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Closed expressions for specific massive multiloop self-energy integrals *

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Abstract

In this paper the class of N loop massive scalar self-energy diagrams with $N+1$ propagators is studied in an arbitrary number of dimensions. As it is known these integrals cannot be expressed in terms of polylogarithms. Here it is shown, however, that they can be described by generalized hypergeometric functions of several variables, namely Lauricella functions. These results represent previous small and large momentum expansions in closed form. Numerical comparisons for the finite part in four dimensions with a two-dimensional integral representation show good agreement.

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1 Introduction

The precision of the experiments testing the electroweak theory is at this moment high and will even improve in the future. For instance, the Z mass measurements could reach a precision of about 5×10^{-5} and the forward Bhabha measurement an accuracy of 2×10^{-4}.

Ideally, one would like to have two-loop and higher-order calculations for various experimental quantities. One then faces the problem of evaluating two-loop diagrams with massive propagators, where several combinations of different masses can arise. Even at the simplest level, i.e. that of scalar two-loop self-energy diagrams, one encounters a serious problem: when no masses vanish the integrals are certainly not expressible in terms of known functions such as polylogarithms. It can be shown that the diagrams in which three particle cuts arise, with all particles massive, lead to the above problem. To circumvent it two approaches have been followed in the literature.

One of them is to find expansions of the self-energy diagrams in terms of powers and possibly logarithms in p^2, where p is the external momentum. Both small and large p^2 expansions have been derived in [1] and [2]. For the intermediate region no series expansions have been given so far. The expansion coefficients are functions of the masses and can be derived with algorithms which are accessible via symbolic manipulation programs. General closed expressions for these coefficients have not been derived.

Another approach is a numerical one. Recently an elegant two-dimensional integral representation has been given [3] for the convergent scalar self-energy diagram with two loops and five propagators, the so-called master diagram. This method has now been extended [4] to the finite parts of all two-loop scalar self-energy diagrams, which are in general divergent. The numerical integration can be carried out. Where analytical results and series expansions exist good agreement is obtained.

This paper aims at enlarging the analytical knowledge of the scalar diagrams in those cases where known functions like polylogarithms cannot represent the result anymore. Since this phenomenon is related to the massive three particle cut, the problem in its simplest form arises in the two-loop self-energy diagram with three massive propagators (fig. 1).

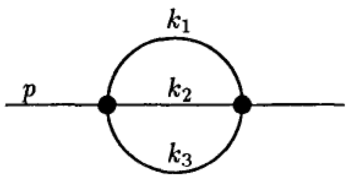

Figure 1: The London transport diagram

In this so-called London transport diagram the only way to cut the diagram results in a three-particle cut. It will be shown that an evaluation of this diagram in

D dimensions is possible leading to a number of series, in fact triple series in variables which are ratios of the three masses and p^2. The series turn out to belong to a special class of generalized hypergeometric series, which has been studied in the mathematical literature. These are the Lauricella functions F_C for which quite some mathematical apparatus is known, like convergence properties and a few analytic continuation formulae. Although the results are of course not expressible in polylogarithms they are at least expressible in this known type of functions which is a significant improvement on previous results. The finite parts in four dimensions are series which are easily evaluated numerically with adequate precision.

In a sense, the results of this paper provide an explicit formula for the coefficients of the large and small p^2 expansions referred to above. The coefficients are themselves series which can be read off explicitly from the Lauricella series. It should be remarked that multiple series for the evaluation of two-loop Feynman diagrams have been studied before [5]. The results in this paper differ in the sense that they are related to known series. Moreover, they are developed to the level of practical numerical evaluation for the finite parts. The expressions are valid in regions where Lauricella series and their continuations converge.

From the way the London transport diagram is evaluated, it will become clear that the N-loop generalization of this diagram can easily be derived. Those diagrams will arise in gauge theories from a reduction of tensor integrals to scalar ones. One again finds a sum of Lauricella functions which now depend on $N+1$ variables since there are $N+1$ ratios between the $N+1$ masses and p^2. Also the generalization to arbitrary powers of the propagators is easily obtained. The latter are relevant for the analytic regularization scheme. Integer powers sometimes occur directly in certain multi-loop diagrams.

The outline of the paper is as follows. Section 2 describes two methods to obtain results for massive diagrams, the x-space method and the Mellin-Barnes representation. The small p^2 result for the two-loop London transport diagram is derived. The next section summarizes the large p^2 result for two loops, the N loop formulae and the case of arbitrary powers of the propagators in the two-loop diagram. Section 4 derives the results near $D = 4$ and presents numerical comparisons with a two-dimensional integration of the finite part of the diagram. The last section closes the paper with concluding remarks.

