

## Close-to-Convex Schlicht Functions

by

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1. Principal results. Known theorems yield the following: if  $\phi(z)$  is a convex schlicht function for  $|z| < R$  and  $f(z)$  is a function analytic for  $|z| < R$  such that

$$\operatorname{Re} \left[ \frac{f'(z)}{\phi'(z)} \right] > 0, \quad |z| < R,$$

then  $f(z)$  is also schlicht for  $|z| < R$ . Since the vectors  $f'$ ,  $\phi'$  never differ in direction by more than  $90^\circ$ , it is natural to call  $f$  close-to-convex:

Definition. Let  $f(z)$  be analytic for  $|z| < R$ . Then  $f(z)$  is close-to-convex for  $|z| < R$  if there exists a function  $\phi(z)$ , convex and schlicht for  $|z| < R$ , such that  $f'(z)/\phi'(z)$  has positive real part for  $|z| < R$ .

When  $R = 1$ , it will be convenient to omit reference to the circular domain of definition. Therefore, a close-to-convex function will mean a function which is close-to-convex for  $|z| < 1$ .

We verify that the close-to-convex functions include several familiar classes of schlicht functions: e.g., the star functions, as well as some less familiar ones: e.g., the functions  $f(z)$  having a Poisson integral representation in terms of a function  $P(e^{i\theta})$  which is monotone in  $\theta$  within each of two complementary arcs of  $|z| = 1$ .

It is of interest to characterize the close-to-convex functions intrinsically, without reference to a

convex function  $\phi$ . Such a characterization is obtained as follows:  $f(z)$  is close-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left[ 1 + z \frac{f''(z)}{f'(z)} \right] d\theta > -\pi$$

when  $\theta_1 < \theta_2$ ,  $z = re^{i\theta}$  and  $r < 1$ .

2. The class of close-to-convex functions. It is known ([6] p. 582, V.) that if  $g(z)$  is analytic in a convex domain  $D$  and

$$(1) \quad \operatorname{Re} [g'(z)] > 0 \quad \text{in } D,$$

then  $g(z)$  is schlicht in  $D$ . If  $\phi(z)$  is a schlicht map of  $|z| < 1$  onto  $D$ , then  $f(z) = g[\phi(z)]$  is also schlicht. Since  $f'(z) = g'(\phi) \phi'(z)$ ,  $f(z)$  satisfies the condition

$$(2) \quad \operatorname{Re} \left[ \frac{f'(z)}{\phi'(z)} \right] > 0 \quad \text{for } |z| < 1.$$

Conversely, if  $f(z)$  satisfies (2), then  $g(z) = f[\phi^{-1}(z)]$  satisfies (1) and  $f(z) = g[\phi(z)]$  is schlicht for  $|z| < 1$ .

Theorem 1. Every close-to-convex function is schlicht.

The class of close-to-convex functions clearly includes the convex functions themselves, as well as the functions  $f(z)$  whose derivative has positive real part in the unit circle. The normalized schlicht functions  $f(z)$  which map the unit circle onto a domain star-shaped with respect to the origin are characterized by the inequality ([2] pp. 92-94):

$$(3) \quad \operatorname{Re} \left[ z \frac{f'(z)}{f(z)} \right] > 0;$$

since ([2] p. 93)

$$(4) \quad \phi(z) = \int_0^z \frac{f(z)}{z} dz$$

is known to be convex, it follows that the star mappings are included in the close-to-convex functions. By specializing the choice of  $\phi(z)$ , one obtains other subclasses:

$$(5) \quad \operatorname{Re} [(z-1)^2 f'(z)] > 0,$$

$$(6) \quad \operatorname{Re} [(z - e^{i\alpha})(z - e^{i\beta}) f'(z)] > 0 \quad (\alpha, \beta \text{ real})$$

$$(7) \quad \operatorname{Re} \left[ \prod_{j=1}^n (z - e^{i\alpha_j})^{k_j} f'(z) \right] > 0,$$

$$0 \leq k_j \leq 1, \quad \sum_{j=1}^n k_j \leq 2,$$

$$0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_n \leq 2\pi.$$

For the class (7)  $\phi$  is a Schwarz - Christoffel mapping; for  $\alpha \neq \beta$ , (6) is a special case of (7), and (5) is a special case of (6).

