Closed-Form Design of Maximally Flat FIR Hilbert Transformers, Differentiators, and Fractional Delayers by Power Series Expansion

Soo-Chang Pei, *Fellow, IEEE,* and Peng-Hua Wang

Abstract—In this paper, novel closed-form designs of the FIR Hilbert transformers, maximally flat digital differentiators and fractional delayers are proposed. The transfer functions of these filters are analytically obtained by expanding some suitable functions into power series. Efficient implementations can be derived from the resultant transfer functions. The weighting coefficients and the building blocks of these filters are explicitly expressed in closed form. The proposed filter structures are more robust to the coefficient quantization than the direct form.

Index Terms—Differentiator, FIR filter, fractional delayer, maximal flatness, transformer.

I. INTRODUCTION

THE DESIGN of the Hilbert transformers (HTs), digital differentiators (DDs), and fractional delayers (FDs) has been widely considered. In [4], explicit expressions for the impulse responses of the Case 3 and Case 4 maximally flat (MF) FIR HTs were derived. These expressions with double summations are computed by using a generalization of the Bernstein polynomial. In the above article, the MF conditions were set at the frequency $\pi/2$ for the Case 3 HTs and set at π for the Case 4 HTs. In [5], another explicit expression for the impulse response of the Case 3 MF HTs with maximal flatness at the frequency $\pi/2$ was obtained. This closed-form impulse response expressed by the product of two binomial coefficients is solved according to the result of the DDs in the same article. The impulse response obtained in [4] is identical to that in [5] but the latter one is more compact.

In [6], the impulse response of the Case 3 maximally linear FIR DDs with maximal flatness at $\omega = 0$ was explicitly solved by using the Crout's method. The expression of the impulse response is represented in recursive form with the sum of the binomial coefficients. In [7], the DDs with maximal linearity at $\omega = \pi$ were approximated by the sum of a Case 3 and a Case 4 FIR filters. The filter coefficients are solved and recursively expressed by using the Crout's method. In order to realize the filter, one half-delay ($z^{-1/2}$) is required. Consequently, an additional filter is needed for implementing this half-delay filter. In [8], an explicit expression for the Case 3 FIR DDs with maximal linearity at $\omega = 0$ was solved. This result is equivalent to the one in [6] but the former one was more compact in mathematical ex-

pression. In [8], the author also showed that the MF FIR DDs could be obtained by differentiating the continuous-time signal which is reconstructed by the samples using the Lagrange interpolation. In [10], the MF FIR DDs were designed with maximal flatness at $\omega = \pi/p$ where p is an integer. This MF FIR DDs were generalized to higher order degrees and presented in [9]. These filters could be regarded as the interpolated FIR (IFIR) filters [11], [12] where the *model filters* were the MF FIR DDs with maximal flatness at $\omega = \pi$ described in [7] with a p-fold stretch. In [13], the Case 3 FIR DDs of the first and the higher order degrees at $\omega = 0$ and at $\omega = \pi/2$ were redesigned based on expressing the frequency responses of the FIR filters as weighted sums of $\sin^n \omega$ with maximal linearity at $\omega = 0$, or $\cos^n \omega$ with maximal linearity at $\omega = \pi/2$. Accordingly, the weighting coefficients can be explicitly solved. Efficient realization structures were derived based on the closed-form transfer functions of the filters. In [5], the impulse responses for the Case 3 FIR DDs with maximal linearity at $\omega = \pi/2$ were derived and explicitly expressed in binomial coefficients. These weighting coefficients are equivalent to the ones proposed in [7], [10] but the former ones were more compact in mathematical expression.

The design of the FDs is a recent well-studied problem. In [14], a comprehensive review for the design of the FIR and allpass FDs was presented. In [15], the design of the MF FIR FDs with maximal flatness at arbitrary frequency was converted to solve a set of linear equations but the solution was not explicitly solved. In [16], the authors derived a closed-form expression of the impulse response for the FIR FDs based on expanding the ideal transfer function into power series. This result may be regarded as the MF FIR FDs with maximal flatness at $\omega = 0$. In [17], the weighting coefficients of FIR FDs of odd length were explicitly solved based on expressing the continuous-time signals as polynomials. The result is also equivalent to the FIR FDs with maximal flatness at $\omega = 0$. The explicit expression for the impulse responses of the MF FIR FDs of arbitrary order with maximal flatness at arbitrary frequency was derived in [18], [19] by solving linear equations expressed in [15].

The filters designed by MF approach may lead to efficient implementation structures. In [20], the linear-phase MF lowpass FIR filters were explicitly derived. The filters were implemented efficiently by structures proposed in [1]. In [2], the linear-phase sharp cutoff lowpass FIR filters were designed where the MF FIR filters served as the building blocks. This technique did not involve iterative approximation and the resultant filters can be designed quickly and implemented efficiently. In [3], the concept of MF building blocks was extended to design the optimal

The authors are with the Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan, R.O.C. (e-mail: pei@cc.ee.ntu.edu.tw).

linear-phase FIR lowpass filters with a prescribed degree of flatness in the passband. Because the weighting coefficients of the linear-phase MF lowpass FIR filters are the binomial coefficients, they can be expressed as cumulative products of simple rational numbers. The dynamic range of the weighting coefficients is greatly reduced. In this paper, we will show that filters designed by the power series expansion will obtain similar efficient realization structures.

In this paper, we derive the transfer functions of the HTs, DDs and FDs by expanding some suitable functions into power series. Although the methods for designing these filters have been widely reported, the proposed approach is different from the existing methods in several ways. First, the method of power series expansion does not express the weighting coefficients as the solution of certain linear equation. These coefficients are obtained without analytically or numerically solving linear equations while the step of explicitly or implicitly solving linear equation appears in many existing method. That is, the power series expansion is a more simple and direct method for designing FIR filters. Second, we will show that the resultant filters obtained by the proposed method are exactly the maximally flat filters which have been widely reported in the literature and reviewed in this section. In other words, we propose a new method to unify the design of the maximally flat filters. However, this approach not only gives new expressions of the filter coefficients, but also reveals new structures of the maximally filters. These new structures facilitate efficient implementation of the maximally flat filters. Third, we will show that resultant weighting coefficients are independent of the filter orders. Based on this property, the filter quality can be gradually improved by adding extra blocks without necessity of updating the coefficients in the previous blocks. That is to say, the proposed method reveal the fact that the maximally flat filters can be implemented by module.

This paper is organized as follows. In Section II, we propose a design of the FIR HTs based on expanding the signum function into power series. The MF FIR HTs proposed in [4] are special cases of our results. In Section III, FIR DDs are designed by expanding some inverse triangular functions into power series. These filters are equivalent to the well-studied MF FIR DDs but the proposed approach provides efficient realization structures which are not easily found from the previous results. In Section IV, the FIR FDs are designed by expanding the ideal transfer function into binomial series. This result is equivalent to the transfer function proposed in [16]. However, our approach will obtain an efficient implementation structure similar to the structures of the HTs and DDs. In Section V, design examples and coefficient quantization effects are provided. Section VI makes a final conclusion.

