).

To prove (3.24), we note that $1/e(|x|)$ is a harmonic function, with

$$\nabla \frac{1}{e(|x|)} = -\frac{x}{|x|^N} \quad , \quad |x| < 1 \quad , \text{ and } \frac{\partial}{\partial \nu} \frac{1}{e(|x|)}\Big|_{|x|=r} = -\frac{1}{r^{N-1}} \quad , \quad r < 1 \quad ,$$

where $\partial/\partial\nu$ is the normal (radial) derivative at the boundary of the ball of radius r. Therefore

$$\begin{aligned}
\left| \int_{\partial B_r} w \, d\sigma \right| &= \left| r^{N-1} \int_{|x|=r} w \frac{\partial}{\partial \nu} \frac{1}{e(|x|)} \, d\sigma \right| \\
&= \left| r^{N-1} \int_{r<|x|<2r_0} \nabla w \cdot \nabla \Big(\frac{1}{e(|x|)}\Big) \, dx \right| \\
&\le r^{N-1} \left(\int_{r<|x|<2r_0} |\nabla w|^2 \, dx \right)^{1/2} \left(\int_{r<|x|<2r_0} \Big|\nabla \Big(\frac{1}{e(|x|)}\Big)\Big|^2 \, dx \right)^{1/2} \\
&\le C r^{N-1} |e(r)|^{-1/2} \|w\|_{H^1(\Omega \setminus \overline{B_r})} \quad .
\end{aligned}$$

This gives

$$\left| \frac{1}{|\partial B_r|} \int_{\partial B_r} w \, d\sigma \right| \le C |e(r)|^{-1/2} \|w\|_{H^1(\Omega \setminus \overline{B_r})} \quad ,$$

which is equivalent to (3.24). □

The following lemma estimates the distance between an arbitrary function in $H^1(\Omega)$ and its "projection" to $H^1_*(\Omega \setminus \overline{B_r})$. Its conclusion is precisely our assertion (3.21).

Lemma 3.7. *Suppose Ω contains a ball of radius $2r_0$, $r_0 < 1$. Assume the spatial dimension is $N = 2$ or 3, and let $e(r)$ be defined by (2.12). For any $v \in H^1(\Omega)$, let $P(v)$ denote the orthogonal projection of w onto $H^1(\Omega) \cap \{ v = 0 \text{ on } B_r \}$ using the $H^1(\Omega)$ inner-product, and define $v_* \in H^1_*(\Omega \setminus \overline{B_r})$ by*

$$v_* = P(v)|_{\Omega \setminus \overline{B_r}} \quad .$$

Then there is a constant C (independent of v and r) such that

$$\|v_* - v\|_{L^2(\Omega \setminus \overline{B_r})} \le C e(r)^{1/2} \|v\|_{H^1(\Omega \setminus \overline{B_r})} \quad , \quad 0 < r < r_0 \quad .$$

Proof. Let $V = v_* - v \in H^1(\Omega \setminus \overline{B_r})$. We already know from (3.19) that

$$\int_{\Omega \setminus \overline{B_r}} \nabla V \cdot \nabla \phi_* \, dx + \int_{\Omega \setminus \overline{B_r}} V \phi_* \, dx = 0 \quad \forall \phi_* \in H^1_*(\Omega \setminus \overline{B_r})$$

or, in the equivalent "strong" formulation

$$-\Delta V + V = 0 \text{ in } \Omega \setminus \overline{B_r} \ , \ \ V = -v \text{ on } \partial B_r \ , \ \ \frac{\partial V}{\partial \nu} = 0 \text{ on } \partial \Omega \quad .$$

We shall prove in Section 3.5 that there exists W in $H^1(\Omega\setminus\overline{B_r})$ such that $\Delta W = 0$ in $\Omega\setminus\overline{B_r}$, $W = v$ on ∂B_r, and

$$\|\frac{\partial W}{\partial \nu}\|_{L^2(\partial\Omega)} \leq Ce(r)\|v(r\,\cdot)\|_{L^2(\partial B_1)} = C\frac{e(r)}{r^{(N-1)/2}}\|v\|_{L^2(\partial B_r)} \;, \tag{3.25}$$

$$\|W\|_{L^2(\Omega\setminus\overline{B_r})} \leq Ce(r)\|v(r\,\cdot)\|_{L^2(\partial B_1)} = C\frac{e(r)}{r^{(N-1)/2}}\|v\|_{L^2(\partial B_r)} \;. \tag{3.26}$$

(see Proposition 3.8). The function $W_1 = V + W$ satisfies

$$-\Delta W_1 + W_1 = W \text{ in } \Omega\setminus\overline{B_r}\;,\;\; \frac{\partial W_1}{\partial\nu} = \frac{\partial W}{\partial\nu} \text{ on } \partial\Omega\;,\;\; W_1 = 0 \text{ on } \partial B_r \;.$$

Multiplication by W_1 and integration by parts gives

$$\begin{aligned}
&\int_{\Omega\setminus\overline{B_r}} |\nabla W_1|^2 + |W_1|^2\,dx \\
&\qquad = \int_{\partial\Omega}\frac{\partial W}{\partial\nu}W_1\,d\sigma + \int_{\Omega\setminus\overline{B_r}} WW_1\,dx \\
&\qquad \leq C\left(\|\frac{\partial W}{\partial\nu}\|_{L^2(\partial\Omega)} + \|W\|_{L^2(\Omega\setminus\overline{B_r})}\right)\times\|W_1\|_{H^1(\Omega\setminus\overline{B_r})} \;,
\end{aligned}$$

whence by (3.25) and (3.26)

$$\begin{aligned}
\|W_1\|_{H^1(\Omega\setminus\overline{B_r})} &\leq C\left(\|\frac{\partial W}{\partial\nu}\|_{L^2(\partial\Omega)} + \|W\|_{L^2(\Omega\setminus\overline{B_r})}\right) \\
&\leq C\frac{e(r)}{r^{(N-1)/2}}\|v\|_{L^2(\partial B_r)} \;.
\end{aligned}$$

Since $V = -W + W_1$, this estimate combines with (3.26) to give

$$\|V\|_{L^2(\Omega\setminus\overline{B_r})} = \|-W + W_1\|_{L^2(\Omega\setminus\overline{B_r})} \leq C\frac{e(r)}{r^{(N-1)/2}}\|v\|_{L^2(\partial B_r)}.$$

Applying Lemma 3.6 we conclude that

$$\|V\|_{L^2(\Omega\setminus\overline{B_r})} \leq Ce(r)^{1/2}\|v\|_{H^1(\Omega\setminus\overline{B_r})} \;,$$

which is exactly the assertion of Lemma 3.7. □

3.5 Some results on harmonic extensions

We used certain estimates on harmonic extensions in Sections 3.2 and 3.4, namely equations (3.10), (3.25), and (3.26). This section provides the proofs. As in Section 3.4, it suffices to consider real-valued functions.

There are (at least) two different approaches. One uses separation of variables, making use of the fact that the desired estimates are on the exterior of a ball. The other uses potential theory; it has the advantage of working just as well when the ball is replaced by a more general inclusion. Rather than stick to one approach, we shall present them both – giving the separation-of-variables-based argument in 2D, and the potential-theory-based argument in 3D.

Proposition 3.8. *Assume Ω contains B_{2r_0}, $r_0 < 1$, and suppose $N = 2$ or $N = 3$. Then there is a constant C (depending only on Ω and r_0) with the following property: for any $r < r_0$, and any $g \in H^{1/2}(\partial B_r)$, there is a solution of*

$$\Delta W = 0 \text{ in } \mathbb{R}^N \setminus \overline{B_r} \ , \ \ W = g \text{ on } \partial B_r$$

such that

$$\|\frac{\partial}{\partial \nu} W\|_{L^2(\partial\Omega)} \leq Ce(r)\|g(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ \ , \tag{3.27}$$

$$\|W\|_{H^{1/2}(\partial\Omega)} \leq Ce(r)\|g(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ \ , \tag{3.28}$$

$$\|W\|_{L^2(\Omega\setminus\overline{B_r})} \leq Ce(r)\|g(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ \ , \tag{3.29}$$

with $e(r)$ defined by (2.12).

Proof for $N = 2$ using separation of variables. Consider the Fourier representation of g:

$$g(r\cos\theta, r\sin\theta) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

The function $g(r\,\cdot)$ is defined on ∂B_1, and

$$c\left(|a_0| + \sum_{n=1}^{\infty}\frac{a_n^2+b_n^2}{n}\right)^{1/2} \leq \|g(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \leq C\left(|a_0| + \sum_{n=1}^{\infty}\frac{a_n^2+b_n^2}{n}\right)^{1/2}$$

(see e.g. [15] for a concise discussion of this well-known fact). The obvious harmonic extension is

$$W = a_0\frac{\log R}{\log r} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n \sin n\theta)r^n R^{-n}$$

where $R = |x|$. We claim it satisfies the desired estimates.

Since high modes decay quickly, our estimates will be driven by the lowest modes. Therefore it is convenient to write $W = W_0 + W_1 + \tilde{W}$ with

$$W_0 = a_0\frac{\log R}{\log r} \ \ , \quad W_1 = (a_1\cos\theta + b_1\sin\theta)rR^{-1} \ \ ,$$

and $\tilde{W} = W - W_0 - W_1$. We will show that each of the functions W_0, W_1, and $\tilde{W}$ satisfies (3.27)–(3.29).