If  $h(z) = \log \phi(z)$  is chosen to be analytic, then the condition that  $\phi$  be convex is expressed by the inequality

$$(8) \quad \operatorname{Re} [1 + z h'(z)] > 0.$$

Accordingly, if  $f(z)$  is analytic for  $|z| < 1$ , then  $f(z)$  is close-to-convex if and only if there exists a function  $h(z)$ , analytic for  $|z| < 1$ , such that

$$(9) \quad \operatorname{Re} [f'(z) e^{-h(z)}] > 0, \quad \operatorname{Re} [1 + z h'(z)] > 0.$$

From the familiar integral representation of a func-

tion with positive real part ([7] p. 185) we obtain the expressions

$$(10) \quad f'(z) = e^{h(z)} \left[ \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\Psi(\theta) + i\alpha \right],$$

$$(11) \quad h'(z) = \frac{1}{z} \left[ \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\chi(\theta) + i\beta - 1 \right],$$

from which an integral representation for  $f(z)$  in terms of two monotone non-decreasing functions  $\Psi(\theta)$  and  $\chi(\theta)$  can be obtained.

3. Intrinsic characterization. Let  $f(z)$  and  $\phi(z)$  be given as in Section 1; let  $p(z) = \arg f'(z)$  and  $q(z) = \arg \phi'(z)$  be chosen to be continuous for  $|z| < 1$ . Since  $f'(z)$  and  $\phi'(z)$  have no roots for  $|z| < 1$ , such a choice is possible. Because of (2), at each  $z$

$$|p(z) - q(z) + 2k\pi| < \frac{1}{2}\pi$$

for some  $k = 0, +1, \dots$ . Because of the continuity of  $p(z)$  and  $q(z)$ ,  $k$  must be independent of  $z$ . If  $p(z)$  is properly chosen,  $k \equiv 0$ , and it will be assumed that such a choice has been made, so that

$$(12) \quad |p(z) - q(z)| < \frac{1}{2}\pi \text{ for } |z| < 1.$$

We now introduce the functions

$$(13) \quad P(r, \theta) = p(re^{i\theta}) + \theta, \quad Q(r, \theta) = q(re^{i\theta}) + \theta,$$

which are defined for  $0 \leq r < 1$  and all real  $\theta$ . Condition (12) becomes

$$(14) \quad |P(r, \theta) - Q(r, \theta)| < \frac{1}{2}\pi.$$

The condition that  $\phi(z)$  be a convex mapping is described by (8) or equivalently by the condition

$$(15) \quad \frac{\partial Q}{\partial \theta} > 0 ;$$

Thus  $Q(r, \theta)$  is monotone increasing in  $\theta$  for each fixed  $r$ . Now, if  $\theta_1 < \theta_2$ ,

$$\begin{aligned} P(r, \theta_1) - P(r, \theta_2) &= [P(r, \theta_1) - Q(r, \theta_1)] \\ &\quad - [P(r, \theta_2) - Q(r, \theta_2)] \\ &\quad + [Q(r, \theta_1) - Q(r, \theta_2)] \\ &< [P(r, \theta_1) - Q(r, \theta_1)] - [P(r, \theta_2) - Q(r, \theta_2)] . \end{aligned}$$

Accordingly, by (14),

$$(16) \quad P(r, \theta_1) - P(r, \theta_2) < \pi \quad \text{for } \theta_1 < \theta_2 .$$

Condition (16) is thus a necessary condition that  $f(z)$  be close-to-convex; it can be expressed in other equivalent forms:

$$(16') \quad \arg f'(re^{i\theta_1}) - \arg f'(re^{i\theta_2}) < \pi + (\theta_2 - \theta_1)$$

for  $\theta_1 < \theta_2$ , provided  $\arg f'(z) = p(z)$  is chosen as above to be continuous for  $|z| < 1$ ;

$$(16'') \quad \int_{\theta_1}^{\theta_2} \operatorname{Re} \left[ 1 + re^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right] d\theta > -\pi,$$

for  $\theta_1 < \theta_2$ . The condition is also sufficient:

Theorem 2. A necessary and sufficient condition that a function  $f(z)$ , analytic and with non-van-

ishing derivative for  $|z| < 1$ , be close-to-convex is that (16'') hold for  $\theta_1 < \theta_2$  and  $r < 1$ .