2 Method

In this section two techniques are briefly described with which the N-loop massive Feynman diagrams will be evaluated. One method uses x-space techniques for massive propagators. The details can be found in [5]. The second method uses the Mellin-Barnes representation for a massive propagator. In this way, massive diagrams can be related to integrals over massless diagrams, where the propagators are raised to an arbitrary power. This technique has been advocated in [6] and already applied to two-loop integrals in [1, 7]. The method will be illustrated for the two-loop self-energy diagram with three propagators. The generalization of this diagram, i.e. two vertices, $N+1$ propagators, N-loops, can be treated in the same way.

For the notation we shall follow the conventions of [8], which means that for every D dimensional loop integration a bracket is introduced

$$< \ldots > = \int \frac{d^D q}{i\pi^2 (2\pi\mu)^{D-4}} (\ldots) , \tag{1}$$

where μ is an arbitrary mass. In this way the one-loop self-energy diagram will be denoted by

$$T_{12}(p^2, m_1^2, m_2^2) = < \frac{1}{(k_1^2 - m_1^2)(k_2^2 - m_2^2)} > , \tag{2}$$

whereas the two-loop self-energy diagram reads

$$T_{123}(p^2, m_1^2, m_2^2, m_3^2) = << \frac{1}{(k_1^2 - m_1^2)(k_2^2 - m_2^2)(k_3^2 - m_3^2)} >> \tag{3}$$

and the N-loop self-energy diagram becomes

$$T_{1 \ldots N+1}(p^2, m_1^2, \ldots, m_{N+1}^2) = < \ldots < \frac{1}{(k_1^2 - m_1^2)} \cdots \frac{1}{(k_{N+1}^2 - m_{N+1}^2)} > \ldots > . \tag{4}$$

In order to present the x-space formalism one uses Euclidean momenta k_i as integration variables and p^2 is considered to be spacelike and $q^2 = -p^2$.
One writes

$$T_{123}(p^2, m_1^2, m_2^2, m_3^2) = -\frac{1}{\pi^4 (2\pi\mu)^{2(D-4)}} \frac{1}{(2\pi)^D} \int d^D x \exp(iqx) \tag{5}$$
$$\int d^D k_1 \frac{\exp(-ik_1 x)}{(k_1^2 + m_1^2)} \int d^D k_2 \frac{\exp(-ik_2 x)}{(k_2^2 + m_2^2)} \int d^D k_3 \frac{\exp(-ik_3 x)}{(k_3^2 + m_3^2)} .$$

Now the Fourier transform of the propagators is introduced

$$\int d^D k_i \frac{\exp(-ik_i x)}{(k_i^2 + m_i^2)} = 2\pi^{\nu+1} (x/2)^{-\nu} m_i^\nu K_\nu(m_i x) , \tag{6}$$

where $\nu = (D-2)/2$ and $K_\nu(m_i x)$ is the modified Bessel function, sometimes called McDonald function. Furthermore we need

$$\int d\hat{x} \exp(iqx) = 2\pi^{\nu+1} (qx/2)^{-\nu} J_\nu(qx) , \tag{7}$$

where $J_\nu(qx)$ is the Bessel function and $d\hat{x}$ denotes the angular integration in D dimensions. The resulting x-space representation reads

$$T_{123}(q^2, m_1^2, m_2^2, m_3^2) = -\frac{2^4}{(2\pi\mu^2)^{2(\nu-1)}} \tag{8}$$
$$\times (m_1 m_2 m_3 / q)^\nu \int_0^\infty dx\, x^{-2\nu+1} J_\nu(qx) K_\nu(m_1 x) K_\nu(m_2 x) K_\nu(m_3 x) ,$$

where x is the radial part of the D dimensional $\vec{x}$. At this point it should be noted that the N loop case has $N+1$ McDonald functions

$$T_{1 \ldots N+1}(q^2, m_1^2, \ldots, m_{N+1}^2) = (-1)^{N+1} \frac{2^{2N}}{(2\pi\mu^2)^{N(\nu-1)}} \tag{9}$$
$$\times (m_1 \ldots m_{N+1} / q)^\nu \int_0^\infty dx\, x^{1-N\nu} J_\nu(qx) K_\nu(m_1 x) \ldots K_\nu(m_{N+1} x) .$$