For W_0, we observe that

$$\|\frac{\partial}{\partial\nu}\log|x|\|_{L^2(\partial\Omega)} \leq C \ , \ \ \|\log|x|\|_{H^{1/2}(\partial\Omega)} \leq C \ , \ \text{ and } \|\log|x|\|_{L^2(\Omega\setminus\overline{B_r})} \leq C \ \ .$$

Therefore (remembering that $e(r) = 1/|\log r|$ when $N = 2$)

$$\begin{aligned}\|\frac{\partial}{\partial\nu}W_0\|_{L^2(\partial\Omega)} + \|W_0\|_{H^{1/2}(\partial\Omega)} + \|W_0\|_{L^2(\Omega\setminus\overline{B_r})} &\leq Ce(r)|a_0| \\ &\leq Ce(r)\|g(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ \ ,\end{aligned}$$

i.e. W_0 satisfies (3.27)–(3.29).

For W_1, we observe that

$$\|\frac{\partial}{\partial\nu}\frac{1}{|x|}\|_{L^2(\partial\Omega)} \le C \quad \text{and} \quad \|\frac{1}{|x|}\|_{H^{1/2}(\partial\Omega)} \le C \ ,$$

so

$$\|\frac{\partial}{\partial\nu}W_1\|_{L^2(\partial\Omega)} + \|W_1\|_{H^{1/2}(\partial\Omega)} \le Cr\|g(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ .$$

For the L^2 norm, suppose $\Omega \subset B_{r_1}$. Then

$$\|\frac{1}{|x|}\|^2_{L^2(\Omega\setminus\overline{B_r})} \le C\int_r^{r_1}\frac{1}{R^2}RdR \le C|\log r| \ ,$$

so

$$\|W_1\|_{L^2(\Omega\setminus\overline{B_r})} \le Cr|\log r|^{1/2}\|g(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ .$$

Since $r \ll r|\log r|^{1/2} \ll e(r)$ as $r \to 0$, we conclude that W_1 satisfies (3.27)–(3.29).

For $\tilde{W} = \sum_{n=2}^{\infty}(a_n\cos n\theta + b_n \sin n\theta)r^nR^{-n}$ we use the fact that Ω contains B_{2r_0} and the hypothesis $r < r_0$ to see that

$$\begin{aligned}\|\frac{\partial\tilde{W}}{\partial\nu}\|_{L^2(\partial\Omega)} &\le C\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,n\left(\frac{r}{2r_0}\right)^n \\ &\le Cr^2\left(\sum_{n=2}^{\infty}\frac{|a_n|^2+|b_n|^2}{n}\right)^{1/2} \\ &\le Cr^2\|g(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ . \end{aligned} \tag{3.30}$$

Similarly

$$\|\tilde{W}\|_{H^{1/2}(\partial\Omega)} \le \|\tilde{W}\|_{H^1(\partial\Omega)} \le Cr^2\|g(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ . \tag{3.31}$$

As for the L^2 norm, we have

$$\begin{aligned}\|\tilde{W}\|^2_{L^2(\Omega\setminus\overline{B_r})} &\le \|\tilde{W}\|^2_{L^2(\mathbb{R}^2\setminus\overline{B_r})} \\ &\le C\sum_{n=2}^{\infty}(|a_n|^2+|b_n|^2)r^{2n}\int_r^{\infty}R^{-2n+1}\,dR \\ &\le Cr^2\sum_{n=2}^{\infty}(|a_n|^2+|b_n|^2)n^{-1} \\ &\le Cr^2\|g(r\,\cdot)\|^2_{H^{-1/2}(\partial B_1)} \ . \end{aligned} \tag{3.32}$$

Since $r^2 \ll r \ll e(r)$, it follows from (3.30)–(3.32) that $\tilde{W}$ satisfies (3.27)–(3.29). □

Proof for $N = 3$ using potential theory. We decompose $g = g_0 + \tilde{g}$, where

$$g_0 = \frac{1}{|\partial B_r|}\int_{\partial B_r} g\,d\sigma = \frac{1}{|\partial B_1|}\int_{\partial B_1} g(r\,\cdot)\,d\sigma$$

is the mean value of g and $\tilde{g}$ has mean value 0. Notice that

$$|g_0| \leq C\|g(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \quad . \tag{3.33}$$

The obvious choice of W is $W_0 + \tilde{W}$, where

$$W_0(x) = g_0 \frac{r}{|x|}$$

and $\tilde{W}$ is the unique solution of

$$\Delta\tilde{W} = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B_r}\ , \ \ \tilde{W} = \tilde{g} \text{ on } \partial B_r\ , \ \ \tilde{W}(x) \to 0 \text{ as } |x| \to \infty \ . \tag{3.34}$$

To show that W satisfies (3.27)–(3.29), we will show that both W_0 and $\tilde{W}$ satisfy these relations.

For W_0, we observe that

$$\|\frac{\partial}{\partial\nu}\frac{1}{|x|}\|_{L^2(\partial\Omega)} \leq C\ , \ \ \|\frac{1}{|x|}\|_{H^{1/2}(\partial\Omega)} \leq C\ , \ \text{ and } \|\frac{1}{|x|}\|_{L^2(\Omega\setminus\overline{B_r})} \leq C \ .$$

Therefore (remembering that $e(r) = r$ when $N = 3$)

$$\begin{aligned}\|\frac{\partial}{\partial\nu}W_0\|_{L^2(\partial\Omega)} + \|W_0\|_{H^{1/2}(\partial\Omega)} + \|W_0\|_{L^2(\Omega\setminus\overline{B_r})} &\leq Ce(r)|g_0| \\ &\leq Ce(r)\|g(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ ,\end{aligned}$$

using (3.33). Thus W_0 satisfies (3.27)–(3.29).

To estimate $\tilde{W}$ we use the following lemma.

Lemma 3.9. *Let B_1 be the unit ball in $\mathbb{R}^3$, and let $h \in H^{1/2}(\partial B_1)$ have mean value 0. Then the solution V of*

$$\Delta V = 0 \textit{ in } \mathbb{R}^3 \setminus \overline{B_1}\ , \ \ V = h \textit{ on } \partial B_1\ , \ \ V(x) \to 0 \textit{ as } |x| \to \infty \tag{3.35}$$

satisfies, for any $R \geq 2$,

$$\|\nabla V\|_{L^\infty(|x|=R)} \leq \frac{C}{R^3}\|h\|_{H^{-1/2}(\partial B_1)} \ , \tag{3.36}$$

$$\|V\|_{L^\infty(|x|=R)} \leq \frac{C}{R^2}\|h\|_{H^{-1/2}(\partial B_1)} \ , \ \textit{and} \tag{3.37}$$

$$\|V\|_{L^2(B_R\setminus\overline{B_1})} \leq C\|h\|_{H^{-1/2}(\partial B_1)} \ , \tag{3.38}$$

with C independent of R.

Given this Lemma, our task is easy. In fact, by definition $\tilde{W}(x) = V(x/r)$ where V solves (3.35) with $h = \tilde{g}(r\,\cdot)$. Since $B_{2r} \subset \Omega \subset B_{r_1}$ for some r_1, the estimates (3.36) – (3.38) imply, by change of variables and elementary manipulation, that

$$\begin{aligned}\|\frac{\partial}{\partial\nu}\tilde{W}\|_{L^2(\partial\Omega)} &\leq Cr^2\|\tilde{g}(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ , \\ \|\tilde{W}\|_{H^{1/2}(\partial\Omega)} &\leq Cr^2\|\tilde{g}(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ , \\ \|\tilde{W}\|_{L^2(\Omega\setminus\overline{B_r})} &\leq Cr^{3/2}\|\tilde{g}(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \ .\end{aligned}$$

Since $\|\tilde{g}(r\,\cdot)\|_{H^{-1/2}(\partial B_1)} \leq C\|g(r\,\cdot)\|_{H^{-1/2}(\partial B_1)}$ and $r^2 \ll r^{3/2} \ll e(r)$ when $N=3$ it follows that $\tilde{W}$ satisfies (3.27)–(3.29). $\square$

Proof of Lemma 3.9. We shall use the double layer potential representation of V. If G is the "free-space" fundamental solution

$$G(x,y) = -\frac{1}{|\partial B_1||x-y|} = -\frac{1}{4\pi|x-y|} \quad ,$$

then the desired representation is $V = D(\phi)$, where

$$\begin{aligned} D(\phi)(x) &= \int_{\partial B_1} \frac{\partial}{\partial \nu_y} G(x,y)\phi(y)\,d\sigma_y \\ &= \frac{1}{4\pi}\int_{\partial B_1} \frac{(y-x)\cdot y}{|x-y|^3}\phi(y)\,d\sigma_y \end{aligned}$$

for $x \in \mathbb{R}^3 \setminus \partial B_1$, and ϕ is an appropriately chosen density. For points $x \in \partial B_1$, and continuous ϕ, this double layer potential gives rise to the following well-known jump condition

$$\begin{aligned} \lim_{x'\to x, x'\in\mathbb{R}^3\setminus\overline{B_1}} D(\phi)(x) &= -\frac{1}{2}\phi(x) + \frac{1}{4\pi}\int_{\partial B_1} \frac{(y-x)\cdot y}{|x-y|^3}\phi(y)\;d\sigma_y \\ &= -\frac{1}{2}\phi(x) + \frac{1}{8\pi}\int_{\partial B_1} \frac{1}{|x-y|}\phi(y)\;d\sigma_y \\ &= (-\frac{1}{2}+T)\phi(x) \quad . \end{aligned} \tag{3.39}$$

The mapping T is a compact linear operator from $L^2(\partial B_1)$ to itself. Since the kernel is symmetric, T is selfadjoint.

