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Closed-Form Direct Kinematics Solution of a New Parallel Minimanipulator

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Closed-Form Direct Kinematics Solution of a New Parallel Minimanipulator

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Abstract

Closed-form direct kinematics solution of a new three-limbed six-degree-of-freedom minimanipulator is presented. Five-bar linkages and inextensible limbs are used in synthesis of the minimanipulator to improve its positional resolution and stiffness. All of the minimanipulator actuators are base-mounted. Kinematic inversion is used to reduce the direct kinematics of the minimanipulator to an eighth-degree polynomial in the square of tangent of half-angle between one of the limbs and the moving platform. Hence, the maximum number of assembly configurations for the minimanipulator is sixteen. Furthermore, it is proved that the sixteen solutions are eight pairs of reflected configurations with respect to the plane passing through the lower ends of the three limbs. A numerical example is also presented and the results are verified by an inverse kinematics analysis.

1 Introduction

In recent years, many researchers have shown a great deal of interest in studying parallel manipulators. Such mechanisms are most suitable for applications in which the requirements for accuracy, rigidity, load-to-weight ratio, and load distribution are more important than the need for a large workspace.

The famous Stewart platform (Stewart, 1965) is probably the first six-degree-of-freedom (six-DOF) parallel mechanism which has been studied in the literature. It consists of a moving platform and a base which are connected by means of six independent limbs. Many researchers have considered the Stewart platform as a robot manipulator (e.g., Fichter and

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MacDowell, 1980; Hunt, 1983; Yang and Lee, 1984; Fichter, 1986). Other types of six-DOF parallel manipulators have been introduced and studied in the literature (e.g., Kohli et al., 1988; Hudgens and Tesar, 1988; Tsai and Tahmasebi, 1991a).

Waldron and Hunt (1987) demonstrated that kinematic behavior of parallel mechanisms has many inverse characteristics to that of serial mechanisms. For example, direct kinematics of a parallel manipulator is much more difficult than its inverse kinematics; whereas, for a serial manipulator, the opposite is true. Dieudonne et al. (1972) applied Newton-Raphson's method to solve direct kinematics of a motion simulator identical to the Stewart platform. Behi (1988) used a similar technique to numerically solve the direct kinematics problem of a parallel mechanism similar to the Stewart platform. Griffis and Duffy (1989) as well as Nanua et al. (1990) studied direct kinematics of special cases of Stewart platform, in which pairs of spherical joints are concentric on either the platform or both the base and the platform. They were able to reduce the problem to an eighth-degree polynomial in the square of a single variable (total degree of sixteen). However, as mentioned by Griffis and Duffy (1989), pairs of concentric spherical joints may very well present design problems. Lin et al. (1990) solved direct kinematics of another class of Stewart platforms, in which there are two concentric spherical joints on the base and two more concentric spherical joints on the platform. The latter class of Stewart platforms suffer from lack of symmetry and concentric spherical joints are still needed in their construction. Other researcher have also been able to obtain closed-form solutions for other special forms of the Stewart platform (e.g., Innocenti and Parenti-Castelli, 1990; Parenti-Castelli and Innocenti, 1990). It is worth mentioning that, to the best of our knowledge, no one has yet been able to obtain a closed-form direct kinematics solution for the general Stewart platform with six independent limbs. Recently, Raghavan (1991) used a numerical technique known as polynomial continuation to show that there are forty solutions for the direct kinematics of the Stewart platform of general geometry. Murthy and Waldron (1990a, 1990b) have been able to relate the direct kinematics of some parallel mechanisms to the inverse kinematics of their serial dual mechanisms.

In this paper, closed-form direct kinematic solution for a six-DOF parallel minimanipulator is presented. The minimanipulator is one of the high-stiffness and high-resolution mechanisms introduced by Tsai and Tahmasebi (1991a, 1991b) for fine position and force control in a hybrid serial-parallel manipulator system. It will be shown that direct kinematics of the minimanipulator involves solving an eighth-degree polynomial in the square of a single variable.

2 The Minimanipulator

Let subscript i in this section and the rest of this work represent numbers 1, 2, and 3 in a cyclic manner. The minimanipulator contains three inextensible limbs, P_iR_i, as shown in Figure 1. The lower end of each limb is connected to a simplified five-bar linkage driver and can be moved freely on the base plate. The desired minimanipulator motion is obtained by moving the lower ends of its three limbs on its base plate. Two-DOF universal joints connect the limbs to the moving platform. The lower ends of the limbs are connected to the drivers through three more universal joints. Note that one of the axes of the upper universal joint is collinear with the limb, while the other axis of the upper universal joint as well as one of the axes of the lower universal joint are always perpendicular to the limb. This arrangement is kinematically equivalent to a limb with a spherical joint at its lower end and a revolute joint at its upper end, as shown in Figure 2. The minimanipulator drivers are shown in Figure 3. Point C_i is the output point of a driver. At point D_i, there is an actuator on each side of the base plate to drive links D_iA_i and D_iB_i. The simplified five-bar drivers are completely symmetric. That is

$$\left|\overline{D_iA_i}\right| = \left|\overline{D_iB_i}\right| = a \tag{1}$$

$$\left|\overline{A_iC_i}\right| = \left|\overline{B_iC_i}\right| = b \tag{2}$$

As a result, coordination between actuator rotations can be easily accomplished. Namely, angular displacement of an output point C_i is obtained by equal actuator rotations, and its radial displacement is obtained by equal and opposite actuator rotations.

Simplified five-bar linkages and inextensible limbs are used to improve positional resolution and stiffness of the minimanipulator. Since the minimanipulator actuators are base-mounted; higher payload capacity, smaller actuator sizes, and lower power dissipation can be obtained. In addition, to achieve even load distribution, the minimanipulator is made completely symmetric.

3 Direct Kinematics

Figure 4 shows a parallel mechanism whose limbs are kinematically equivalent to those of an actual minimanipulator. The equivalent limb configuration will be used for analysis, because the spherical-and-revolute limb (Figure 2) is easier to analyze than the universal-and-universal limb (Figure 1). The lower ends of the limbs (points R_1, R_2, and R_3) are connected to two-DOF drivers. The upper end of the limbs (points P_1, P_2, and P_3) are connected to the platform through revolute joints. Note that the joint axes at points P_1, P_2, and P_3 are parallel to lines P_2P_3, P_1P_3, and P_1P_2, respectively.

Let us define the fixed base reference frame (XYZ) and the moving platform reference frame (UVW) in detail. The base reference frame is shown in Figure 3. The origin of the base reference frame (point O) is placed at the centroid of triangle $D_1D_2D_3$. The positive X-axis is parallel to and points in the direction of vector $\overline{D_2D_3}$. The positive Y-axis points from point O to point D_1. The Z-axis is defined by the right-hand-rule. Similarly, the origin of the platform reference frame (point G) is placed at the centroid of triangle $P_1P_2P_3$ (see Figure 4). The positive U-axis is parallel to and points in the direction of vector $\overline{P_2P_3}$. The positive V-axis points from point O to point P_1. The W-axis is defined by the right-hand-rule. To keep the minimanipulator symmetric, both triangles $D_1D_2D_3$ and $P_1P_2P_3$ are made equilateral.

In Figure 5, θ_i and ϕ_i (driver input angles) are the angles from the positive X-axis to the vectors $\overline{D_iB_i}$ and $\overline{D_iA_i}$, respectively, measured about the positive Z-axis. D_iB_i and D_iA_i are the the input links of the driver and vector $\overline{D_iX_{i,1}}$ is parallel to the positive X-axis. In the direct kinematics problem, angles θ_i and ϕ_i (i=1,2,3) are given. Coordinates of points P_1, P_2, and P_3 in the base reference frame are to be found. These three points completely define the platform position and orientation.