II. DESIGN OF FIR HILBERT TRANSFORMERS

The ideal frequency response of HTs is expressed as

$$H_{\text{IHT}}(\omega) = \begin{cases} j, & \text{for } -\pi < \omega < 0 \\ -j, & \text{for } 0 < \omega < \pi. \end{cases} \qquad (1)$$

This frequency response may be regarded as the signum function $\text{sgn}\,\omega$ multiplied by a constant factor $-j$ where the signum function is defined by

$$\text{sgn}\,\omega = \begin{cases} -1, & \text{for } -\pi < \omega < 0 \\ 1, & \text{for } 0 < \omega < \pi. \end{cases} \qquad (2)$$

Therefore, instead of designing a filter with frequency response expressed in (1) directly, the design of the HTs can be regarded as designing a filter with the signum frequency response. This may be accomplished by expressing the signum function in Fourier series, and represented as [21]

$$\frac{\pi}{4}\text{sgn}\,\omega = \sum_{m=0}^{\infty} (2m+1)^{-1} \sin\left[(2m+1)\omega\right] - \pi < \omega < \pi. \qquad (3)$$

The above sine series is equivalent to the idea Case 3 HT [22].

Instead of expanding the original $\text{sgn}\,\omega$, another approach is to express it as a composite function. An interesting property of $\text{sgn}\,\omega$ is that

$$\text{sgn}\,\omega = \text{sgn}(\sin\,\omega) = \text{sgn}\left[\sin\left(\omega/2\right)\right]$$

for $-\pi < \omega < \pi$. Consequently, if $\text{sgn}\,x$ is approximated by a polynomial in x and $\sin\,\omega$ or $\sin\left(\omega/2\right)$ is substituted for x, we obtain an FIR filter. On the other hand, if $\text{sgn}\,x$ is approximated by a rational function, an IIR filter is obtained. In this paper, $\text{sgn}\,x$ is approximated by a polynomial. This is achieved by truncating a power series of $\text{sgn}\,x$ to finite terms.

In order to expand $\text{sgn}\,x$ into power series, we begin with the following representation of $\text{sgn}\,x$

$$\text{sgn}\,x = \frac{x}{\sqrt{x^2}} = xf(x^2), \qquad x \neq 0 \qquad (4)$$

where $f(u) = 1/\sqrt{u}$. The Taylor series of $f(u)$ at center c is expressed by

$$f(u) = \frac{1}{\sqrt{c}}\left[1 + \sum_{m=1}^{\infty} \frac{(2m-1)!!}{(2m)!!}\left(1 - \frac{u}{c}\right)^m\right]$$

where the double factorial is defined as follows:

$$(2m-1)!! = 1 \cdot 3 \cdot 5 \cdots (2m-1)(2m)!! = 2 \cdot 4 \cdot 6 \cdots (2m)$$

Consequently, the signum function $\text{sgn}\,x$ is expressed by

$$\text{sgn}\,x = \frac{x}{\sqrt{c}}\left[1 + \sum_{m=1}^{\infty} \frac{(2m-1)!!}{(2m)!!}\left(1 - \frac{x^2}{c}\right)^m\right]. \qquad (5)$$

The range of the expanding center c is restricted for its convergence. The series expressed in (5) converges for $-1 < (1 - x^2/c) < 1$, that is, $c > x^2/2$. Since $\sin\,\omega$ or $\sin\,\omega/2$ will be substituted for x, c has to be chosen larger than $1/2$. On the other hand, the expanding center c in the x-domain is associated to a frequency center ω_c in the ω-domain with the relationship of $c = \sin^2\,\omega_c$ or $c = \sin^2\left(\omega_c/2\right)$. Therefore, c must be smaller than unity to obtain a real value of ω_c. Accordingly, the value of c is restricted by
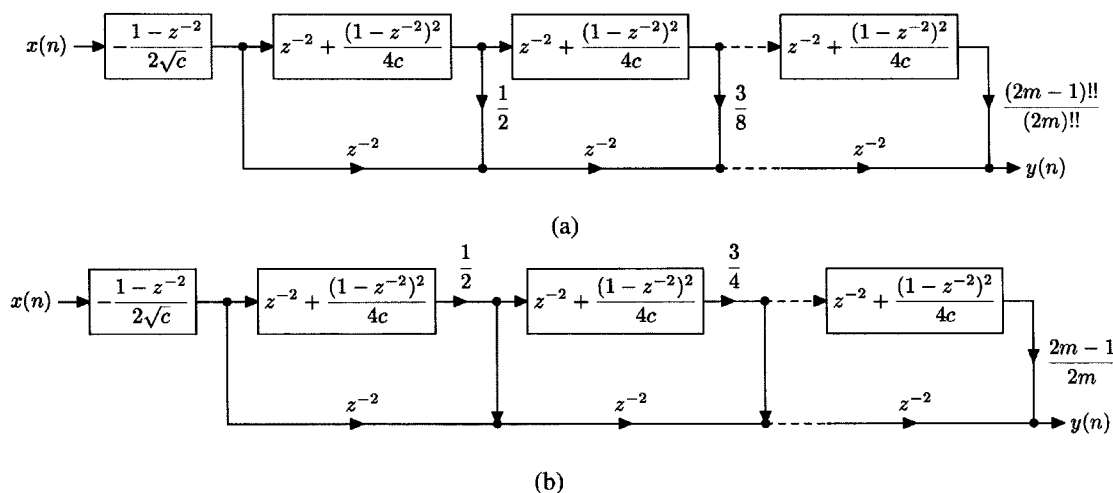
$$\tfrac{1}{2} < c \leq 1. \qquad (6)$$

Fig. 1. Implementation structures for the Case 3 FIR HTs. The Case 4 filters are obtained by substituting the z^{-1} for z^{-2} in each block. (a) Original structure derived from power series expansion. (b) Modified structure.

To design the FIR HTs, $\sin \omega$ or $\sin (\omega/2)$ will be substituted for x. In the following sections, we will discuss these two cases separately. If $\sin \omega$ or $\sin (\omega/2)$ is substituted for x, we obtain the Case 3 or Case 4 FIR filters, respectively. Detailed investigation on designing and implementing these filters is provided in each sections.

A. Design of Case 3 FIR Hilbert Transformers

To obtain the Case 3 FIR filters, we substitute $\sin \omega$ for x in (5) and truncate the series up to the first M terms. The signum function is now approximated by a sinusoid power series and expressed as

$$\text{sgn}\,\omega \approx \frac{\sin \omega}{\sqrt{c}} \left[1 + \sum_{m=1}^{M} \frac{(2m-1)!!}{(2m)!!} \left(1 - \frac{\sin^2 \omega}{c} \right)^m \right].$$
(7)

Multiplying (7) by $-j$ and substituting $(z - z^{-1})/2j$ for $\sin \omega$, the transfer functions for the zero phase Case 3 FIR HTs are expressed as

$$H_{\text{HT}}(z; c) = -\frac{z - z^{-1}}{2\sqrt{c}} \left\{ 1 + \sum_{m=1}^{M} \frac{(2m-1)!!}{(2m)!!} \right.$$
$$\left. \cdot \left[1 + \frac{1}{c} \left(\frac{z - z^{-1}}{2} \right)^2 \right]^m \right\}.$$
(8)

To obtain the causal transfer functions, $H_{\text{HT}}(z; c)$ is multiplied by z^{-2M-1}, and the resultant transfer functions of the Case 3 FIR HTs of the $(4M+2)$th-order are represented by

$$\tilde{H}_{\text{HT}}(z; c) = -\frac{1 - z^{-2}}{2\sqrt{c}} \left\{ z^{-2M} + \sum_{m=1}^{M} \frac{(2m-1)!!}{(2m)!!} z^{-2(M-m)} \right.$$
$$\left. \cdot \left[z^{-2} + \frac{1}{c} \left(\frac{1 - z^{-2}}{2} \right)^2 \right]^m \right\}.$$
(9)

The expanding center of c in x-domain is associated with another expanding center α_c in the ω-domain. The relation between c and α_c is represented as

$$c = \sin^2 \alpha_c.$$

Since c is restricted by $1/2 < c < 1$, α_c is within the range of $[\pi/4, \pi/2]$. That is, the ideal frequency response is approximated well within middle frequency band.