The necessity being established above, it remains to prove the sufficiency. Given  $f(z)$ , we choose  $p(z) = \arg f'(z)$  to be continuous and then define  $P(r, \theta)$  by (13). The condition (16'') is then replaced by (16). In addition,

$$(17) \quad P(r, \theta + 2\pi) - P(r, \theta) = 2\pi,$$

since  $p(z)$  has period  $2\pi$  with respect to  $\theta$ .

Lemma. Let  $t(\theta)$  be a real function of  $\theta$  for  $-\infty < \theta < \infty$  such that

$$(18) \quad t(\theta + 2\pi) - t(\theta) = 2\pi,$$

$$(19) \quad t(\theta_1) - t(\theta_2) < \pi \quad \text{for } \theta_1 < \theta_2.$$

Then there exists a real function  $s(\theta)$  which is monotonic non-decreasing and satisfies the conditions

$$(20) \quad s(\theta + 2\pi) - s(\theta) = 2\pi,$$

$$(21) \quad |s(\theta) - t(\theta)| \leq \frac{1}{2} \pi.$$

Proof. Let

$$s(\theta) = \text{l. u. b. } t(\theta') - \frac{1}{2} \pi.$$

$$\theta' < \theta$$

Then  $s(\theta)$  is non-decreasing. By (19),  $t(\theta')$  is bounded above, for  $\theta' < \theta$ , by  $t(\theta) + \pi$ . Hence the l. u. b. is finite and

$$s(\theta) \leq t(\theta) + \frac{1}{2} \pi.$$

Furthermore, since  $t(\theta) \leq \underline{1. u. b.} t(\theta')$  for  $\theta' \leq \theta$ ,

$$s(\theta) \geq t(\theta) - \frac{1}{2} \pi.$$

Hence (21) is proved; (20) follows from (18). The lemma is thus established.

We now set  $P(\rho, \theta) = t(\theta)$ , for a fixed  $\rho < 1$ , and apply the lemma, denoting the corresponding function  $s(\theta)$  by  $s(\rho, \theta)$ . For  $r < \rho$  we define

$$(22) \quad q_\rho(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(\rho^2 - r^2)(s(\rho, \alpha) - \alpha)}{\rho^2 + r^2 - 2\rho r \cos(\alpha - \theta)} d\alpha,$$

so that  $q_\rho(r, \theta)$  is harmonic for  $r < \rho$ . Moreover, the function

$$(23) \quad Q_\rho(r, \theta) = q_\rho(r, \theta) + \theta$$

is monotonically increasing in  $\theta$  for each fixed  $r < \rho$ . For, if  $\theta_1 < \theta_2$ ,

$$\begin{aligned} & Q_\rho(r, \theta_2) - Q_\rho(r, \theta_1) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(\rho^2 - r^2)(s(\rho, \alpha + \theta_2) - s(\rho, \alpha + \theta_1))}{\rho^2 + r^2 - 2\rho r \cos \alpha} d\alpha \end{aligned}$$

and, since  $s(\rho, \alpha)$  is non-decreasing in  $\alpha$ , the right hand side is positive or 0. Hence  $Q_\rho(r, \theta)$  is non-decreasing in  $\theta$ , so that  $\partial Q_\rho / \partial \theta \geq 0$ . Since this derivative is harmonic, the equality is ruled out, so that  $Q_\rho(r, \theta)$  is strictly increasing.