3.1 Coordinates of Point $\mathbf{R_i}$

Let the distance from point O to each one of points D_1, D_2, and D_3 (see Figure 3) be equal to d. Then, X and Y coordinates of point D_i in the base reference frame are given by

$$X_{D,i} = d\cos\beta_i \tag{3}$$

$$Y_{D,i} = d\sin\beta_i \tag{4}$$

where

$$\beta_i = \frac{\pi}{2} + (i-1)\frac{2\pi}{3} \tag{5}$$

As shown in Figure 5, let the angle from the positive X-axis to vector $\overline{A_iB_i}$ measured about the positive Z-axis be equal to γ_i. Then

$$\gamma_i = Atan2(\sin\theta_i - \sin\phi_i, \cos\theta_i - \cos\phi_i) \tag{6}$$

where $Atan2$ is a single-valued function which calculates arc tangent of $(\sin\theta_i - \sin\phi_i)/(\cos\theta_i - \cos\phi_i)$ but uses the signs of both $(\sin\theta_i - \sin\phi_i)$ and $(\cos\theta_i - \cos\phi_i)$ to determine the quadrant in which γ_i lies. In addition, let the angle from vector $\overline{A_iB_i}$ to vector $\overline{A_iC_i}$ measured about the positive Z-axis be equal to δ_i. Then

$$\delta_i = \pm\left|\cos^{-1}\frac{\sqrt{2a^2 - 2a^2\cos(\phi_i - \theta_i)}}{2b}\right| \tag{7}$$

Note that vector $\overline{A_iX_{i,2}}$ is parallel to the positive X-axis in Figure 5. The X and Y coordinates of a driver output point C_i can be found from the following equations.

$$X_{C,i} = X_{D,i} + a\cos\phi_i + b\cos(\gamma_i + \delta_i) \tag{8}$$

$$Y_{C,i} = Y_{D,i} + a\sin\phi_i + b\sin(\gamma_i + \delta_i) \tag{9}$$

The above expressions for X and Y coordinates of C_i are similar to those presented by Bajpai and Roth (1986). Mathematically, there are two solutions for each output point C_i. However, only one of these solutions is feasible. The other solution can be obtained only if the simplified five-bar driver is disassembled and reassembled. A simple algorithm can be developed to determine the feasible solution for any given input angles.

Vector $\overline{C_iR_i}$ is perpendicular to the base plate. Once the coordinates of point C_i are known, the coordinates of point R_i can be found easily by adding a constant to the Z-component.

3.2 Angles between the Limbs and the Platform

As shown in Figure 6, let η_i be the angle from vector $\overline{GP_i}$ to vector $\overline{P_iR_i}$ measured about a vector $\overline{j_i}$ which is collinear with the axis of the revolute joint at point P_i and points in the direction of vector $\overline{P_{i+2}P_{i+1}}$.[1] Also, let α_i be the angle from the positive U-axis to vector $\overline{GP_i}$ about the positive W-axis. Angle α_i can be found from

$$\alpha_i = \frac{\pi}{2} + (i-1)\frac{2\pi}{3} \tag{10}$$

In the next step, angle η_i will be found from a kinematic inversion. If r is the length of each limb, the coordinates of point R_i in the moving reference frame UVW are

$$\begin{aligned} U_{R,i} &= rC\alpha_i C\eta_i + U_{P,i} && (11)\\ V_{R,i} &= rS\alpha_i C\eta_i + V_{P,i} && (12)\\ W_{R,i} &= -rS\eta_i && (13) \end{aligned}$$

where $C\alpha_i = \cos\alpha_i, S\alpha_i = \sin\alpha_i, C\eta_i = \cos\eta_i$, and $S\eta_i = \sin\eta_i$. Note that $U_{P,i}$ and $V_{P,i}$ are known quantities which can be found from

$$\begin{aligned} U_{P,i} &= pC\alpha_i && (14)\\ V_{P,i} &= pS\alpha_i && (15) \end{aligned}$$

where p is the length of vector $\overline{GP_i}$ (see Figure 4).

The coordinates of point R_i in the base reference frame have already been found and the length of vector $\overline{R_iR_{i+1}}$, which is denoted by l_{i+2}, is known. We can write

$$(U_{R,i} - U_{R,i+1})^2 + (V_{R,i} - V_{R,i+1})^2 + (W_{R,i} - W_{R,i+1})^2 = l_{i+2}^2 \tag{16}$$

[1]The subscripts are cyclic. If $i = 2$, $i+2$ represents 1. If $i = 3$, $i+1$ and $i+2$ represent 1 and 2, respectively.

Substituting equations (11) - (13) into equation (16) and simplifying, we get

$$A_i S\eta_i S\eta_{i+1} + B_i C\eta_i C\eta_{i+1} + D_i C\eta_i + D_i C\eta_{i+1} + E_i = 0 \qquad (17)$$

where $A_i = -2r^2, B_i = r^2, D_i = 3rp$, and $E_i = 2r^2 + 3p^2 - l_{i+2}^2$. Let $t_i = \tan(\eta_i/2)$. Then $\cos\eta_i = (1 - t_i^2)/(1 + t_i^2)$, and $\sin\eta_i = 2t_i/(1 + t_i^2)$. Substituting these expressions into equation (17) and simplifying, we get

$$F_i t_i^2 t_{i+1}^2 + G_i t_i^2 + G_i t_{i+1}^2 + I_i t_i t_{i+1} + J_i = 0 \qquad (18)$$

where $F_i = 3r^2 + 3p^2 - 6rp - l_{i+2}^2, G_i = r^2 + 3p^2 - l_{i+2}^2, I_i = -8r^2$, and $J_i = 3r^2 + 3p^2 + 6rp - l_{i+2}^2$. Equation (18) can be rewritten, for $i = 1, 2, 3$, in the following forms

$$(F_1 t_1^2 + G_1)t_2^2 + (I_1 t_1)t_2 + (G_1 t_1^2 + J_1) = 0 \qquad (19)$$

$$(F_2 t_3^2 + G_2)t_2^2 + (I_2 t_3)t_2 + (G_2 t_3^2 + J_2) = 0 \qquad (20)$$

$$(F_3 t_1^2 + G_3)t_3^2 + (I_3 t_1)t_3 + (G_3 t_1^2 + J_3) = 0 \qquad (21)$$

We can think of equations (19) and (20) as two equations in the variable t_2. Vanishing of their eliminant results in the following equation (Salmon, 1964). Refer to Appendix A for details.

$$N_1 t_3^4 + N_2 t_3^3 + N_3 t_3^2 + N_4 t_3 + N_5 = 0 \qquad (22)$$

where

$$\begin{aligned} N_1 &= Q_1 t_1^4 + Q_2 t_1^2 + Q_3 \\ N_2 &= Q_4 t_1^3 + Q_5 t_1 \\ N_3 &= Q_6 t_1^4 + Q_7 t_1^2 + Q_8 \\ N_4 &= Q_9 t_1^3 + Q_{10} t_1 \\ N_5 &= Q_{11} t_1^4 + Q_{12} t_1^2 + Q_{13} \end{aligned}$$

where Q's are constants which can be found from the following relationships.