In order to investigate the realization of the transfer function, denote $S_1(z) = -(1 - z^{-2})/2\sqrt{c}$, $C_1(z) = z^{-2} + (1 - z^{-2})^2/4c$, and $a(m) = (2m - 1)!!/(2m)!!$ for $m = 1, 2, \ldots, M$. The transfer functions $\tilde{H}_{\text{HT}}(z; c)$ are rewritten as

$$\tilde{H}_{\text{HT}}(z; c)$$
$$= S_1(z) \left[z^{-2M} + \sum_{m=1}^{M} a(m) z^{-2(M-m)} C_1^m(z) \right].$$
(10)

Based on (10), an implementation structure is shown in Fig. 1(a). This structure consists of M building blocks of $C_1(z)$ and one block of $S_1(z)$. There are $(6M+2)$ delay elements and M weighting coefficients for this realization scheme. However, there may be extra M multiplications in the M blocks of $C_1(z)$ if we do not pose any limit on c. Therefore, the number of multiplications are $2M$ for a $2M$th-order FIR HTs. However, for some special choices of c, the multiplications may be halved since $C_1(z)$ may be realized without any multiplication. A useful property is that the weighting coefficients are independent of the filter order. Accordingly, the higher order filter can be obtained by cascading a lower order filter with $C_1(z)$ without changing the low order filter coefficients.

A modified realization structure is obtained based on the observation that the weighting coefficients $a(m)$ can be written as

$$a(m) = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2m-1}{2m}.$$

That is, the weighting coefficients $a(m)$ are products of simple rational numbers. According to this observation, this leads to an implementation structure shown in Fig. 1(b). The dynamic
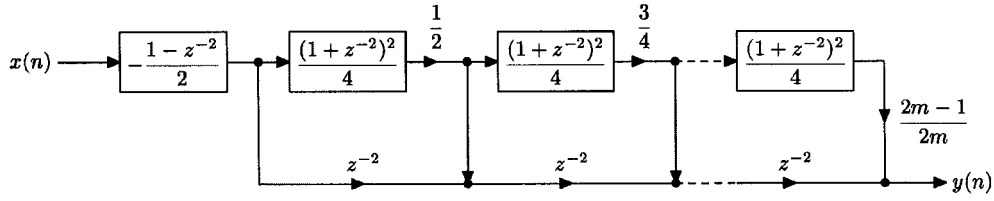
Fig. 2.   Efficient implementation structure of the Case 3 MF FIR HTs with maximal flatness at $\omega = \pi/2$. Case 4 MF FIR HTs with maximal flatness at $\omega = \pi$ are obtained by substituting the z^{-1} for z^{-2}.

range of weighting coefficients are now reduced. In fact, the coefficients are all within the range of $[0.5, 1]$.

The special case of $c = 1$ reduces to a very simple transfer function of the Case 3 FIR HTs. The transfer function is expressed as

$$\tilde{H}_{\mathrm{HT}}(z; 1) = -\frac{1 - z^{-2}}{2} \left[z^{-2M} + \sum_{m=1}^{M} \cdot \frac{(2m-1)!!}{(2m)!!} z^{-2(M-m)} \left(\frac{1 + z^{-2}}{2} \right)^{2m} \right]. \quad (11)$$

There are $(6M + 2)$ delay elements and M multipliers for implementation. Refer to the frequency responses of the MF Case 3 FIR HTs derived in [4]

$$A(\omega) = \sum_{i=0}^{N/2-1} \frac{(N-1)!! 2^{i-N/2+1}}{(2i+1)!!(N/2-i-1)!} \cdot \sin^{2i+1} \omega (1 - \sin^2 \omega)^{N/2-i-1}$$

and the magnitude responses of our proposed FIR HTs in (11) are represented as

$$\hat{H}_{\mathrm{HT}}(\omega; 1) = \sin \omega \left[1 + \sum_{m=1}^{M} \frac{(2m-1)!!}{(2m)!!} (1 - \sin^2 \omega)^m \right]$$

it is easy to show the following property.

Property 1: $A(\omega) = \hat{H}_{\mathrm{HT}}(\omega; 1)$ for $N = 2M + 2$.

That is, this special case is equivalent to the MF Case 3 FIR HTs proposed in [4].

B. Design of Case 4 FIR Hilbert Transformers

The Case 4 FIR filters are obtained by substituting $\sin(\omega/2)$ for x in (5) and truncate the resultant series up to the first M terms. The signum function is now approximated by

$$\mathrm{sgn}\omega \approx \frac{\sin(\omega/2)}{\sqrt{c}} \cdot \left\{ 1 + \sum_{m=1}^{M} \frac{(2m-1)!!}{(2m)!!} \left[1 - \frac{\sin^2(\omega/2)}{c} \right]^m \right\}. \quad (12)$$

Multiplying this equation by $-j$ and substituting $(z^{1/2} - z^{-1/2})/2j$ for $\sin(\omega/2)$, the transfer functions for the Case 4 FIR HTs are expressed by

$$G_{\mathrm{HT}}(z; c) = -\frac{z^{1/2} - z^{-1/2}}{2\sqrt{c}} \left\{ 1 + \sum_{m=1}^{M} \frac{(2m-1)!!}{(2m)!!} \cdot \left[1 + \frac{1}{c} \left(\frac{z^{1/2} - z^{-1/2}}{2} \right)^2 \right]^m \right\}. \quad (13)$$

To obtain the causal transfer functions, $G_{\mathrm{HT}}(z; c)$ are multiplied by $z^{-M-1/2}$. Therefore, the transfer functions of the $(2M + 1)$th-order Case 4 FIR HTs are represented by

$$\tilde{G}_{\mathrm{HT}}(z; c) = -\frac{1 - z^{-1}}{2\sqrt{c}} \left\{ z^{-M} + \sum_{m=1}^{M} \frac{(2m-1)!!}{(2m)!!} \cdot z^{-(M-m)} \left[z^{-1} + \frac{1}{c} \left(\frac{1 - z^{-1}}{2} \right)^2 \right]^m \right\}. \quad (14)$$

The relation between c and the expanding center β_c in the frequency domain is expressed by

$$c = \sin^2 (\beta_c/2).$$

Since c is restricted by $1/2 < c < 1$, β_c is within the range of $[\pi/2, \pi]$. Therefore, the ideal frequency response is approximated well within the high frequency band.

The implementation is similar to that of the Case 3 filter except different building blocks. Denote $S_2(z) = -(1 - z^{-1})/2\sqrt{c}$, $C_2(z) = z^{-1} + (1 - z^{-1})^2/4c$, and $a(m) = (2m - 1)!!/(2m)!!$ for $m = 1, 2, \ldots, M$. The transfer functions $\tilde{G}_{\mathrm{HT}}(z; c)$ are rewritten as

$$\tilde{G}_{\mathrm{HT}}(z; c) = S_2(z) \left[z^{-M} + \sum_{m=1}^{M} a(m) z^{-(M-m)} C_2^m(z) \right]. \quad (15)$$

Equation (15) implies that $(3M + 1)$ delay elements and M weighting coefficients are needed for implementation. There may be extra M multiplications in the M blocks of $C_2(z)$ for the general choice of c. Therefore, the number for multiplications are $2M$ for the Mth block FIR HTs and it needs more multipliers than the direct form. However, for some special choices of c, the number of multipliers may be halved since $C_2(z)$ may be realized without any multiplication. The weighting coefficients are identical to the ones in the Case 3 HTs. Accordingly, a higher order filter can still be constructed by cascading a lower order filter and $C_2(z)$ without altering the low order filter coefficients. A modified implementation with reduced dynamic range is shown in Fig. 2.

