Closed-Form Boundary State Feedbacks for a Class of 1-D Partial Integro-Differential Equations

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***Abstract*—In this paper, a problem of boundary stabilization of a class of linear parabolic partial integro-differential equations (P(I)DEs) in one dimension is considered using the method of backstepping, avoiding spatial discretization required in previous efforts. The problem is formulated as a design of an integral operator whose kernel is required to satisfy a hyperbolic P(I)DE. The kernel P(I)DE is then converted into an equivalent integral equation and by applying the method of successive approximations, the equation's well posedness and the kernel's smoothness are established. It is shown how to extend this approach to design optimally stabilizing controllers. An adaptation mechanism is developed to reduce the conservativeness of the inverse optimal controller, and the performance bounds are derived. For a broad range of physically motivated special cases feedback laws are constructed explicitly and the closed-loop solutions are found in closed form. A numerical scheme for the kernel P(I)DE is proposed; its numerical effort compares favorably with that associated with operator Riccati equations.**

***Index Terms*—Backstepping, boundary control, distributed parameter systems, Lyapunov function, partial differential equations (PDEs), stabilization.**

I. Introduction

A. Motivation

After about three decades of development, partial differential equation (PDE) control theory, and boundary control in particular, consists of a wealth of mathematically impressive results that solve stabilization and optimal control problems. Two of the main driving principles in this development have been *generality* and *extending* the existing finite dimensional results. The latter objective has led to extending (at least) two of the basic control theoretic results to PDEs: pole placement and optimal/robust control. While these efforts have been successful, by following the extremely general finite-dimensional path ($\dot{x} = Ax + Bu$, where A and B can be any matrices), they have diverted the attention from structure-specific opportunities that exist in PDEs. Such opportunities have recently started to be capitalized on in the elegant work on distributed control of spatially invariant systems by Bamieh *et al.* [5].

Structure is particularly pronounced in *boundary control* problems for PDEs. Backstepping techniques for parabolic systems, focused on revealing and exploiting this structure, have been under development since about 1998 [3], [4], [7]–[10], but have been limited to discretized formulations until a recent breakthrough by Liu [23]. In this paper we design continuum backstepping controllers that require the solution of a linear Klein–Gordon-type hyperbolic PDE on a triangular domain, an easier task than solving an operator Riccati equation or executing the procedure for pole placement.

Even more important than practical/numerical benefits are conceptual benefits of the continuum backstepping approach. For a broad class of practically relevant one-dimensional PDEs, the control design problem can be solved in closed form, obtaining an explicit expression for the gain kernel of the boundary control law. This is the main novelty of this paper. The explicit controller allows to go a step further and calculate the closed-loop solutions explicitly/analytically, which not only directly answers questions related to well posedness but also provides insight into how the control has affected the eigenvalues and eigenvectors of the system.

This paper introduces backstepping as a structure-specific paradigm for parabolic PDEs (at least within the class considered) and demonstrates its capability to incorporate optimality in addition to stabilization.

B. Brief Summary of Prior Literature

The prior work on stabilization of general parabolic equations includes, among others, the results of Lasiecka and Triggiani [33] and [21] who developed a general framework for the structural assignment of eigenvalues in parabolic problems through the use of semigroup theory. Separating the open loop system into a finite-dimensional unstable part and an infinite-dimensional stable part, they apply feedback boundary control that stabilizes the unstable part while leaving the stable part stable. A linear quadratic regulator (LQR) approach in [22] is also applicable to this problem. A unified treatment of both interior and boundary observations/control generalized to semilinear problems can be found in [2]. Nambu [25] developed auxiliary functional observers to stabilize diffusion equations using boundary observation and feedback. Stabilizability by boundary control in the optimal control setting is discussed by Bensoussan *et al.* [6]. For the general Pritchard–Salamon class of state–space systems a number of frequency-domain results has been established on stabilization during the last decade (see, e.g., [15] and [24] for a survey). The placement of finitely many eigenvalues was generalized to the case of moving infinitely many eigenvalues by Russel [29]. The stabilization problem can be also approached using the abstract theory of boundary control systems developed by Fattorini [18] that results in a dynamical feedback controller

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(see the remarks in [16, Sec. 3.5]). Extensive surveys on the controllability and stabilizability theory of linear partial differential equations can be found in [28], [22].

C. Problem Formulation

We consider the following class of linear parabolic partial integro-differential equations (P(I)DEs):

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + b(x)u_x(x,t) + \lambda(x)u(x,t) + g(x)u(0,t) + \int_0^x f(x,y)u(y,t)\,dy \quad (1)$$

for $x \in (0,1), t > 0$, with boundary conditions[1]

$$u_x(0,t) = qu(0,t) \quad (2)$$

$$u(1,t) = U(t) \quad \text{or} \quad u_x(1,t) = U(t) \quad (3)$$

and under the assumption

$$\varepsilon > 0, \qquad q \in \mathbb{R}, b, \lambda, g \in C^1([0,1]), f \in C^1([0,1]\times[0,1]) \quad (4)$$

where the $U(t)$ is the control input. The control objective is to stabilize the equilibrium $u(x,t) \equiv 0$.

The (1) is in fact a P(I)DE, but for convenience we abuse the terminology and call it a PDE. The problem is formulated as a design of an *integral operator state transformation* whose kernel is shown to satisfy a well posed hyperbolic PDE. The kernel well posedness for $b = g = f = 1/q = 0$ was shown by Liu [23], whose result has inspired the efforts that have led to this paper. Integral operator transformations for linear parabolic PDEs can be traced as far back as the papers of Colton [14] and Seidman [30] who were studying solvability and open-loop controllability of the problem with $b = g = f = 1/q = 0$. More recent considerations include [12].

D. Contribution of this Paper

1) Kernel Smoothness: The class of PDEs (1)–(2) was studied by Balogh and Krstic [4] who applied a finite-dimensional backstepping to its discretized version, obtaining a boundary control with bounded but discontinuous gain kernel. In this paper we do not use discretization. We prove that the gain kernel is C^2 smooth. This cannot be seen, much less proved, from discretization.

2) Explicit Solutions: The most striking feature of the approach is that for a broad class of physically relevant parabolic 1-D PDEs the control laws can be obtained in *closed form*, which is not the case with the existing methods (LQR, pole placement) [16], [22], [33]. *One* of the subclasses for which explicit controllers can be obtained is

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + b\,u_x(x,t) + \lambda_0 u(x,t) + g_0 e^{\alpha x}u(0,t) \quad (5)$$

$$u_x(0,t) = qu(0,t) \quad (6)$$

for arbitrary values of the six parameters $\varepsilon, b, \lambda_0, g_0, \alpha, q$. The explicit controllers, in turn, allow us to find explicit closed-loop solutions. Some of the PDEs within the class (5) and (6) are not even explicitly solvable in open loop, however, we solve them in closed form in the presence of the backstepping control law. While we prove closed-loop well posedness for the general class separately from calculating the explicit solutions for the subclasses, these solutions serve to illustrate how well posedness issues can be pretty much bypassed by a structure-specific control approach, allowing the designer to concentrate on control-relevant issues like the feedback algorithm and its performance.

3) Optimality: An important result of the proposed method from the practical point of view is that in addition to solving the stabilization problem for (1)–(2), it also naturally leads to controllers that are inverse optimal.[2] Unlike LQR controllers for PDEs [22], our controllers avoid the problem of having to solve operator Riccati equations. They minimize cost functionals that penalize both the control and the state. We also propose an adaptive control law to reduce the control effort and derive explicit bounds on the state and control.

4) Numerical Advantage: An effective computational scheme, based on the Ablowitz–Kruskal–Ladik method [1], commonly applied to the Klein–Gordon PDE arising in quantum mechanics and the study of solitary waves, is suggested for approximating the gain kernel when it can not be obtained in closed form. We use it to make a numerical comparison of our design and the traditional LQR approach. As numerical simulations show, the proposed approach requires much less computational effort while exhibiting comparable performance.

E. Simplification

Before we start, without loss of generality we set

$$b(x) \equiv 0$$

since it can be eliminated from the equation with the transformation (see, e.g., [31])

$$u(x,t) \mapsto u(x,t)e^{-\frac{1}{2\varepsilon}\int_0^x b(\tau)\,d\tau} \quad (7)$$

and the appropriate changes of the parameters

$$\lambda(x) \mapsto \lambda(x) + \frac{b'(x)}{2} + \frac{b^2(x)}{4\varepsilon} \quad g(x) \mapsto g(x)e^{-\int_0^x \frac{b(\tau)}{2\varepsilon}\,d\tau}$$

$$q \mapsto q - \frac{b(0)}{2\varepsilon} \quad f(x,y) \mapsto f(x,y)e^{-\int_y^x \frac{b(\tau)}{2\varepsilon}\,d\tau}. \quad (8)$$

F. Organization

This paper is organized as follows. In Section II, we obtain a hyperbolic PDE governing the kernel. This PDE is converted to the equivalent integral equation in Section III. In Section IV, we find a unique solution of this equation and establish its properties, and in Section V, we state the properties of the closed-loop system. In Section VI, we show how to modify the controller to solve an inverse optimal problem. The control effort is reduced

[1]The case of Dirichlet boundary condition at the zero end can be handled by setting $q = +\infty$.

[2]Optimality has important practical consequences of guaranteeing infinite gain margins and 60° phase margins.

by using adaptation in Section VII. The explicit controllers are constructed for many special cases in Section VIII. The numerical simulations are presented in Section IX.

II. PDE for the Kernel

We look for a backstepping-like coordinate transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\,dy \tag{9}$$

that transforms system (1)–(3) into the system

$$w_t(x,t) = \varepsilon w_{xx}(x,t) - cw(x,t), \qquad x \in (0,1) \tag{10}$$
$$w_x(0,t) = qw(0,t) \tag{11}$$
$$w(1,t) = 0 \text{ or } w_x(1,t) = 0 \tag{12}$$

which is exponentially stable for $c \geq \varepsilon\bar{q}^2$ (respectively, $c \geq \varepsilon\bar{q}^2 + \varepsilon/2$) where $\bar{q} = \max\{0, -q\}$. The free parameter c can be used to set the desired rate of stability. Once we find the transformation (9) (namely $k(x,y)$), the boundary condition (12) gives the controller in the form

$$u(1,t) = U(t) = \int_0^1 k_1(y)u(y,t)\,dy \tag{13}$$

for the Dirichlet actuation and

$$u_x(1,t) = U(t) = k_1(1)u(1,t) + \int_0^1 k_2(y)u(y,t)\,dy \tag{14}$$

for the Neumann actuation. Here, we denoted

$$k_1(y) = k(1,y) \quad k_2(y) = k_x(1,y). \tag{15}$$

Differentiating (9), we get[3]

$$\begin{aligned} w_t(x,t) &= u_t(x,t) - \int_0^x k(x,y)\left\{\varepsilon u_{yy}(y,t) + \lambda(y)u(y,t)\right.\\ &\quad \left. + g(y)u(0,t) + \int_0^y f(y,\xi)u(\xi,t)\,d\xi\right\} dy \\ &= u_t(x,t) - \varepsilon k(x,x)u_x(x,t) + \varepsilon k(x,0)u_x(0,t) \\ &\quad + \varepsilon k_y(x,x)u(x,t) - \varepsilon k_y(x,0)u(0,t) \\ &\quad - \int_0^x (\varepsilon k_{yy}(x,y) + \lambda(y))u(y,t)\,dy \\ &\quad - u(0,t)\int_0^x k(x,y)g(y)\,dy \\ &\quad - \int_0^x u(y,t)\left(\int_y^x k(x,\xi)f(\xi,y)\,d\xi\right) dy \end{aligned} \tag{16}$$

$$\begin{aligned} w_{xx}(x,t) &= u_{xx}(x,t) - u(x)\frac{d}{dx}k(x,x) - k(x,x)u_x(x,t) \\ &\quad - k_x(x,x)u(x) - \int_0^x k_{xx}(x,y)u(y,t)\,dy. \end{aligned} \tag{17}$$

[3]We use the following notation: $k_x(x,x) = k_x(x,y)|_{y=x}$, $k_y(x,x) = k_y(x,y)|_{y=x}$, $(d)/(dx)k(x,x) = k_x(x,x) + k_y(x,x)$.