$$Q_1 = F_1^2 G_2^2 - 2F_1 F_2 G_1 G_2 + F_2^2 G_1^2$$

$$
\begin{aligned}
Q_2 &= 2F_2^2G_1J_1 + F_2G_2I_1^2 + 2F_1G_1G_2^2 - 2F_1F_2G_2J_1 - 2F_2G_1^2G_2 \\
Q_3 &= F_2^2J_1^2 + G_1^2G_2^2 - 2F_2G_1G_2J_1 \\
Q_4 &= -F_1G_2I_1I_2 - F_2G_1I_1I_2 \\
Q_5 &= -F_2I_1I_2J_1 - G_1G_2I_1I_2 \\
Q_6 &= 2F_1^2G_2J_2 + F_1G_1I_2^2 + 2F_2G_1^2G_2 - 2F_1F_2G_1J_2 - 2F_1G_1G_2^2 \\
Q_7 &= F_2I_1^2J_2 + 4F_1G_1G_2J_2 + F_1I_2^2J_1 + 4F_2G_1G_2J_1 + G_1^2I_2^2 + G_2^2I_1^2 - \\
&\quad 2F_1F_2J_1J_2 - 2F_2G_1^2J_2 - 2F_1G_2^2J_1 - 2G_1^2G_2^2 \\
Q_8 &= 2G_1^2G_2J_2 + 2F_2G_2J_1^2 + G_1I_2^2J_1 - 2F_2G_1J_1J_2 - 2G_1G_2^2J_1 \\
Q_9 &= -F_1I_1I_2J_2 - G_1G_2I_1I_2 \\
Q_{10} &= -G_1I_1I_2J_2 - G_2I_1I_2J_1 \\
Q_{11} &= F_1^2J_2^2 + G_1^2G_2^2 - 2F_1G_1G_2J_2 \\
Q_{12} &= 2F_1G_1J_2^2 + G_2I_1^2J_2 + 2G_1G_2^2J_1 - 2F_1G_2J_1J_2 - 2G_1^2G_2J_2 \\
Q_{13} &= G_1^2J_2^2 + G_2^2J_1^2 - 2G_2G_1J_1J_2
\end{aligned}
$$

Similarly, we can think of equations (21) and (22) as two equations in the variable t_3. Vanishing of their eliminant results in (Salmon, 1964)

$$
\begin{aligned}
&-N_5S_1[(N_3S_1 - N_1S_3)(S_2^2 - S_1S_3) - S_1S_2(-N_2S_3 + N_3S_2 + N_4S_1) + \\
&S_1^2(N_4S_2 + N_5S_1)] + (N_2S_1 - N_1S_2)[N_5S_2(S_2^2 - S_1S_3) + \\
&S_3^2(-N_2S_3 + N_3S_2 + N_4S_1) - S_2(N_4S_2 + N_5S_1)S_3] + \\
&N_4S_1[-S_1S_3(-N_2S_3 + N_3S_2 + N_4S_1) + S_2S_3(N_3S_1 - N_1S_3) + N_5S_1^2S_2] - \\
&(N_3S_1 - N_1S_3)[S_3^2(N_3S_1 - N_1S_3) - S_1(N_4S_2 + N_5S_1)S_3 + N_5S_1S_2^2] = 0 \qquad (23)
\end{aligned}
$$

where S's are the coefficients of equation (21). Namely, $S_1 = F_3t_1^2 + G_3, S_2 = I_3t_1$, and $S_3 = G_3t_1^2 + J_3$. Equation (23) is an eighth-degree polynomial in square of t_1 (see Appendix B for details). It follows that there are sixteen possible solutions for each angle η_i and therefore sixteen solutions for the direct kinematics of the minimanipulator. The elimination procedure described above is similar to those used by Nanua et al. (1990), Griffis and Duffy

(1989), and Innocenti and Parenti-Castelli (1990) for solving direct kinematics of special forms of the Stewart platform.

3.3 Coordinates of Point G

In the next step, coordinates of point G in the base reference frame will be found. Let h_i be the length of vector $\overline{GR_i}$. Using equations (11) - (13), we can write

$$h_i^2 = (rC\alpha_i C\eta_i + U_{P,i})^2 + (rS\alpha_i C\eta_i + V_{P,i})^2 + r^2 S\eta_i^2 \tag{24}$$

Sixteen η_i values result in only eight values for h_i^2. Such values can be used in the following equation.

$$(X_G - X_{R,i})^2 + (Y_G - Y_{R,i})^2 + (Z_G - k)^2 = h_i^2 \tag{25}$$

where k is the Z-coordinate of point R_i. If equation (25) is written for $i = 1$ and $i = 2$ and the latter equation is subtracted from the former one, the following relation is obtained.

$$2(X_{R,2} - X_{R,1})X_G + 2(Y_{R,2} - Y_{R,1})Y_G = h_1^2 - h_2^2 - X_{R,1}^2 - Y_{R,1}^2 + X_{R,2}^2 + Y_{R,2}^2 \tag{26}$$

Similarly, If equation (25) is written for $i = 1$ and $i = 3$ and the latter equation is subtracted from the former one, the following relation is obtained.

$$2(X_{R,3} - X_{R,1})X_G + 2(Y_{R,3} - Y_{R,1})Y_G = h_1^2 - h_3^2 - X_{R,1}^2 - Y_{R,1}^2 + X_{R,3}^2 + Y_{R,3}^2 \tag{27}$$

The above two equations can be solved for X_G and Y_G. Eight h_i^2 values result in eight solutions for (X_G, Y_G). These solutions can be substituted back in equation (25) to find Z_G. For each (X_G, Y_G) solution, two values for Z_G are obtained. Therefore, there are eight pairs of (sixteen) solutions for point G. In each pair, one element is the mirror image of the other element with respect to the plane passing through points R_1, R_2, and R_3 ($Z = k$ plane).

3.4 Coordinates of Point $\mathrm{P_i}$

In the final step, coordinates of point P_i in the base reference frame are found. We can write

$$(X_{P,i} - X_{R,i})^2 + (Y_{P,i} - Y_{R,i})^2 + (Z_{P,i} - k)^2 = r^2 \tag{28}$$

$$(X_{P,i} - X_G)^2 + (Y_{P,i} - Y_G)^2 + (Z_{P,i} - Z_G)^2 = p^2 \tag{29}$$

Subtracting equation (29) from equation (28) results in

$$(X_G - X_{R,i})X_{P,i} + (Y_G - Y_{R,i})Y_{P,i} + (Z_G - k)Z_{P,i} = \frac{1}{2}(r^2 - p^2 + X_G^2 + Y_G^2 + Z_G^2 - X_{R,i}^2 - Y_{R,i}^2 - k^2) \tag{30}$$

Vector $\overline{P_iR_i}$ is perpendicular to vector $\overline{P_{i+1}P_{i+2}}$. Hence

$$(X_{P,i} - X_{R,i})(X_{P,i+1} - X_{P,i+2}) + (Y_{P,i} - Y_{R,i})(Y_{P,i+1} - Y_{P,i+2}) + (Z_{P,i} - k)(Z_{P,i+1} - Z_{P,i+2}) = 0 \tag{31}$$

Also, vector $\overline{GP_i}$ is perpendicular to vector $\overline{P_{i+1}P_{i+2}}$. Thus

$$(X_{P,i} - X_G)(X_{P,i+1} - X_{P,i+2}) + (Y_{P,i} - Y_G)(Y_{P,i+1} - Y_{P,i+2}) + (Z_{P,i} - Z_G)(Z_{P,i+1} - Z_{P,i+2}) = 0 \tag{32}$$

Subtracting equation (31) from equation (32) results in

$$(X_{R,i} - X_G)X_{P,i+1} + (X_G - X_{R,i})X_{P,i+2} + (Y_{R,i} - Y_G)Y_{P,i+1} + (Y_G - Y_{R,i})Y_{P,i+2} + (k - Z_G)Z_{P,i+1} + (Z_G - k)Z_{P,i+2} = 0 \tag{33}$$

Point G is the centroid of triangle $P_1P_2P_3$. Hence

$$X_{P,1} + X_{P,2} + X_{P,3} = 3X_G \tag{34}$$

$$Y_{P,1} + Y_{P,2} + Y_{P,3} = 3Y_G \tag{35}$$

$$Z_{P,1} + Z_{P,2} + Z_{P,3} = 3Z_G \tag{36}$$

Equations (30) and (33) can be written for $i = 1, 2, 3$. These equations plus equations (34), (35), and (36) represent nine linearly-independent equations in nine unknowns, $X_{P,1}$, $Y_{P,1}$, $Z_{P,1}$, $X_{P,2}$, $Y_{P,2}$, $Z_{P,2}$, $X_{P,3}$, $Y_{P,3}$, and $Z_{P,3}$. Solving this system of equations results in the following expressions for the unknowns.