We discuss some additional properties of the operator T. If τ_x is the tangent vector field on ∂B_1 given by $\tau_x = (x_2, -x_1, 0)$, then

$$\nabla_x\left(\frac{1}{|x-y|}\right)\cdot\tau_x = \frac{(y-x)\cdot\tau_x}{|x-y|^3} = \frac{y\cdot\tau_x}{|x-y|^3} = -\frac{x\cdot\tau_y}{|x-y|^3} = -\nabla_y\left(\frac{1}{|x-y|}\right)\cdot\tau_y \quad .$$

It follows, after integration by parts, that

$$\frac{\partial}{\partial\theta_1}T\phi(x) = T\left(\frac{\partial}{\partial\theta_1}\phi\right)(x)$$

where θ_1, $0 \leq \theta_1 < 2\pi$ denotes the azimuthal angle of the standard spherical coordinate system $(\cos\theta_1\sin\theta_2, \sin\theta_1\sin\theta_2, \cos\theta_2)$. Varying the coordinate system, and using the fact that T maps L^2 into itself, we conclude that T maps $H^1(\partial B_1)$ to itself. Using interpolation we conclude that T maps $H^{1/2}(\partial B_1)$ to itself. It follows, since T is L^2-selfadjoint, that T also maps $H^{-1/2}(\partial B_1)$ (the dual of $H^{1/2}(\partial B_1)$) to itself. It is well-known that

$$\mathrm{Ker}\{-\frac{1}{2}+T\} = \{\text{ constants }\}$$

in any of these spaces (see [6] for this assertion in L^2, from which the assertions in $H^{1/2}$ and $H^{-1/2}$ follow easily). Moreover, the full space (L^2, $H^{1/2}$, or $H^{-1/2}$ respectively) may be decomposed as

$$\mathrm{Ker}\{-\frac{1}{2}+T\}\oplus\mathrm{Range}\{-\frac{1}{2}+T\} \quad .$$

Due to the L^2-orthogonality of this decomposition (remember: T is selfadjoint) it now follows that

$$(-\frac{1}{2}+T)\phi = h \quad , \quad h\in L^2(\partial B_1) \quad ,$$

has a solution $\phi\in L^2(\partial B_1)$ iff $\int_{\partial B_1} h = 0$, and furthermore, if we require that $\int_{\partial B_1}\phi = 0$ then

$$\|\phi\|_{L^2(\partial B_1)} \le C\|h\|_{L^2(\partial B_1)} \quad .$$

A similar existence statement and estimate holds with $L^2(\partial B_1)$ replaced by $H^{\pm 1/2}(\partial B_1)$.

We claim that the solution of (3.35) is

$$V(x) = D(\phi)(x) = \frac{1}{4\pi}\int_{\partial B_1}\frac{(y-x)\cdot y}{|x-y|^3}\phi(y)\,d\sigma_y \quad \text{for } x\in\mathbb{R}^3\setminus\overline{B}_1 \quad , \tag{3.40}$$

where ϕ is the solution of $(-\frac{1}{2}+T)\phi = h$. When h is continuous this statement is classical: if h is continuous so is ϕ (see e.g. [6] Proposition 3.14), so (3.39) shows that $D(\phi) = h$ at ∂B_1; moreover it is obvious that $D(\phi)(x)\to 0$ as $|x|\to\infty$. The validity of (3.40) for all $h\in H^{1/2}(\partial B_1)$ with mean value 0 follows easily, by a density argument.

We now estimate V in terms of ϕ. For any $x\in\mathbb{R}^3\setminus B_2$, let $h_x(\cdot)$ be the function

$$h_x(y) = \frac{(y-x)\cdot y}{|x-y|^3} \quad , \quad y\in\partial B_1 \quad .$$

It is easy to see that

$$\|h_x\|_{H^{1/2}(\partial B_1)} \le \|h_x\|_{H^1(\partial B_1)} \le C\frac{1}{|x|^2} \quad , \tag{3.41}$$

with C independent of $x\in\mathbb{R}^3\setminus B_2$. Similarly, for any $x\in\mathbb{R}^3\setminus B_2$ let $H_x(\cdot)$ be the vector-valued function

$$H_x(y) = \nabla_x h_x(y) = 3\frac{(y-x)(y-x)\cdot y}{|x-y|^5} - \frac{y}{|x-y|^3} \quad , \quad y\in\partial B_1 \quad .$$

It is easy to see that

$$\|H_x\|_{H^{1/2}(\partial B_1)} \le \|H_x\|_{H^1(\partial B_1)} \le C\frac{1}{|x|^3} \tag{3.42}$$

with C independent of $x\in\mathbb{R}^3\setminus B_2$. Using (3.41) we see that the double layer potential

$$D(\phi)(x) = \frac{1}{4\pi}\int_{\partial B_1}\frac{(y-x)\cdot y}{|x-y|^3}\phi(y)\,d\sigma_y$$

satisfies

$$\|D(\phi)\|_{L^2(B_R\setminus\overline{B_2})} \leq C\|\phi\|_{H^{-1/2}(\partial B_1)} \quad ,$$
$$\|D(\phi)\|_{L^\infty(\{|x|=R\})} \leq \frac{C}{R^2}\|\phi\|_{H^{-1/2}(\partial B_1)} \quad .$$

Using (3.42) we also have

$$\|\nabla D(\phi)\|_{L^\infty(\{|x|=R\})} \leq \frac{C}{R^3}\|\phi\|_{H^{-1/2}(\partial B_1)} \quad .$$

By the $H^{-1/2}(\partial B_1)$ boundedness of $(-\frac{1}{2}+T)^{-1}$, these estimates imply

$$\|V\|_{L^2(B_R\setminus\overline{B_2})} \leq C\|h\|_{H^{-1/2}(\partial B_1)} \quad ,$$
$$\|V\|_{L^\infty(\{|x|=R\})} \leq \frac{C}{R^2}\|h\|_{H^{-1/2}(\partial B_1)} \quad ,$$
$$\|\nabla V\|_{L^\infty(\{|x|=R\})} \leq \frac{C}{R^3}\|h\|_{H^{-1/2}(\partial B_1)} \quad ,$$

for any $R \geq 2$. This proves (3.36) and (3.37).

To establish (3.38), and thus complete the proof of the lemma, we only need to show that

$$\|V\|_{L^2(B_2\setminus\overline{B_1})} \leq C\|h\|_{H^{-1/2}(\partial B_1)} \quad .$$

It suffices to show that

$$\|V\|_{L^2(B_2\setminus\overline{B_1})} \leq C\left(\|h\|_{H^{-1/2}(\partial B_1)} + \|V\|_{H^{-1/2}(\partial B_2)}\right) \quad , \tag{3.43}$$

since the second term on the right is estimated by (3.37) with $R = 2$.

We use a standard duality argument to prove (3.43). Let w solve

$$\Delta w = V \text{ in } B_2\setminus\overline{B_1} \quad \text{with} \quad w = 0 \quad \text{on } \partial B_2 \cup \partial B_1 \quad .$$

It satisfies

$$\|w\|_{H^2(B_2\setminus\overline{B_1})} \leq C\|V\|_{L^2(B_2\setminus\overline{B_1})} \quad ,$$

and thus

$$\|\frac{\partial}{\partial\nu}w\|_{H^{1/2}(\partial B_1)} + \|\frac{\partial}{\partial\nu}w\|_{H^{1/2}(\partial B_2)} \leq C\|V\|_{L^2(B_2\setminus\overline{B_1})} \quad .$$

We therefore calculate

$$\begin{aligned}
\int_{B_2\setminus\overline{B_1}} V^2\,dx &= \int_{B_2\setminus\overline{B_1}} V\Delta w\,dx \\
&= \int_{\partial B_2} V\frac{\partial}{\partial\nu}w\,dx - \int_{\partial B_1} V\frac{\partial}{\partial\nu}w\,dx \\
&\leq \|V\|_{H^{-1/2}(\partial B_2)}\|\frac{\partial}{\partial\nu}w\|_{H^{1/2}(\partial B_2)} \\
&\qquad + \|h\|_{H^{-1/2}(\partial B_1)}\|\frac{\partial}{\partial\nu}w\|_{H^{1/2}(\partial B_1)} \\
&\leq C\|V\|_{L^2(B_2\setminus\overline{B_1})}\left(\|h\|_{H^{-1/2}(\partial B_1)} + \|V\|_{H^{-1/2}(\partial B_2)}\right) \quad ,
\end{aligned}$$

whence

$$\|V\|_{L^2(B_2\setminus\overline{B_1})} \leq C\left(\|h\|_{H^{-1/2}(\partial B_1)} + \|V\|_{H^{-1/2}(\partial B_2)}\right) \quad .$$

This verifies (3.43), completing the proof of the lemma. □

4 Numerical results

The main goal of this section is to demonstrate the sharpness of our estimates. After briefly reviewing the task at hand, we begin with a discussion of the "cloak-busting" inclusions whose existence was announced in Section 2.5. Then we show that for these cloak-busting inclusions the estimate (2.11) is sharp. We also examine the performance of the near-cloak as a function of the loss parameter β, and we study the degree to which the fields outside the cloak emulate those of a uniform domain.

To describe the formulas used in our computations, complex notation is very convenient. For all of our computations we take the background solution u_0 to be a plane wave, $u_0(x) = e^{i\omega x_2}$, propagating in the x_2 direction. This u_0 is the solution of

$$\begin{cases} \Delta u_0 + \omega^2 u_0 = 0 & \text{in } \Omega \ , \\ \dfrac{\partial u_0}{\partial \nu} = \psi & \text{on } \partial\Omega \ , \end{cases} \tag{4.1}$$

with

$$\psi = i\omega e^{i\omega x_2}\nu_2 \quad . \tag{4.2}$$

Throughout this section, B_R denotes the ball of radius R centered at the origin, the domain Ω is chosen to be $\Omega \doteq B_2$, and all calculations are done at frequency $\omega = 1$. (Note that these choices make (4.1) well-posed, since -1 is not an eigenvalue of the Neumann Laplacian on B_2.)