We now choose an analytic function  $h_\rho(z)$  whose imaginary part is  $q_\rho(r, \theta)$  and such that  $\text{Re}[h_\rho(0)] = 0$ . Then set

$$(24) \quad \phi_\rho(z) = \int_0^z e^{h_\rho(z)} dz,$$

so that

$$(25) \quad \phi_\rho(0) = 0, \quad |\phi'_\rho(0)| = 1.$$

The function  $\phi_\rho(z)$  is then analytic for  $|z| < \rho$ . Moreover,

$$(26) \quad \operatorname{Re} \left[ 1 + z \frac{\phi''_\rho(z)}{\phi'_\rho(z)} \right] = \frac{\partial Q_\rho}{\partial \theta} > 0, \quad |z| < \rho.$$

Hence  $\phi_\rho(z)$  is a convex function for  $|z| < \rho$ . Furthermore, since

$$|P(\rho, \theta) - s(\rho, \theta)| \leq \frac{1}{2} \pi,$$

we conclude from (22) and the Poisson integral for  $p(r, \theta)$  in terms of  $p(\rho, \theta)$  that

$$|P(r, \theta) - Q_\rho(r, \theta)| < \frac{1}{2} \pi \quad \text{for } r < \rho.$$

Accordingly,

$$(27) \quad \operatorname{Re} \left[ \frac{f'(z)}{\phi'_\rho(z)} \right] > 0 \quad \text{for } |z| < \rho,$$

so that  $f(z)$  is close-to-convex for  $|z| < \rho$ . It remains to show that we can pass to the limit,  $\rho \rightarrow 1$ , and get a unique function  $\phi(z)$  for  $|z| < 1$ .

If we choose the sequence  $\rho_n = 1 - \frac{1}{n}$ , then the corresponding functions  $\phi_{\rho_n}(z)$  are defined for an increasing sequence of domains. For each fixed  $n$ , the functions  $\phi_{\rho_m}(z)$  for  $m \geq n$  form a normal family for  $|z| < \rho_n$ ; this follows from the normality of the family of normalized schlicht functions and condition (25).



Hence a subsequence converges uniformly in this domain. By applying the diagonal process in the familiar fashion, we obtain a subsequence of  $\phi_{\rho_n}(z)$  which converges uniformly in each circle  $|z| < \rho < 1$  and hence has as limit a unique function  $\phi(z)$ , analytic for  $|z| < 1$ . Since the  $\phi_{\rho}(z)$  are schlicht and convex,  $\phi(z)$  must also be so. Since (27) holds for  $\rho = \rho_n$ , we conclude that

$$(28) \quad \operatorname{Re} \left[ \frac{f'(z)}{\phi'(z)} \right] > 0 \quad \text{for } |z| < 1;$$

i. e.,  $f(z)$  is close-to-convex.

4. Geometric interpretation. The condition (16'') or its equivalent, condition (16), has the following geometric meaning:  $w = f(z)$  maps each circle  $z = re^{i\theta}$  ( $r$  fixed and  $r < 1$ ) onto a simple closed curve whose unit tangent vector  $T = i \exp [iP(r, \theta)]$  either rotates in a counterclock-wise direction, as  $\theta$  increases, or else rotates clockwise in such a manner that  $\arg T = P + \frac{1}{2}\pi$  never drops to a value  $\pi$  radians below a previous value; i. e.,  $\Delta \arg T$  exceeds  $-\pi$ , as  $\theta$  increases. This is illustrated in Fig. 1. Here  $\arg T_2 - \arg T_1$  is only slightly

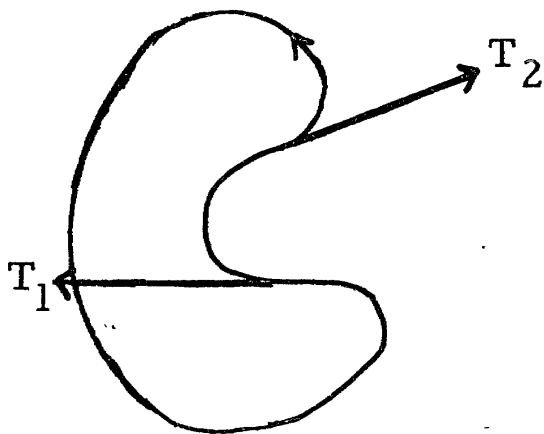


Fig. 1.

greater than  $-\pi$ . Thus such a "hairpin turn" is permitted, provided one does not make a complete reversal of direction.