Substituting (16) and (17) into (10)–(11) and using (1)–(2) (with $b(x) \equiv 0$), we obtain the following equation:

$$\begin{aligned} 0 &= \int_0^x \{\varepsilon k_{xx}(x,y) - \varepsilon k_{yy}(x,y) - (\lambda(y)+c)k(x,y) \\ &\quad + f(x,y)\}u(y,t)\,dy \\ &\quad - \int_0^x u(y,t)\int_y^x k(x,\xi)f(\xi,y)\,d\xi\,dy \\ &\quad + \left\{\lambda(x) + c + 2\varepsilon\frac{d}{dx}k(x,x)\right\}u(x,t) + \varepsilon qk(x,0)u(0,t) \\ &\quad + \left\{g(x) - \int_0^x k(x,y)g(y)\,dy - \varepsilon k_y(x,0)\right\}u(0,t). \end{aligned} \tag{18}$$

For this equation to be verified for all $u(x,t)$, the following PDE for $k(x,y)$ must be satisfied:

$$\begin{aligned} \varepsilon k_{xx}(x,y) - \varepsilon k_{yy}(x,y) &= (\lambda(y)+c)k(x,y) - f(x,y) \\ &\quad + \int_y^x k(x,\xi)f(\xi,y)\,d\xi \end{aligned} \tag{19}$$

for $(x,y) \in \mathcal{T}$ with boundary conditions

$$\varepsilon k_y(x,0) = \varepsilon qk(x,0) + g(x) - \int_0^x k(x,y)g(y)\,dy \tag{20}$$
$$k(x,x) = -\frac{1}{2\varepsilon}\int_0^x (\lambda(y)+c)\,dy. \tag{21}$$

Here, we denote $\mathcal{T} = \{x,y : 0 < y < x < 1\}$. Note that one of the boundary conditions is on the characteristic (Goursat-type) and the other is nonlocal, i.e., contains an integral term. We will prove well posedness of (19)–(21) in the next two sections.[4]

III. Converting the PDE Into an Integral Equation

We derive now an integral equation equivalent to the system (19)–(21). We introduce the standard change of variables [31]

$$\xi = x + y \quad \eta = x - y \tag{22}$$

and denote

$$G(\xi,\eta) = k(x,y) = k\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) \tag{23}$$

transforming problem (19)–(21) to the following PDE:

$$\begin{aligned} 4\varepsilon G_{\xi\eta}(\xi,\eta) &= a\left(\frac{\xi-\eta}{2}\right)G(\xi,\eta) - f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) \\ &\quad + \int_{(\xi-\eta)/2}^{(\xi+\eta)/2} G\left(\frac{\xi+\eta}{2}+\tau, \frac{\xi+\eta}{2}-\tau\right) f\left(\tau, \frac{\xi-\eta}{2}\right) d\tau \end{aligned} \tag{24}$$

[4]General books on PDEs (see, e.g., [31]) do establish well posedness of second-order hyperbolic PDEs, but not with integral terms and these boundary conditions.

for $(\xi,\eta)\in\mathcal{T}_1$ with boundary conditions

$$\varepsilon G_\xi(\xi,\xi)=\varepsilon G_\eta(\xi,\xi)+\varepsilon qG(\xi,\xi)+g(\xi) - \int_0^\xi G(\xi+\tau,\xi-\tau)g(\tau)\,d\tau \tag{25}$$

$$G(\xi,0)=-\frac{1}{4\varepsilon}\int_0^\xi a\left(\frac{\tau}{2}\right)\,d\tau. \tag{26}$$

Here, we introduced $\mathcal{T}_1=\{\xi,\eta : 0<\xi<2, 0<\eta<\min(\xi,2-\xi)\}$ and $a(\tau)=\lambda(\tau)+c$.

Integrating (24) with respect to η from 0 to η, and using (26), we obtain

$$\begin{aligned}G_\xi(\xi,\eta)=&-\frac{1}{4\varepsilon}a\left(\frac{\xi}{2}\right)+\frac{1}{4\varepsilon}\int_0^\eta a\left(\frac{\xi-s}{2}\right)G(\xi,s)\,ds\\&+\frac{1}{4\varepsilon}\int_0^\eta\int_\xi^{\xi+\eta-s}G(\tau,s)\\&f\left(\frac{\tau-s}{2},\xi-\frac{\tau+s}{2}\right)d\tau\,ds\\&-\frac{1}{4\varepsilon}\int_0^\eta f\left(\frac{\xi+\tau}{2},\frac{\xi-\tau}{2}\right)d\tau.\end{aligned} \tag{27}$$

Integrating (27) with respect to ξ from η to ξ gives

$$\begin{aligned}G(\xi,\eta)=&\,G(\eta,\eta)-\frac{1}{4\varepsilon}\int_\eta^\xi a\left(\frac{\tau}{2}\right)d\tau\\&-\frac{1}{4\varepsilon}\int_\eta^\xi\int_0^\eta f\left(\frac{s+\tau}{2},\frac{s-\tau}{2}\right)d\tau\,ds\\&+\frac{1}{4\varepsilon}\int_\eta^\xi\int_0^\eta\int_\mu^{\mu+\eta-s}G(\tau,s)\\&f\left(\frac{\tau-s}{2},\mu-\frac{\tau+s}{2}\right)d\tau\,ds\,d\mu\\&+\frac{1}{4\varepsilon}\int_\eta^\xi\int_0^\eta a\left(\frac{\tau-s}{2}\right)G(\tau,s)\,ds\,d\tau.\end{aligned} \tag{28}$$

To find $G(\eta,\eta)$, we use (25) to write

$$\begin{aligned}\frac{d}{d\xi}G(\xi,\xi)&=G_\xi(\xi,\xi)+G_\eta(\xi,\xi)\\&=2G_\xi(\xi,\xi)-qG(\xi,\xi)-\frac{1}{\varepsilon}g(\xi)\\&\quad+\frac{1}{\varepsilon}\int_0^\xi G(\xi+s,\xi-s)g(s)\,ds.\end{aligned} \tag{29}$$

Using (27) with $\eta=\xi$, we can write (29) in the form of differential equation for $G(\xi,\xi)$

$$\begin{aligned}\frac{d}{d\xi}G(\xi,\xi)=&-qG(\xi,\xi)-\frac{1}{2\varepsilon}\int_0^\xi f\left(\frac{\xi+\tau}{2},\frac{\xi-\tau}{2}\right)d\tau\\&-\frac{1}{2\varepsilon}a\left(\frac{\xi}{2}\right)+\frac{1}{2\varepsilon}\int_0^\xi a\left(\frac{\xi-s}{2}\right)G(\xi,s)\,ds\\&+\frac{1}{2\varepsilon}\int_0^\xi\int_\xi^{2\xi-s}G(\tau,s)f\left(\frac{\tau-s}{2},\xi-\frac{\tau+s}{2}\right)d\tau\,ds\\&-\frac{1}{\varepsilon}g(\xi)+\frac{1}{\varepsilon}\int_0^\xi G(\xi+s,\xi-s)g(s)\,ds.\end{aligned} \tag{30}$$

Integrating (30) using the variation of constants formula and substituting the result into (28), we obtain an integral equation for G

$$G(\xi,\eta)=G_0(\xi,\eta)+F[G](\xi,\eta) \tag{31}$$

where G_0 and $F[G]$ are given by

$$\begin{aligned}G_0(\xi,\eta)=&-\frac{1}{4\varepsilon}\int_\eta^\xi a\left(\frac{\tau}{2}\right)d\tau-\frac{1}{2\varepsilon}\int_0^\eta e^{q(\tau-\eta)}\\&\times\left[a\left(\frac{\tau}{2}\right)+2g(\tau)\right]d\tau\\&-\frac{1}{4\varepsilon}\int_\eta^\xi\int_0^\eta f\left(\frac{s+\tau}{2},\frac{s-\tau}{2}\right)d\tau\,ds\\&-\frac{1}{2\varepsilon}\int_0^\eta e^{q(\tau-\eta)}\int_0^\tau f\left(\frac{\tau+s}{2},\frac{\tau-s}{2}\right)ds\,d\tau\end{aligned} \tag{32}$$

$$\begin{aligned}F[G](\xi,\eta)=&\,\frac{1}{2\varepsilon}\int_0^\eta e^{q(\tau-\eta)}\int_0^\tau a\left(\frac{\tau-s}{2}\right)G(\tau,s)\,ds\,d\tau\\&+\frac{1}{4\varepsilon}\int_\eta^\xi\int_0^\eta a\left(\frac{\tau-s}{2}\right)G(\tau,s)\,ds\,d\tau\\&+\frac{1}{2\varepsilon}\int_0^\eta\int_s^{2\eta-s}e^{q\left(\frac{\tau+s}{2}-\eta\right)}g\left(\frac{\tau-s}{2}\right)G(\tau,s)\,d\tau\,ds\\&+\frac{1}{4\varepsilon}\int_\eta^\xi\int_0^\eta\int_\mu^{\mu+\eta-s}\\&f\left(\frac{\tau-s}{2},\mu-\frac{\tau+s}{2}\right)G(\tau,s)\,d\tau\,ds\,d\mu\\&+\frac{1}{2\varepsilon}\int_0^\eta e^{q(\mu-\eta)}\int_0^\mu\int_\mu^{2\mu-s}f\left(\frac{\tau-s}{2},\mu-\frac{\tau+s}{2}\right)\\&\times G(\tau,s)\,d\tau\,ds\,d\mu.\end{aligned} \tag{33}$$

Thus, we have proved the following statement.

Lemma 1: Any $G(\xi,\eta)$ satisfying (24)–(26) also satisfies integral (31). $\square$

IV. Analysis of the Integral Equation by a Successive Approximation Series

Using the result of the previous section we can now compute a uniform bound on the solutions by the method of successive approximations.[5]

With G_0 defined in (32), let

$$G_{n+1} = F[G_n], \qquad n = 0,1,2,\ldots \tag{34}$$

and denote

$$\begin{aligned} \bar\lambda &= \sup_{x\in[0,1]} |\lambda(x)| \quad \bar g = \sup_{x\in[0,1]} |g(x)| \\ \bar f &= \sup_{(x,y)\in[0,1]\times[0,1]} |f(x,y)|. \end{aligned} \tag{35}$$

We estimate now $G_n(\xi,\eta)$

$$\begin{aligned} |G_0(\xi,\eta)| &\le \frac{1}{4\varepsilon}(\bar\lambda + c)(\xi-\eta) + \frac{1}{2\varepsilon}(\bar\lambda + c + 2\bar g)\eta \\ &\quad + \frac{1}{4\varepsilon}\bar f\eta^2 + \frac{1}{4\varepsilon}\bar f(\xi-\eta)\eta \\ &\le \frac{1}{\varepsilon}(\bar\lambda + c + \bar f + \bar g)(1+e^{-q}) \equiv M. \end{aligned} \tag{36}$$

Suppose that

$$|G_n(\xi,\eta)| \le M^{n+1}\frac{(\xi+\eta)^n}{n!}. \tag{37}$$

Then, we have the following estimate.

$$\begin{aligned} |G_{n+1}| &\le \frac{M^{n+1}}{4\varepsilon}\frac{1}{n!}\Bigg\{(\bar\lambda + c)\int_\eta^\xi\int_0^\eta (\tau+s)^n\,ds\,d\tau \\ &\quad + 2(\bar\lambda + c)\int_0^\eta e^{q(\tau-\eta)}\int_0^\tau (\tau+s)^n\,ds\,d\tau \\ &\quad + \bar f\int_\eta^\xi\int_0^\eta\int_\mu^{\mu+\eta-s}(\tau+s)^n\,d\tau\,ds\,d\mu \\ &\quad + 2\bar g\int_0^\eta\int_s^{2\eta-s} e^{q\left(\frac{\tau+s}{2}-\eta\right)}(\tau+s)^n\,d\tau\,ds + 2\bar f \\ &\quad \times\int_0^\eta e^{q(\mu-\eta)}\int_0^\mu\int_\mu^{2\mu-s}(\tau+s)^n\,d\tau\,ds\,d\mu\Bigg\} \\ &\le \frac{M^{n+1}}{4\varepsilon}\frac{1}{n!}\{(\bar\lambda + c)(2+2(1+e^{-q})) \\ &\quad + 2\bar f + 4(1+e^{-q})\bar g \\ &\quad + 2(1+e^{-q})\bar f\}\frac{(\xi+\eta)^{n+1}}{n+1} \\ &\le M^{n+2}\frac{(\xi+\eta)^{n+1}}{(n+1)!}. \end{aligned} \tag{38}$$

[5]The results of this section can be also obtained using the argument that the operator F is compact and does not have (-1) as eigenvalue, which means that the operator on the right hand side of (31) is bounded invertible. However, we use the successive approximations approach because it can be used for finding an approximate solution to the kernel by symbolic calculation, because the expressions for successive terms are used to derive explicit controllers in Section VIII, and because this approach yields a quantitative bound on the kernel (such a bound on the kernel of the inverse transformation is needed in the inverse optimal design in Section VI).