We denote by u_ρ the solution to the following problem

$$\begin{cases} \operatorname{div}(A_\rho \nabla u_\rho) + \omega^2 q_\rho u_\rho = 0 & \text{in } B_2 \ , \\ \dfrac{\partial u_\rho}{\partial \nu} = \psi & \text{on } \partial B_2 \ , \end{cases} \tag{4.3}$$

where $A_\rho(y), q_\rho(y)$ are given by

$$\begin{cases} A_\rho = q_\rho = 1 & \text{for } 2\rho < |x| \leq 2 \ , \\ A_\rho = 1,\ q_\rho = 1 + i\beta & \text{for } \rho < |x| \leq 2\rho \ , \\ A_\rho, q_\rho \ > 0 \text{ arbitrary} & \text{for } |x| \leq \rho \ , \end{cases} \tag{4.4}$$

with $\beta > 0$. In principle the value of $A_\rho(y)$ inside B_ρ could be any symmetric positive-definite matrix, but for simplicity we take both A_ρ and q_ρ to be *scalar constants* in B_ρ. When there is no danger of confusion, we will sometimes abuse notation by writing A_ρ, q_ρ for the (arbitrary, constant) values of the coefficients in B_ρ (in particular, we have done this in (4.4)).

Our near-cloaks are obtained by change-of-variables using the map F, defined by

$$F = \begin{cases} \frac{x}{2\rho} & \text{if } |x| \le 2\rho \\ \left(\frac{1-2\rho}{1-\rho} + \frac{1}{2(1-\rho)}|x|\right)\frac{x}{|x|} & \text{if } 2\rho \le |x| \le 2 \end{cases} \tag{4.5}$$

(in the notation of Section 3 this is $F_{2\rho}$). Note that F maps B_ρ to $B_{\frac{1}{2}}$, $B_{2\rho}$ to B_1, and the annulus $B_2 \setminus B_{2\rho}$ to the annulus $B_2 \setminus B_1$. The "push-forward" of u_ρ, i.e. the function $U_\rho(y) \doteq u_\rho(F^{-1}(y))$, satisfies

$$\begin{cases} \operatorname{div}(F_*(A_\rho)\nabla U_\rho) + \omega^2 F_*(q_\rho)U_\rho = 0 & \text{in } B_2 \ , \\ (F_*(A_\rho)\nabla U_\rho)\cdot \nu = \psi & \text{in } \partial B_2 \ , \end{cases} \tag{4.6}$$

where ψ is as before. Taking into account the special form (4.4) of the coefficients under consideration, and the fact that A_ρ and q_ρ are scalar constants in B_ρ, the pushed-forward coefficients $F_*(A_\rho, q_\rho) \doteq (F_*(A_\rho), F_*(q_\rho))$ are given

$$\text{in 2D by} \begin{cases} \left.\begin{array}{l} F_*(A_\rho)(y) = \frac{DF(x)DF^T(x)}{\det DF(x)}\big|_{x=F^{-1}(y)} \ , \\ F_*(q_\rho)(y) = \frac{1}{\det DF(x)}\big|_{x=F^{-1}(y)} \end{array}\right\} & \text{for } 1 < |y| \le 2 \\ F_*(A_\rho)(y) = 1 \ , \ F_*(q_\rho)(y) = 4\rho^2(1+i\beta) & \text{for } \frac{1}{2} < |y| \le 1 \\ \left.\begin{array}{l} F_*(A_\rho)(y) = A_\rho \ , \\ F_*(q_\rho)(y) = 4\rho^2 q_\rho \end{array}\right\} & \text{for } |y| \le \frac{1}{2} \end{cases} \tag{4.7}$$

and

$$\text{in 3D by} \begin{cases} \left.\begin{array}{l} F_*(A_\rho)(y) = \frac{DF(x)DF^T(x)}{\det DF(x)}\big|_{x=F^{-1}(y)} \ , \\ F_*(q_\rho)(y) = \frac{1}{\det DF(x)}\big|_{x=F^{-1}(y)} \end{array}\right\} & \text{for } 1 < |y| \le 2 \\ F_*(A_\rho)(y) = 2\rho \ , \ F_*(q_\rho)(y) = 8\rho^3(1+i\beta) & \text{for } \frac{1}{2} < |y| \le 1 \\ \left.\begin{array}{l} F_*(A_\rho)(y) = 2\rho A_\rho \ , \\ F_*(q_\rho)(y) = 8\rho^3 q_\rho \end{array}\right\} & \text{for } |y| \le \frac{1}{2} \end{cases} \tag{4.8}$$

We shall write v_ρ for the solution of the problem (4.3) in the particular case when $\beta = 0$. Thus, v_ρ solves

$$\begin{cases} \operatorname{div}(A'_\rho \nabla v_\rho) + \omega^2 q'_\rho v_\rho = 0 & \text{in } B_2 \ , \\ \dfrac{\partial v_\rho}{\partial n} = \psi & \text{in } \partial B_2 \ , \end{cases} \tag{4.9}$$

where A'_ρ, q'_ρ are piecewise constant functions given by

$$(4.10) \qquad \begin{cases} A'_\rho = q'_\rho = 1 & \text{for } \rho < |x| \leq 2 \\ A'_\rho, q'_\rho > 0 \text{ arbitrary} & \text{for } |x| \leq \rho \ . \end{cases}$$

The corresponding pushed forward problem and pushed forward coefficients are described by (4.6) and (4.7)/(4.8) with $\beta = 0$, for 2D/3D, respectively.

We recall the following representations, in 2D and 3D, of the plane wave solution of (4.1), $u_0 = e^{i\omega x_2}$:

$$(4.11) \qquad u_0(r,\theta) = \sum_{k=-\infty}^{k=+\infty} J_k(\omega r) e^{ik\theta}, \quad \text{in 2D}$$

$$(4.12) \qquad u_0(r,\theta,\phi) = 4\pi \sum_{l=0}^{\infty} i^l j_l(\omega r) \sum_{|m|\leq l} \overline{Y_l^m}(\frac{\pi}{2},\frac{\pi}{2}) Y_l^m(\theta,\phi), \quad \text{in 3D}$$

where here and in what follows, $i^2 = -1$, $\bar{z}$ denotes the complex conjugate of z, J_k and j_l are the classical Bessel and spherical Bessel functions, respectively (see for instance [24]) and for each $l \geq 0$, $Y_l^m(\theta,\phi)$ with $|m| \leq l$ are the $2l+1$-orthonormal spherical harmonics of degree l and order m, (see for instance [18]). The explicit (dual) presence of the angle $\pi/2$ in the 3D formula stems from the fact that the propagation direction (the x_2 direction) corresponds to azimuthal and polar angle $\pi/2$. From (4.2), (4.11) and (4.12) we get that the flux ψ (defined on $r = 2$) can be written as

$$(4.13) \qquad \left\{ \begin{array}{l} \psi(\theta) = \sum_k \hat{\psi}_k e^{ik\theta}, \quad \text{with} \\ \hat{\psi}_k = \omega J'_k(2\omega) \end{array} \right\} \text{in 2D} \ ,$$

$$(4.14) \qquad \left\{ \begin{array}{l} \psi(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{|m|\leq l} \hat{\psi}_l^m Y_l^m(\theta,\phi), \quad \text{with} \\ \hat{\psi}_l^m = 4\pi\omega i^l j'_l(2\omega) \overline{Y_l^m}(\frac{\pi}{2},\frac{\pi}{2}) \end{array} \right\} \text{in 3D} \ .$$

4.1 Cloak-busting inclusions

We turn now to the identification of “cloak-busting” inclusions, elaborating on the discussion in Section 2.5. It is natural to begin with the 2D setting. Using separation of variables, we may express the solution v_ρ of problem (4.9) as follows:

$$(4.15) \qquad \begin{cases} v_\rho(r,\theta) = \sum_k \alpha_k J_k\left(\omega r \sqrt{\frac{q'_\rho}{A'_\rho}}\right) e^{ik\theta} & \text{if } r \leq \rho \ , \\ v_\rho(r,\theta) = \sum_k \left(\beta_k J_k(\omega r) + \gamma_k H_k^{(1)}(\omega r)\right) e^{ik\theta} & \text{if } \rho < r \leq 2 \ . \end{cases}$$

From the appropriate transmission conditions for problem (4.9), i.e., continuity of v_ρ and $(A'_\rho \nabla v_\rho) \cdot \nu$ across ∂B_ρ, and the Neumann condition for v_ρ on ∂B_2, we arrive at the following necessary and sufficient condition for the well-posedness of the problem (4.9):

(4.16)
$$0 \neq D_k(A'_\rho, q'_\rho) \doteq J_k\left(\omega\rho\sqrt{\frac{q'_\rho}{A'_\rho}}\right)\left(J'_k(2\omega)(H_k^{(1)})'(\omega\rho) - (H_k^{(1)})'(2\omega)J'_k(\omega\rho)\right)$$
$$- \sqrt{A'_\rho q'_\rho}\,J'_k\left(\omega\rho\sqrt{\frac{q'_\rho}{A'_\rho}}\right)\left(J'_k(2\omega)H_k^{(1)}(\omega\rho) - (H_k^{(1)})'(2\omega)J_k(\omega\rho)\right) \quad ,$$

for all integers k. Note that, due to well known properties of the Bessel functions, it suffices to require that (4.16) hold for all nonnegative integers.