5. Extremal aspects. For each function  $f(z)$ , analytic and with non-vanishing derivative for  $|z| < 1$ , we make the definition:

$$(29) \quad c[f] = \underset{\phi}{\text{g. l. b.}} \left[ \underset{|z| < 1}{\text{l. u. b.}} |\arg f'(z) - \arg \phi'(z)| \right],$$

where  $\phi$  ranges over the class of all convex schlicht functions for  $|z| < 1$ ; for each  $z$ , the arguments of  $f'$  and  $\phi'$  are to be chosen to give the absolute value of the difference its smallest value. In general,  $c[f] \leq \pi$ . If  $c[f] < \pi$ , then as in the preceding section one can compute  $c[f]$  by restricting  $\phi$  by the conditions (25); the restricted family is normal and accordingly there exists a convex  $\phi$  such that

$$(30) \quad |\arg f'(z) - \arg \phi'(z)| \leq c[f], \quad |z| < 1;$$

$c[f]$  is the smallest constant for which such a  $\phi$  can be found. The function  $\phi$  in (30) can be termed a "best convex approximation to  $f(z)$ ".

If  $c[f] = 0$ , then  $f$  must itself be convex; if  $c[f] \leq \frac{1}{2}\pi$ , then  $f$  is close-to-convex.

The constant  $c[f]$  and a corresponding extremal  $\phi$  satisfying (30) can be found directly by the procedure of Section 3. We introduce the function  $P(r, \theta) = \arg f'(z) + \theta$ . Then

$$(31) \quad c[f] = \min \left( \frac{1}{2} \text{l. u. b.} [P(r, \theta_1) - P(r, \theta_2)], \pi \right),$$

where the l. u. b. is taken over all  $r, \theta_1, \theta_2$  for which

$\theta_1 < \theta_2$  and  $r < 1$ . For, if  $c[f] < \pi$ , then we choose  $\phi$  satisfying (30) and let  $Q(r, \theta) = \arg \phi'(z) + \theta$ ; as in Section 3,  $\arg f'$  and  $\arg \phi'$  can be chosen so that

$$(32) \quad |P(r, \theta) - Q(r, \theta)| \leq c[f].$$

The method of derivation of (16) then yields the relation

$$(33) \quad P(r, \theta_1) - P(r, \theta_2) \leq 2c[f] \text{ for } \theta_1 < \theta_2.$$

On the other hand, if

$$\frac{1}{2} \text{l.u. b. } [P(r, \theta_1) - P(r, \theta_2)] = \alpha < \pi,$$

then the proof of Theorem 2 can be repeated to yield a convex function  $\phi$  such that

$$(34) \quad |\arg f'(z) - \arg \phi'| \leq \alpha.$$

Hence

$$c[f] \leq \alpha \leq c[f]$$

and (31) is proved.

If  $f(z)$  is analytic with non-vanishing derivative for  $|z| < R$ , we can define  $c_R[f]$ , for example by (31) with  $r$  restricted to be less than  $R$ . Then  $c_R[f]$  will be a monotone non-decreasing function of  $R$ . For example, if  $f(z) = e^z$ , we find

$$c_r[e^z] = 0, \quad r \leq 1$$

$$c_r[e^z] = \sqrt{r^2 - 1} - \arccos \frac{1}{r}, \quad r \geq 1.$$

The equation  $c_r = \frac{1}{2}\pi$  is satisfied for  $r$  slightly less than  $\frac{1}{2}$ ; this gives the largest circle  $|z| = a$  within which  $e^z$  is close-to-convex;  $e^z$  remains schlicht for  $|z| < \pi$ , is convex only for  $|z| < 1$ .

For  $w = (z - 1)^2$ , we find

$$c_r [(z - 1)^2] = 0, \quad r \leq \frac{1}{2},$$

$$c_r [(z - 1)^2] = \cos^{-1} \frac{2r^2 + 1}{3r} + \tan^{-1} \frac{\sqrt{4r^2 - 1}}{3r(1 - r^2)}, \quad \frac{1}{2} \leq r < 1.$$

As  $r \rightarrow 1$ ,  $c_r \rightarrow \frac{1}{2}\pi$ , so that  $c_1 [(z - 1)^2] = \frac{1}{2}\pi$ . Accordingly, this function is convex for  $|z| < 1/2$ , is close-to-convex for  $r < 1$ ; the latter domain is the largest circular domain, with center at 0, in which the function is schlicht.