So, by induction, (37) is proved. Note also that $G_n(\xi,\eta)$ is $C^2(\mathcal{T}_1)$ which follows from (32)–(33) with the assumption (4). Therefore, the series

$$G(\xi,\eta) = \sum_{n=0}^{\infty} G_n(\xi,\eta) \tag{39}$$

converges absolutely and uniformly in $\mathcal{T}_1$ and its sum G is a twice continuously differentiable solution of (31) with a bound $|G(\xi,\eta)| \le M\exp(M(\xi+\eta))$. The uniqueness of this solution can be proved by the following argument. Suppose $G'(\xi,\eta)$ and $G''(\xi,\eta)$ are two different solutions of (31). Then $\Delta G(\xi,\eta) = G'(\xi,\eta) - G''(\xi,\eta)$ satisfies the homogeneous integral (33) in which G_n and G_{n+1} are changed to ΔG. Using the above result of boundedness we have $|\Delta G(\xi,\eta)| \le 2Me^{2M}$. Using this inequality in the homogeneous integral equation and following the same estimates as in (38), we get that $\Delta G(\xi,\eta)$ satisfies for all n

$$|\Delta G(\xi,\eta)| \le 2M^{n+1}e^{2M}\frac{(\xi+\eta)^n}{n!} \to 0 \text{ as } n\to\infty. \tag{40}$$

Thus, $\Delta G \equiv 0$, which means that (38) is a unique solution to (31). By direct substitution we can check that it is also a unique (by Lemma 1) solution to PDE (24)–(26). Thus, we proved the following result, which generalizes [23].

Theorem 2: The (19) with boundary conditions (20)–(21) has a unique $C^2(\mathcal{T})$ solution. The bound on the solution is

$$|k(x,y)| \le Me^{2Mx} \tag{41}$$

where M is given by (36). $\square$

To prove stability we need to prove that the transformation (9) is invertible. The proof that for (9) an inverse transformation with bounded kernel exists can be found in [3], [23], and can be also inferred from [26, p. 254]. The other way to prove it is to directly find and analyze the PDE for the kernel of the inverse transformation. We take this route because we need the inverse kernel for further quantitative analysis. Let us denote the kernel of the inverse transformation by $l(x,y)$. The transformation itself has the form

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)\,dy. \tag{42}$$

Substituting (42) into (10)–(12) and using (1)–(3), we obtain the following PDE governing $l(x,y)$:

$$\begin{aligned} \varepsilon l_{xx}(x,y) - \varepsilon l_{yy}(x,y) &= -(\lambda(x)+c)l(x,y) - f(x,y) \\ &\quad - \int_y^x l(\tau,y)f(x,\tau)\,d\tau \end{aligned} \tag{43}$$

for $(x,y)\in\mathcal{T}$ with boundary conditions

$$\varepsilon l_y(x,0) = \varepsilon q l(x,0) + g(x) \tag{44}$$

$$l(x,x) = -\frac{1}{2\varepsilon}\int_0^x (\lambda(y)+c)\,dy. \tag{45}$$

This hyperbolic PDE is a little bit simpler than the one for k (the boundary condition does not contain an integral term), but has a very similar structure. So, we can apply the same approach of converting the PDE to an integral equation and using a method of successive approximations to show that the inverse kernel exists and has the same properties as we proved for the direct kernel.

Theorem 3: The (43) with boundary conditions (44)–(45) has a unique $C^2(\mathcal{T})$ solution. The bound on the solution is

$$|l(x,y)| \le Me^{2Mx} \tag{46}$$

where M is given by (36). □

V. Properties of the Closed-Loop System

Theorems 2 and 3 establish the equivalence of norms of u and w in both L_2 and H_1. From the properties of the damped heat (10)–(12) (discussed in some detail in [4] and [23]), exponential stability in both L_2 and H_1 follows. Furthermore, it can be proved (see, e.g., [3]) that if the kernels (15) are bounded than the system (1)–(2) with a boundary condition (13) or (14) is well posed. Thus, we get the following main result.

Theorem 4: For any initial data $u_0(x) \in L_2(0,1)$ (respectively, $H_1(0,1)$) that satisfy the compatibility conditions

$$u_{0x}(0) = qu_0(0) \quad u_0(1) = \int_0^1 k_1(y)u_0(y)\,dy \tag{47}$$

system (1)–(2) with Dirichlet boundary control (13) has a unique classical solution $u(x,t) \in C^{2,1}((0,1)\times(0,\infty))$ and is exponentially stable at the origin $u(x,t) \equiv 0$

$$\|u(t)\|_{\mathcal{L}} \le Ce^{-(c-\varepsilon\bar{q}^2)t}\|u_0\|_{\mathcal{L}} \tag{48}$$

where C is a positive constant independent of u_0 and $\mathcal{L}$ is either L_2 or H_1.

For any initial data $u_0(x) \in L_2(0,1)$ (respectively, $H_1(0,1)$) that satisfy the compatibility conditions $u_{0x}(0) = qu_0(0)$

$$u_{0x}(1) = k_1(1)u_0(1) + \int_0^1 k_2(y)u_0(y)\,dy \tag{49}$$

system (1)–(2) with Neumann boundary control (14) has a unique classical solution $u(x,t) \in C^{2,1}((0,1)\times(0,\infty))$ and is exponentially stable at the origin, $u(x,t) \equiv 0$

$$\|u(t)\|_{\mathcal{L}} \le Ce^{-(c-\varepsilon\bar{q}^2-1/2)t}\|u_0\|_{\mathcal{L}}. \tag{50}$$

□

With the backstepping method employing a target systems in the simple heat equation form, it becomes possible to write the solution of the closed-loop system (1)–(2), (13) explicitly, in terms of the initial condition $u_0(x)$ and the kernels $k(x,y)$ and $l(x,y)$. We show how this is done for $q = +\infty$, since in this case the solution to the target system (10)–(12) can be written in the most compact way

$$w(x,t) = 2\sum_{n=1}^{\infty} e^{-(c+\varepsilon\pi^2n^2)t}\sin(\sqrt{\varepsilon}\pi nx)\times \\ \times \int_0^1 w_0(\xi)\sin(\sqrt{\varepsilon}\pi n\xi)\,d\xi. \tag{51}$$

The initial condition $w_0(x)$ can be calculated explicitly from $u_0(x)$ using the transformation (9)

$$w_0(x) = u_0(x) - \int_0^x k(x,y)u_0(y)\,dy. \tag{52}$$

Substituting (51) and (52) into the inverse transformation (42), and changing the order of integration, we obtain the following result, which we specialize in a subsequent development for a subclass of PDEs.

Lemma 5: For $q = +\infty$, the unique solution to the closed-loop system (1)–(2), (13) is given by

$$u(x,t) = \sum_{n=1}^{\infty} e^{-(c+\varepsilon\pi^2n^2)t}\phi_n(x)\int_0^1 \psi_n(\xi)u_0(\xi)\,d\xi \tag{53}$$

where

$$\phi_n(x) = 2\left(\sin(\sqrt{\varepsilon}\pi nx) + \int_0^x l(x,y)\sin(\sqrt{\varepsilon}\pi ny)\,dy\right) \tag{54}$$

$$\psi_n(x) = \sin(\sqrt{\varepsilon}\pi nx) - \int_x^1 k(y,x)\sin(\sqrt{\varepsilon}\pi ny)\,dy. \tag{55}$$

□

One can directly see from (53) that the backstepping controller has moved the eigenvalues from their open-loop (unstable) locations into locations of the damped heat equation $-(c+\varepsilon\pi^2n^2)$. It is interesting to note, that although infinitely many eigenvalues cannot be arbitrarily assigned, our controller is able to assign all of them to the particular location $-(c+\varepsilon\pi^2n^2)$. The eigenfunctions of the closed-loop system are assigned to $\phi_n(x)$.

1) Comparison With Pole Placement: The results of this section should be evaluated in comparison to the well-known results by Triggiani [33] which represent pole placement for PDEs. In [33] one first finds the unstable eigenvalues and the corresponding eigenvectors of the open-loop system. Suppose there are n of them. Then one solves the auxiliary elliptic problem and compute n integrals (which are L_2 inner products of the elliptic solution and the unstable modes). After that one solves an n-dimensional matrix equation to find the desired kernel. In contrast, our method, for this 1-D parabolic class, consists of successive integrations (symbolic or numerical) to obtain several terms of the series (38). Another option is to solve the kernel PDE (19)–(21) directly by an effective numerical procedure (see Section IX). Most importantly, as we shall show in Section VIII, our method in many cases leads to closed-form controllers. Another issue is the familiar nonrobustness of pole placement controllers. Even in the finite-dimensional case, it is known that backstepping is more robust than pole placement because it converts a system into a tridiagonal Hessenberg form, rather than a companion form. Furthermore, as we show next, our controllers can be modified (without recalculating the gain kernel) to have robustness margins.

VI. Inverse Optimal Stabilization

In this section, we show how to solve an inverse optimal stabilization problem (pursued for finite-dimensional systems in [20]) for (1)–(3). We design a controller that not only stabilizes

(1)–(2) [as (13) or (14) do] but also minimizes some meaningful cost functional. The stabilizing controllers designed in the previous sections do not possess this property.

For our main result, stated next, we remind the reader that[6]

$$w(1)=u(1)-\int_0^1 k_1(y)u(y)\,dy. \tag{56}$$

We point out that, in this section, w satisfies a heat equation (10)–(11) with a much more complicated boundary condition at $x=1$; hence, w should be understood primarily as a coordinate transformation from u, or a short way of writing (9). The kernel PDE does not change, so the same $k(x,y)$ and $l(x,y)$ are used in this section as in the stabilization problem.