Our "cloak-busting" inclusions correspond to choices of A'_ρ, q'_ρ such that $D_k(A'_\rho, q'_\rho) = 0$ for some $k \in \mathbb{Z}$. Such coefficients make the problem (4.9) ill-posed (i.e. they make $-\omega^2$ an eigenvalue), despite the fact that (4.1) is well-posed by hypothesis. For such inclusions near-cloaking is clearly not achieved in the lossless case. We will not attempt to classify all solutions of $D_k(A'_\rho, q'_\rho) = 0$; rather, we examine selected solutions that are easy to identify and analyze.

For $k = 0$ we make the choice $A'_\rho = q'_\rho$ and obtain the following positive solutions of $D_0(A'_\rho, q'_\rho) = 0$:

$$A'_\rho = q'_\rho = \frac{J_0(\omega\rho)\left((H_0^{(1)})'(2\omega)J'_0(\omega\rho) - (H_0^{(1)})'(\omega\rho)J'_0(2\omega)\right)}{J'_0(\omega\rho)\left((H_0^{(1)})'(2\omega)J_0(\omega\rho) - H_0^{(1)}(\omega\rho)J'_0(2\omega)\right)} \quad . \tag{4.17}$$

Here we have used the fact that

$$0 \neq (H_k^{(1)})'(2\omega)J_k(\omega\rho) - J'_k(2\omega)H_k^{(1)}(\omega\rho) \text{ for } k \in \mathbb{Z} \quad , \tag{4.18}$$

when ρ is sufficiently small. The non-vanishing condition (4.18) is a direct consequence of classical results about the asymptotic behavior of Bessel functions, and the fact that $J'_k(2\omega) \neq 0$ (since the problem (4.1) is wellposed by assumption). It is quite easy to see that the right hand side of (4.17) is real (both numerator and denominator are pure imaginary) and due to the asymptotic behavior of Bessel functions it is actually positive for ρ sufficiently small.

To find real positive solutions of $D_k(A'_\rho, q'_\rho) = 0$ for some $k > 0$ we take a different approach. Given k, we start by choosing a real number $z^* > 0$ such that

$$J_k(z^*)J'_k(z^*) < 0 \quad , \tag{4.19}$$

then we make choice
$$q'_\rho = (z^*)^2 A'_\rho/(\omega\rho)^2 \quad .$$
It is easy to verify that with this choice of q'_ρ, $D_k(A'_\rho, q'_\rho) = 0$ when

$$A'_\rho = \frac{\omega\rho J_k(z^*)\left((H_k^{(1)})'(2\omega)J'_k(\omega\rho) - (H_k^{(1)})'(\omega\rho)J'_k(2\omega)\right)}{z^* J'_k(z^*)\left((H_k^{(1)})'(2\omega)J_k(\omega\rho) - H_k^{(1)}(\omega\rho)J'_k(2\omega)\right)} \quad . \tag{4.20}$$

Due to the condition (4.18) this A'_ρ is well defined, and it is easily seen to be real. Because of the asymptotics of the Bessel functions, and the fact that $J_k(z^*)$ and $J'_k(z^*)$ have opposite signs, we may conclude that A'_ρ and q'_ρ are positive.

Figure 4.1 shows the pushed-forward values $F_*(A'_\rho), F_*(q'_\rho)$ when $k=0$, using (4.17) and (4.7). When the coefficients in $B_{1/2}$ take these values the lossless version of our construction (4.10) is resonant, i.e. $-\omega^2$ is a Neumann eigenvalue of the ρ-inclusion problem. Notice that in this case $F_*(A'_\rho) \to \infty$ as $\rho \to 0$. Thus, in the "physical" (pushed-forward) variables, these cloak-busting inclusions have extreme physical properties in the limit $\rho \to 0$.

Figure 4.2 gives the analogous picture for $k=1$: it shows $F_*(A'_\rho)$ and $F_*(q'_\rho)$ when (A'_ρ, q'_ρ) are the particular solutions of $D_1(A'_\rho, q'_\rho) = 0$ given by (4.20) (for a specific choice of z^* satisfying (4.19)). Notice that in this case $F_*(A'_\rho)$ and $F_*(q'_\rho)$ have finite, nonzero limits as $\rho \to 0$. Thus, in the "physical" (pushed-forward) variables, these cloak-busting inclusions do *not* have extreme physical properties. We wonder how a lossless singular cloak of the type considered in [8, 21] would perform when faced with such an inclusion.

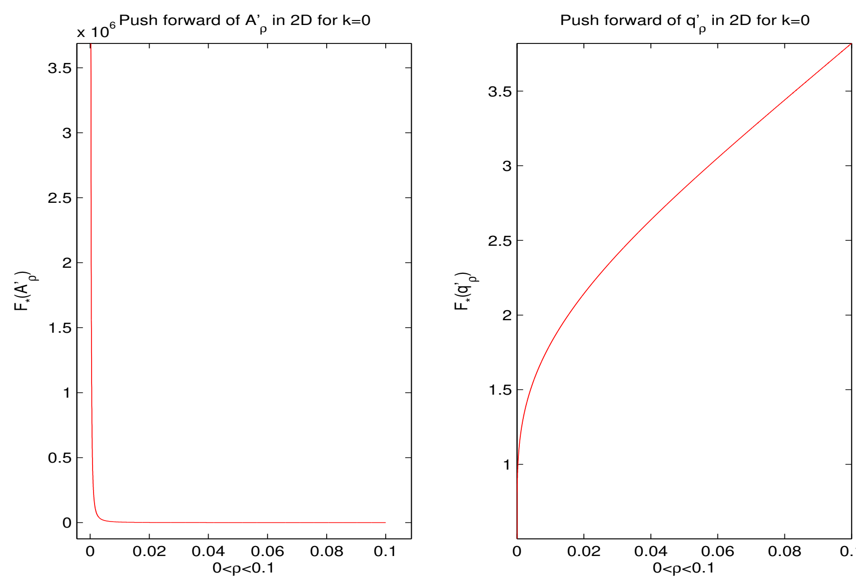

FIGURE 4.1. The $k=0$ cloak-busting inclusions in 2D: $F_*(A'_\rho) = A'_\rho$ and $F_*(q'_\rho) = 4\rho^2 q'_\rho$ with $A'_\rho = q'_\rho$ given by (4.17).

We turn now to the 3D setting. The situation is not very different, so we shall be relatively brief. Separation of variables yields the following expression for the

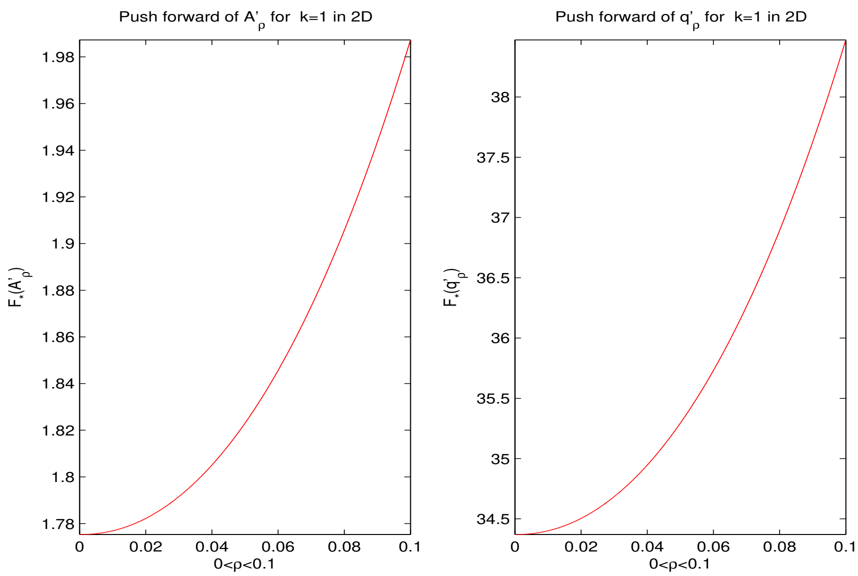

FIGURE 4.2. The $k=1$ cloak-busting inclusions in 2D: $F_*(A'_\rho)=A'_\rho$ and $F_*(q'_\rho)=4\rho^2 q'_\rho$ when A'_ρ is given by (4.20) with $k=1$ and $q'_\rho = (z^*)^2 A'_\rho/(\omega\rho)^2$.

solution v_ρ of the lossless problem (4.9):
(4.21)

$$\begin{cases} v_\rho(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{|m|\le l} \alpha_l^m j_l\left(\omega r\sqrt{\frac{q'_\rho}{A'_\rho}}\right) Y_l^m(\theta,\phi) & \text{if } r\le\rho \\ v_\rho(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{|m|\le l}\left(R_l^m j_l(\omega r) + S_l^m h_l^{(1)}(\omega r)\right) Y_l^m(\theta,\phi) & \text{if } \rho<r\le 2 \end{cases}$$

where $h_l^{(1)} = j_l + iy_l$ denotes the first kind spherical Hankel function. Arguing as for 2D, one finds the following necessary and sufficient condition for the well-posedness of the problem (4.9) in 3D:

(4.22)

$$\begin{aligned} 0 \neq D_l(A'_\rho, q'_\rho) \doteq\; & j_l\left(\omega\rho\sqrt{\frac{q'_\rho}{A'_\rho}}\right)\left(j'_l(2\omega)(h_l^{(1)})'(\omega\rho) - (h_l^{(1)})'(2\omega) j'_l(\omega\rho)\right) \\ & - \sqrt{A'_\rho q'_\rho}\; j'_l\left(\omega\rho\sqrt{\frac{q'_\rho}{A'_\rho}}\right)\left(j'_l(2\omega) h_l^{(1)}(\omega\rho) - (h_l^{(1)})'(2\omega) j_l(\omega\rho)\right) \quad , \end{aligned}$$

for all positive l.