It is natural to ask whether  $\frac{1}{2}\pi$  is the best possible value for  $c[f]$  in the sense that it is the smallest value which guarantees schlichtness. For each  $\alpha$ ,  $\frac{1}{2}\pi < \alpha \leq \pi$ , we can indeed construct a function  $f$  such that  $c_r[f] = \alpha$ , but  $f$  is not schlicht for  $|z| < r$ . To this end we let

$$f_\epsilon(z) = (1 - z)^2 + \epsilon, \quad f_\epsilon(0) = 1, \quad |z| < 1, \quad 0 \leq \epsilon \leq 1.$$

We then find that  $c_1[f_\epsilon(z)]$  is continuous in  $\epsilon$  and equals  $\frac{1}{2}\pi$  for  $\epsilon = 0$ , equals  $\pi$  for  $\epsilon = 1$ . Accordingly,  $c_1[f_\epsilon]$  takes on every value  $\alpha$  between  $\frac{1}{2}\pi$  and  $\pi$  as  $\epsilon$  goes from 0 to 1, but  $f_\epsilon(z)$  is not schlicht for  $|z| < 1$  and  $\epsilon > 0$ .

As remarked above, the largest  $r$  for which

$c_r[f] = 0$  gives the largest circle  $|z| = r$  within which  $f(z)$  is convex; it is known that for normalized schlicht functions this  $r$  is  $> 2 - \sqrt{3}$  ([2] p. 92) and can equal this value. It would be of interest to obtain a similar lower bound for the largest circle within which  $f$  is close-to-convex.

6. A sub-class of the close-to-convex functions.  
In a previous paper ([5]), the author demonstrated that the functions  $f(z)$  representable for  $|z| < 1$  by a Poisson integral

$$(35) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} h(\theta) d\theta,$$

in which  $h(\theta)$  is monotone non-decreasing, are schlicht for  $|z| < 1$ . It was also shown that each such  $f(z)$  satisfies

$$(36) \quad \text{Im} [(z - 1)^2 f'(z)] > 0,$$

from which we conclude that  $f(z)$  is close-to-convex.

We now show that these conclusions remain valid if we assume that  $h(\theta)$  is monotone non-decreasing in one interval of  $\theta$  and monotone non-increasing for the remaining values:

Theorem 3. Let  $h(\theta)$  be defined and non-constant for  $\pi < \theta < 2\pi$ ; let  $h(\theta)$  be monotone non-decreasing for  $0 < \theta < \pi$  and monotone non-increasing for  $\pi < \theta < 2\pi$ . Then (35) defines a function  $f(z)$  which is schlicht and close-to-convex for  $|z| < 1$ .

Proof: We shall verify that

$$\text{Re} \left[ \frac{f'(z)}{\phi'(z)} \right] > 0, \quad \phi(z) = \log \frac{z-1}{z+1},$$

i. e., that (6) holds with  $\alpha = 0$  and  $\beta = \pi$ . As in [5], we find that

$$\begin{aligned} \operatorname{Re} [(z^2 - 1) f'(z)] &= \operatorname{Re} \left[ \frac{i}{\pi} \int_0^{2\pi} \frac{(z+1)(1-e^{i\theta})}{e^{i\theta} - z} dh(\theta) \right] \\ &= \frac{1}{\pi} \int_0^{2\pi} \frac{\sin \theta (1+r)}{|e^{i\theta} - z|^2} dh(\theta). \end{aligned}$$

Since  $h(\theta)$  is non-decreasing for  $0 < \theta < \pi$ , where  $\sin \theta > 0$ , and non-increasing for  $\pi < \theta < 2\pi$ , where  $\sin \theta < 0$ , the integral is positive as asserted and  $f(z)$  is close-to-convex.