Theorem 6: For any $c>\varepsilon(2\bar{q}^2+1)$ and $\beta\geq 2$, the control law

$$u_x^*(1,t)=-\frac{\beta}{R}\left(u(1,t)-\int_0^1 k_1(y)u(y,t)\,dy\right) \tag{57}$$

stabilizes (1)–(2) in $L_2(0,1)$ and minimizes the cost functional

$$J(u)=\int_0^{\infty}(Q(u)+R\,u_x(1,t)^2)\,dt \tag{58}$$

with

$$R=\left(|l(1,1)|+\int_0^1 l_x(1,y)^2\,dy+\bar{q}\right)^{-1}>0 \tag{59}$$

and

$$\begin{aligned}Q(u)=\frac{1}{R}\beta^2 w^2(1)+2\beta\Big\{&\bar{q}w^2(0)+w(1)\int_0^1 l_x(1,y)w(y)\,dy\\ &+l(1,1)w^2(1)+\frac{c}{\varepsilon}\int_0^1 w^2\,dx+\int_0^1 w_x^2\,dx\Big\}\end{aligned} \tag{60}$$

where

$$\begin{aligned}Q(u)\geq \frac{1}{R}\beta(\beta-2)w^2(1)+\frac{2\beta}{\varepsilon}\left[c-\varepsilon(2\bar{q}^2+1)\right]\int_0^1 w^2\,dx\\ +\beta\int_0^1 w_x^2\,dx.\end{aligned} \tag{61}$$

□

Proof: Taking a Lyapunov function in the form

$$V=\frac{1}{2}\int_0^1 w^2(x,t)\,dx \tag{62}$$

we get

$$\begin{aligned}\dot{V}&=\varepsilon\int_0^1 w w_{xx}\,dx-c\int_0^1 w^2\,dx\\ &=\varepsilon w(1)u_x(1)-\varepsilon w(1)\\ &\quad\times\int_0^1 l_x(1,y)w(y)\,dy-\varepsilon l(1,1)w^2(1)\\ &\quad-\varepsilon\bar{q}\,w^2(0)-\varepsilon\int_0^1 w_x^2\,dx-c\int_0^1 w^2\,dx.\end{aligned} \tag{63}$$

[6]In this and the next section, we drop t-dependence for clarity, so $w(1)\equiv w(1,t)$, etc.

Using (63), we can write $Q(u)$ in the form

$$Q(u)=\frac{1}{R}\beta^2 w^2(1)+2\beta\left(w(1)u_x(1)-\frac{1}{\varepsilon}\dot{V}\right). \tag{64}$$

Substituting now (64) into the cost functional (58), we obtain

$$\begin{aligned}J(u)&=\int_0^{\infty}\Big[\frac{\beta^2}{R}w^2(1)+2\beta\\ &\quad\times\left(w(1)u_x(1)-\frac{\dot{V}}{\varepsilon}\right)+Ru_x(1)^2\Big]\,dt\\ &=\frac{2\beta}{\varepsilon}(V(0)-V(\infty))+R\int_0^{\infty}\left[u_x(1)+\frac{\beta}{R}w(1)\right]^2 dt\\ &=\frac{2\beta}{\varepsilon}(V(0)-V(\infty))+R\int_0^{\infty}[u_x(1)-u_x^*(1)]^2 dt.\end{aligned} \tag{65}$$

Using (59) and (60), the Cauchy–Shwartz inequality, and Agmon's inequality

$$\max_{x\in[0,1]} w^2(x)\leq w^2(1)+2\sqrt{\int_0^1 w^2\,dx}\sqrt{\int_0^1 w_x^2\,dx} \tag{66}$$

we get the following estimate:

$$\begin{aligned}Q(u)&\geq\frac{1}{R}\beta(\beta-2)w^2(1)+\frac{2\beta}{R}w^2(1)\\ &\quad+\frac{2\beta c}{\varepsilon}\int_0^1 w^2\,dx+2\beta\int_0^1 w_x^2\,dx\\ &\quad-2\beta\Big(|l(1,1)|w^2(1)+\bar{q}w^2(0)\\ &\quad+|w(1)|\int_0^1|l_x(1,y)w(y)|\,dy\Big)\\ &\geq\frac{1}{R}\beta(\beta-2)w^2(1)+\frac{2\beta}{R}w^2(1)\\ &\quad+\frac{2\beta c}{\varepsilon}\int_0^1 w^2\,dx+2\beta\int_0^1 w_x^2\,dx\\ &\quad-2\beta\Big(|l(1,1)|w^2(1)+\bar{q}w^2(1)\\ &\quad+2\bar{q}^2\int_0^1 w^2\,dx+\frac{1}{2}\int_0^1 {w_x}^2\,dx\\ &\quad+\int_0^1 w^2\,dx+w^2(1)\int_0^1 l_x(1,y)^2\,dy\Big)\\ &\geq\frac{1}{R}\beta(\beta-2)w^2(1)+2\beta w^2(1)\\ &\quad\times\left(\frac{1}{R}-|l(1,1)|-\bar{q}-\int_0^1 l_x(1,y)^2\,dy\right)\\ &\quad+\frac{2\beta}{\varepsilon}[c-\varepsilon(2\bar{q}^2+1)]\\ &\quad\times\int_0^1 w^2\,dx+\beta\int_0^1 w_x^2\,dx\\ &\geq\frac{1}{R}\beta(\beta-2)w^2(1)+\frac{2\beta}{\varepsilon}\\ &\quad\times[c-\varepsilon(2\bar{q}^2+1)]\int_0^1 w^2\,dx+\beta\int_0^1 w_x^2\,dx.\end{aligned} \tag{67}$$

So, Q is a positive–definite functional which makes (58) a reasonable cost which puts penalty on both states and the control. From (64), we now have

$$\dot{V} = \frac{\beta\varepsilon}{2R} w^2(1) + \varepsilon w(1) u_x(1) - \frac{\varepsilon}{2\beta} Q$$
$$\leq -\left[c - \varepsilon(2\bar{q}^2 + 1)\right] \int_0^1 w^2 \, dx - \frac{\varepsilon}{2} \int_0^1 w_x^2 \, dx \quad (68)$$

which proves that controller $u_x^*(1)$ stabilizes the system (10)–(11) [and, thus, the original system (1)–(2)]. Setting now $V(\infty) = 0$ in (65) completes the proof. ■

1) Meaning of Inverse Optimality: The inverse optimality result is of considerable significance but its meaning is not obvious. We explain it next. A prevalent method one would pursue for the class of systems considered in this paper is the design of an infinite-dimensional LQR. This approach would result in stability and good performance but would require the solution of an infinite dimensional Riccati equation. On the other hand, the inverse optimality result for our 1-D parabolic class, which is given explicitly and does not require the solution of a Riccati equation,[7] may appear somewhat limited in that it does not solve the optimal control problem for an *arbitrary* cost functional, but, instead, solves it for a very specialized cost functional (58)–(60). However, this cost functional is shown to be positive definite and bounded from below by L_2 and H_1 norms, thus constraining both the (spatial) energy of the transients and the (spatial) peaks of the transients. We also remind the reader of the often overlooked infinite gain margin of inverse optimal controllers (not possessed by ordinary stabilizing controllers), which allows the gain to be of any size, provided it is above a minimal stabilizing value (note that β in Theorem 6 is an arbitrary number greater than 2). The other benefit of inverse optimality is the 60° phase margin (the ability to tolerate a certain class of passive but possibly large actuator unmodeled dynamics), which although not stated here, is provable.

2) Computational Comparison Between LQR and Inverse Optimal Approach: Instead of requiring the solution of an operator Riccati equation, the inverse optimal approach for our 1-D parabolic class requires a solution of a *linear* hyperbolic PDE, an object conceptually less general than an operator Riccati equation (which is *quadratic*), and numerically easier to solve. The hyperbolic PDE (19)–(21) can be solved in several ways: in closed form (see Section VIII) for a subclass of plants; by truncating the series (38), although it is hard to say *a priori* how many terms should be kept; and directly, using a technique from [1] (see Section IX). In Section IX, we show that the last approach results in more than an order of magnitude in savings of computational time over solving a Riccati equation (for comparable performance), showing the advantage of a structure specific approach.

[7] In fact, the Riccati equation is explicitly solved. The solution is $(Id - \mathcal{K})^*(Id - \mathcal{K})$, where Id is the identity operator, $\mathcal{K}$ is the operator corresponding to the transformation $u(x) \mapsto \int_0^x k(x,\xi)u(\xi)d\xi$, and $*$ denotes the adjoint of the operator. The Riccati equation, whose solution this is, is difficult to write without extensive additional notation.

VII. Reducing the Control Effort Through Adaptation

The controller gain in (57) contains R that is rather conservative. We show how the control effort can be reduced by using adaptation to start with a zero gain and raise it on line to a stabilizing value. The result is given by the following theorem.

Theorem 7: The controller

$$u_x(1,t) = -\hat{\theta} w(1,t)$$
$$= -\hat{\theta}\left(u(1,t) - \int_0^1 k_1(y) u(y,t)\, dy\right) \quad (69)$$

with an update law

$$\dot{\hat{\theta}} = \gamma w^2(1,t) \quad \hat{\theta}(0) = 0 \quad \gamma > 0 \quad (70)$$

applied to system (1)–(2) guarantees the following bounds for the state $u(x,t)$:

$$\sup_{t\geq 0} \int_0^1 u^2(x,t)\, dx \leq (1+\bar{l})^2 \left\{ \frac{2\varepsilon}{\gamma R^2} + (1+\bar{k})^2 \int_0^1 u^2(x,0)\, dx \right\} \quad (71)$$

$$\int_0^\infty \int_0^1 u^2(x,t)\, dx\, dt \leq \frac{(1+\bar{l})^2}{2[c - \varepsilon(2\bar{q}^2+1)]} \left\{ \frac{2\varepsilon}{\gamma R^2} + (1+\bar{k})^2 \int_0^1 u^2(x,0)\, dx \right\} \quad (72)$$

and the following bound for control $u_x(1,t)$:

$$\int_0^\infty u_x^2(1,t)\, dt \leq \frac{2\gamma R}{\varepsilon^2} \left((1+\bar{k})^2 \int_0^1 u^2(x,0)\, dx + \frac{4\varepsilon}{\gamma R^2} \right)^2 \quad (73)$$

where $\bar{l} = \sup_{(x,y)\in\mathcal{T}} l(x,y)$ and $\bar{k} = \sup_{(x,y)\in\mathcal{T}} k(x,y)$. □

Proof: Denote $\tilde{\theta} = \theta - \hat{\theta}$

$$\theta = \frac{\beta}{2R} = \frac{\beta}{2}\left(|l(1,1)| + \int_0^1 l_x(1,y)^2\, dy + \bar{q} \right). \quad (74)$$

Taking Lyapunov function

$$V_1 = \frac{1}{2}\int_0^1 w^2(x,t)\, dx + \frac{\varepsilon}{2\gamma}\tilde{\theta}^2 \quad (75)$$

and, using (68), we get

$$\dot{V}_1 = \varepsilon w(1)(\hat{\theta} w(1) + u_x(1)) - \frac{\varepsilon Q}{2\beta} + \tilde{\theta}\left(\varepsilon w^2(1) - \frac{\varepsilon}{\gamma}\dot{\hat{\theta}}\right). \quad (76)$$

Choosing the controller and the update law as in (69) and (70), respectively, we obtain

$$\dot{V}_1 \le -\frac{\varepsilon}{2R}(\beta-2)w^2(1) - \frac{\varepsilon}{2}\int_0^1 w_x^2\,dx - [c-\varepsilon(2\bar{q}^2+1)]\int_0^1 w^2\,dx. \quad (77)$$

From (75) and (77), the following estimates easily follow:

$$\sup_{t\ge 0}\int_0^1 w^2(x,t)\,dx \le \int_0^1 w^2(x,0)\,dx + \frac{\varepsilon}{\gamma}\tilde{\theta}^2(0) \quad (78)$$

$$\int_0^\infty\int_0^1 w^2(x,t)\,dx\,dt \le \frac{\int_0^1 w^2(x,0)\,dx + \frac{\varepsilon}{\gamma}\tilde{\theta}^2(0)}{2[c-\varepsilon(2\bar{q}^2+1)]} \quad (79)$$

$$\begin{aligned}\int_0^\infty u_x^2(1,t)\,dt &\le \int_0^\infty \hat{\theta}^2(t)w^2(1,t)\,dt \\ &\le 2\,(\theta^2 + \sup_{t\ge 0}\tilde{\theta}^2(t))\int_0^\infty w^2(1,t)\,dt \\ &\le 2\left(2\theta^2 + \frac{\gamma}{\varepsilon}\int_0^1 w^2(x,0)\,dx\right)\frac{R}{\varepsilon(\beta-2)} \times\left(\int_0^1 w^2(x,0)\,dx + \frac{\varepsilon}{\gamma}\theta^2\right) \\ &\le \frac{\gamma R}{\varepsilon^2(\beta-2)}\left(\int_0^1 w^2(x,0)\,dx + \frac{\varepsilon\beta^2}{2\gamma R^2}\right)^2. \end{aligned} \quad (80)$$

Using (9) and (42), one can get the relationship between the norms of u and w

$$\|u\|^2 \le (1+\bar{l})^2\|w\|^2 \quad \|w\|^2 \le (1+\bar{k})^2\|u\|^2 \quad (81)$$

where $\bar{l} = \sup_{(x,y)\in\mathcal{T}} l(x,y), \bar{k} = \sup_{(x,y)\in\mathcal{T}} k(x,y)$. Now (78)–(80) with (81) and $\beta = 2\sqrt{2}$ give estimates (71)–(73), respectively. ■

The adaptive controllers are not inverse optimal, but they perform better than stabilizing controllers and use less gain than nonadaptive inverse optimal ones. As the numerical simulations in Sections VIII and IX will show, actual savings of the control effort can be quite large; for the considered settings, the adaptive controller uses several dozen times less gain than the nonadaptive one would.