The special case of $c = 1$ leads to a simple transfer function of the Case 4 FIR HTs. The transfer function is expressed as

$$\tilde{G}_{\mathrm{HT}}(z; 1) = -\frac{1 - z^{-1}}{2} \left[z^{-M} + \sum_{m=1}^{M} \frac{(2m-1)!!}{(2m)!!} \cdot z^{-(M-m)} \left(\frac{1 + z^{-1}}{2} \right)^{2m} \right]. \quad (16)$$

There are $(3M + 1)$ delay elements and M multipliers needed for implementation. This special case is also equivalent to the MF Case 4 FIR HTs at $\omega = \pi$ proposed in [4].

III. DESIGN OF MAXIMALLY FLAT DIGITAL DIFFERENTIATORS

The ideal frequency response of the DDs is

$$H_{\text{ID}}(\omega) = j\omega, \qquad -\pi \le \omega \le \pi. \tag{17}$$

The design of the DDs is similar to the design of the HTs. The filters with frequency response $H_{\text{ID}}(\omega)/j = \omega$ are designed and the resultant filters are multiplied by j to obtain the DDs. The Fourier series of ω is written by [21]

$$-\frac{1}{2}\omega = \sum_{m=1}^{\infty} (-1)^m m^{-1} \sin(m\omega), \qquad -\pi < \omega < \pi. \tag{18}$$

The weighting coefficients are equivalent to the ideal impulse response of the DDs.

The proposed design of DDs by power series is based on the relation between the trigonometric functions and their inverse functions. All the designed DDs are MF FIR filters which may appear in open literature reviewed in Section I, but the power series approach reveals some interesting structures which are useful for designing efficient implementation structures.

A. Design of Case 4 Maximally Linear FIR Differentiators at $\omega = 0$

It may be a natural approach for the design of the DDs to use the relationship between $\sin \omega$ and $\arcsin \omega$, since $\arcsin \sin k\omega = k\omega$ over some frequency band where k is equal to 1 or $1/2$ in this paper. The power series of the inverse sine function can be expressed by [24]:

$$\arcsin x = x + \sum_{m=1}^{\infty} \frac{(2m-1)!!}{(2m)!!(2m+1)} x^{2m+1}, \qquad x^2 \le 1. \tag{19}$$

Substituting $\sin \omega$ for x and truncating the series up to first M terms, we obtain the following approximation for ω

$$\omega \approx \sin \omega + \sum_{m=1}^{M} \frac{(2m-1)!!}{(2m)!!(2m+1)} \sin^{2m+1} \omega. \tag{20}$$

In [13], the authors designed a first-order DD for the low frequency band by expressing the FIR DDs as a sum of the power of $\sin \omega$. The weighting coefficients are explicitly solved and the transfer functions could be written by (20). The authors also pointed out the relationship between the design of the DDs and the series expansion of the inverse sine function. The frequency response of this filter exhibits good linearity near $\omega = 0$. However, the approximation to ω expressed in (20) is valid only in the range of $[-\pi/2, \pi/2]$. We can find out this fact by considering the composite function of $\arcsin \sin \omega$ in the range of $[0, \pi/2]$. Note that $\arcsin x$ is increasing for $-1 \le x \le 1$ when ω is increased from 0 to $\pi/2$, $\sin \omega$ is increasing from 0 to 1 and $\arcsin \sin \omega$ is increasing from 0 to $\pi/2$. However, when ω is increased from $\pi/2$ to π, $\sin \omega$ is decreasing from 1 to 0 and thus $\arcsin \sin \omega$ is decreasing from $\pi/2$ to 0. That is, $\arcsin \sin \omega$

is approximated to ω in $[0, \pi/2)$ rather than the full frequency band by the power series expressed in (20).

To design the fullband DDs, we may substitute $\sin(\omega/2)$ for x in (19) and use the equality of

$$\omega = 2\arcsin \sin(\omega/2).$$

Then the approximation to ω is expressed by

$$\omega \approx 2\sin \frac{\omega}{2} \left[1 + \sum_{m=1}^{M} \frac{(2m-1)!!}{(2m)!!(2m+1)} \sin^{2m} \frac{\omega}{2} \right]. \tag{21}$$

The associated causal transfer function is expressed by

$$\tilde{H}_{\text{D1}}(z) = (1 - z^{-1}) \left\{ z^{-M} + \sum_{m=1}^{M} \frac{(2m-1)!!}{(2m)!!(2m+1)} \right.$$
$$\left. \cdot z^{-(M-m)} \left[-\frac{(1-z^{-1})^2}{4} \right]^m \right\}. \tag{22}$$

The DDs with the transfer functions expressed in (22) are Case 4 FIR filters of the $(2M + 1)$th order. There are $(3M + 1)$ delay elements and M multipliers needed to implement the DDs expressed in (22). However, there is a modified implementation structure with a new set of weighting coefficients expressed as

$$h'_{\text{D1}}(m) = \frac{h_{\text{D1}}(m)}{h_{\text{D1}}(m-1)} = \frac{(2m-1)^2}{2m(2m+1)}$$

where $h_{\text{D1}}(m) = (2m-1)!!/[(2m)!!(2m+1)]$ are the weighting coefficients of the original implementation. The block diagram of the modified structure is shown in Fig. 3.

B. Design of Case 3 Maximally Linear FIR Differentiators at $\omega = 0$

The Case 3 MF FIR DDs are designed by expanding a different function into the power series from the original inverse sine function discussed in Section III-A. This prototype function is expressed by

$$\frac{2\arcsin x}{\sqrt{1-x^2}} = 2x \left[1 + \sum_{m=1}^{\infty} \frac{(2m)!!}{(2m+1)!!} x^{2m} \right]. \tag{23}$$

In [23], the above equality was established by changing variable in the *Gregory series*. In [13], the function

$$\frac{2\arcsin x}{\sqrt{1-x^2}}$$

was a intermediate result to expand $\arcsin^2 x$ into power series. However, (23) may be derived by differentiating the power series of $\arcsin^2 x$ with respect to x and is available in some mathematical handbooks, such as [24].

To design the DDs by using (23), we substitute $\sin(\omega/2)$ for x and rearrange some terms. After truncating (23) up to the first M terms, we obtain the following approximating formula of ω as

$$\omega \approx \sin \omega \left[1 + \sum_{m=1}^{M} \frac{(2m)!!}{(2m+1)!!} \left(\frac{1-\cos \omega}{2} \right)^m \right]. \tag{24}$$
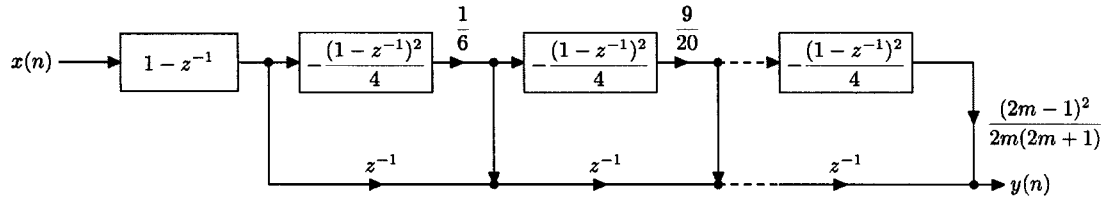
Fig. 3.   Efficient implementation structure of the Case 4 MF FIR DDs with maximal linearity at $\omega = 0$.
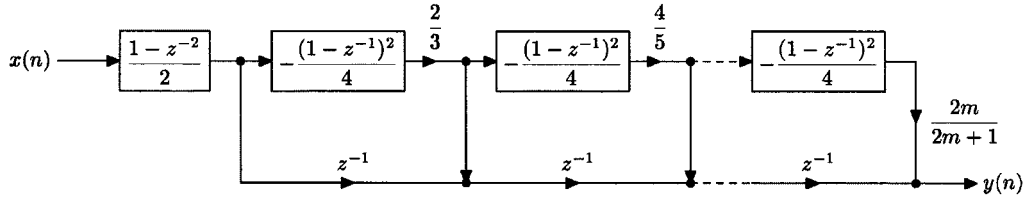

Fig. 4.   Efficient implementation structure of the Case 3 MF FIR DDs with maximal linearity at $\omega = 0$.