By applying a suitable linear transformation, we can extend the theorem to the case in which  $h$  is non-decreasing from 0 to  $\alpha$  and non-increasing from  $\alpha$  to  $2\pi$ , also to the case of functions defined in the half-plane  $y > 0$  by a Poisson integral

$$\frac{1}{\pi i} \int_{-\infty}^{\infty} \left\{ \frac{1}{t-z} - \frac{t}{1+t^2} \right\} h(t) dt.$$

Each function  $u + iv = f(z)$  can be shown to map onto a domain  $D$  which is convex in the sense that, if  $(u, v_1)$  and  $(u, v_2)$  are in  $D$ , then so also is the line segment joining these points. When  $h$  is a step-function,  $D$  is bounded by rays and two lines, all parallel to the  $v$ -axis; examples are given on pp. 605-609 of [6].

7. Mapping by non-analytic functions. It would be of interest to generalize the preceding discussion to mappings

$$(37) \quad u = F(x, y), \quad v = G(x, y), \quad x^2 + y^2 < 1,$$

where  $F$  and  $G$  are of class  $C^1$  and the Jacobian

$$(38) \quad J = \frac{\partial(F, G)}{\partial(x, y)}$$

is positive throughout. In conversations with the author, C. J. Titus has conjectured that if the mapping has the property that each circle  $x^2 + y^2 = r^2 < 1$  is mapped onto a path  $C$  whose tangent never turns back through  $\pi$ , as in Section 4, then it must be one-to-one. One can also ask whether the geometric condition just stated is sufficient to guarantee that a mapping (37) defined only for  $x^2 + y^2 = r^2$  is one-to-one; simple counter-examples show that this is not the case. However, C. J. Titus has conjectured that, if one also requires that the image curve has non-negative circulation (in a properly defined sense), then it must indeed be a simple closed curve.

The theorem of the preceding section does have a natural extension to mappings (37):

Theorem 4. Let  $F(x, y)$  be continuous for  $x^2 + y^2 < 1$  and let  $F(\cos \theta, \sin \theta) = h(\theta)$  be non-decreasing for  $0 < \theta < \pi$ , non-increasing for  $\pi < \theta < 2\pi$ . Let  $G(x, y)$  be defined for  $x^2 + y^2 < 1$  in such a manner that equations (37) define a mapping which is locally a homeomorphism. Then (37) defines a homeomorphism of the set:  $x^2 + y^2 < 1$  into the  $uv$ -plane.

Proof: Since the mapping is locally a homeomorphism, the level curves of  $F(x, y)$  must form a regular curve-family  $H$  filling the domain:  $x^2 + y^2 < 1$  ([3], p. 155). From the results of [3], it follows that each curve  $C$  of the family  $H$  can be parametrized by equations  $x = x(t)$ ,  $y = y(t)$ ,  $-\infty < t < \infty$ , so that  $x^2 + y^2 \rightarrow 1$  as  $t \rightarrow \infty$ . We denote by  $L[C, +]$  (or  $L[C, -]$ ) the set of limit points of sequences  $(x(t_n), y(t_n))$  as  $t_n \rightarrow \infty$  (or  $t_n \rightarrow -\infty$ ). Then  $L[C, +]$  must be

an arc of  $x^2 + y^2 = 1$  or a single point; since  $F(x, y)$  is continuous,  $L[C, +]$  can be an arc only when  $F(\cos \theta, \sin \theta) = h(\theta)$  is constant along the arc. From the monotonicity of  $h$  it follows that, if  $C_1, C_2, C_3$  are curves of  $H$  on which  $F$  has the respective values  $k_1, k_2, k_3$ , with  $k_1 < k_2 < k_3$ , then  $C_2$  separates  $C_1$  from  $C_3$  in  $x^2 + y^2 < 1$ . Hence in general, for every triple  $C_1, C_2, C_3$  in  $H$ , one curve separates the other two. By the main theorem (p. 11) of [4], it follows that  $H$  must have the structure of a family of parallel lines: i. e., there is a homeomorphism of  $x^2 + y^2 < 1$  onto itself transforming  $H$  onto the lines  $y = \text{const.}$ . In the new coordinates, the function  $F$  becomes a function  $F_0(y)$  which is strictly monotone, while  $G$  becomes a function  $G_0(x, y)$  which is strictly monotone in  $x$  for each  $y$ . Accordingly,  $u = F_0(y)$ ,  $v = G_0(x, y)$  defines a homeomorphism and therefore a similar conclusion holds for (37).



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December 1952