Although the results of this and previous section are stated for Neumann boundary control, our approach is not restricted to it. One way to address the Dirichlet case is to use (69) to express $u(1,t)$

$$u(1,t) = -\frac{1}{\hat{\theta}}u_x(1,t) + \int_0^1 k_1(y)u(y,t)\,dy \quad (82)$$

with the dynamics of $\hat{\theta}$ given by (70). The only restriction is that $\hat{\theta}(0) > 0$. Of course, this controller puts a penalty on $u_x(1,t)$ instead of $u(1,t)$, so it can not be called optimal as a Dirichlet controller. To get a true inverse optimal Dirichlet controller we write similarly to (69)–(70)

$$u(1,t) = -\hat{\theta}w_x(1,t) \quad \dot{\hat{\theta}} = \gamma w_x^2(1,t) \quad (83)$$

or, using the expression for $w_x(1,t)$

$$w_x(1,t) = u_x(1,t) - k_1(1)u(1,t) - \int_0^1 k_2(y)u(y,t)\,dy \quad (84)$$

the controller can be written as

$$u(1,t) = -\frac{\hat{\theta}}{1+\hat{\theta}\int_0^1(\lambda(y)+c)\,dy} \times\left(u_x(1,t) - \int_0^1 k_2(y)u(y,t)\,dy\right). \quad (85)$$

The initial value of the adaptive gain $\hat{\theta}(0)$ can be taken zero in meaningful stabilization problems where $\lambda(x) \ge 0$. Note the structural similarity of the controllers (82) and (85). Both employ an integral operator of u measured for all $x \in [0,1)$ and a measurement of u_x at $x = 1$. The optimality of the controller (85) (in case of constant $\hat{\theta}$) can be proved along the same lines as in the proof of Theorem 7.

VIII. Explicit Construction of Feedback Laws

We show now how our approach can be used to obtain explicit solutions for $k(x,y)$. We present closed-form solutions to four distinct parabolic problems and then show how to combine them for a rather general class of plants.

Apart from the obvious practical benefit of having a closed-form control law, the explicit $k(x,y)$ allows us to find explicit solutions for the closed-loop system, which offers insight into how control affects eigenvalues and eigenfunctions. Another possible usage is in testing numerical schemes, similar to the role analytical solutions to Burgers' equation play in computational fluid dynamics.

A. Unstable Heat Equation

Let $\lambda(x) \equiv \lambda_0 = \mathrm{const}, g(x) \equiv 0, f(x,y) \equiv 0, q = +\infty$. In this case, system (1)–(2) takes the form of the unstable heat equation

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + \lambda_0 u(x,t), \qquad x \in (0,1) \quad (86)$$

$$u(0,t) = 0. \quad (87)$$

The open-loop system (86)–(87) (with $u(1,t) = 0$ or $u_x(1,t) = 0$) is unstable with arbitrarily many unstable eigenvalues (for large λ_0/ε). Although this constant coefficient problem may appear easy, the explicit (closed-form) boundary stabilization result in the case of arbitrary ε, λ_0 is not available in the literature.

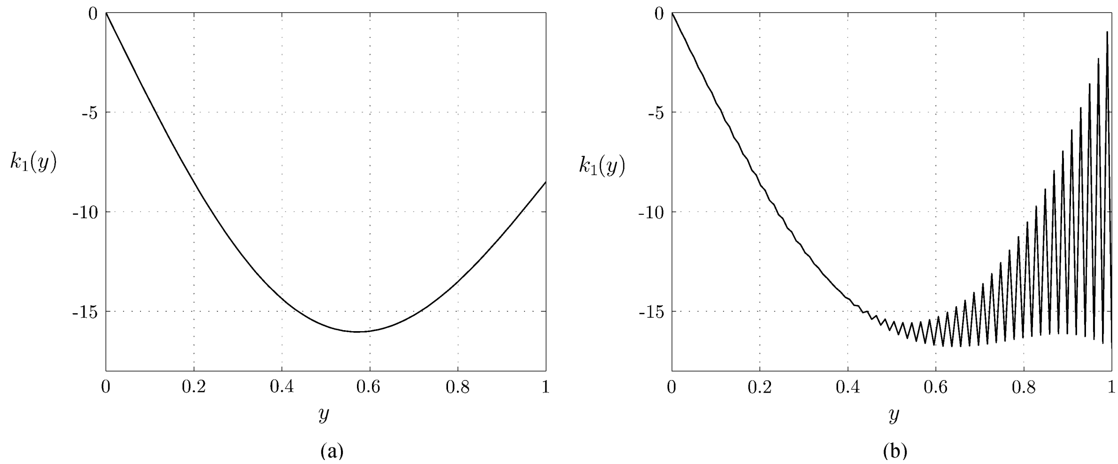

Fig. 1. Stabilizing gain kernel $k_1(y)$ for unstable heat (86)–(87) with $\lambda = 17$. (a) Obtained by formula (96). (b) Obtained by 100-step backstepping from [3].

The backstepping methods for solving the boundary control problem for (86)–(87) have been considered in [10], [3], and [23]. In [10], the stabilizing controller was constructed in a closed form, but only for $\lambda_0 < 3\pi^2\varepsilon/4$ (i.e., for the case of one unstable eigenvalue). In [3], finite-dimensional backstepping was applied to the discretized version of (86)–(87) for arbitrary λ_0 and shown to be convergent in L_∞. In [23], the kernel PDE was derived and shown to be well posed.

We will thoroughly explore this problem to illustrate all the results of Sections II–VII.

The kernel PDE (19)–(21), in this case, takes the following form:

$$k_{xx}(x,y) - k_{yy}(x,y) = \lambda k(x,y), \qquad (x,y) \in \mathcal{T} \quad (88)$$

$$k(x,0) = 0 \quad (89)$$

$$k(x,x) = -\frac{\lambda x}{2} \quad (90)$$

where we denote $\lambda = (\lambda_0 + c)/\varepsilon$. Let us solve this equation directly by the method of successive approximations. Integral (31) for G becomes

$$G(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta) + \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta G(\tau,s)\,ds\,d\tau. \quad (91)$$

Now, set

$$G_0 = -\frac{\lambda}{4}(\xi-\eta), G_{n+1} = \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta G_n(\tau,s)\,ds\,d\tau. \quad (92)$$

Fortunately, we can find the general term G_n in closed form

$$G_n(\xi,\eta) = -\frac{(\xi-\eta)\,\xi^n\eta^n}{(n!)^2(n+1)}\left(\frac{\lambda}{4}\right)^{n+1}. \quad (93)$$

Now, we can calculate the series (38):

$$G(\xi,\eta) = \sum_{n=0}^{\infty} G_n(\xi,\eta) = -\frac{\lambda(\xi-\eta)}{2}\frac{I_1(\sqrt{\lambda\xi\eta})}{\sqrt{\lambda\xi\eta}} \quad (94)$$

where I_1 is a modified Bessel function of order one. Writing (94) in terms of x, y gives the following solution for $k(x,y)$:

$$k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} \quad (95)$$

which gives the gain kernels

$$k_1(y) = k(1,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}}. \quad (96)$$

and

$$k_2(y) = k_x(1,y) = -\lambda y \frac{I_2\left(\sqrt{\lambda(1-y^2)}\right)}{1-y^2}. \quad (97)$$

A comparison of this result and the kernel obtained by finite-dimensional backstepping [3] is presented in Fig. 1. Both kernels can be used to successfully stabilize the system (even the discontinuous one on the right). We simply point out that applying infinite-dimensional backstepping transformation to the plant (86)–(87) first and then dealing with the resulting kernel PDE leads to a *smooth kernel*, whereas the approach [3] of discretizing the plant (86)–(87) first and then applying finite-dimensional backstepping results in a discontinuous oscillating kernel (which is still stabilizing).

In Fig. 2, the kernel $k_1(y)$ is plotted for several values of λ. We see that the maximum of the absolute value of the kernel moves to the left as λ grows. We can actually calculate the area

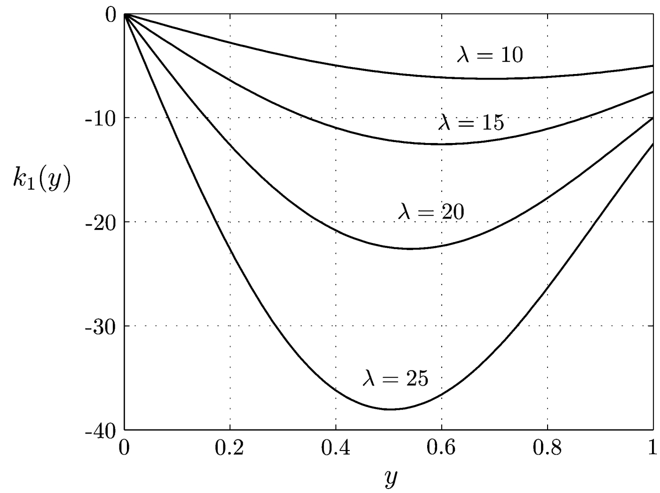

Fig. 2. Dependence of the gain kernel on the level of instability.

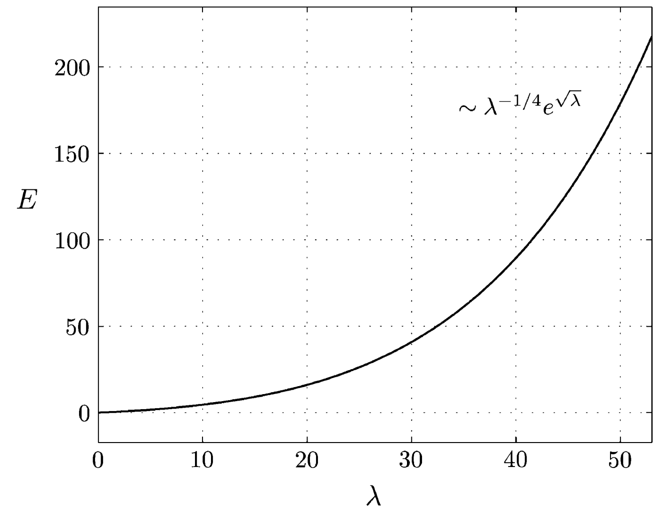

Fig. 3. Amount of total gain as a function of λ.

under the curves in Fig. 2 and estimate an amount of total gain effort required

$$E = \int_0^1 |k_1(y)|\, dy = \lambda \int_0^1 y \frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}}\, dy = \int_0^{\sqrt{\lambda}} I_1(z)\, dz = I_0\left(\sqrt{\lambda}\right) - 1. \quad (98)$$

In Fig. 3, the dependence of total gain on the level of instability is shown. As $\lambda \to \infty, E$ behaves as $E \sim e^{\sqrt{\lambda}}/(\sqrt{2\pi}\, \lambda^{1/4})$.