$$X_{P,i} = [3(Y_{R,i+2} - Y_{R,i+1})(Z_G - k)^2 + 2(Y_{R,i+2} - Y_{R,i+1})Y_{R,i}^2 + (Y_{R,i+2}^2 - 6Y_G Y_{R,i+2} - Y_{R,i+1}^2 + 6Y_G Y_{R,i+1} + X_{R,i+2}^2 - 6X_G X_{R,i+2} - X_{R,i+1}^2 + 6X_G X_{R,i+1})Y_{R,i} - Y_{R,i+1}Y_{R,i+2}^2 + (Y_{R,i+1}^2 + 3Y_G^2 + 2X_{R,i}^2 + X_{R,i+1}^2 - 6X_G X_{R,i+1} + 3X_G^2 - 3r^2 + 3p^2)Y_{R,i+2} + (-3Y_G^2 - 2X_{R,i}^2 - X_{R,i+2}^2 + 6X_G X_{R,i+2} - 3X_G^2 + 3r^2 - 3p^2)Y_{R,i+1} / \{4[X_{R,i+1}(Y_{R,i} - Y_{R,i+2}) - X_{R,i+2}Y_{R,i} + X_{R,i}Y_{R,i+2} + (X_{R,i+2} - X_{R,i})Y_{R,i+1}]\} \tag{37}$$

$$Y_{P,i} = [3(\mathrm{X_{R,i+1}} - \mathrm{X_{R,i+2}})(\mathrm{Z_G} - k)^2 + 2(\mathrm{X_{R,i+1}} - \mathrm{X_{R,i+2}})\,\mathrm{Y_{R,i}}^2 + (\mathrm{X_{R,i+1}} - \mathrm{X_{R,i}})\,\mathrm{Y_{R,i+2}}^2 +$$
$$6(\mathrm{X_{R,i}} - \mathrm{X_{R,i+1}})\,\mathrm{Y_G}\,\mathrm{Y_{R,i+2}} + (\mathrm{X_{R,i}} - \mathrm{X_{R,i+2}})\,\mathrm{Y_{R,i+1}}^2 + 6(\mathrm{X_{R,i+2}} - \mathrm{X_{R,i}})\,\mathrm{Y_G}\,\mathrm{Y_{R,i+1}} +$$
$$3(\mathrm{X_{R,i+1}} - \mathrm{X_{R,i+2}})\,\mathrm{Y_G}^2 + 2(\mathrm{X_{R,i+1}} - \mathrm{X_{R,i+2}})\,\mathrm{X_{R,i}}^2 +$$
$$(-\mathrm{X_{R,i+2}}^2 + 6\,\mathrm{X_G}\,\mathrm{X_{R,i+2}} + \mathrm{X_{R,i+1}}^2 - 6\,\mathrm{X_G}\,\mathrm{X_{R,i+1}})\,\mathrm{X_{R,i}} + \mathrm{X_{R,i+1}}\,\mathrm{X_{R,i+2}}^2 +$$
$$(-\mathrm{X_{R,i+1}}^2 - 3\,\mathrm{X_G}^2 + 3\,r^2 - 3\,p^2)\,\mathrm{X_{R,i+2}} + 3(\mathrm{X_G}^2 - r^2 + p^2)\,\mathrm{X_{R,i+1}}]/$$
$$\{4[\mathrm{X_{R,i+1}}(\mathrm{Y_{R,i}} - \mathrm{Y_{R,i+2}}) - \mathrm{X_{R,i+2}}\,\mathrm{Y_{R,i}} + \mathrm{X_{R,i}}\,\mathrm{Y_{R,i+2}} + (\mathrm{X_{R,i+2}} - \mathrm{X_{R,i}})\,\mathrm{Y_{R,i+1}}]\} \qquad (38)$$

$$Z_{P,i} = \{[5(\mathrm{X_{R,i+1}} - \mathrm{X_{R,i+2}})\,\mathrm{Y_{R,i}} + (5\,\mathrm{X_{R,i}} - 2\,\mathrm{X_{R,i+1}} - 3\,\mathrm{X_G})\,\mathrm{Y_{R,i+2}} +$$
$$(-5\,\mathrm{X_{R,i}} + 2\,\mathrm{X_{R,i+2}} + 3\,\mathrm{X_G})\,\mathrm{Y_{R,i+1}} + 3(\mathrm{X_{R,i+2}} - \mathrm{X_{R,i+1}})\,\mathrm{Y_G}]\,(\mathrm{Z_G} - k)^2 +$$
$$4k\,[(\mathrm{X_{R,i+1}} - \mathrm{X_{R,i+2}})\,\mathrm{Y_{R,i}} + (\mathrm{X_{R,i}} - \mathrm{X_{R,i+1}})\,\mathrm{Y_{R,i+2}} + (\mathrm{X_{R,i+2}} - \mathrm{X_{R,i}})\,\mathrm{Y_{R,i+1}}]\,(\mathrm{Z_G} - k) +$$
$$2[(\mathrm{X_{R,i+1}} - \mathrm{X_G})\,\mathrm{Y_{R,i+2}} + (\mathrm{X_G} - \mathrm{X_{R,i+2}})\,\mathrm{Y_{R,i+1}} + (\mathrm{X_{R,i+2}} - \mathrm{X_{R,i+1}})\,\mathrm{Y_G}]\,\mathrm{Y_{R,i}}^2 +$$
$$[(\mathrm{X_{R,i+1}} - \mathrm{X_G})\,\mathrm{Y_{R,i+2}}^2 + 6(\mathrm{X_G} - \mathrm{X_{R,i+1}})\,\mathrm{Y_G}\,\mathrm{Y_{R,i+2}} + (\mathrm{X_G} - \mathrm{X_{R,i+2}})\,\mathrm{Y_{R,i+1}}^2 +$$
$$6(\mathrm{X_{R,i+2}} - \mathrm{X_G})\,\mathrm{Y_G}\,\mathrm{Y_{R,i+1}} + 5(\mathrm{X_{R,i+1}} - \mathrm{X_{R,i+2}})\,\mathrm{Y_G}^2 + (\mathrm{X_{R,i+1}} - \mathrm{X_G})\,\mathrm{X_{R,i+2}}^2 +$$
$$(-\mathrm{X_{R,i+1}}^2 + \mathrm{X_G}^2 + r^2 - p^2)\,\mathrm{X_{R,i+2}} + \mathrm{X_G}\,\mathrm{X_{R,i+1}}^2 + (-\mathrm{X_G}^2 - r^2 + p^2)\,\mathrm{X_{R,i+1}}]\,\mathrm{Y_{R,i}} +$$
$$[(\mathrm{X_G} - \mathrm{X_{R,i}})\,\mathrm{Y_{R,i+1}} + (\mathrm{X_{R,i}} - \mathrm{X_{R,i+1}})\,\mathrm{Y_G}]\,\mathrm{Y_{R,i+2}}^2 +$$
$$[(\mathrm{X_{R,i}} - \mathrm{X_G})\,\mathrm{Y_{R,i+1}}^2 + (-\mathrm{X_{R,i}} + 4\,\mathrm{X_{R,i+1}} - 3\,\mathrm{X_G})\,\mathrm{Y_G}^2 + 2(\mathrm{X_{R,i+1}} - \mathrm{X_G})\,\mathrm{X_{R,i}}^2 +$$
$$(\mathrm{X_{R,i+1}}^2 - 6\,\mathrm{X_G}\,\mathrm{X_{R,i+1}} + 5\,\mathrm{X_G}^2 - r^2 + p^2)\,\mathrm{X_{R,i}} - \mathrm{X_G}\,\mathrm{X_{R,i+1}}^2 + 2(2\,\mathrm{X_G}^2 - r^2 + p^2)\,\mathrm{X_{R,i+1}} -$$
$$3\,\mathrm{X_G}^3 + 3(r^2 - p^2)\,\mathrm{X_G}]\,\mathrm{Y_{R,i+2}} + (\mathrm{X_{R,i+2}} - \mathrm{X_{R,i}})\,\mathrm{Y_G}\,\mathrm{Y_{R,i+1}}^2 +$$
$$[(\mathrm{X_{R,i}} - 4\,\mathrm{X_{R,i+2}} + 3\,\mathrm{X_G})\,\mathrm{Y_G}^2 + 2(\mathrm{X_G} - \mathrm{X_{R,i+2}})\,\mathrm{X_{R,i}}^2 +$$
$$(-\mathrm{X_{R,i+2}}^2 + 6\,\mathrm{X_G}\,\mathrm{X_{R,i+2}} - 5\,\mathrm{X_G}^2 + r^2 - p^2)\,\mathrm{X_{R,i}} + \mathrm{X_G}\,\mathrm{X_{R,i+2}}^2 +$$
$$2(-2\,\mathrm{X_G}^2 + r^2 - p^2)\,\mathrm{X_{R,i+2}} + 3\,\mathrm{X_G}^3 + 3(p^2 - r^2)\,\mathrm{X_G}]\,\mathrm{Y_{R,i+1}} +$$
$$3(\mathrm{X_{R,i+2}} - \mathrm{X_{R,i+1}})\,\mathrm{Y_G}^3 + [2(\mathrm{X_{R,i+2}} - \mathrm{X_{R,i+1}})\,\mathrm{X_{R,i}}^2 +$$
$$(\mathrm{X_{R,i+2}}^2 - 6\,\mathrm{X_G}\,\mathrm{X_{R,i+2}} - \mathrm{X_{R,i+1}}^2 + 6\,\mathrm{X_G}\,\mathrm{X_{R,i+1}})\,\mathrm{X_{R,i}} - \mathrm{X_{R,i+1}}\,\mathrm{X_{R,i+2}}^2 +$$
$$(\mathrm{X_{R,i+1}}^2 + 3\,\mathrm{X_G}^2 - 3\,r^2 + 3\,p^2)\,\mathrm{X_{R,i+2}} + 3(-\mathrm{X_G}^2 + r^2 - p^2)\,\mathrm{X_{R,i+1}}]\,\mathrm{Y_G}\}/$$
$$\{4[(\mathrm{X_{R,i+1}} - \mathrm{X_{R,i+2}})\,\mathrm{Y_{R,i}} + (\mathrm{X_{R,i}} - \mathrm{X_{R,i+1}})\,\mathrm{Y_{R,i+2}} + (\mathrm{X_{R,i+2}} - \mathrm{X_{R,i}})\,\mathrm{Y_{R,i+1}}]\,(\mathrm{Z_G} - k)\} \qquad (39)$$