In order to carry out the x integration one can find explicit expressions for the cases with up to two K_ν functions, see e.g [9]. For the other cases one should expand the Bessel function J_ν and a number of K_ν functions, using

$$
\begin{aligned}
J_\nu(qx) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(1+\nu+n)\,n!}\left(\frac{qx}{2}\right)^{2n+\nu}, \qquad (10)\\
K_\nu(mx) &= \frac{\Gamma(\nu)\Gamma(1-\nu)}{2} \qquad (11)\\
&\times \sum_{k=0}^{\infty}\left(\frac{1}{\Gamma(1-\nu+k)k!}\left(\frac{mx}{2}\right)^{2k-\nu} - \frac{1}{\Gamma(1+\nu+k)k!}\left(\frac{mx}{2}\right)^{2k+\nu}\right).
\end{aligned}
$$

It is convenient to choose for the latter $K_\nu(m_1x), \ldots, K_\nu(m_Nx)$, so one is left with an integral

$$
\int_0^\infty dx\, x^{1+\nu+2t} K_\nu(m_{N+1}x) = 2^{\nu+2t} m_{N+1}^{-2-\nu-2t}\,\Gamma(1+\nu+t)\Gamma(1+t). \qquad (12)
$$

Applying this to (8) we find the following series

$$
\begin{aligned}
T_{123}(p^2, m_1^2, m_2^2, m_3^2) &= -m_3^2\left(\frac{m_3^2}{4\pi\mu^2}\right)^{2\nu-2}(\Gamma(\nu)\Gamma(1-\nu))^2 \qquad (13)\\
\sum_{m,n,k=0}^{\infty} &\left\{ z_1^\nu z_2^\nu \frac{\Gamma(1+m+n+k)\Gamma(1+\nu+m+n+k)}{\Gamma(1+\nu+m)\Gamma(1+\nu+n)\Gamma(1+\nu+k)}\right.\\
&- z_1^\nu \frac{\Gamma(1+m+n+k)\Gamma(1-\nu+m+n+k)}{\Gamma(1+\nu+m)\Gamma(1-\nu+n)\Gamma(1+\nu+k)}\\
&- z_2^\nu \frac{\Gamma(1+m+n+k)\Gamma(1-\nu+m+n+k)}{\Gamma(1-\nu+m)\Gamma(1+\nu+n)\Gamma(1+\nu+k)}\\
&\left.+ \frac{\Gamma(1-\nu+m+n+k)\Gamma(1-2\nu+m+n+k)}{\Gamma(1-\nu+m)\Gamma(1-\nu+n)\Gamma(1+\nu+k)}\right\} \frac{z_1^m z_2^n z_3^k}{m!n!k!},
\end{aligned}
$$

where $z_i = m_i^2/m_3^2$, $i = 1, 2$, $z_3 = -q^2/m_3^2 = p^2/m_3^2$. In z_3 the substitution to the usual spacelike p^2 is made.

In the above series we recognize a special instance of the Lauricella functions [10, 11, 12, 13] defined by

$$
F_C^{(n)}(a, b; c_1, \ldots, c_n; z_1, \ldots, z_n) = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{(a)_{k_1+\ldots+k_n}(b)_{k_1+\ldots+k_n}}{(c_1)_{k_1}\cdots(c_n)_{k_n}} \frac{z_1^{k_1}\cdots z_n^{k_n}}{k_1!\cdots k_n!}, \qquad (14)
$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$. The defining multiple series converges for

$$
\sqrt{|z_1|} + \ldots + \sqrt{|z_n|} < 1. \qquad (15)
$$

Now it is apparent that $T_{123}(p^2, m_1^2, m_2^2, m_3^2)$ can be written as a combination of four Lauricella functions

$$T_{123}(p^2,m_1^2,m_2^2,m_3^2) = -m_3^2\left(\frac{m_3^2}{4\pi\mu^2}\right)^{2(\nu-1)} \times \tag{16}$$
$$\Big\{z_1^\nu z_2^\nu\,\Gamma^2(-\nu)\,F_C^{(3)}(1,1+\nu;1+\nu,1+\nu,1+\nu;z_1,z_2,z_3)$$
$$-z_1^\nu\,\Gamma^2(-\nu)\,F_C^{(3)}(1,1-\nu;1+\nu,1-\nu,1+\nu;z_1,z_2,z_3)$$
$$-z_2^\nu\,\Gamma^2(-\nu)\,F_C^{(3)}(1,1-\nu;1-\nu,1+\nu,1+\nu;z_1,z_2,z_3)$$
$$-\Gamma(\nu)\Gamma(-\nu)\Gamma(1-2\nu)\,F_C^{(3)}(1-2\nu,1-\nu;1-\nu,1-\nu,1+\nu;z_1,z_2,z_3)\Big\}\,.$$

The individual series above converge for $m_1+m_2+\sqrt{|p^2|}<m_3$. Collecting powers in p^2, however, the total sum converges due to analyticity up to the next singularity on the physical sheet given by the threshold condition

$$|p^2|\,<\,(m_1+m_2+m_3)^2\,, \tag{17}$$

provided that the coefficients themselves do exist, which is the case for

$$m_1+m_2\,<\,m_3\,. \tag{18}$$

In the annulus $(m_1+m_2-m_3)^2\le|p^2|<(m_1+m_2+m_3)^2$ some numerical care is required because of compensations between the coefficients, which are inevitable for the above reason.