With both $k(x,y)$ and $l(x,y)$ being explicit, we calculate $\phi_n(x)$ and $\psi_n(x)$ in (54)–(55) and get the following result.

Theorem 8: The solution to the closed-loop system (86)–(87), (13) with $k_1(y)$ given by (96) is

$$u(x,t) = \sum_{n=1}^{\infty} e^{-(c+\varepsilon\pi^2 n^2)t}\, \frac{2\sqrt{\varepsilon}\pi n \sin\sqrt{\lambda+\varepsilon\pi^2 n^2}x}{\sqrt{\lambda+\varepsilon\pi^2 n^2}} \times \int_0^1 \left(\int_\xi^1 \xi \frac{I_1(\sqrt{\lambda(\tau^2-\xi^2)})}{\sqrt{\lambda(\tau^2-\xi^2)}} \sin(\sqrt{\varepsilon}\pi n\tau)\, d\tau + \sin(\sqrt{\varepsilon}\pi n\xi) \right) u_0(\xi)\, d\xi. \quad (99)$$

□

This result is proved using Lemma 5. In particular, the integral in (54) is solved explicitly with the help of [27]. It can also be shown [27] that $(\sqrt{\varepsilon}\pi n)/(\sqrt{\lambda+\varepsilon\pi^2 n^2}) \sin\sqrt{\lambda+\varepsilon\pi^2 n^2}x$ and $\sin(\sqrt{\varepsilon}\pi n\xi) + \int_\xi^1 \xi(I_1(\sqrt{\lambda(\tau^2-\xi^2)}))/(\sqrt{\lambda(\tau^2-\xi^2)}) \sin(\sqrt{\varepsilon}\pi n\tau)\, d\tau$ are orthonormal bases.

Now, let us construct an inverse-optimal controller. First, we need to find the kernel of the inverse transformation. Noticing that in our case $l(x,y) = -k(x,y)$ when λ is replaced by $-\lambda$, we immediately obtain

$$l(x,y) = -\lambda y \frac{J_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)}} \quad (100)$$

where J_1 is the usual (nonmodified) Bessel function of the first order. Since $l(x,y)$ is now available in closed form, we can get more careful estimates than in Section VI

$$\int_0^1 |l_x(1,y)|\, dy = \int_0^1 \frac{\lambda y}{1-y^2} \left| J_2\left(\sqrt{\lambda(1-y^2)}\right)\right|\, dy = \lambda \int_0^{\sqrt{\lambda}} \frac{|J_2(z)|}{z}\, dz \le \lambda + 1. \quad (101)$$

So, we obtain from (63)

$$\dot{V} \le w(1)\left(u_x(1) + \frac{1}{2}(\lambda+1)^2 w(1) + \frac{\lambda}{2} w(1)\right) - \frac{1}{2}\int_0^1 w_x^2\, dx - c\int_0^1 w^2 dx. \quad (102)$$

Taking now

$$u_x(1,t) = -\beta \frac{(\lambda+1)^2+\lambda}{2} \left(u(1,t) + \int_0^1 \lambda y \frac{I_1(\sqrt{\lambda(1-y^2)})}{\sqrt{\lambda(1-y^2)}} u(y,t)\, dy \right) \quad (103)$$

for $\beta \ge 2$ we get the Neumann controller that solves an inverse optimal stabilization problem. The Dirichlet controller can be obtained using (85) and (97). Let us summarize the results in the following theorem.

Theorem 9: Controller (13) with the kernel (96) stabilizes the unstable heat (86)–(87). The controller (103), while stabilizing

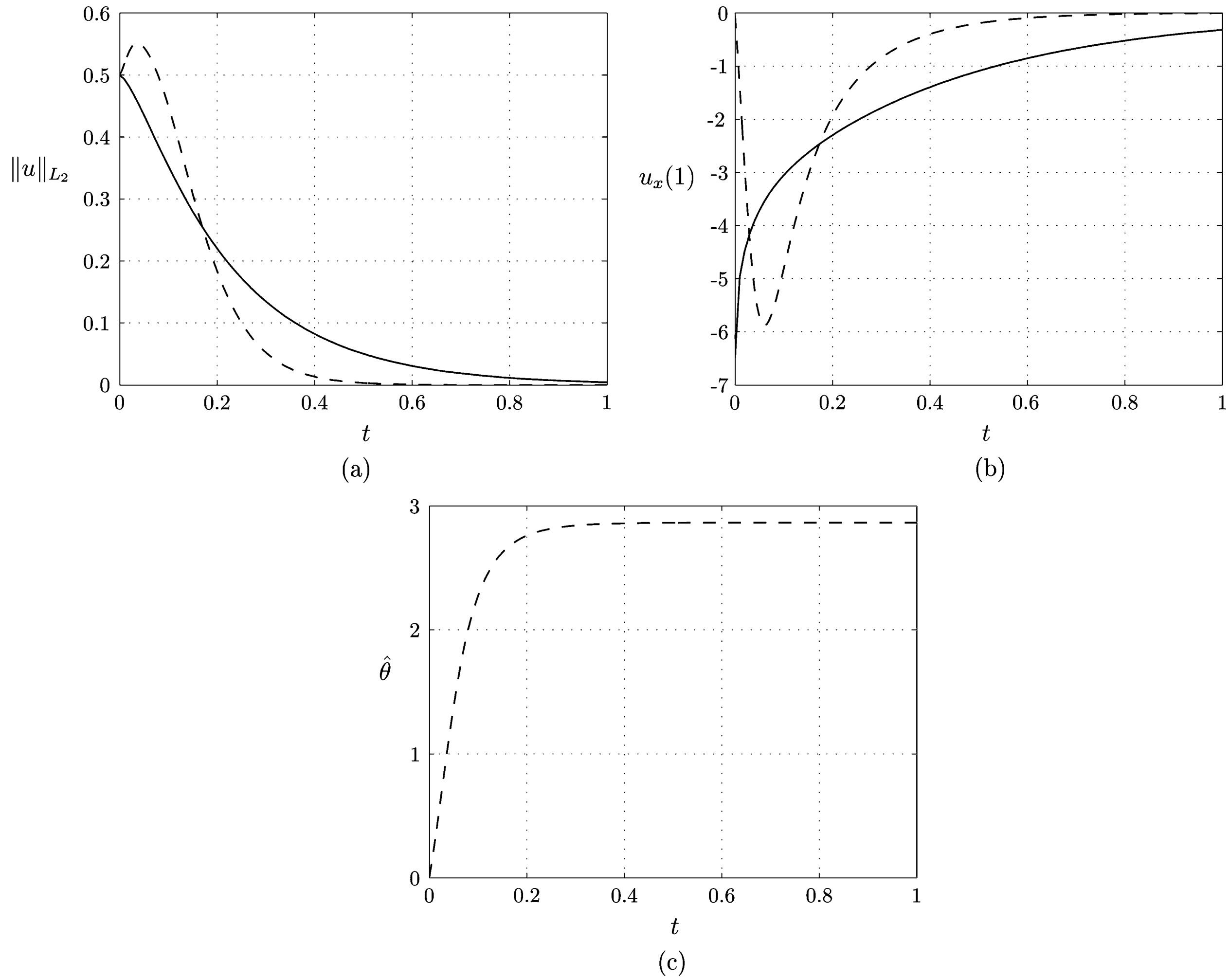

Fig. 4. Comparison between stabilizing (solid) and inverse optimal (dashed) controllers. (a) L_2-norm of the state. (b) Control effort. (c) Adaptive gain.

(86)–(87), also minimizes the cost functional (58) with $R^{-1} = ((\lambda+1)^2+\lambda)/2$ and Q given by (60). □

The penalty on the control in this case seems to become smaller as λ increases, but in fact only the ratio between $Q(u)$ and R matters, which can not be estimated in our design. This is another difference with the LQR approach: in LQR the penalties on the state and the control are constant and in our design these penalties are free to change. In both approaches the optimal value of the functional changes with λ and grows unbounded as $\lambda \to \infty$.

The comparison of the stabilizing and adaptive inverse optimal controller is presented in Fig. 4. System (86)–(87) was simulated with $\lambda_0 = 10, \varepsilon = 1, c = 0$, and the initial condition $u_0(x) = \sin(\pi x)$. The adaptation gain was taken as $\gamma = 2$. We can see that, compared to the stabilizing controller, the adaptive inverse optimal controller achieves better performance with less control effort. We can also estimate how much gain it actually saves compared to a nonadaptive controller. As (103) shows, the gain of the nonadaptive controller should be greater than or equal to $(\lambda+1)^2+\lambda = 131$, while the adaptive gain is less than or equal to 2.85 at all times [Fig. 4(c)]. So the adaptive controller used $131/2.85 = 46$ times less gain.

For the case of a homogeneous Neumann boundary condition at $x = 0$ for the (86) it is easy to repeat all the steps we have done for the Dirichlet case and get the following closed-form solution for the kernel:

$$k(x,y) = -\lambda x \frac{I_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)}}. \tag{104}$$

Note that the leading factor here is x, versus y in (95). The maximum of the absolute value of the kernel is reached at $x = 0$. This makes sense because the control has to react the most aggressively to perturbations that are the farthest from it. The total gain can be calculated to be [27]

$$\begin{aligned} E &= \int_0^1 |k_1(y)|\, dy = \lambda \int_0^1 \frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}}\, dy \\ &= \cosh(\sqrt{\lambda}) - 1. \end{aligned} \tag{105}$$

B. Heat Equation With Destabilizing Boundary Condition

We now consider a more complicated system

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + \lambda_0 u(x,t) \tag{106}$$
$$u_x(0,t) = q u(0,t) \tag{107}$$

where the boundary condition on the uncontrolled end is mixed and can cause instability for $q < 0$. This type of boundary condition appears for example in a solid propellant rocket model [9] and can also arise due to transformation (7). The results of the previous subsection apply when $q = 0$ or $q = +\infty$. We will use them here to get the solution for an arbitrary q.

The gain kernel PDE takes the following form:

$$k_{xx}(x,y) - k_{yy}(x,y) = \lambda k(x,y) \tag{108}$$
$$k_y(x,0) = qk(x,0) \tag{109}$$
$$k(x,x) = -\frac{\lambda x}{2} \tag{110}$$

where $\lambda = (\lambda_0 + c)/\varepsilon$. We propose to search for a solution in the following form:

$$k(x,y) = -\lambda x \frac{I_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)}} + \int_0^{x-y} I_0(\sqrt{\lambda(x+y)(x-y-\tau)})\rho(\tau)\, d\tau. \tag{111}$$

Here, the first term is a solution to (108) and (110) with $q = 0$, which has been obtained in the previous subsection. The second term is just one of the solutions to (108), ρ being a function to be determined. We can see now that (111) is a solution to (108), (110) and we need only to choose $\rho(\tau)$ so that (109) is satisfied. Substituting (111) into (109) we obtain the following integral equation for $\rho(x)$:

$$\int_0^x \rho(\tau)\left(\frac{\lambda}{2}\tau \frac{I_1(\sqrt{\lambda x(x-\tau)})}{\sqrt{\lambda x(x-\tau)}} + qI_0(\sqrt{\lambda x(x-\tau)})\right) d\tau = -\rho(x) + \sqrt{\lambda} q I_1(\sqrt{\lambda}x). \tag{112}$$

To solve this equation, we apply the Laplace transform with respect to x to both sides of (112) and get

$$\rho(s) + \int_0^\infty e^{-s\xi}\int_0^\xi \rho(\tau)\left(\frac{\lambda}{2}\tau\frac{I_1(\sqrt{\lambda\xi(\xi-\tau)})}{\sqrt{\lambda\xi(\xi-\tau)}} + qI_0\left(\sqrt{\lambda\xi(\xi-\tau)}\right)\right) d\tau\, d\xi = q\frac{s-\sqrt{s^2-\lambda}}{\sqrt{s^2-\lambda}}. \tag{113}$$

After changing the order of integration, calculating the inner integral, and introducing a new variable $s' = (s + \sqrt{s^2-\lambda})/2$ we obtain:

$$\int_0^\infty \rho(\tau)e^{-s'\tau}\, d\tau = q\frac{s-\sqrt{s^2-\lambda}}{q+\sqrt{s^2-\lambda}}. \tag{114}$$

Now, using the relation $s = s' + \lambda/(4s')$, we get

$$\rho(s') = \frac{2q\lambda}{(2s'+q)^2 - (\lambda+q^2)}. \tag{115}$$

Taking the inverse Laplace transform gives

$$\rho(x) = \frac{q\lambda}{\sqrt{\lambda+q^2}}e^{-qx/2}\sinh\frac{\sqrt{\lambda+q^2}}{2}x. \tag{116}$$

Substituting (116) into (111), we get the expression for the gain kernel given in quadratures.