Note that Variables $X_{P,i}$ and $Y_{P,i}$ are functions of X_G, Y_G, and $(Z_G - k)^2$. It was shown in section 3.3 that there are only eight values for each one of these quantities. Hence, only eight values for $(X_{P,i}, Y_{P,i})$ exist. The above equation for $Z_{P,i}$ can be rewritten as

$$Z_{P,i} = k + \Upsilon_1(Z_G - k) + \Upsilon_2(Z_G - k)^{-1} \tag{40}$$

where Υ_1 and Υ_2 are functions of (X_G, Y_G) and the coordinates of points R_1, R_2, and R_3. As a result, there are eight values for each variable Υ_1 and Υ_2. In section 3.3, we showed that there are eight pairs of opposite and equal values for $(Z_G - k)$. Therefore, equation (40) is equivalent to $Z_{P,i} = k \pm \Upsilon_3$ where eight values for Υ_3 exist. The preceeding discussion shows that there are eight pairs of (sixteen) solutions for $(X_{P,i}, Y_{P,i}, Z_{P,i})$. In each pair, one element is the mirror image of the other element with respect to the $Z = k$ plane.

3.5 Numerical Example

In this example, the above procedure is used to determine the coordinates of points P_1, P_2, and P_3 for a given set of driver input angles. Let the minimanipulator dimensions be

$$a = 1, b = 2, d = 1.443, p = 3.175, r = 5, k = 0.125$$

In addition, let the driver input angles (in degrees) be

$$\theta_1 = 90.0, \theta_2 = 70.0, \theta_3 = 300.0$$

$$\phi_1 = 210.0, \phi_2 = 170.0, \phi_3 = 60.0$$

Then, equation (23) reduces to

$$t_1^{16} - 35.9507t_1^{14} + 700.3896t_1^{12} - 6635.7398t_1^{10} + 21284.2651t_1^8 - 50544.2635t_1^6 + 387426.1233t_1^4 + 524669.7298t_1^2 + 226.9895 = 0 \tag{41}$$

The sixteen solutions for t_1 are

±j2.02797, ±j1.8918, ±1.6923, ±1.691, ±2.7575, ±2.8221,

3.6273±j2.0096, -3.6273±j2.0096

where j=$\sqrt{-1}$. The eight real solutions yield the values shown in Table 1 for angles η_1, η_2, and η_3 (in degrees) and the coordinates of points G, P_1, P_2, and P_3.

Table 1 - Solutions of the Sample Problem

No.	1	2	3	4
η_1	118.8422	-118.8422	118.8016	-118.8016
η_2	119.7530	-119.7530	119.7972	-119.7972
η_3	55.0319	-55.0319	-156.8897	156.8897
X_G	-1.8203	-1.8203	1.5689	1.5689
Y_G	1.6418	1.6418	0.1605	0.1605
Z_G	4.4640	-4.2140	1.2035	-0.9535
$X_{P,1}$	-2.0971	-2.0971	2.9905	2.9905
$Y_{P,1}$	4.8017	4.8017	2.5909	2.5909
$Z_{P,1}$	4.6104	-4.3604	-0.2647	0.5147
$X_{P,2}$	-4.4205	-4.4205	0.6697	0.6697
$Y_{P,2}$	-0.1808	-0.1808	-2.3918	-2.3918
$Z_{P,2}$	4.4473	-4.1973	-0.4580	0.7080
$X_{P,3}$	1.0569	1.0569	1.0464	1.0464
$Y_{P,3}$	0.3044	0.3044	0.2824	0.2824
$Z_{P,3}$	4.3342	-4.0842	4.3333	-4.0833

Table 1 - Continued

No.	5	6	7	8
η_1	140.1345	-140.1345	140.9769	-140.9769
η_2	142.3654	-142.3654	141.7002	-141.7002
η_3	44.1586	-44.1586	-140.1406	140.1406
X_G	-2.8656	-2.8656	0.5987	0.5987
Y_G	1.9494	1.9494	0.6790	0.6790
Z_G	3.2128	-2.9628	0.2659	-0.0159
$X_{P,1}$	-5.4744	-5.4744	-0.3373	-0.3373
$Y_{P,1}$	0.1892	0.1892	-1.8827	-1.8827
$Z_{P,1}$	2.7899	-2.5399	-1.3606	1.6106
$X_{P,2}$	-3.1036	-3.1036	2.0231	2.0231
$Y_{P,2}$	5.1058	5.1058	3.0836	3.0836
$Z_{P,2}$	3.4648	-3.2148	-1.2414	1.4914
$X_{P,3}$	-0.0187	-0.0187	0.1103	0.1103
$Y_{P,3}$	0.5531	0.5531	0.8359	0.8359
$Z_{P,3}$	3.3837	-3.1337	3.3996	-3.1496

The above results have been verified by performing an inverse kinematics analysis. Note that pairs of solutions for point P_i are symmetric with respect to the $Z = 0.125$ plane, as predicted.

4 Summary

In this paper, closed-form solution for direct kinematics of a new three-limbed six-degree-of-freedom minimanipulator is presented. It is shown that the maximum number of solutions for direct kinematics of the minimanipulator is sixteen. To obtain these solutions, only an eighth-degree polynomial in the square of a single variable has to be solved. It is also proved that the sixteen solutions are eight pairs of reflected configurations with respect to the plane passing through the lower ends of the three limbs. The results of a numerical example are verified by an inverse kinematics analysis.