The mass m_3 plays a prominent role in expression (16) and consequently for the convergence region (18), but may be interchanged freely with the other masses due to the symmetry of the diagram.

The corresponding N-loop integral can be solved in a similar way. The explicit result is given in the next section.

In case we consider a propagator raised to a power α we need the corresponding Fourier transform

$$\int d^Dk_i\frac{\exp(-ik_ix)}{(k_i^2+m_i^2)^\alpha}=\frac{1}{\Gamma(\alpha)}2\pi^{\nu+1}(x/2)^{-\nu-1+\alpha}m_i^{\nu+1-\alpha}K_{\nu+1-\alpha}(m_ix)\,. \tag{19}$$

The result for such a type of diagram will be given in the next section.

All results may alternatively be derived by use of the Mellin-Barnes representation for a massive propagator

$$\frac{1}{(k^2-m^2)^\alpha}=\frac{1}{\Gamma(\alpha)}\frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty}ds\,\frac{(-m^2)^s}{(k^2)^{\alpha+s}}\Gamma(-s)\Gamma(\alpha+s)\,, \tag{20}$$

where the integration contour in the s plane must separate the series of poles of $\Gamma(-s)$ on the right from the series of poles of $\Gamma(s+\alpha)$ on the left. In the expression for $T_{123}(p^2,m_1^2,m_2^2,m_3^2)$ we apply (20) with $\alpha=1$ to all propagators, thereby relating the general massive case to the massless one, but with the propagators raised to powers $1+s_1, 1+s_2, 1+s_3$.

The corresponding result is well-known, see e.g.[2]

$$T_{123}(p^2,0,0,0;1+s_1,1+s_2,1+s_3) = -(4\pi\mu^2)^{4-D}\, e^{-i\pi(s_1+s_2+s_3)}\,(-p^2)^{D-3-s_1-s_2-s_3}$$
$$\times\frac{\Gamma(3-D+s_1+s_2+s_3)\Gamma(D/2-1-s_1)\Gamma(D/2-1-s_2)\Gamma(D/2-1-s_3)}{\Gamma(1+s_1)\Gamma(1+s_2)\Gamma(1+s_3)\Gamma(3D/2-3-s_1-s_2-s_3)}. \quad (21)$$

Insertion of this expression leads to the following integral representation

$$T_{123}(p^2,m_1^2,m_2^2,m_3^2) = -(4\pi\mu^2)^{4-D}\times \quad (22)$$
$$\frac{1}{(2\pi i)^3}\iiint_{-i\infty}^{+i\infty} ds_1\,ds_2\,ds_3\,\frac{(m_1^2)^{s_1}(m_2^2)^{s_2}(m_3^2)^{s_3}}{(-p^2)^{3-D+s_1+s_2+s_3}}\Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3)$$
$$\times\,\Gamma(3-D+s_1+s_2+s_3)\,\frac{\Gamma(D/2-1-s_1)\Gamma(D/2-1-s_2)\Gamma(D/2-1-s_3)}{\Gamma(3\,D/2-3-s_1-s_2-s_3)}.$$

We close the integration contours of s_1 and s_2 to the right and the one of s_3 to the left. Summation over the residua of the poles in the Γ-functions at

$$\begin{aligned} s_1 &= k_1,\ D/2-1+k_1 && \text{for } k_1=0,1,\dots;\\ s_2 &= k_2,\ D/2-1+k_2 && \text{for } k_2=0,1,\dots;\\ s_3 &= -s_1-s_2+D-3-k_3 && \text{for } k_3=0,1,\dots \end{aligned}$$

again yields the same four Lauricella series. The convergence of these multiple series, governed by (15), serves, a posteriori, as a criterion as to whether it was really justified to close the contours in the specified manner [14].

Note that one may also close all of the contours to the right. The result thus obtained is the same as the one we shall derive in section 3 by analytically continuing the Lauricella functions to the region $|p^2| > (m_1+m_2+m_3)^2$.

Associated with a Lauricella function $F_C^{(n)}$ in the variables $z_1,\dots,z_n$ there is a system of n coupled partial differential equations in these variables [12]. It has 2^n independent solutions, all of them expressible in terms of $F_C^{(n)}$ functions in the same variables but with different parameters and prefactors. It is interesting to note that all four $F_C^{(3)}$-terms in (16) belong to the same system and are just those out of the possible eight solutions which are regular at $p^2=0$, that is with no z_3-dependent factor in front. Thus $T_{123}(p^2,m_1^2,m_2^2,m_3^2)$ as a whole obeys the same system of three partial differential equations. This can also be demonstrated directly with the help of the integral representation (22).

