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CLOSED EXPRESSION FOR THE ELECTRICAL FIELD OF A TWO-DIMENSIONAL GAUSSIAN CHARGE

by

M. Bassetti* and G.A. Erskine

ABSTRACT

For the simulation of beam-beam interaction one needs to evaluate the electric field of a two-dimensional Gaussian charge distribution, preferably by means of a fast computer program. This paper shows how this can be done by using the complex error function. This connection with the complex error function has previously been established, for one co-ordinate plane only, by B.W. Montague[1).

*Visitor from Laboratori Nazionali di Frascati

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1. Derivation

Let us consider the charge density function

$$\rho(x,y) = \frac{Q}{2\pi\ \sigma_x\ \sigma_y}\ e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)} .$$

An expression for the associated potential $\phi(x,y)$ which satisfies the equation

$$\nabla^2\phi = \frac{\rho}{\varepsilon_o}$$

can be written[2]:

$$\phi(x,y) = \frac{Q}{4\pi\ \varepsilon_o} \int_o^\infty \frac{e^{-\left[\frac{x^2}{2\sigma_x^2 + q} + \frac{y^2}{2\sigma_y^2 + q}\right]}}{\sqrt{2\sigma_x^2 + q}\ \sqrt{2\sigma_y^2 + q}}\ dq .$$

To establish the connection between this problem and the complex error function it is necessary to change the integration variable. If we suppose that $\sigma_x > \sigma_y$, set

$$t^2 = \frac{2\sigma_y^2 + q}{2\sigma_x^2 + q} ,$$

and define

$$r = \frac{\sigma_y}{\sigma_x} ,$$

$$a = \frac{x}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \quad \text{and} \quad b = \frac{y}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} , \tag{1}$$

we obtain, after some algebra,

$$\phi(x,y) = \frac{-Q}{2\pi\ \varepsilon_o} \int_r^1 \frac{e^{\left[a^2(t^2 - 1) + b^2\left(1 - \frac{1}{t^2}\right)\right]}}{t^2 - 1}\ dt . \tag{2}$$

If E_x and E_y are the components of the electric field, it follows from (1) and (2) that

$$E_x - iE_y = -\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y} = \frac{-1}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}\left(\frac{\partial\phi}{\partial a} - i\frac{\partial\phi}{\partial b}\right)$$

$$= \frac{Q}{\pi\varepsilon_o\sqrt{2(\sigma_x^2 - \sigma_y^2)}}\int_r^1\left(a - \frac{ib}{t^2}\right)e^{\left[a^2(t^2-1) + b^2\left(1 - \frac{1}{t^2}\right)\right]}dt \qquad (3)$$

$$= \frac{Q}{\pi\varepsilon_o\sqrt{2(\sigma_x^2 - \sigma_y^2)}}e^{-(a+ib)^2}\int_r^1\left(a - \frac{ib}{t^2}\right)e^{\left(at + i\frac{b}{t}\right)^2}dt$$

$$= \frac{Q}{\pi\varepsilon_o\sqrt{2(\sigma_x^2 - \sigma_y^2)}}e^{-(a+ib)^2}\int_{ar + i\frac{b}{r}}^{a+ib}e^{\zeta^2}d\zeta \quad , \qquad (4)$$

where the path of integration is defined by

$$\zeta = at + i\frac{b}{t} \quad , \quad (r \le t \le 1) \quad .$$

Making use of the definition of the complex error function w(z), namely[3)]

$$w(z) = e^{-z^2}\left[1 + \frac{2i}{\sqrt{\pi}}\int_o^z e^{\zeta^2}d\zeta\right] \quad , \qquad (5)$$

where the path of integration is arbitrary, we obtain from (4) and (5):

$$E_x - iE_y = -i\frac{Q}{2\varepsilon_o\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\left[w(a+ib) - e^{\left[-(a+ib)^2+\left(ar + i\frac{b}{r}\right)^2\right]}w\left(ar + i\frac{b}{r}\right)\right] \quad , \qquad (6)$$

and hence:

$$E_x = \frac{Q}{2\varepsilon_o\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\,\mathrm{Im}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{\left[-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right]} w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right], \quad (7)$$

$$E_y = \frac{Q}{2\varepsilon_o\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\,\mathrm{Re}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{\left[-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right]} w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right], \quad (8)$$

A computer program[4)] for the evaluation of w(z) is available in the CERN program library. Using this program, the evaluation of E_x and E_y from the expressions (7) and (8) is usually considerably faster than the use of numerical quadrature in expression (3).

2. Notes and acknowledgements

The connection between the expressions for the electric field and two integrals equivalent to (3) was established by one of the authors (M.B.) some fifteen years ago, prior to the construction of the ADONE storage ring.

We should like to thank P.M. Hanney for her help with the programming.

3. References

1) B.W. Montague, Calculation of luminosity and beam-beam detuning in coasting beam interaction regions, CERN/ISR-GS/75-36.

2) S. Kheifets, Potential of a three-dimensional Gauss bunch, PETRA Note 119 (1.10.76).

3) M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions (National Bureau of Standards, Washington, 1966), formula 7.1.3.

4) K.S. Kölbig, Complex error function, Program C 335, CERN, Geneva (1970).

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