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Appendix A

Derivation of Equation (22)

Let

$$K_1 = F_1 t_1^2 + G_1, L_1 = I_1 t_1, M_1 = G_1 t_1^2 + J_1$$

and

$$K_2 = F_2 t_3^2 + G_2, L_2 = I_2 t_3, M_2 = G_2 t_3^2 + J_2$$

then equations (19) and (20) can be written as

$$K_1 t_2^2 + L_1 t_2 + M_1 \tag{42}$$

and

$$K_2 t_2^2 + L_2 t_2 + M_2 \tag{43}$$

Multiplying equation (42) by K_2 and equation (43) by K_1, and subtracting, we get

$$(L_1 K_2 - L_2 K_1) t_2 + (M_1 K_2 - M_2 K_1) = 0 \tag{44}$$

Multiplying equation (42) by M_2 and equation (43) by M_1, subtracting, and dividing by t_2, we get

$$(K_1 M_2 - K_2 M_1) t_2 + (L_1 M_2 - L_2 M_1) = 0 \tag{45}$$

Equations (44) and (45) represent two linear equations in one unknown. Vanishing of their eliminant means

$$\begin{vmatrix} L_1 K_2 - L_2 K_1 & M_1 K_2 - M_2 K_1 \\ K_1 M_2 - K_2 M_1 & L_1 M_2 - L_2 M_1 \end{vmatrix} = 0 \tag{46}$$

Expanding equation (46) and substituting the expressions for K_1, L_1, M_1, K_2, L_2, and M_2 into the resulting equation, we get

$$\mathrm{F_1}^2\,\mathrm{G_2}^2\,\mathrm{t_1}^4\,\mathrm{t_3}^4 - 2\,\mathrm{F_1}\,\mathrm{F_2}\,\mathrm{G_1}\,\mathrm{G_2}\,\mathrm{t_1}^4\,\mathrm{t_3}^4 + \mathrm{F_2}^2\,\mathrm{G_1}^2\,\mathrm{t_1}^4\,\mathrm{t_3}^4 - 2\,\mathrm{F_1}\,\mathrm{F_2}\,\mathrm{G_2}\,\mathrm{J_1}\,\mathrm{t_1}^2\,\mathrm{t_3}^4 +$$

$$2\,\mathrm{F_2}^2\,\mathrm{G_1}\,\mathrm{J_1}\,\mathrm{t_1}^2\,\mathrm{t_3}^4 + \mathrm{F_2}\,\mathrm{G_2}\,\mathrm{I_1}^2\,\mathrm{t_1}^2\,\mathrm{t_3}^4 + 2\,\mathrm{F_1}\,\mathrm{G_1}\,\mathrm{G_2}^2\,\mathrm{t_1}^2\,\mathrm{t_3}^4 - 2\,\mathrm{F_2}\,\mathrm{G_1}^2\,\mathrm{G_2}\,\mathrm{t_1}^2\,\mathrm{t_3}^4 +$$

$$\mathrm{F_2}^2\,\mathrm{J_1}^2\,\mathrm{t_3}^4 - 2\,\mathrm{F_2}\,\mathrm{G_1}\,\mathrm{G_2}\,\mathrm{J_1}\,\mathrm{t_3}^4 + \mathrm{G_1}^2\,\mathrm{G_2}^2\,\mathrm{t_3}^4 - \mathrm{F_1}\,\mathrm{G_2}\,\mathrm{I_1}\,\mathrm{I_2}\,\mathrm{t_1}^3\,\mathrm{t_3}^3 -$$

$$F_2\,G_1\,I_1\,I_2\,t_1^{\,3}\,t_3^{\,3} - F_2\,I_1\,I_2\,J_1\,t_1\,t_3^{\,3} - G_1\,G_2\,I_1\,I_2\,t_1\,t_3^{\,3} + 2\,F_1^{\,2}\,G_2\,J_2\,t_1^{\,4}\,t_3^{\,2} -$$
$$2\,F_1\,F_2\,G_1\,J_2\,t_1^{\,4}\,t_3^{\,2} + F_1\,G_1\,I_2^{\,2}\,t_1^{\,4}\,t_3^{\,2} - 2\,F_1\,G_1\,G_2^{\,2}\,t_1^{\,4}\,t_3^{\,2} + 2\,F_2\,G_1^{\,2}\,G_2\,t_1^{\,4}\,t_3^{\,2} -$$
$$2\,F_1\,F_2\,J_1\,J_2\,t_1^{\,2}\,t_3^{\,2} + F_2\,I_1^{\,2}\,J_2\,t_1^{\,2}\,t_3^{\,2} + 4\,F_1\,G_1\,G_2\,J_2\,t_1^{\,2}\,t_3^{\,2} - 2\,F_2\,G_1^{\,2}\,J_2\,t_1^{\,2}\,t_3^{\,2} +$$
$$F_1\,I_2^{\,2}\,J_1\,t_1^{\,2}\,t_3^{\,2} - 2\,F_1\,G_2^{\,2}\,J_1\,t_1^{\,2}\,t_3^{\,2} + 4\,F_2\,G_1\,G_2\,J_1\,t_1^{\,2}\,t_3^{\,2} + G_1^{\,2}\,I_2^{\,2}\,t_1^{\,2}\,t_3^{\,2} +$$
$$G_2^{\,2}\,I_1^{\,2}\,t_1^{\,2}\,t_3^{\,2} - 2\,G_1^{\,2}\,G_2^{\,2}\,t_1^{\,2}\,t_3^{\,2} - 2\,F_2\,G_1\,J_1\,J_2\,t_3^{\,2} + 2\,G_1^{\,2}\,G_2\,J_2\,t_3^{\,2} +$$
$$2\,F_2\,G_2\,J_1^{\,2}\,t_3^{\,2} + G_1\,I_2^{\,2}\,J_1\,t_3^{\,2} - 2\,G_1\,G_2^{\,2}\,J_1\,t_3^{\,2} - F_1\,I_1\,I_2\,J_2\,t_1^{\,3}\,t_3 -$$
$$G_1\,G_2\,I_1\,I_2\,t_1^{\,3}\,t_3 - G_1\,I_1\,I_2\,J_2\,t_1\,t_3 - G_2\,I_1\,I_2\,J_1\,t_1\,t_3 + F_1^{\,2}\,J_2^{\,2}\,t_1^{\,4} -$$
$$2\,F_1\,G_1\,G_2\,J_2\,t_1^{\,4} + G_1^{\,2}\,G_2^{\,2}\,t_1^{\,4} + 2\,F_1\,G_1\,J_2^{\,2}\,t_1^{\,2} - 2\,F_1\,G_2\,J_1\,J_2\,t_1^{\,2} +$$
$$G_2\,I_1^{\,2}\,J_2\,t_1^{\,2} - 2\,G_1^{\,2}\,G_2\,J_2\,t_1^{\,2} + 2\,G_1\,G_2^{\,2}\,J_1\,t_1^{\,2} + G_1^{\,2}\,J_2^{\,2} -$$
$$2\,G_1\,G_2\,J_1\,J_2 + G_2^{\,2}\,J_1^{\,2} = 0 \qquad (47)$$

Factoring the above equation results in equation (22). The above method is introduced by Salmon (1964).