Our 3D "cloak-busting inclusions" are associated with choices of A'_ρ, q'_ρ such that $D_l(A'_\rho, q'_\rho) = 0$ for some l. As before, our goal is not to classify all solutions of $D_l(A'_\rho, q'_\rho) = 0$, but rather to explore some examples. For $l = 0$ we make the choice $A'_\rho = q'_\rho$ and obtain (using well-known results about the asymptotics of the spherical Bessel functions) the following positive solution of $D_0(A'_\rho, q'_\rho) = 0$:

$$A'_\rho = q'_\rho = \frac{j_0(\omega\rho)\left((h_0^{(1)})'(2\omega)\, j'_0(\omega\rho) - (h_0^{(1)})'(\omega\rho)\, j'_0(2\omega)\right)}{j'_0(\omega\rho)\left((h_0^{(1)})'(2\omega)\, j_0(\omega\rho) - h_0^{(1)}(\omega\rho)\, j'_0(2\omega)\right)} \quad . \tag{4.23}$$

For any $l > 0$, we make the choice

$$q'_\rho = (z^*)^2 \frac{A'_\rho}{(\omega\rho)^2} \text{ where } z^* \text{ is such that } j_l(z^*) \cdot j'_l(z^*) < 0 \tag{4.24}$$

and we find that $D_l(A'_\rho, q'_\rho) = 0$ and $A'_\rho > 0, q'_\rho > 0$ when

$$A'_\rho = \frac{\omega\rho\, j_l(z^*)\left((h_l^{(1)})'(2\omega)\, j'_l(\omega\rho) - (h_l^{(1)})'(\omega\rho)\, j'_l(2\omega)\right)}{z^*\, j'_l(z^*)\left((h_l^{(1)})'(2\omega)\, j_l(\omega\rho) - h_l^{(1)}(\omega\rho)\, j'_l(2\omega)\right)} \quad . \tag{4.25}$$

Figure 4.3 shows the pushed-forward values $F_*(A'_\rho)$ and $F_*(q'_\rho))$ of our $l = 0$ example, when A'_ρ, q'_ρ are given by (4.23). The push-forward in this 3D setting is given by (4.8). Notice that in this case $F_*(A'_\rho) \to \infty$ while $F_*(q'_\rho) \to 0$ as $\rho \to 0$. Thus, in the "physical" (pushed-forward) variables, both coefficients associated with these 3D cloak-busting inclusions become extreme as $\rho \to 0$.

When $l = 1$ and A'_ρ, q'_ρ are given by (4.24)-(4.25), both $F_*(A'_\rho)$ and $F_*(q'_\rho)$ tend to 0 as $\rho \to 0$ (not shown). We did not find any examples in 3D analogous to the one shown in Figure 4.2, where the push-forwards both remain bounded as $\rho \to \infty$. This suggests (but does not prove) that in the 3D setting, all cloak-busting inclusions have extreme physical properties in the physical (pushed-forward) variables.

4.2 Sharpness of Theorem 3.1

We turn now to the optimality of our results concerning the performance of our near-cloak. According to Theorem 3.1, when $\rho \ll 1$ and $\beta \sim \rho^{-2}$ we have

$$||u_\rho - u_0||_{H^{1/2}(\partial B_2)} \le \begin{cases} \dfrac{C}{|\log(\rho)|}||\psi||_{H^{-1/2}(\partial B_2)} & \text{in 2D} \\[2ex] C\rho||\psi||_{H^{-1/2}(\partial B_2)} & \text{in 3D} \end{cases} \tag{4.26}$$

where u_ρ is the solution of (4.3), u_0 is the solution of (4.1), and the constant C is independent of the coefficients A_ρ, q_ρ in B_ρ. To assess the sharpness of this estimate, we focus (as already noted) on the case when u_0 is the plane wave $e^{i\omega x_2}$,

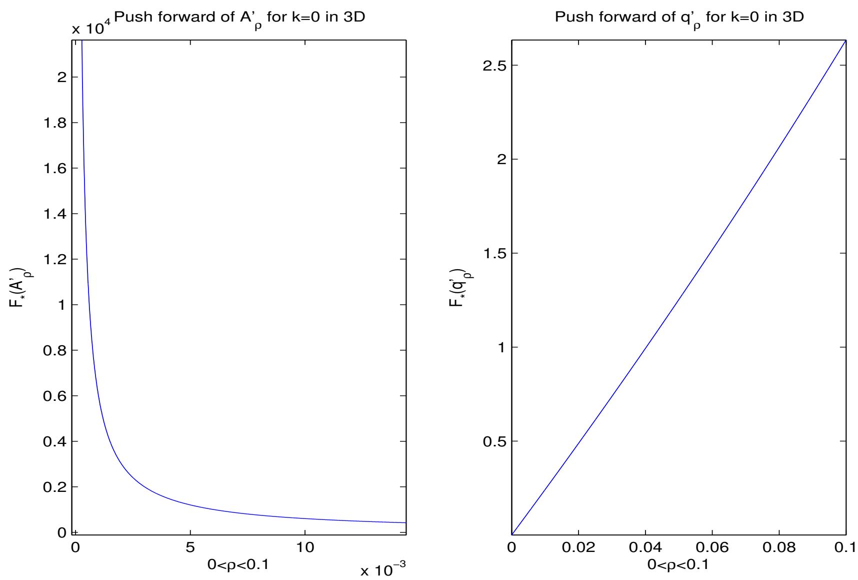

FIGURE 4.3. The $l=0$ cloak-busting inclusions in 3D: $F_*(A'_\rho)=2\rho A'_\rho$ and $F_*(q'_\rho)=8\rho^3 q'_\rho$ when $A'_\rho = q'_\rho$ are given by (4.23).

i.e. when $\psi = i\omega e^{i\omega x_2}\nu_2$. Let $E_\rho(\beta)$ be defined by

$$E_\rho(\beta) = \begin{cases} \dfrac{|\log(\rho)|\cdot||u_\rho - u_0||_{H^{\frac{1}{2}}(\partial B_2)}}{||\psi||_{H^{-1/2}(\partial B_2)}} & \text{in 2D} \\[2ex] \dfrac{||u_\rho - u_0||_{H^{\frac{1}{2}}(\partial B_2)}}{\rho||\psi||_{H^{-1/2}(\partial B_2)}} & \text{in 3D.} \end{cases} \tag{4.27}$$

The assertion of (4.26) is thus that $E_\rho(\beta) \le C$ when $\beta \sim \rho^{-2}$.

To approximate u_ρ numerically we used separation of variables with finitely many modes. In 2D we used the modes $e^{ik\theta}$ with $-30 \le k \le 30$; in 3D we used the modes $Y_l^m(\theta,\phi)$ with $0 \le l \le 30$ and $|m| \le l$. Thus the plane wave u_0 was approximated by

$$u_0(r,\theta) \approx u_0^{appr}(r,\theta) = \sum_{k=-30}^{k=+30} J_k(\omega r)e^{ik\theta} \quad \text{in 2D} \quad ,$$

$$u_0(r,\theta,\phi) \approx u_0^{appr}(r,\theta,\phi) = 4\pi \sum_{l=0}^{30} i^l j_l(\omega r) \sum_{|m|\le l} \overline{Y_l^m}(\frac{\pi}{2},\frac{\pi}{2})Y_l^m(\theta,\phi) \quad \text{in 3D} \quad ,$$

and the solution u_ρ of (4.3) was approximated by similar finite sums.

Figures 4.4 and 4.5 show the dependence of E_ρ on β in the 2D and 3D cases respectively. In the top frames of each figure E_ρ is plotted as a function of β, for three different values of ρ: $\rho = 10^{-3}$, $\rho = 10^{-5}$, $\rho = 10^{-7}$; the bottom frames show zoomed-in versions near the optimal values of β (which are just beyond the range of the top frames). For all these plots the values of A_ρ and q_ρ in B_ρ were our mode-0 cloak-busting inclusions, given by (4.17) for 2D and (4.23) for 3D. Similar results were obtained (not shown) for mode-1 cloak-busting inclusions, given by (4.20) in 2D and (4.24)-(4.25) in 3D. These are natural test problems, since for such A_ρ, q_ρ the structure is resonant (roughly: $E_\rho = \infty$) when $\beta = 0$.

Theorem 3.1 asserts that E_ρ is bounded by a constant (independent of A_ρ and q_ρ) when $\beta \sim \rho^{-2}$. Figures 4.4 and 4.5 confirm this; in addition, the lower plots suggest that the optimal value of β (at least for our mode-0 cloak-busting examples) is about $c\rho^{-2}$ with $c \approx 2.5$ in 2D and $c \approx 4$ in 3D. As β decreases from this optimal value the value of E_ρ increases, becoming very much larger when $\beta \ll \rho^{-2}$. Thus, a value of β on the order of ρ^{-2} is *required* to control the resonance associated with a cloak-busting inclusion. The situation for β larger than the optimal value is different: making β very large does no real harm. Indeed, our calculations (not shown) indicate that E_β remains finite as $\beta \to \infty$. This is consistent with the results in [17], where estimates similar to ours are obtained using a Dirichlet boundary condition (roughly the same as our setting with $\beta = \infty$).