The corresponding causal transfer function is expressed as

$$\tilde{H}_{\text{D2}}(z) = \frac{1 - z^{-2}}{2} \left\{ z^{-M} + \sum_{m=1}^{M} \frac{(2m)!!}{(2m+1)!!} \right.$$
$$\left. \cdot z^{-(M-m)} \left[-\frac{(1-z^{-1})^2}{4} \right]^m \right\}. \quad (25)$$

The DDs with the transfer functions expressed in (25) are the Case 3 FIR filters of the $(2M+2)$th-order. There are $(3M+2)$ delay elements and M multipliers needed to implement the DDs according to (25). These DDs are equivalent to the MF Case 3 FIR DDs explicitly expressed in [8]. Consider the frequency responses of the MF Case 3 FIR DDs derived in [8]

$$C(\omega) = \sum_{n=1}^{N} \frac{2(-1)^{n+1} N! N!}{n(N+n)!(N-n)!} \sin(n\omega)$$

and the magnitude responses of our proposed FIR DDs in (24) are represented as

$$\hat{H}_{\text{D2}}(\omega) = \sum_{m=0}^{M-1} \frac{2^{2m} m! m!}{(2m+1)!} \sin^{2m}(\omega/2)$$

we can prove the following property.

Property 2: $C(\omega) = \hat{H}_{\text{D2}}(\omega)$ for $N = M$.

Like the modified structure of the HTs described in Section II a new modified implementation is obtained with the weighting coefficients expressed as

$$h'_{\text{D2}}(m) = \frac{h_{\text{D2}}(m)}{h_{\text{D2}}(m-1)} = \frac{2m}{2m+1}$$

where $h_{\text{D2}}(m) = (2m)!!/(2m+1)!!$ are the weighting coefficients of the original implementation structure. The block diagram of the modified structure is shown in Fig. 4.

IV. Implementation of the Maximally Flat FIR Fractional Delayers

The ideal transfer function of the FDs with desired delay τ is

$$H_{\text{IFD}}(z) = z^{-\tau}. \quad (26)$$

In [16], the function in (26) was expanded into the binomial series, truncated, and then simplified algebraically to obtain the impulse response. If the power series of $H_{\text{IFD}}(z)$ is truncated to the first $M+1$ terms, it may be written as [24]

$$H_{\text{FD}}(z) = 1 + \sum_{m=1}^{M} \frac{\tau(\tau-1)\cdots(\tau-m+1)}{m!} (z^{-1} - 1)^m. \quad (27)$$

This transfer function of the FIR FDs is an intermediate and simplified result in [16]. In fact, the transfer function expressed in (27) is equivalent to the MF FIR FDs explicitly obtained in [18] with maximal flatness at $\omega = 0$. To show this fact, consider the transfer function of the Nth-order MF FIR FDs derived in [18]

$$D(z) = \sum_{n=0}^{N} \frac{\tau(\tau-1)\cdots(\tau-N)}{\tau-n} \frac{(-1)^{N-n}}{n!(N-n)!} z^{-n}. $$

The following property holds.

Property 3: $D(z) = H_{\text{FD}}(z)$ for $M = N$.

The transfer function in (27) reveals a new structure of the FIR FDs. The major advantage of this structure is that the weighting coefficients are independent of the filter order. If we represent the weighting coefficients as cumulative products of rational numbers and write the coefficients as

$$h'_{\text{FD}}(m) = \frac{h_{\text{FD}}(m)}{h_{\text{FD}}(m-1)} = \frac{\tau-m+1}{m}$$

where $h_{\text{FD}}(m) = \tau(\tau-1)\cdots(\tau-m+1)/m!$ are the original weighting coefficients, an efficient implementation structure similar to the ones of the MF HTs and DDs is obtained. This realization structure is shown in Fig. 5.

V. Coefficient Quantization Effects on Filter Performance

Fig. 6 shows the magnitude responses of the Case 3 FIR HTs expressed in (9) with $M = 5$, 10, 15, and 30. The expanding center c is $\sin^2 0.4\pi$ for the four filters. These filters exhibit good approximation around $\omega = 0.5\pi$. The bandwidth is increasing by increasing M. The effect of the expanding center
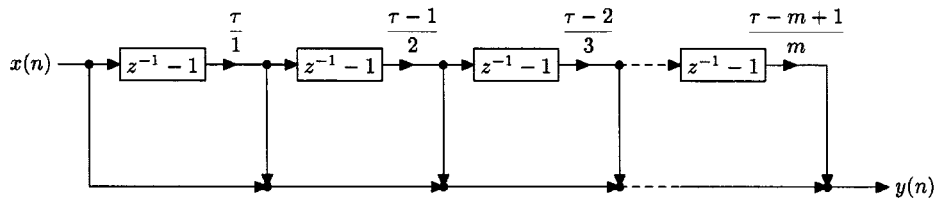
Fig. 5. Efficient implementation structure for the MF FIR FDs with maximal flatness at $\omega = 0$.
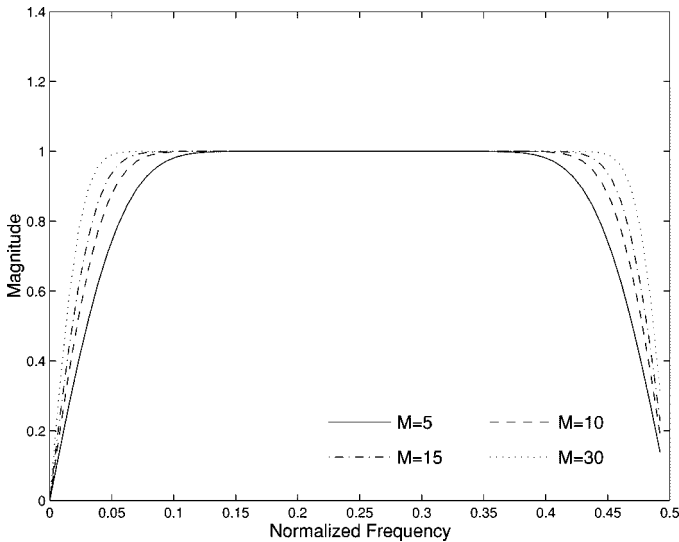