Theorem 10: The solution to (108)–(110) is

$$k(x,y) = \frac{q\lambda}{\sqrt{\lambda+q^2}}\int_0^{x-y} e^{-q\tau/2}I_0(\sqrt{\lambda(x+y)(x-y-\tau)}) \times \sinh\left(\frac{\sqrt{\lambda+q^2}}{2}\tau\right)d\tau - \lambda x\frac{I_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)}}. \tag{117}$$

□

C. Explicit Solution for a Family of Nonconstant $\Lambda(x)$

Consider now the system

$$u_t(x,t) = u_{xx}(x,t) + \lambda_\sigma(x)u(x,t), \qquad x\in(0,1) \tag{118}$$
$$u(0,t) = 0 \tag{119}$$

where $\lambda_\sigma(x)$ is given by

$$\lambda_\sigma(x) = \frac{2\sigma^2}{\cosh^2(\sigma(x-x_0))}. \tag{120}$$

This $\lambda_\sigma(x)$ parameterizes a family of "one-peak" functions. The maximum of $\lambda_\sigma(x)$ is $2\varepsilon\sigma^2$ and is achieved at $x = x_0$. The parameters σ and x_0 can be chosen to give the maximum an arbitrary value and location. Examples of $\lambda_\sigma(x)$ for different values of σ and x_0 are shown in Fig. 5. The "sharpness" of the peak is not arbitrary and is given by $\lambda''_{\max} = -\lambda^2_{\max}/\varepsilon$. Despite the strange-looking expression for $\lambda_\sigma(x)$, the system (118)–(119) can approximate the linearized model of a chemical tubular reactor very well (see [8] and the references therein), which is open-loop unstable.

Our result on stabilization of (118)–(119) is given by the following theorem.

Theorem 11: The controller

$$u(1,t) = -\int_0^1 \sigma e^{\sigma\tanh\sigma x_0(1-y)} \times [\tanh\sigma x_0 - \tanh(\sigma(x_0-y))]\, u(y,t)\, dy \tag{121}$$

stabilizes the system (118)–(119). □

Proof: The kernel PDE for (118)–(119) is

$$k_{xx}(x,y) - k_{yy}(x,y) = \lambda_\sigma(y)k(x,y) \tag{122}$$
$$k(x,0) = 0 \tag{123}$$
$$k(x,x) = -\frac{1}{2}\int_0^x \lambda_\sigma(\tau)\, d\tau. \tag{124}$$

Postulating $k(x,y) = X(x)Y(y)$, we have the following set of ODEs:

$$X''(x) = \mu X(x) \tag{125}$$
$$Y''(y) = Y(y)(\mu + 2X(y)Y'(y) + 2X'(y)Y(y)) \tag{126}$$
$$Y(0) = 0 \tag{127}$$

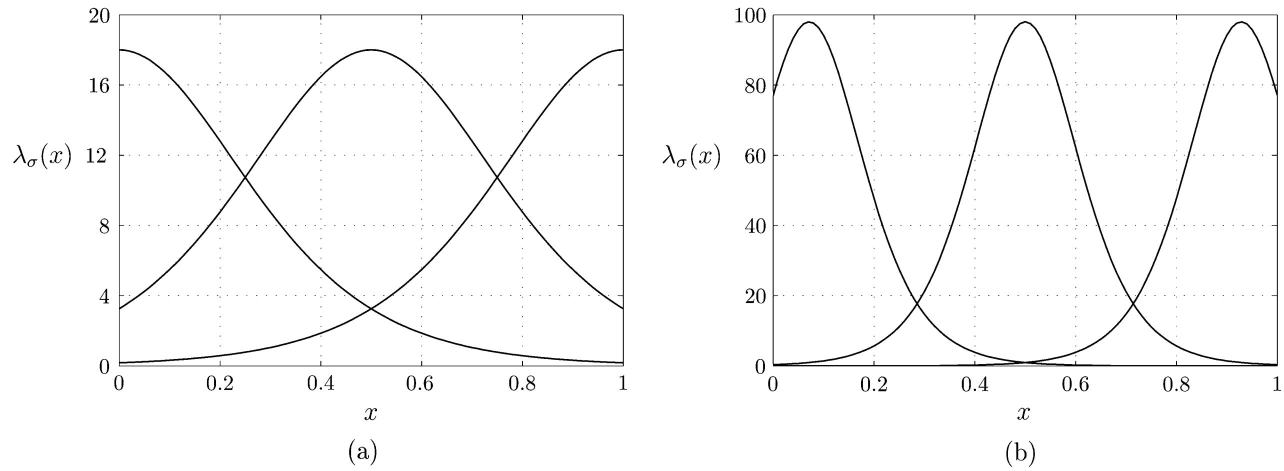

Fig. 5. "One-peak" $\lambda_\sigma(x)$ for different values of σ and x_0. (a) $\sigma = 3$. (b) $\sigma = 7$.

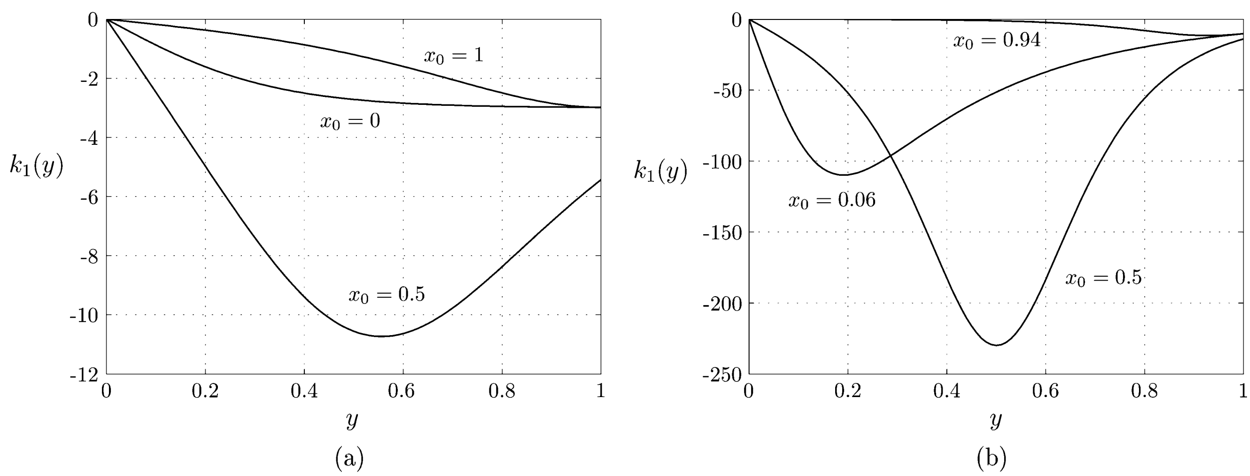

Fig. 6. Gain kernel $k_1(y)$ for (118)–(119). (a) $\sigma = 3$. (b) $\sigma = 7$.

where μ is an arbitrary parameter. Let us choose $X(x) = e^{\sqrt{\mu}x}$ and substitute it into (126). We get

$$Y''(y) - 2e^{\sqrt{\mu}y}Y(y)(Y'(y) + \sqrt{\mu}Y(y)) - \mu Y(y) = 0. \quad (128)$$

Changing variables $Y(y) = Z(y)e^{-\sqrt{\mu}y}$, we arrive at the following ODE:

$$Z''(y) - 2Z'(y)Z(y) - 2\sqrt{\mu}Z'(y) = 0 \quad (129)$$
$$Z(0) = 0 \quad (130)$$
$$Z'(0) = \mu - \sigma^2. \quad (131)$$

Here, we introduced an arbitrary parameter σ. The solution to (129)–(131) is

$$Z(y) = -\sigma\left(\tanh(\sigma(y - x_0)) + \tanh\sigma x_0\right) \quad (132)$$

where $\tanh\sigma x_0 = \sqrt{\mu}/\sigma$. Now, we can check that

$$\lambda_\sigma(x) = -2Z'(x) = \frac{2\sigma^2}{\cosh^2(\sigma(x - x_0))} \quad (133)$$

which gives (120). Using (132), we obtain the kernel

$$k(x, y) = -\sigma e^{\sigma\tanh\sigma x_0(x-y)} \times[\tanh\sigma x_0 - \tanh(\sigma(x_0 - y))]. \quad (134)$$

Setting $x = 1$ in (134) concludes the proof. ■

In Fig. 6, the stabilizing kernels corresponding to $\lambda_\sigma(x)$ from Fig. 5 are shown. We can see that the control effort depends very much on the location of the peak of $\lambda_\sigma(x)$, which has an obvious explanation. When the peak is close to $x = 1$, the controller's influence is very high, when it is close to $x = 0$, the boundary condition (119) helps to stabilize, so the worst case is the peak somewhere in the middle of the domain.

The results of this section can be extended to the case of the mixed boundary condition (2) using the method of Section VIII-E.

D. Solid Propellant Rocket Model

Consider the following system:

$$u_t(x, t) = u_{xx}(x, t) + g_0 e^{\alpha x}u(0, t) \quad (135)$$
$$u_x(0, t) = 0. \quad (136)$$

Here, g_0 and α are arbitrary constants. This equation represents a linearized model of unstable burning in solid propellant rockets (for more details, see [9] and the references therein). This system is unstable (with $u(1)=0$) for any $g_0>2, \alpha \geq 0$.

The PDE (19)–(21) takes the form

$$k_{xx}(x,y) - k_{yy}(x,y) = 0 \tag{137}$$

$$k_y(x,0) = g_0 e^{\alpha x} - g_0 \int_0^x k(x,y) e^{\alpha y}\, dy \tag{138}$$

$$k(x,x) = 0. \tag{139}$$

The structure of (137)–(139) suggests to search for the solution in the following form:

$$k(x,y) = C_1 e^{\alpha_1 (x-y)} + C_2 e^{\alpha_2 (x-y)}. \tag{140}$$

Substituting (140) into (137)–(139), we determine the constants $C_1, C_2, \alpha_1, \alpha_2$ and thus obtain the solution

$$k(x,y) = -\frac{g_0}{g} e^{\frac{\alpha}{2}(x-y)} \sinh\left(g(x-y)\right) \tag{141}$$

where $g = (g_0 + \alpha^2/4)^{1/2}$. We arrive at the following result.

Theorem 12: The controller

$$u(1,t) = -\int_0^1 \frac{g_0 e^{\frac{\alpha}{2}(1-y)}}{g} \sinh\left(g(1-y)\right) u(y,t)\, dy \tag{142}$$

exponentially stabilizes the zero solution of (135)–(136). □

Again, the closed-loop solutions can be obtained explicitly. For example, for $g(x) = g_0$ one can get

$$u = 2\sum_{n=0}^{\infty} e^{-\mu_n^2 t} \left(\cos\left(\mu_n x\right) - \frac{g_0}{g_0+\mu_n^2}\right) \int_0^1 u_0(\xi) \times \left\{\cos\left(\mu_n \xi\right) + (-1)^n \frac{\sqrt{g_0}}{\mu_n} \sinh(\sqrt{g_0}(1-\xi))\right\} d\xi \tag{143}$$

where $\mu_n = \pi n + \pi/2$.