Appendix B

Derivation and Expansion of Equation (23)

Equations (22) and (21) can be rewritten in the following forms

$$N_1 t_3^4 + N_2 t_3^3 + N_3 t_3^2 + N_4 t_3 + N_5 = 0 \tag{48}$$

$$S_1 t_3^2 + S_2 t_3 + S_3 = 0 \tag{49}$$

Multiplying equation (48) by S_1 and equation (49) by $N_1 t_3^2$, and subtracting, we get

$$(N_2 S_1 - N_1 S_2) t_3^3 + (N_3 S_1 - N_1 S_3) t_3^2 + N_4 S_1 t_3 + N_5 S_1 = 0 \tag{50}$$

Multiplying equation (48) by $S_1 t_3 + S_2$ and equation (49) by $N_1 t_3^3 + N_2 t_3^2$, and subtracting, we get

$$(N_3 S_1 - N_1 S_3) t_3^3 + (N_4 S_1 + N_3 S_2 - N_2 S_3) t_3^2 + (N_5 S_1 + N_4 S_2) t_3 + N_5 S_2 = 0 \tag{51}$$

Multiplying equation (49) by t_3, we get

$$S_1 t_3^3 + S_2 t_3^2 + S_3 t_3 = 0 \tag{52}$$

We can think of equations (50), (51), (52), and (49) as four linear equations in three unknowns t_3^3, t_3^2, and t_3. Vanishing of their eliminant means

$$\begin{vmatrix} N_2 S_1 - N_1 S_2 & N_3 S_1 - N_1 S_3 & N_4 S_1 & N_5 S_1 \\ N_3 S_1 - N_1 S_3 & N_4 S_1 + N_3 S_2 - N_2 S_3 & N_5 S_1 + N_4 S_2 & N_5 S_2 \\ S_1 & S_2 & S_3 & 0 \\ 0 & S_1 & S_2 & S_3 \end{vmatrix} = 0 \tag{53}$$

Expansion of equation (53) results in equation (23). If we substitute the expressions given in section 3.2 for $S_1, S_2, S_3, N_1, N_2, N_3, N_4$, and N_5 and expand equation (23), we get

$$(-\mathrm{F_3}^2\,\mathrm{G_3}^2\,\mathrm{Q_6}^2 + 2\,\mathrm{F_3}^3\,\mathrm{G_3}\,\mathrm{Q_{11}}\,\mathrm{Q_6} + 2\,\mathrm{F_3}\,\mathrm{G_3}^3\,\mathrm{Q_1}\,\mathrm{Q_6} - \mathrm{F_3}^4\,\mathrm{Q_{11}}^2 -$$

$$2\,\mathrm{F_3}^2\,\mathrm{G_3}^2\,\mathrm{Q_1}\,\mathrm{Q_{11}} - \mathrm{G_3}^4\,\mathrm{Q_1}^2)\,\mathrm{t_1}^{16} +$$

$$(-F_3^3\,G_3\,Q_9^2 + F_3^2\,G_3\,I_3\,Q_6\,Q_9 + 2\,F_3^2\,G_3^2\,Q_4\,Q_9 + F_3^3\,I_3\,Q_{11}\,Q_9-$$
$$3\,F_3\,G_3^2\,I_3\,Q_1\,Q_9 - 2\,F_3^2\,G_3^2\,Q_6\,Q_7 + 2\,F_3^3\,G_3\,Q_{11}\,Q_7 + 2\,F_3\,G_3^3\,Q_1\,Q_7-$$
$$2\,F_3^2\,G_3\,J_3\,Q_6^2 - 2\,F_3\,G_3^3\,Q_6^2 + F_3\,G_3^2\,I_3\,Q_4\,Q_6 + 2\,F_3\,G_3^3\,Q_2\,Q_6+$$
$$2\,F_3^3\,G_3\,Q_{12}\,Q_6 + 2\,F_3^3\,J_3\,Q_{11}\,Q_6 - F_3^2\,I_3^2\,Q_{11}\,Q_6 + 6\,F_3^2\,G_3^2\,Q_{11}\,Q_6+$$
$$6\,F_3\,G_3^2\,J_3\,Q_1\,Q_6 - G_3^2\,I_3^2\,Q_1\,Q_6 + 2\,G_3^4\,Q_1\,Q_6 - F_3\,G_3^3\,Q_4^2-$$
$$3\,F_3^2\,G_3\,I_3\,Q_{11}\,Q_4 + G_3^3\,I_3\,Q_1\,Q_4 - 2\,F_3^2\,G_3^2\,Q_{11}\,Q_2 - 2\,G_3^4\,Q_1\,Q_2-$$
$$2\,F_3^4\,Q_{11}\,Q_{12} - 2\,F_3^2\,G_3^2\,Q_1\,Q_{12} - 4\,F_3^3\,G_3\,Q_{11}^2 - 4\,F_3^2\,G_3\,J_3\,Q_1\,Q_{11}+$$
$$4\,F_3\,G_3\,I_3^2\,Q_1\,Q_{11} - 4\,F_3\,G_3^3\,Q_1\,Q_{11} - 4\,G_3^3\,J_3\,Q_1^2)\,t_1^{14}+$$
$$(-F_3^3\,J_3\,Q_9^2 - 3\,F_3^2\,G_3^2\,Q_9^2 + F_3^2\,G_3\,I_3\,Q_7\,Q_9 + F_3^2\,I_3\,J_3\,Q_6\,Q_9+$$
$$2\,F_3\,G_3^2\,I_3\,Q_6\,Q_9 + 2\,F_3^2\,G_3^2\,Q_5\,Q_9 + 4\,F_3^2\,G_3\,J_3\,Q_4\,Q_9 - F_3\,G_3\,I_3^2\,Q_4\,Q_9+$$
$$4\,F_3\,G_3^3\,Q_4\,Q_9 - 3\,F_3\,G_3^2\,I_3\,Q_2\,Q_9 + F_3^3\,I_3\,Q_{12}\,Q_9 + 3\,F_3^2\,G_3\,I_3\,Q_{11}\,Q_9-$$
$$2\,F_3^3\,G_3\,Q_{10}\,Q_9 - 6\,F_3\,G_3\,I_3\,J_3\,Q_1\,Q_9 + G_3\,I_3^3\,Q_1\,Q_9 - 3\,G_3^3\,I_3\,Q_1\,Q_9-$$
$$2\,F_3^2\,G_3^2\,Q_6\,Q_8 + 2\,F_3^3\,G_3\,Q_{11}\,Q_8 + 2\,F_3\,G_3^3\,Q_1\,Q_8 - F_3^2\,G_3^2\,Q_7^2-$$
$$4\,F_3^2\,G_3\,J_3\,Q_6\,Q_7 - 4\,F_3\,G_3^3\,Q_6\,Q_7 + F_3\,G_3^2\,I_3\,Q_4\,Q_7 + 2\,F_3\,G_3^3\,Q_2\,Q_7+$$
$$2\,F_3^3\,G_3\,Q_{12}\,Q_7 + 2\,F_3^3\,J_3\,Q_{11}\,Q_7 - F_3^2\,I_3^2\,Q_{11}\,Q_7 + 6\,F_3^2\,G_3^2\,Q_{11}\,Q_7+$$
$$6\,F_3\,G_3^2\,J_3\,Q_1\,Q_7 - G_3^2\,I_3^2\,Q_1\,Q_7 + 2\,G_3^4\,Q_1\,Q_7 - F_3^2\,J_3^2\,Q_6^2-$$
$$4\,F_3\,G_3^2\,J_3\,Q_6^2 - G_3^4\,Q_6^2 + F_3\,G_3^2\,I_3\,Q_5\,Q_6 + 2\,F_3\,G_3\,I_3\,J_3\,Q_4\,Q_6+$$
$$G_3^3\,I_3\,Q_4\,Q_6 + 2\,F_3\,G_3^3\,Q_3\,Q_6 + 6\,F_3\,G_3^2\,J_3\,Q_2\,Q_6 - G_3^2\,I_3^2\,Q_2\,Q_6+$$
$$2\,G_3^4\,Q_2\,Q_6 + 2\,F_3^3\,G_3\,Q_{13}\,Q_6 + 2\,F_3^3\,J_3\,Q_{12}\,Q_6 - F_3^2\,I_3^2\,Q_{12}\,Q_6+$$
$$6\,F_3^2\,G_3^2\,Q_{12}\,Q_6 + 6\,F_3^2\,G_3\,J_3\,Q_{11}\,Q_6 - 2\,F_3\,G_3\,I_3^2\,Q_{11}\,Q_6 + 6\,F_3\,G_3^3\,Q_{11}\,Q_6+$$
$$F_3^2\,G_3\,I_3\,Q_{10}\,Q_6 + 6\,F_3\,G_3\,J_3^2\,Q_1\,Q_6 - 2\,G_3\,I_3^2\,J_3\,Q_1\,Q_6 + 6\,G_3^3\,J_3\,Q_1\,Q_6-$$
$$2\,F_3\,G_3^3\,Q_4\,Q_5 - 3\,F_3^2\,G_3\,I_3\,Q_{11}\,Q_5 + G_3^3\,I_3\,Q_1\,Q_5 - 3\,F_3\,G_3^2\,J_3\,Q_4^2-$$
$$G_3^4\,Q_4^2 + G_3^3\,I_3\,Q_2\,Q_4 - 3\,F_3^2\,G_3\,I_3\,Q_{12}\,Q_4 - 3\,F_3^2\,I_3\,J_3\,Q_{11}\,Q_4+$$
$$F_3\,I_3^3\,Q_{11}\,Q_4 - 6\,F_3\,G_3^2\,I_3\,Q_{11}\,Q_4 + 2\,F_3^2\,G_3^2\,Q_{10}\,Q_4 + 3\,G_3^2\,I_3\,J_3\,Q_1\,Q_4-$$
$$2\,F_3^2\,G_3^2\,Q_{11}\,Q_3 - 2\,G_3^4\,Q_1\,Q_3 - G_3^4\,Q_2^2 - 2\,F_3^2\,G_3^2\,Q_{12}\,Q_2-$$