3 General results

In this section we derive by analytic continuation the result for the London transport diagram in the region $|p^2| > (m_1+m_2+m_3)^2$. Then the same diagram with arbitrary powers of the propagators is considered for small p^2, which leads again to four Lauricella functions. Finally the N-loop case is presented in terms of 2^N Lauricella functions.

A Lauricella function in the arguments z_i for $i = 1, \ldots, n$ can be analytically continued to a sum of two Lauricella functions in the arguments $x_i = z_i/z_n$ for $i = 1, \ldots, n-1$ and $x_n = 1/z_n$ by the following relation [11]

$$
\begin{aligned}
&F_C^{(n)}(a, b; c_1, \ldots, c_n; z_1, \ldots, z_n) = \qquad (23)\\
&\frac{\Gamma(c_n)\Gamma(b-a)}{\Gamma(b)\Gamma(c_n-a)}(-z_n)^{-a} F_C^{(n)}(a, 1+a-c_n; c_1, \ldots, c_{n-1}, 1-b+a; x_1, \ldots, x_n)\\
&+\frac{\Gamma(c_n)\Gamma(a-b)}{\Gamma(a)\Gamma(c_n-b)}(-z_n)^{-b} F_C^{(n)}(b, 1+b-c_n; c_1, \ldots, c_{n-1}, 1-a+b; x_1, \ldots, x_n)\,.
\end{aligned}
$$

Applied to (16) this yields, as one coefficient vanishes, a total of seven transformed Lauricella functions

$$
\begin{aligned}
&T_{123}(p^2, m_1^2, m_2^2, m_3^2) = -\left(\frac{-p^2}{4\pi\mu^2}\right)^{2\nu-2}(-p^2)\times \qquad (24)\\
&\Big\{(-x_1)^\nu(-x_3)^\nu\Gamma^2(-\nu)\; F_C^{(3)}(1, 1-\nu; 1+\nu, 1-\nu, 1+\nu; x_1, x_2, x_3)\\
&+(-x_2)^\nu(-x_3)^\nu\Gamma^2(-\nu)\; F_C^{(3)}(1, 1-\nu; 1-\nu, 1+\nu, 1+\nu; x_1, x_2, x_3)\\
&+(-x_1)^\nu(-x_2)^\nu\Gamma^2(-\nu)\; F_C^{(3)}(1, 1-\nu; 1+\nu, 1+\nu, 1-\nu; x_1, x_2, x_3)\\
&+(-x_1)^\nu\frac{\Gamma^2(\nu)\Gamma(-\nu)\Gamma(1-\nu)}{\Gamma(2\nu)}\; F_C^{(3)}(1-\nu, 1-2\nu; 1+\nu, 1-\nu, 1-\nu; x_1, x_2, x_3)\\
&+(-x_2)^\nu\frac{\Gamma^2(\nu)\Gamma(-\nu)\Gamma(1-\nu)}{\Gamma(2\nu)}\; F_C^{(3)}(1-\nu, 1-2\nu; 1-\nu, 1+\nu, 1-\nu; x_1, x_2, x_3)\\
&+(-x_3)^\nu\frac{\Gamma^2(\nu)\Gamma(-\nu)\Gamma(1-\nu)}{\Gamma(2\nu)}\; F_C^{(3)}(1-\nu, 1-2\nu; 1-\nu, 1-\nu, 1+\nu; x_1, x_2, x_3)\\
&+\frac{\Gamma^3(\nu)\Gamma(1-2\nu)}{\Gamma(3\nu)}\; F_C^{(3)}(1-3\nu, 1-2\nu; 1-\nu, 1-\nu, 1-\nu; x_1, x_2, x_3)\Big\}\,.
\end{aligned}
$$

Now, this expression is valid for

$$|p^2| > (m_1 + m_2 + m_3)^2\,. \qquad (25)$$

Again, $T_{123}(p^2, m_1^2, m_2^2, m_3^2)$ obeys in the transformed variables the same system of partial differential equations as the individual terms.

One may wonder what the relation is between the large p^2 expansion of (24) and that given in [2]. In the latter approach the various terms in the p^2 expansion are obtained from the expansion of subgraphs. The subgraphs are obtained by distributing the momentum p over the propagators in all possible ways. In the case of the two-loop London transport diagram one has the following subgraphs: the diagram itself, the three diagrams where one internal line is removed and the three diagrams where two internal lines have been removed.