E. Combining Previous Results

The solutions presented in Sections VIII-A–D can be combined to obtain the explicit results even for more complex systems. For example, consider the system

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + \left(\lambda_\sigma(x) + \lambda_0\right) u(x,t) \tag{144}$$

$$u(0,t) = 0. \tag{145}$$

Denote by $k^\sigma(x,y)$ and $k^\lambda(x,y)$ the (closed-form) control gains for the (118)–(119) and (86)–(87), respectively. The transformation

$$\bar{w}(x,t) = u(x,t) - \int_0^x k^\sigma(x,y) u(y,t)\, dy \tag{146}$$

maps (144)–(145) into the intermediate system

$$\bar{w}_t(x,t) = \varepsilon \bar{w}_{xx}(x,t) + \lambda_0 \bar{w}(x,t) \tag{147}$$

$$\bar{w}(0,t) = 0. \tag{148}$$

Now, we apply to (147)–(148) the transformation

$$w(x,t) = \bar{w}(x,t) - \int_0^x k^\lambda(x,y) \bar{w}(y,t)\, dy \tag{149}$$

to map it into the system

$$w_t(x,t) = \varepsilon w_{xx}(x,t) - c w(x,t) \tag{150}$$

$$w(0,t) = 0 \tag{151}$$

$$w(1,t) = 0. \tag{152}$$

Using (149) and (146), we derive the transformation directly from $u(x,t)$ into $w(x,t)$

$$\begin{aligned} w(x,t) &= u(x,t) - \int_0^x k^\sigma(x,y) u(y,t)\, dy \\ &\quad - \int_0^x k^\lambda(x,y) \left(u(y,t) - \int_0^y k^\sigma(y,\xi) u(\xi,t)\, d\xi\right) dy \\ &= u(x,t) - \int_0^x k^c(x,y) u(y,t)\, dy \end{aligned} \tag{153}$$

where k^c stands for the combined kernel

$$k^c(x,y) = k^\sigma(x,y) + k^\lambda(x,y) - \int_y^x k^\lambda(x,\xi) k^\sigma(\xi,y)\, d\xi. \tag{154}$$

For example, for $\lambda(x) = \lambda + 2\sigma^2/\cosh^2(\sigma x)$, one gets the closed-form solution

$$k^c(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} - \sigma \tanh(\sigma y) I_0\left(\sqrt{\lambda(x^2-y^2)}\right). \tag{155}$$

In the same fashion, one can obtain explicit stabilizing controllers for even more complicated plants. For the following six-parameter family of plants we are able to obtain the explicit controllers using the results of Sections VIII.A, B, and D:

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + b u_x(x,t) + \lambda_0 u(x,t) + g_0 e^{\alpha x} u(0,t) \tag{156}$$

$$u_x(0,t) = q u(0,t). \tag{157}$$

IX. Numerical Results

In this section, we present the results of numerical simulations for the plant (1)–(2) with Dirichlet boundary control. The parameters of the system were taken to be $\varepsilon = 1, \lambda(x) =$

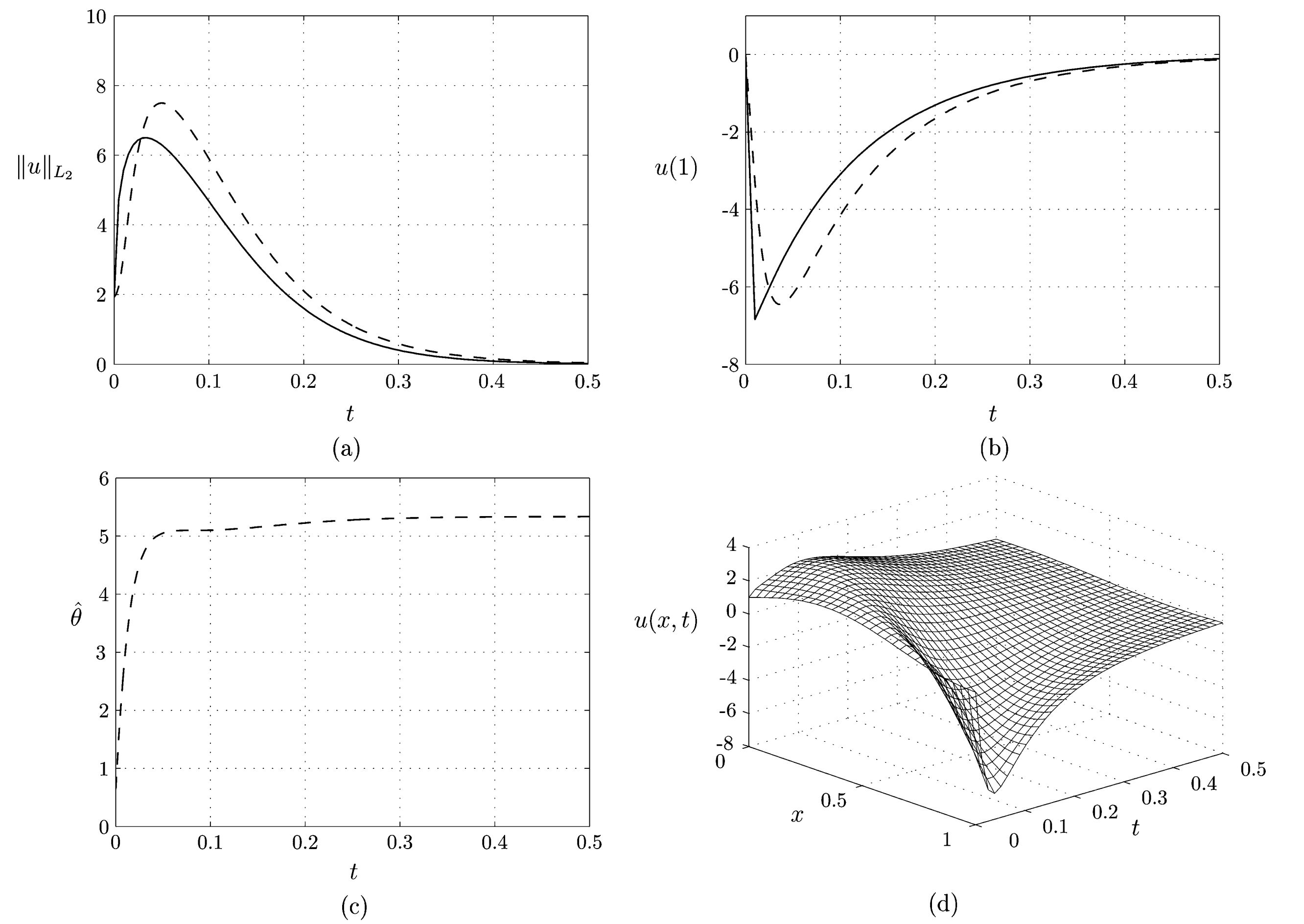

Fig. 7. Comparison between LQR (solid) and inverse optimal (dashed) controllers. (a) L_2-norm of the state. (b) Control effort. (c) Adaptive gain. (d) State for inverse optimal controller.

$14 - 16(x - 1/2)^2, q = 2$, and initial condition $u_0(x) = 1 + \sin(3\pi x/2)$. With these parameters the system has one unstable eigenvalue at 7.8.

Instead of calculating the series (38), we directly numerically solve the PDE (19)–(21). This PDE resembles the Klein–Gordon PDE [17] but it contains additional terms, evolves on a different domain and has more complicated boundary conditions. The scheme for this equation should be selected carefully since the term with $\lambda(x)$ can cause numerical instability. We suggest the following second order accuracy Ablowitz–Kruskal–Ladik [1] scheme which we modified to suit the geometry and boundary conditions of the kernel PDE

$$k_j^{i+1} = -k_j^{i-1} + k_{j+1}^i + k_{j-1}^i + a_j h^2 \frac{k_{j+1}^i + k_{j-1}^i}{2} \tag{158}$$

$$k_{i+1}^{i+1} = k_i^i - \frac{h}{4}(a_i + a_{i+1}), \qquad k_1^1 = 0 \tag{159}$$

$$k_1^{i+1} = -k_1^{i-1} + 2k_2^i \frac{1 + a_1 h^2/2}{1 + qh(1 + a_1 h^2/2)}. \tag{160}$$

Here, $k_j^i = k((i-1)h, (j-1)h), i = 2, \ldots, N, j = 2, \ldots, i-1, a_i = (\lambda((i-1)h) + c)/\varepsilon, h = 1/N, N$ is the number of steps. The key feature of this scheme is the discretization of the $a(y)k(x,y)$-term, averaging $k(x,y)$ in the y-direction.

The terms with $g(x)$ and $f(x,y)$ can also be incorporated into the scheme. We do not consider them here because proving stability of the modified numerical scheme in this case is beyond the scope of this paper.

A static LQR controller (with unity penalties on the state and control) was implemented using the most popular Galerkin approximation, although more advanced techniques exist [11]. We used the discretization with $N = 100$ steps. Computation of the gain kernel for the backstepping controller using the scheme (158)–(160) took as much as 20 times less time than for the LQR/Riccati kernel. This suggests that our method may be of even more interest in 2-D and 3-D problems where the cost of solving operator Riccati equation becomes nearly prohibitive [11]. Symbolic computation using the series (38) is also possible; four terms $G_0, \ldots, G_3$ are sufficient for practical implementation giving relative error 0.6% with respect to the numerical solution.

The results of the simulations of the closed-loop system are presented in Fig. 7. The system was discretized using a BTCS finite difference method with 100 steps. We used the Dirichlet adaptive controller with $c = 0, \gamma = 1, \hat{\theta}(0) = 0.5$. Fig. 7(c) shows the evolution of the adaptive gain, its maximum value turns out to be about 50 times less than the conservative (constant) inverse optimal gain. The control effort and the L_2 norm of the closed-loop state are shown in Fig. 7(a) and (b), respectively. We see that LQR controller shows just slightly better performance.

X. Conclusion

The backstepping technique might be best appreciated by readers familiar with the historical developments of finite-dimensional *nonlinear* control. The first systematic nonlinear control methods were the methods of optimal control, which require the "solution" of Hamilton–Jacobi–Bellman (HJB) nonlinear PDEs. The breakthrough in nonlinear control came with the differential geometric theory and feedback linearization, which recognized the structure of nonlinear control systems and exploited it using coordinate transformations and feedback cancelations. (Backstepping boundary control, incidentally, uses the same approach—a linear integral transformation plus boundary feedback.) In the same way that nonlinear PDEs (HJB) are more complex than what ordinary differential equation control problems call for, operator Riccati equations are more than what linear boundary control calls for in the 1-D parabolic class in this paper. In summary, backstepping, with its linear hyperbolic PDE for the gain kernel, is unique in not exceeding the complexity of the PDE control problem that it is solving.

Only the state feedback problem was considered here due to size limitations. In a companion paper, we develop a boundary observer theory for the same class of PDEs, which is dual to the state feedback control presented here. Output injection kernels are developed in a similar manner as $k(x,y)$ here to achieve exponentially convergent estimation for open-loop systems and output-feedback stabilization for systems with boundary feedback. Due to space limits, we also restricted our attention to a 1-D class of PDEs. Systems on multidimensional domains of specific geometry are very much tractable by backstepping and, in the most general case, lead to interesting ultra-hyperbolic PDEs for the gain kernel. Finally, in the field of nonlinear PDEs, for which successful control designs assuming in-domain actuation have been developed by Christofides [13], the results presented in this paper provide a breakthrough toward solving boundary control problems using Volterra series feedbacks.

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