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CLOSED BOUNDED SETS IN INDUCTIVE LIMITS OF $\mathcal{K}$-SPACES

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A web W in a vector space F is a countable family of balanced subsets of F, arranged in "layers". The first layer of the web consists of a sequence $(A_p : p = 1, 2, \ldots)$ whose union absorbs each point of F. For each set A_p of the first layer there is a sequence $(A_{pq} : q = 1, 2, \ldots)$ of sets, called the sequence determined by A_p, such that

$$A_{pq} + A_{pq} \subset A_p \text{ for each } q,$$
$$\bigcup\{A_{pq} : q = 1, 2, \ldots\} \text{ absorb each point of } A_p.$$

Further layers are made up in a corresponding way so that each set of the k-th layer is indexed by a finite row of k integers and at each step the above mentioned two conditions are satisfied. Suppose that one chooses a set A_p from the first layer, then a set A_{pq} of the sequence determined by A_p and so on. The resulting sequence $\mathcal{S} = (A_p, A_{pq}, A_{pqr}, \ldots)$ is called a strand. Whenever we are dealing with only one strand we can simplify the notation by writing $W_1 = A_p$, $W_2 = A_{pq}$ etc.; thus $\mathcal{S} = (W_k)$ is a strand where for each k, W_k is a set in the k-th layer. We will work only with locally convex spaces and also assume that each member for a web is absolutely convex.

Let $S = (W_n)$ be a strand. Consider $x_n \in W_n$ and the series $\sum_{n=1}^{\infty} x_n$. The space F is webbed if the series $\sum_{n=1}^{\infty} x_n$ is convergent for any choice of x_n in W_n. The standard references for webs in locally convex spaces are [1], [2] and [3].

It is shown in [5] that $E = \operatorname{indlim} E_n$, where all spaces E_n are Fréchet, is regular iff it is locally complete. In [6] T. Gilsdorf substitutes the Fréchet spaces E_n by webbed $\mathcal{K}$-spaces and proves that E is regular iff it is locally Baire. In this paper the same substitution is used to generalize results by Qui Jing Hui, see [7], on closed bounded sets in inductive limits.

Theorem 1. *Let (E, τ) be a webbed locally convex space. Let $B \subset E$ be closed. If B is a Baire disk then B is bounded.*

Proof. Denote by η the topology in E_B induced by the system of neighborhoods of zero $\{(1/nB) \cap V\colon\ V \in \tau, V \text{ closed, and } n \in \mathbf{N}\}$.

First. Let's prove that (E_B, η) is a webbed space. Let W be a web in E consisting of the sets $A_{pq} \ldots r$. We will construct a web in E_B in the following way: the first layer will be $\{A_p \cap B\colon p = 1, 2, \ldots\}$, the second layer $\{A_{pq} \cap (\frac{1}{2}B)\colon q = 1, 2 \ldots\}$ and so on. Take a string (W_n') in E_B, consider any $x_n \in W_n'$ and the series $\sum_{n=1}^{\infty} x_n$. Note that for each W_n' there exist a W_n such that $W_n' \subset W_n \subset E$ where

$$W_1' = A_p \cap B \subset A_p = W_1 \subset E$$
$$W_2' = A_{pq} \cap B \subset A_{pq} = W_2 \subset E$$

and so on.

Since E is webbed for any choice of x_m in W_m the series $\sum_{m=1}^{\infty} x_m$ is convergent in E therefore also for any x_m in W_m'. That means that the sequences $y_m = \sum_{n=1}^{m} x_n$ converges in E so [Theorem 3.2.4 p. 59,8] it converges in (E_B, η). Then E_B, η is webbed.