Fig. 6. Magnitude responses of the Case 3 FIR HTs with various M and expanding center $c = \sin^2 0.4\pi$.
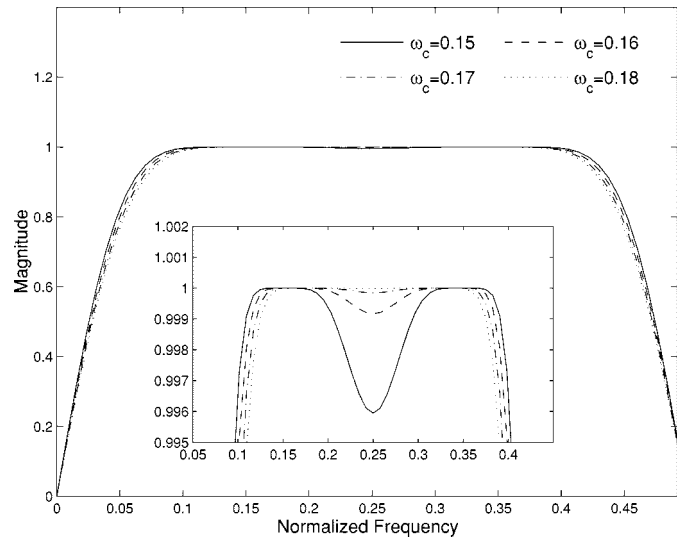

Fig. 7. Magnitude responses of the Case 3 FIR HTs with various expanding centers and $M = 5$.

c on the frequency response of the Case 3 FIR HTs is shown in Fig. 7. The filters of $M = 5$ with $\alpha_c = 0.3\pi$, 0.32π, 0.34π, and 0.36π are designed. These central frequencies are associated to the expanding centers at $c = \sin^2 0.3\pi$, $\sin^2 0.32\pi$, $\sin^2 0.34\pi$, and $\sin^2 0.36\pi$. The inset shows the detailed frequency responses around $\omega = 0.5\pi$. These plots indicate that the bandwidth is increasing but the magnitude response at $\omega = 0.5\pi$ is degraded for smaller α_c.

Fig. 8 shows the rounding effect of the coefficients on the magnitude responses of the Case 3 FIR HTs of $M = 10$ and $c = 1$ which is corresponding to the MF FIR HTs around $\omega = 0.5\pi$. The closed-form impulse response of the direct form structure is obtained in [4]. The original and the modified implementations derived by the power series expansion are shown in Fig. 1. If the impulse response of the direct form is rounded to 4 bits, the corresponding magnitude response is not flat for the direct form implementation. However, the magnitude responses remain flat after rounding the weighting coefficients for the proposed structures. The inset shows the detailed magnitude responses around $\omega = 0.5\pi$. After rounding to 4 bits, the magnitude response of the original structure is better than the one of the modified structure for the larger bandwidth. The performance evaluation on the rounding effect is evaluated by the *root mean squared error* (RMSE) σ over a frequency band of $[\omega_1, \omega_2]$ and expressed as

$$\sigma^2 = \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} |H(\omega) - H_d(\omega)|^2 \, d\omega \qquad (28)$$
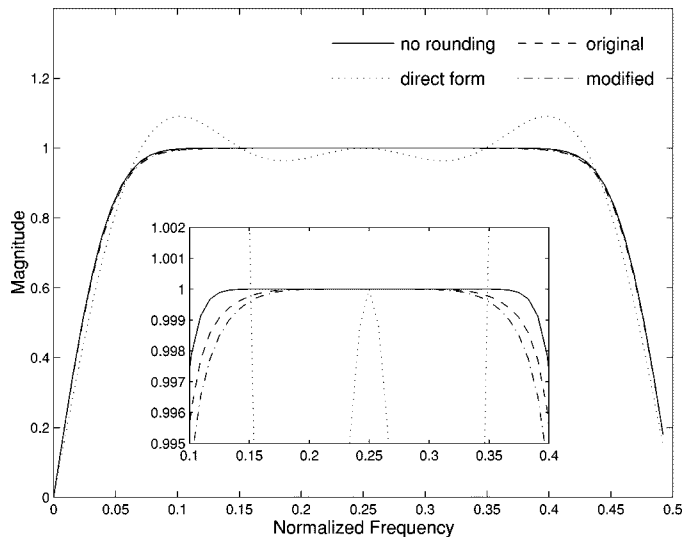

Fig. 8. Magnitude responses of the Case 3 FIR HTs of $M = 10$ and $c = 1$ with coefficients rounded to 4 bits.

where $H(\omega)$ is the designed frequency response and $H_d(\omega)$ is the desired one. Fig. 9 shows the RMSEs for the MF FIR HTs with various M. The RMSEs are evaluated within the interval of $[0.2\pi, 0.8\pi]$, i.e., $\omega_1 = 0.2\pi$ and $\omega_2 = 0.8\pi$ in (28). The impulse responses of the direct form and the weighting coefficients of the proposed structures are all rounded to 4 or 8 bits. The curves indicate the errors of the direct form implementation are larger than those of the proposed structures. The original implementation is better than the modified structure. The error is
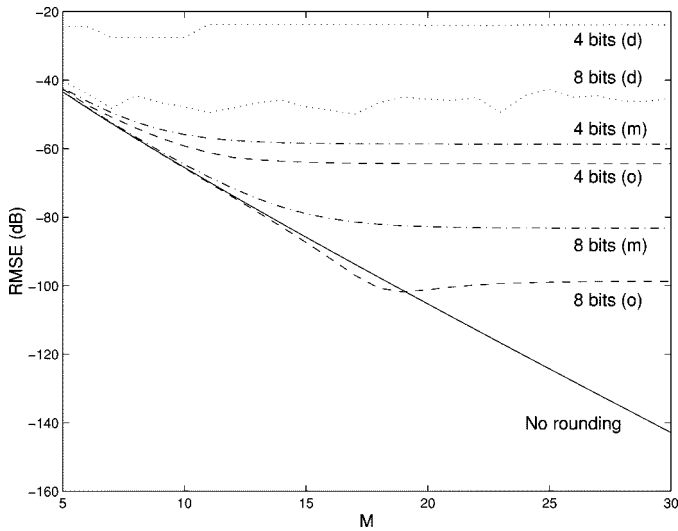
Fig. 9. RMSEs of the Case 3 FIR HTs of different implementation structures versus M. (d), (o), and (m) stand for the error curves of the direct form, original implementation derived by the power series, and the modified structure, respectively.
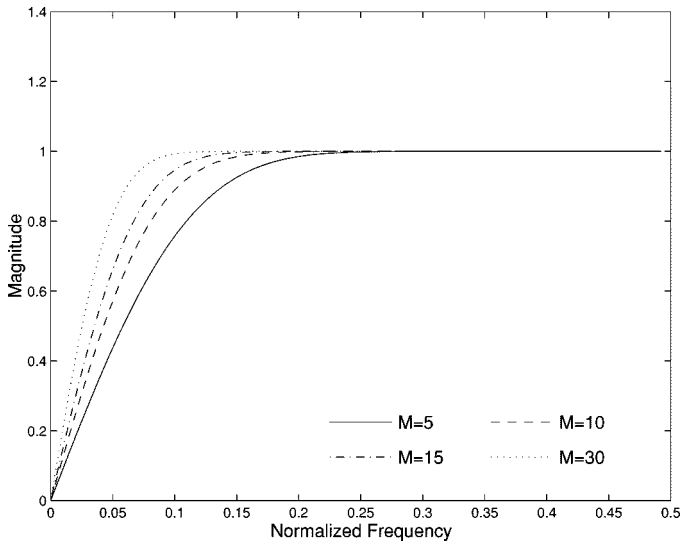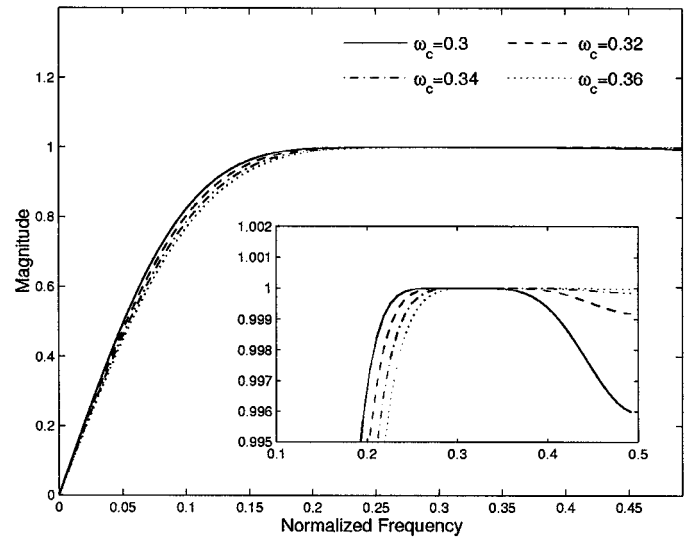

Fig. 11. Magnitude responses of the Case 4 FIR HTs with various expanding centers and $M = 5$.