Figure 4.6 shows the behavior of E_ρ as a function of ρ, when $\beta = (2\rho)^{-2}$. The left frame shows the behavior in 2D the right in 3D. The continuous line and the dashed line in the left frame correspond to our mode-0 and mode-1 cloak-busting inclusions, given by (4.17) and (4.20) respectively. The right frame uses the same convention: the continuous line and the dashed line correspond to our 3D mode-0 and mode-1 cloak-busting inclusions, given by by (4.23) and (4.24)-(4.25) respectively. The figure shows quite clearly that when $\beta = c\rho^{-2}$, $E_\rho(\beta)$ has a finite (nonzero) limit as $\rho \to 0$. This confirms the sharpness of our estimate (4.26).

Finally we examine the degree to which the fields outside the cloak emulate those of a uniform domain. To this end, we observe that our approximate solution of the PDE $u_\rho^{(appr)}$ and its push-forward $U_\rho^{(appr)}$ are given by finite Fourier sums. Therefore they extend naturally beyond B_2. Their (common) extension is the solution of an exterior problem (for the operator $\Delta + \omega^2$) with the Cauchy data $(u_\rho|_{r=2}, \frac{\partial u_\rho}{\partial \nu}|_{r=2}) = (u_\rho|_{r=2}, \psi)$. Abusing notation slightly, we write u_ρ or U_ρ for the *extended* function (dropping even the superscript *appr*).

Consider the L^∞ plane wave residual at radius $R \geq 2$, defined by

$$P(R,\rho) = \frac{||(U_\rho - u_0)_{r=R}||_{L^\infty(0,2\pi)}}{||\psi||_{H^{-\frac{1}{2}}(\partial B_2)}} \tag{4.28}$$

with $u_0(x) = e^{i\omega x_2}$. If the cloaking were perfect then the plane wave residual would vanish. The first frame of Figure 4.7 shows $P(R, 10^{-5})$ as a function of $10 < R <$

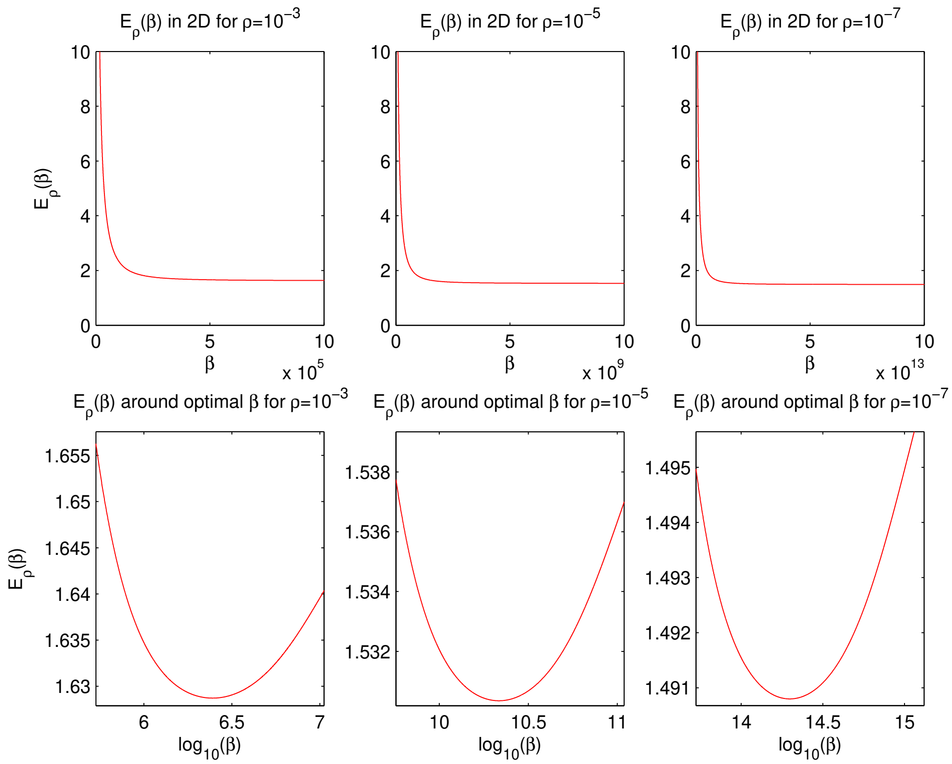

FIGURE 4.4. The influence of the loss parameter β in 2D. The lower frames indicate that the optimal $\beta \approx 10^{4/10}\rho^{-2} \approx 2.5\rho^{-2}$.

100 in 2D. The second frame of Figure 4.7 shows

$$f(\rho) \doteq |\log(2\rho)|P(2,\rho) \tag{4.29}$$

as a function of ρ. (These figures show the 2D case, with $\beta = (2\rho)^{-2}$, for our mode-0 cloak-busting inclusion (4.17); the situation in 3D is similar.) Note from Figure 4.7 that f approaches a constant as $\rho \to 0$, consistent with the sharpness of our estimate (4.26).

Figures 4.8 and Figure 4.9 show contour plots of the real part (2D) and the projection onto the plane $z = 0$ of the real part (3D) of the extended pushed forward solution U_ρ. Figures 4.10 and 4.11 are zoomed-in versions of Figure 4.8 and Figure 4.9. In these examples we have taken $\beta = (2\rho)^{-2}$, and we focus on the mode-0 cloak-busting inclusions, given by (4.17) in 2D and (4.23) in 3D. Each figure shows the behavior for four different values of ρ. Since the near-cloak is not very effective in 2D, Figures 4.8 and 4.10 use relatively small values of ρ, namely 10^{-1}, 10^{-2}, 10^{-4}, and 10^{-6}. Since the near-cloak is more effective in 3D, we use much larger values of ρ for Figures 4.9 and 4.11, namely 0.5, 10^{-1}, 10^{-2}, and 10^{-3}. The

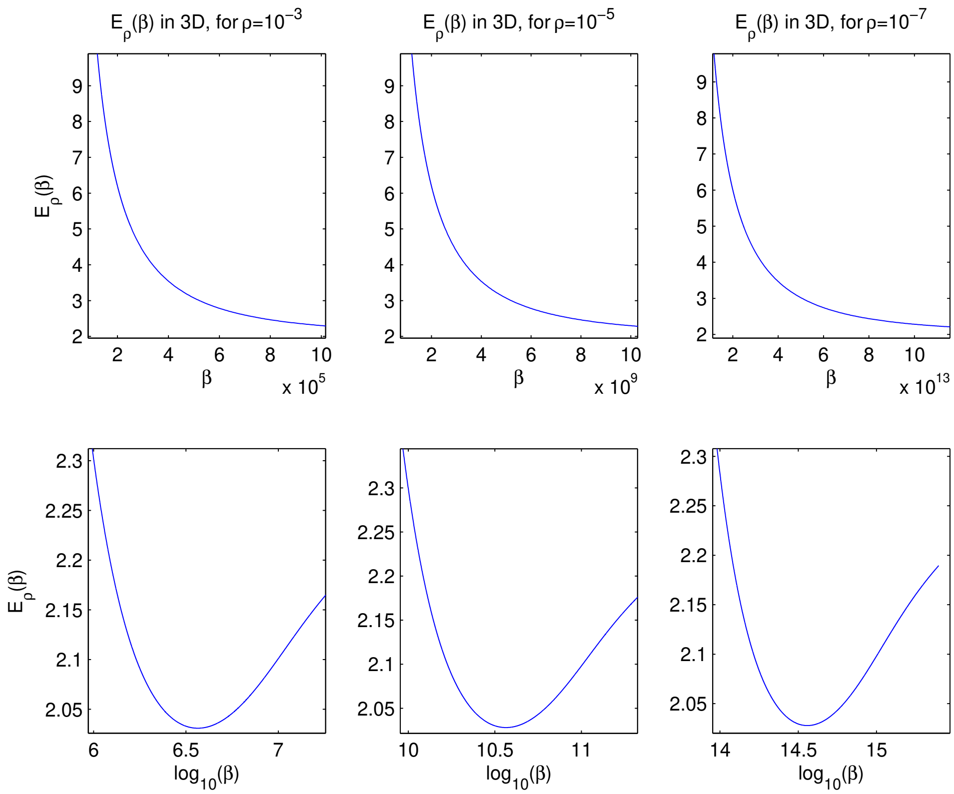

FIGURE 4.5. The influence of the loss parameter β in 3D. The lower frames indicate that the optimal $\beta \approx 10^{6/10}\rho^{-2} \approx 4\rho^{-2}$.

figures show that when ρ is sufficiently small, the extended solution U_ρ is close to the plane wave u_0 away from B_2, i.e. we get approximate cloaking in the far field. Each frame of Figure 4.8 achieves roughly the same degree of approximate cloaking as the corresponding frame of Figure 4.9. This reflects the very different performance of our near-cloaks in 2D (where the deviation from perfect cloaking is of order $1/|\log\rho|$) versus 3D (where the deviation is of order ρ).

In summary, the actual performance of our near-cloak is completely consistent with the estimate of Theorem 3.1, in the sense that (a) the loss parameter β must be at least of order ρ^{-2} for the conclusion of the Theorem to be valid, and (b) with such a loss parameter, the Theorem correctly estimates the performance of the near-cloak for our cloak-busting choices of A_ρ and q_ρ.