Now the map *id*: $(E_B, \eta) \to (E_B, p_B)$ is continuous, (E_B, η) is webbed, and (E_B, p_B) is a Baire space. Hence it is also open by [Theorem 3.2, p. 59,2]. That means $id(B \cap V)$ is a neighborhood of 0 in (E_B, p_B). So there exists $\lambda > 0$ such that $\lambda B \subset B \cap V \subset V$, i.e. B is bounded in E. □

Lemma. *Let (E, τ) be a locally convex space. Let $B \subset E$ be closed. If (E, τ) is a $\mathcal{K}$-space then (E_B, η) is a $\mathcal{K}$-space.*

Proof. Let $x_n \to 0$ in (E_B, η). Then $x_n \to 0$ in (E, τ) because *id*: $(E_B, \eta) \to (E, \tau)$ is continuous. Hence there is a subsequence (x_{n_k}) of (x_n) and $x \in E$ such that $\sum_{k=1}^{\infty} x_{n_k} = x$. Let $y_m = \sum_{k=1}^{m} x_{n_k}$. Then (y_m) is a sequence of elements in E_B, and it converges in (E, τ) hence it also converges in $(E_B, \tau / E_B)$, so [Theorem 3.2.4, p. 59,8] y_m converges to x in (E_B, η). Since B is closed in (E, τ) and (y_m) is a bounded sequence in (E_B, τ) we have for some λ that $x \in \lambda B \subset E_B$. This means (E_B, η) satisfies property $\mathcal{K}$. □

Let $E_1 \subset E_2 \subset \ldots$ be a sequence of locally convex spaces with all identity maps: $(E_n, \tau_n) \to (E_{n+1}, \tau_{n+1})$ continuous and $E = \operatorname{indlim} E_n$.

Theorem 2. *Let each space (E_n, τ_n) be a webbed $\mathcal{K}$-space. Let a set B be an absolutely convex subset of E_n. If B is bounded and closed in some (E_m, τ_m), where $m > n$, then B is bounded and closed in (E_n, τ_n).*

P r o o f. Since B is closed in (E_m, τ_m) and *id*: $(E_n, \tau_n) \to (E_m, \tau_m)$ is continuous, B is also closed in (E_n, τ_n). Now B is bounded and closed in (E_m, τ_m), where (E_m, τ_m) is webbed and has property $\mathcal{K}$, so following the proof of Lemma, (E_B, p_B) also has property $\mathcal{K}$. By [4] any metrizable $\mathcal{K}$-space is Baire. We conclude that (E_B, p_B) is Baire and B is a Baire disk. By Theorem 1, B is bounded in (E_n, τ_n). □

Theorem 3. *Let each space (E_n, τ_n) be a webbed $\mathcal{K}$-space. Let a set B be absolutely convex, bounded, and closed in $(E, \tau) =$ indlim(E_n, τ_n). If (E, τ) is locally Baire and $B \subset E_n$ for some n in $\mathbf{N}$ then B is bounded and closed in (E_n, τ_n).*

P r o o f. It follows from Theorem 2 and Theorem 3 in [6]. □

References

[1] *De Wilde, M.*: Closed Graph Theorems and Webbed Spaces, Pitman, 1978.
[2] *Köthe, G.*: Topological Vector Spaces II, Springer Verlag, 1979.
[3] *Robertson, W.*: On the Closed Graph Theorem and Spaces with Webs, Proc. London Math. Soc. *24* (1972), 692–738.
[4] *Antosik, P. Schwartz, C.*: Matrix Methods in Analysis, Springer Verlag Lectures Notes *Vol. 1113* (1985).
[5] *Kučera, J., Bosch, C.*: Bounded Sets in Fast Complete Inductive Limits, Int. J. Math. *7(3)* (1984), 615–617.
[6] *Gilsdorf, T.*: Regular Inductive Limits of $\mathcal{K}$-spaces, preprint.
[7] *Qui Jing Hui*: Dieudonné-Schwartz Theorem in Inductive Limits of Metrizable Spaces II, preprint.
[8] *Jarchow H.*: Locally Convex Spaces, B.G. Teubner-Stuttgart, 1981.

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