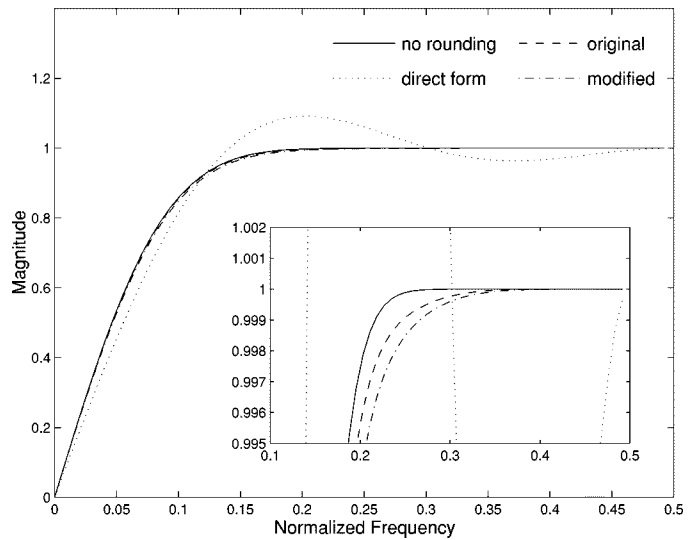Fig. 10. Magnitude responses of the Case 4 FIR HTs with various M and expanding center $c = \sin^2(0.75\pi/2)$.

Fig. 12. Magnitude responses of the Case 4 FIR HTs of $M = 10$ and $c = 1$ with coefficients rounded to 4 bits.

not decreasing for increasing M since some weighting coefficients are too small to be represented by finite bits.

Fig. 10 shows the magnitude responses of the Case 3 FIR HTs expressed in (14) with $M = 5$, 10, 15, and 30. The expanding center c is chosen to be equal to $\sin^2(0.75\pi/2)$ for the four filters. These filters exhibit good approximation around $\omega = \pi$. The bandwidth is increasing by increasing M. The effect of the expanding center c on the frequency response of the Case 4 FIR HTs is illustrated in Fig. 11. The filters of $M = 5$ with $\alpha_c = 0.6\pi$, 0.64π, 0.68π, and 0.72π are designed. These frequencies are associated to the expanding centers of $c = \sin^2 0.3\pi$, $\sin^2 0.32\pi$, $\sin^2 0.34\pi$, and $\sin^2 0.36\pi$, respectively. The inset shows the detailed frequency responses around $\omega = \pi$. These plots indicate that the bandwidth is increasing but the magnitude response at $\omega = \pi$ is degraded for a small α_c.

Fig. 12 shows the rounding effects on the magnitude response of the Case 4 FIR HTs of $M = 10$ with $c = 1$ which is cor-

responding to the MF FIR HT around $\omega = \pi$. The impulse response of the direct form implementation is also obtained in [4]. The original and the modified implementation structures derived by the power series expansion are shown in Fig. 1. Similar to the Case 3 HTs, the corresponding magnitude response is not flat if the impulse response is rounded to 4 bits in the direct form implementation. The magnitude responses keep flat after rounding the weighting coefficients for the proposed structure. The inset shows the detailed magnitude responses around $\omega = 0.5\pi$. Fig. 13 shows the RMSEs for the MF FIR HTs with various M. The RMSEs are evaluated within the interval of $[0.35\pi, \pi]$. The impulse responses of the direct form filter and the weighting coefficients of the proposed structures are rounded to 4 or 8 bits. These curves indicate the errors of the direct forms are larger than those of the proposed ones. The original implementations are better than the modified ones.

Fig. 14 shows the rounding effect on the magnitude response of the Case 4 FIR DDs of $M = 10$ which are associated to the
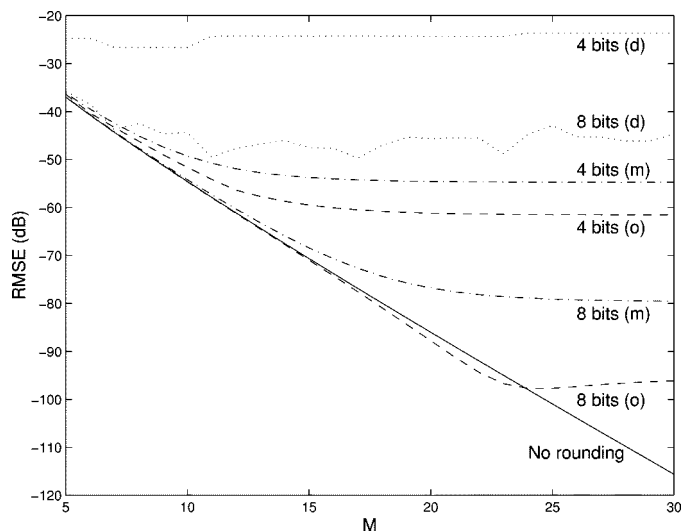
Fig. 13. RMSEs of the Case 4 FIR HTs of different implementation structures vs. M. (d), (o) and (m) stand for the error curves of the direct form, original implementation derived by the power series and the modified structure, respectively.
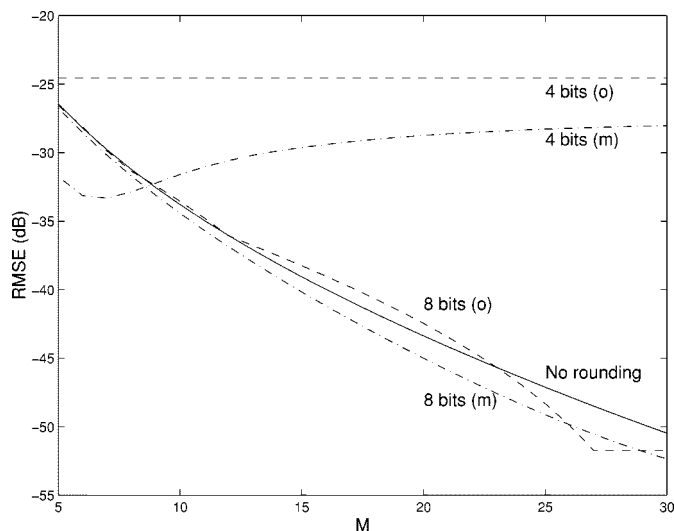