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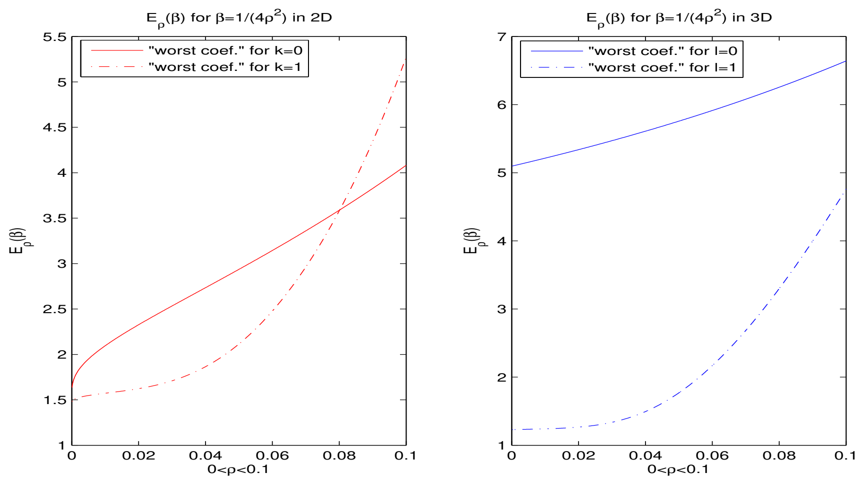

FIGURE 4.6. $E_\rho(\beta)$, with $\beta = (2\rho)^{-2}$, as a function of ρ, for our mode-0 and mode-1 cloak-busting inclusions. The phrase "worst coef" in the inset refers to the cloak-busting values of A_ρ, q_ρ.

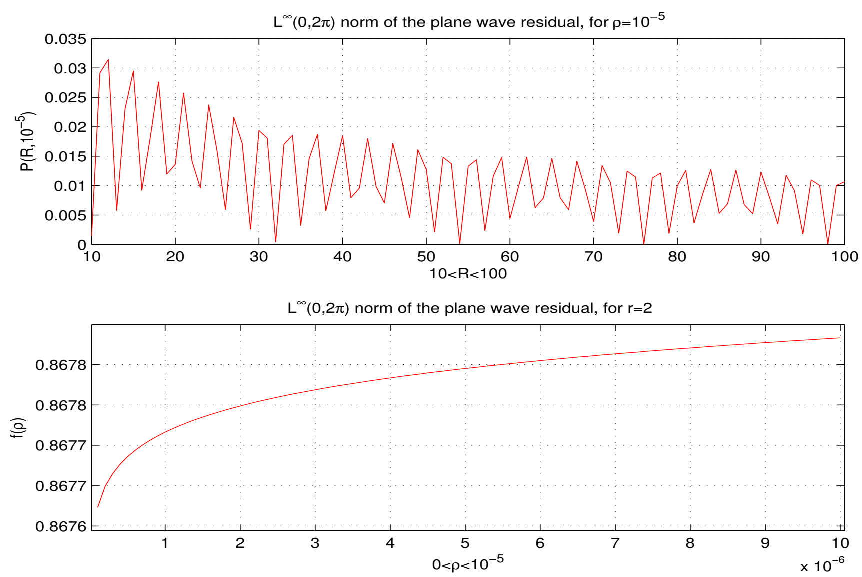

FIGURE 4.7. Upper frame: the plane wave residual, defined by (4.28), as a function of R when $\rho = 10^{-5}$. Lower frame: the function f, defined by (4.29), as a function of ρ.

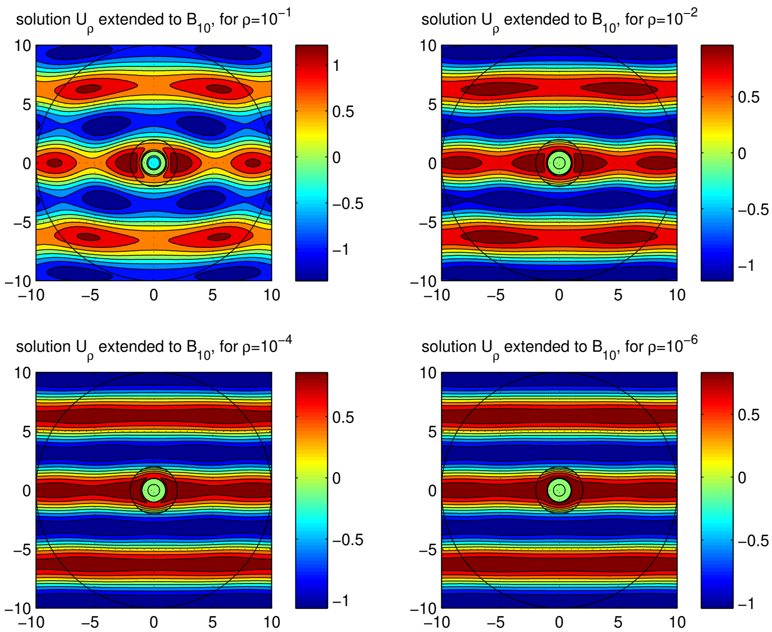

FIGURE 4.8. The 2D extended pushed forward solution U_ρ on B_{10}

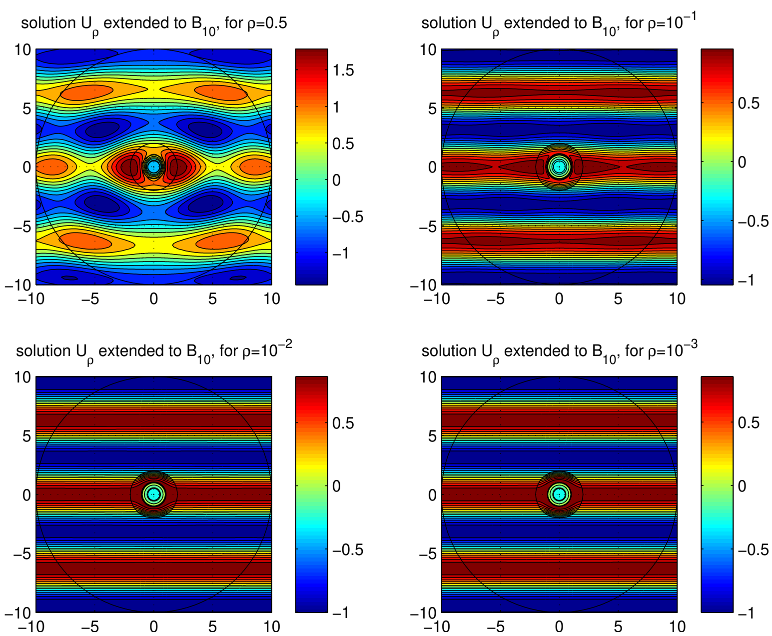

FIGURE 4.9. The 3D extended pushed forward solution U_ρ on B_{10}

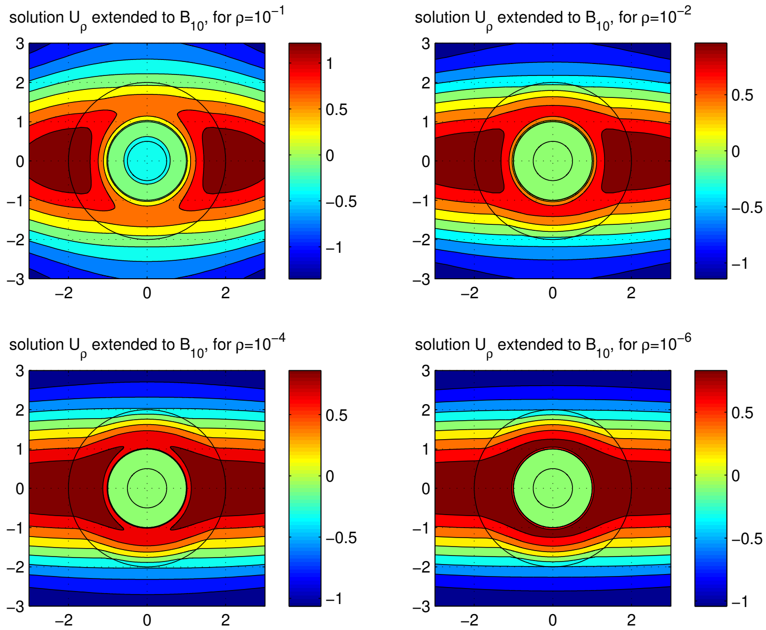

FIGURE 4.10. The 2D extended pushed forward solution U_ρ on B_3

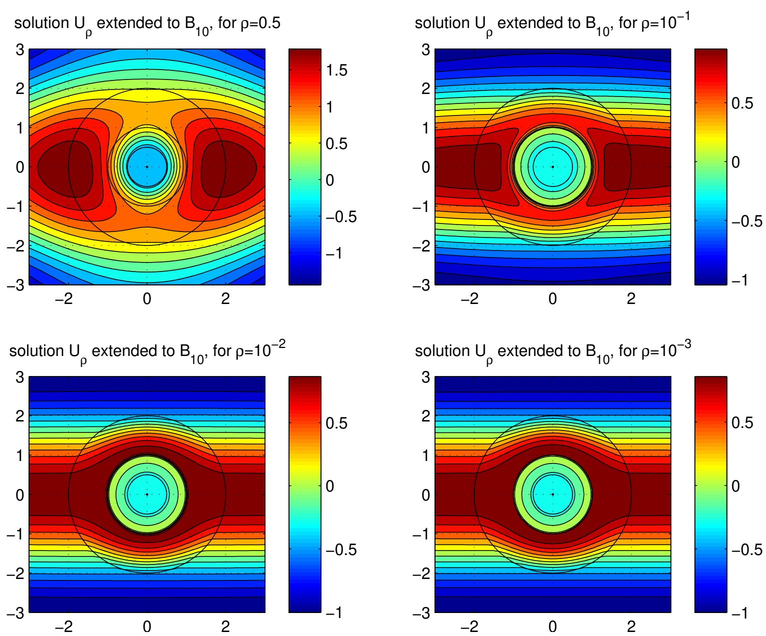

FIGURE 4.11. The 3D extended pushed forward solution U_ρ on B_3

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