Following the analysis of [2] one can easily find the first term of each of the contributing series. For the subgraph representing the whole diagram the first term in the series should be the massless diagram. This series then corresponds to the last term in (24). The series which originates from the subgraph where two lines have been removed, e.g. 1 and 2, starts with the product of two massive tadpoles. They contribute a factor $(m_1^2 m_2^2)^\nu$ which can be identified with the third term in (24). The remaining subgraphs are obtained by removing one internal line, e.g. line 3. This yields a series starting with a massive tadpole proportional to $(m_3^2)^\nu$. This is the sixth term in (24). Thus the seven series in (24) can be related directly to the seven subgraphs which are required for the method of [2].

In order to derive the London transport diagram with arbitrary powers α_i of the propagators one proceeds as in section 2, but now using McDonald K functions with an index like in (19). The result is a direct generalization of (16)

$$
\begin{aligned}
&T_{123}(p^2, m_1^2, m_2^2, m_3^2; \alpha_1, \alpha_2, \alpha_3) = \qquad (26)\\
&e^{-i\pi(\alpha_1+\alpha_2+\alpha_3)} \left(\frac{1}{4\pi\mu^2}\right)^{2(\nu-1)} (m_3^2)^{2\nu+2-\alpha_1-\alpha_2-\alpha_3} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \times \\
&\Big\{ z_1^{\nu+1-\alpha_1} z_2^{\nu+1-\alpha_2} \Gamma(-\nu-1+\alpha_1)\Gamma(-\nu-1+\alpha_2)\Gamma(\alpha_3) \\
&\quad \times F_C^{(3)}(\alpha_3, \nu+1; \nu+2-\alpha_1, \nu+2-\alpha_2, \nu+1; z_1, z_2, z_3) \\
&+ z_1^{\nu+1-\alpha_1} \frac{\Gamma(-\nu-1+\alpha_1)\Gamma(\nu+1-\alpha_2)\Gamma(-\nu-1+\alpha_2+\alpha_3)\Gamma(\alpha_2)}{\Gamma(\nu+1)} \\
&\quad \times F_C^{(3)}(-\nu-1+\alpha_2+\alpha_3, \alpha_2; \nu+2-\alpha_1, -\nu+\alpha_2, \nu+1; z_1, z_2, z_3) \\
&+ z_2^{\nu+1-\alpha_2} \frac{\Gamma(\nu+1-\alpha_1)\Gamma(-\nu-1+\alpha_2)\Gamma(-\nu-1+\alpha_1+\alpha_3)\Gamma(\alpha_1)}{\Gamma(\nu+1)} \\
&\quad \times F_C^{(3)}(-\nu-1+\alpha_1+\alpha_3, \alpha_1; -\nu+\alpha_1, \nu+2-\alpha_2, \nu+1; z_1, z_2, z_3) \\
&+ \frac{\Gamma(\nu+1-\alpha_1)\Gamma(\nu+1-\alpha_2)\Gamma(-\nu-1+\alpha_1+\alpha_2)\Gamma(-2\nu-2+\alpha)}{\Gamma(\nu+1)} \\
&\quad \times F_C^{(3)}(-2\nu-2+\alpha, -\nu-1+\alpha_1+\alpha_2; -\nu+\alpha_1, -\nu+\alpha_2, \nu+1; z_1, z_2, z_3) \Big\} ,
\end{aligned}
$$

where $\alpha = \alpha_1 + \alpha_2 + \alpha_3$.

Finally the London transport diagram is easily generalized to N loops. This implies that more K functions occur, which also have to be expanded. There will now be 2^N series again representing Lauricella functions in $N+1$ variables.

$$
\begin{aligned}
&T_{1\ldots N+1}(p^2, m_1^2, \ldots, m_{N+1}^2) = (-1)^{N+1} \left(\frac{m_{N+1}^2}{4\pi\mu^2}\right)^{N(\nu-1)} \left(m_{N+1}^2\right)^{N-1} \times \qquad (27)\\
&\sum_{k=0}^{N} \frac{\Gamma(1+\nu-k\nu)\Gamma(1-k\nu)}{\Gamma(1+\nu)} \Gamma^k(\nu)\Gamma^{N-k}(-\nu) \sum_{i_k>\ldots>i_1=1}^{N} \frac{z_1^\nu \cdots z_N^\nu}{z_{i_1}^\nu \cdots z_{i_k}^\nu} \\
&\quad \times F_C^{(N+1)}(1+\nu-k\nu, 1-k\nu; 1+\nu, \ldots, \underbrace{1-\nu}_{i_1}, \ldots, \underbrace{1-\nu}_{i_k}, \ldots, 1+\nu, 1+\nu; \\
&\qquad\qquad z_1, \ldots, z_{i_1}, \ldots, z_{i_k}, \ldots, z_N, z_{N+1}) .
\end{aligned}
$$

The analytic continuation to the region $|p^2| > (m_1 + \ldots + m_{N+1})^2$ is straightforward and leads to $2^{N+1} - 1$ Lauricella functions. Here again the series correspond to a division into subgraphs which the method of [2] uses to derive the large p^2 expansion.