Fig. 15. RMSEs of the Case 4 FIR DDs of different implementation structures v.s. M. (o) and (m) stand for the error curves of the original implementation derived by the power series, and the modified structure, respectively.
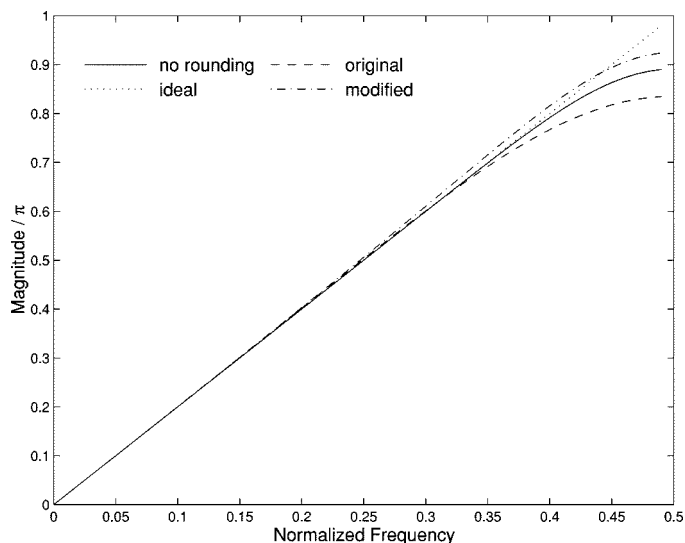

Fig. 14. Magnitude responses of the Case 4 FIR DDs of $M = 10$ with coefficients rounded to 4 bits.
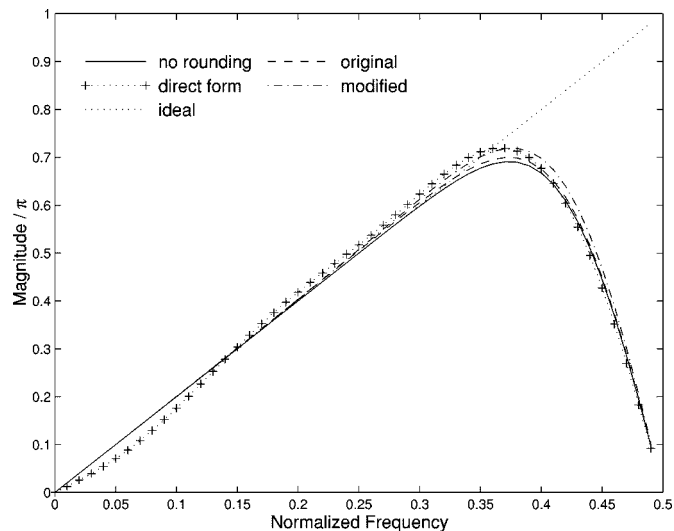

Fig. 16. Magnitude responses of the Case 3 FIR DDs of $M = 10$ with coefficients rounded to 4 bits.

MF FIR DDs with maximal flatness at $\omega = 0$. The magnitude response of the modified structure with coefficients rounded to 4 bits is more closer to the ideal one than the magnitude response of the original structure. Fig. 15 shows the RMSEs for the MF FIR DDs of various M. The RMSEs are evaluated within the interval of $[0, 0.9\pi]$ where the coefficients of the original and the modified implementation structures are rounded to 4 or 8 bits. The errors of the original structures are larger than those of the modified structures. It is interesting that the RMSEs of the modified structure with coefficients rounded to 8 bits are smaller than those without rounding.

Fig. 16 shows the rounding effect on the magnitude responses of the Case 3 FIR DDs of $M = 10$ which are corresponding to the MF FIR DDs with maximal linearity at $\omega = 0$. The closed-form impulse response of the direct form structure is obtained in [8]. If the impulse response of the direct form is rounded to 4 bits, the corresponding magnitude response is

not linear around $\omega = 0$. However, the magnitude responses keep linear after rounding the weighting coefficients for the proposed structures. Fig. 17 shows the RMSEs for the MF FIR DDs of various M. The RMSEs are evaluated in the interval of $[0, 0.75\pi]$. The impulse responses of the direct form structure and the weighting coefficients of the proposed structures are rounded to 4 or 8 bits. The curves indicate the errors of the direct forms are larger than those of the proposed structures.

VI. CONCLUSIONS

A novel design of the FIR HTs, DDs, and FDs based on power series expansion is proposed and the filter coefficients are explicitly expressed in closed form. The weighting coefficients are able to be represented by products of simple rational numbers. Based on the simple forms of the weighting coefficients, efficient implementation structures are derived. The magnitude responses of the filters are less sensitive than those of the direct
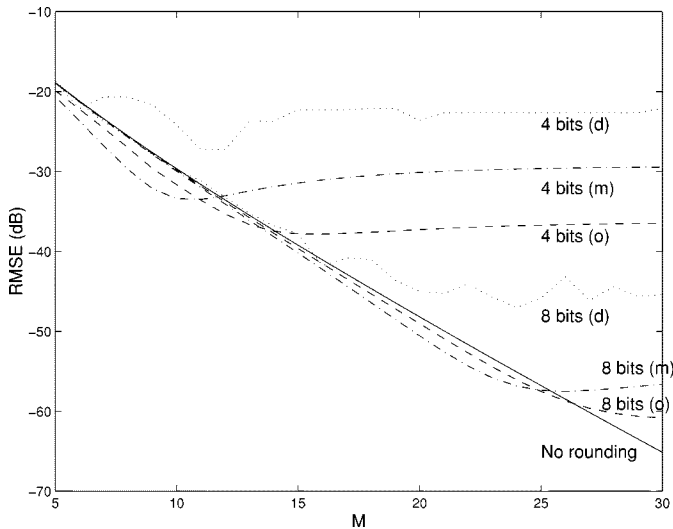
Fig. 17.   RMSEs of the Case 3 FIR DDs of different implementation structures v.s. M. (d), (o), and (m) stand for the error curves of the direct form, original implementation derived by the power series and the modified structure, respectively.

form structures when the filter coefficients are rounded to finite bits. The finite word length effect on the filer performance is investigated and evaluated by the RMSE criterion.

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Soo-Chang Pei (S'71–M'86–SM'89–F'00) was born in Soo-Auo, Taiwan in 1949. He received the B.S.E.E. degree from National Taiwan University, Taipei, Taiwan, in 1970 and the M.S.E.E. and Ph.D. degrees from the University of California, Santa Barbara, in 1972 and 1975, respectively.

He was an Engineering Officer in the Chinese Navy Shipyard from 1970 to 1971. From 1971 to 1975, he was a Research Assistant at the University of California, Santa Barbara. He was the Professor and Chairman in the EE Department of the Tatung Institute of Technology Taipei, Taiwan, and the National Taiwan University, from 1981 to 1983 and 1995 to 1998, respectively. He is currently a Professor in the EE Department at the National Taiwan University. His research interests include digital signal processing, image processing, optical information processing, and laser holography.

Dr. Pei received the National Sun Yet-Sen Academic Achievement Award in Engineering in 1984, the Distinguished Research Award from the National Science Council from 1990 to 1998, the Outstanding Electrical Engineering Professor Award from the Chinese Institute of Electrical Engineering in 1998, and the Academic Achievement Award in Engineering from the Ministry of Education in 1998. He has been President of the Chinese Image Processing and Pattern Recognition Society in Taiwan from 1996 to 1998, and is a member of Eta Kappa Nu and the Optical Society of America.

Peng-Hua Wang received the Bachelors and Master's degrees in electrical engineering from the National Tsing-Hua University, Hsinchu, Taiwan, R. O. C., in 1992 and 1994, respectively. He is currently working toward the Ph.D. degree in electrical engineering at the National Taiwan University, Taipei, Taiwan, R. O. C..

His current research interests are in the area of digital filter design, fast algorithm for signal processing, and VLSI architecture design.