All results can equivalently be derived using the Mellin-Barnes representation and are easily extended to arbitrary powers of the propagators. Furthermore, the diagram as a whole once again obeys the same system of partial differential equations as the individual terms.

4 Expansion in δ and numerical comparisons for the finite part

In the following the general D dimensional expressions for the 2-loop London transport diagram will be expanded in $\delta = (4-D)/2 = 1-\nu$, both for small and large $|p^2|$. Then a numerical comparison for the finite part will be made with a completely independent numerical calculation. The latter approach, as described in [4], is an extension of Kreimer's method [3] to represent a self-energy diagram by a two-dimensional integral.

The following combination of the general massive case with massless cases is chosen in such a way that the infinite parts cancel

$$\begin{aligned} T_{123N}(p^2,m_1^2,m_2^2,m_3^2) &= T_{123}(p^2,m_1^2,m_2^2,m_3^2) - T_{123}(p^2,m_1^2,0,m_3^2) \\ &\quad -T_{123}(p^2,0,m_2^2,m_3^2) + T_{123}(p^2,0,0,m_3^2)\,. \end{aligned} \tag{28}$$

It is this combination which will be calculated in two independent ways.

An analytic form is obtained by expansion of the Lauricella functions and their coefficients in δ, where the first and the second logarithmic derivatives of the Γ-function occur at integer arguments

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}\,, \tag{29}$$

$$\psi'(n+1) = \zeta(2) - \sum_{k=1}^{n} \frac{1}{k^2}\,, \tag{30}$$

with the Euler constant γ and $\zeta(2) = \pi^2/6$.

The $1/\delta^2$ and $1/\delta$ terms indeed drop out in the result and a finite combination of various multiple series remains. A good check is provided by the cancellation of γ in this finite expression.

For small $|p^2|$, i.e. the region $|p^2| < m_3^2$, one finds

$$\begin{aligned} &T_{123N}(p^2,m_1^2,m_2^2,m_3^2)/m_3^2 = \\ &-\sum_{m,n=1,k=0}^{\infty} \frac{(m+n+k-2)!(m+n+k-1)!}{(m-1)!m!(n-1)!n!(k+1)!k!} z_1^m z_2^n z_3^k \\ &\qquad \Big[\{\psi(m+n+k)+\psi(m+n+k-1)-\psi(m)-\psi(m+1)+\log(z_1)\} \\ &\qquad \times \{\psi(m+n+k)+\psi(m+n+k-1)-\psi(n)-\psi(n+1)+\log(z_2)\} \\ &\qquad + \psi'(m+n+k)+\psi'(m+n+k-1)\Big]\,. \end{aligned} \tag{31}$$

For large $|p^2|$, i.e. the region $|p^2| > (m_3+m_2+m_1)^2$, one obtains

$$
\begin{aligned}
& T_{123N}(p^2, m_1^2, m_2^2, m_3^2)/p^2 = \qquad (32)\\
& \qquad x_1 x_2 \{-1+\log(-x_1)\}\{-1+\log(-x_2)\} \\
& + \sum_{\substack{m,n=1\\ m+n>2}}^{\infty} \frac{(m+n-3)!(m+n-2)!}{(m-1)m!(n-1)!n!} x_1^m x_2^n \\
& \qquad \{\psi(m+n-2)+\psi(m+n-1)-\psi(m)-\psi(m+1)+\log(-x_1) \\
& \qquad +\psi(m+n-2)+\psi(m+n-1)-\psi(n)-\psi(n+1)+\log(-x_2)\} \\
& + \sum_{m,n,k=1}^{\infty} \frac{(m+n+k-3)!(m+n+k-2)!}{(m-1)!m!(n-1)!n!(k-1)!k!} x_1^m x_2^n x_3^k \\
& \qquad \Big[\{\psi(m+n+k-2)+\psi(m+n+k-1)-\psi(m)-\psi(m+1)+\log(-x_1)\} \\
& \qquad \times \{\psi(m+n+k-2)+\psi(m+n+k-1)-\psi(n)-\psi(n+1)+\log(-x_2)\} \\
& \qquad + \{\psi(m+n+k-2)+\psi(m+n+k-1)-\psi(m)-\psi(m+1)+\log(-x_1) \\
& \qquad\quad +\psi(m+n+k-2)+\psi(m+n+k-1)-\psi(n)-\psi(n+1)+\log(-x_2)\} \\
& \qquad \times \{\psi(m+n+k-2)+\psi(m+n+k-1)-\psi(k)-\psi(k+1)+\log(-x_3)\} \\
& \qquad + 3\,\psi'(m+n+k-2)+3\,\psi'(m+n+k-1)-6\,\zeta(2) \Big] .
\end{aligned}
$$

Numerical evaluation of both the series and the double integral representation reveals that already few terms of the series suffice to reach agreement up to four digits. Some representative numbers for the real and imaginary parts of $T_{123}(p^2, m_1^2, m_2^2, m_3^2)$ are given in tables 1 and 2.

Other checks on the result (16) are easily performed. For the massive vacuum diagram with $p^2 = 0$ one obtains four Appell F_4 functions, which can be found in [1]. For $m_1 = m_2 = 0$ one has one Gauss ${}_2F_1$ function, which after expansion in δ reduces to a known result [15]. In the case of two equal masses, e.g. $m_1 = m_2 = m$, the triple sums of (16) and (24) can be reduced to double sums by means of Gauss' summation formula for the hypergeometric series. Setting also $m_3 = m$ leads, after some manipulation, to a result given in [7].

5 Conclusions

The results in this paper establish a connection, hitherto unknown, between self-energy integrals and Lauricella functions. Since these functions have been studied to some extend in the mathematical literature this opens the way for the application of established techniques, like contour integral representations, for the purpose of analytic continuation, or partial differential equations. By this means it may become possible to obain explicit series expansions for the still outstanding mass configurations in the small p^2-domain, also. It is likely that for more complicated diagrams related types of generalized hypergeometric functions of several variables will be required.

As the given series representations converge rapidly they are well suited for practical applications in the context of precision calculations in non-abelian massive gauge theories.

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p^2	m_1	m_2	m_3	*series*	*integral*
9	3	3	10	-7.31298	-7.31375
20	2	3	10	-4.14938	-4.14940
30	1	2	10	-0.81176	-0.81180
49	1	1	10	-0.31675	-0.31686
50	3	4	15	-7.94710	-7.94563
25	2	2	10	-2.33538	-2.33508
100	3	4	20	-6.01715	-6.01861
150	3	4	20	-6.39036	-6.39151
150	5	5	25	-14.5339	-14.5338
200	5	5	25	-15.0523	-15.0448
-9	3	3	10	-6.93244	-6.93241
-20	2	3	10	-3.63591	-3.63600
-25	2	2	10	-1.94285	-1.94289
-30	1	2	10	-0.63068	-0.63070
-49	1	1	10	-0.19503	-0.19503
-50	3	4	15	-6.82703	-6.82719
-100	3	4	20	-4.94851	-4.94863
-150	3	4	20	-4.75060	-4.75072
-150	5	5	25	-12.1817	-12.1820
-200	5	5	25	-11.8791	-11.8793

Table 1: Comparison of the small p^2 expansion (31) with the double integral evaluation.

p^2	m_1	m_2	m_3	*series*		*integral*	
				real	imag	real	imag
80	2	3	2	0.58743	-11.2628	0.58728	-11.2623
100	3	3	3	-1.28285	-20.8996	-1.28329	-20.8972
100	2	3	4	-0.32864	-11.8459	-0.32853	-11.8456
150	3	4	4	-1.26795	-26.4912	-1.26873	-26.4899
150	2	4	3	1.56625	-11.9689	1.56585	-11.9698
150	3	3	4	0.98658	-16.1021	0.98624	-16.1012
150	3	4	5	-4.76259	-29.6012	-4.7634	-29.5995
200	2	3	4	1.69608	-6.02417	1.69618	-6.02429
200	3	4	4	1.86356	-20.3985	1.86392	-20.3977
250	4	4	4	2.64395	-27.6090	2.64366	-27.6093
-50	2	2	2	-3.72813	0	-3.72819	0
-100	2	4	3	-7.18368	0	-7.18384	0
-100	3	3	3	-8.79127	0	-8.79142	0
-120	3	4	3	-12.4310	0	-12.4312	0
-150	3	3	4	-7.08305	0	-7.08318	0
-150	3	4	4	-10.8565	0	-10.8567	0
-150	3	4	3	-11.3619	0	-11.3621	0
-200	3	4	4	-9.64611	0	-9.64630	0
-200	2	3	4	-3.30186	0	-3.30195	0
-250	4	4	4	-13.5719	0	-13.5721	0

Table 2: Comparison of the large p^2 expansion (32) with the double integral evaluation.