$4\,F_3^2\,G_3\,J_3\,Q_{11}\,Q_2+4\,F_3\,G_3\,I_3^2\,Q_{11}\,Q_2-4\,F_3\,G_3^3\,Q_{11}\,Q_2-8\,G_3^3\,J_3\,Q_1\,Q_2-$

$2\,F_3^4\,Q_{11}\,Q_{13}-2\,F_3^2\,G_3^2\,Q_1\,Q_{13}-F_3^4\,Q_{12}^2-8\,F_3^3\,G_3\,Q_{11}\,Q_{12}-$

$4\,F_3^2\,G_3\,J_3\,Q_1\,Q_{12}+4\,F_3\,G_3\,I_3^2\,Q_1\,Q_{12}-4\,F_3\,G_3^3\,Q_1\,Q_{12}-6\,F_3^2\,G_3^2\,Q_{11}^2+$

$F_3^3\,I_3\,Q_{10}\,Q_{11}-2\,F_3^2\,J_3^2\,Q_1\,Q_{11}+4\,F_3\,I_3^2\,J_3\,Q_1\,Q_{11}-8\,F_3\,G_3^2\,J_3\,Q_1\,Q_{11}-$

$I_3^4\,Q_1\,Q_{11}+4\,G_3^2\,I_3^2\,Q_1\,Q_{11}-2\,G_3^4\,Q_1\,Q_{11}-3\,F_3\,G_3^2\,I_3\,Q_1\,Q_{10}-$

$6\,G_3^2\,J_3^2\,Q_1^2)\,t_1^{12}+$

$(-3\,F_3^2\,G_3\,J_3\,Q_9^2-3\,F_3\,G_3^3\,Q_9^2+F_3^2\,G_3\,I_3\,Q_8\,Q_9+F_3^2\,I_3\,J_3\,Q_7\,Q_9+$

$2\,F_3\,G_3^2\,I_3\,Q_7\,Q_9+2\,F_3\,G_3\,I_3\,J_3\,Q_6\,Q_9+G_3^3\,I_3\,Q_6\,Q_9+4\,F_3^2\,G_3\,J_3\,Q_5\,Q_9-$

$F_3\,G_3\,I_3^2\,Q_5\,Q_9+4\,F_3\,G_3^3\,Q_5\,Q_9+2\,F_3^2\,J_3^2\,Q_4\,Q_9-F_3\,I_3^2\,J_3\,Q_4\,Q_9+$

$8\,F_3\,G_3^2\,J_3\,Q_4\,Q_9-G_3^2\,I_3^2\,Q_4\,Q_9+2\,G_3^4\,Q_4\,Q_9-3\,F_3\,G_3^2\,I_3\,Q_3\,Q_9-$

$6\,F_3\,G_3\,I_3\,J_3\,Q_2\,Q_9+G_3\,I_3^3\,Q_2\,Q_9-3\,G_3^3\,I_3\,Q_2\,Q_9+F_3^3\,I_3\,Q_{13}\,Q_9+$

$3\,F_3^2\,G_3\,I_3\,Q_{12}\,Q_9+3\,F_3\,G_3^2\,I_3\,Q_{11}\,Q_9-2\,F_3^3\,J_3\,Q_{10}\,Q_9-6\,F_3^2\,G_3^2\,Q_{10}\,Q_9-$

$3\,F_3\,I_3\,J_3^2\,Q_1\,Q_9+I_3^3\,J_3\,Q_1\,Q_9-6\,G_3^2\,I_3\,J_3\,Q_1\,Q_9-2\,F_3^2\,G_3^2\,Q_7\,Q_8-$

$4\,F_3^2\,G_3\,J_3\,Q_6\,Q_8-4\,F_3\,G_3^3\,Q_6\,Q_8+F_3\,G_3^2\,I_3\,Q_4\,Q_8+2\,F_3\,G_3^3\,Q_2\,Q_8+$

$2\,F_3^3\,G_3\,Q_{12}\,Q_8+2\,F_3^3\,J_3\,Q_{11}\,Q_8-F_3^2\,I_3^2\,Q_{11}\,Q_8+6\,F_3^2\,G_3^2\,Q_{11}\,Q_8+$

$6\,F_3\,G_3^2\,J_3\,Q_1\,Q_8-G_3^2\,I_3^2\,Q_1\,Q_8+2\,G_3^4\,Q_1\,Q_8-2\,F_3^2\,G_3\,J_3\,Q_7^2-$

$2\,F_3\,G_3^3\,Q_7^2-2\,F_3^2\,J_3^2\,Q_6\,Q_7-8\,F_3\,G_3^2\,J_3\,Q_6\,Q_7-2\,G_3^4\,Q_6\,Q_7+$

$F_3\,G_3^2\,I_3\,Q_5\,Q_7+2\,F_3\,G_3\,I_3\,J_3\,Q_4\,Q_7+G_3^3\,I_3\,Q_4\,Q_7+2\,F_3\,G_3^3\,Q_3\,Q_7+$

$6\,F_3\,G_3^2\,J_3\,Q_2\,Q_7-G_3^2\,I_3^2\,Q_2\,Q_7+2\,G_3^4\,Q_2\,Q_7+2\,F_3^3\,G_3\,Q_{13}\,Q_7+$

$2\,F_3^3\,J_3\,Q_{12}\,Q_7-F_3^2\,I_3^2\,Q_{12}\,Q_7+6\,F_3^2\,G_3^2\,Q_{12}\,Q_7+6\,F_3^2\,G_3\,J_3\,Q_{11}\,Q_7-$

$2\,F_3\,G_3\,I_3^2\,Q_{11}\,Q_7+6\,F_3\,G_3^3\,Q_{11}\,Q_7+F_3^2\,G_3\,I_3\,Q_{10}\,Q_7+6\,F_3\,G_3\,J_3^2\,Q_1\,Q_7-$

$2\,G_3\,I_3^2\,J_3\,Q_1\,Q_7+6\,G_3^3\,J_3\,Q_1\,Q_7-2\,F_3\,G_3\,J_3^2\,Q_6^2-2\,G_3^3\,J_3\,Q_6^2+$

$2\,F_3\,G_3\,I_3\,J_3\,Q_5\,Q_6+G_3^3\,I_3\,Q_5\,Q_6+F_3\,I_3\,J_3^2\,Q_4\,Q_6+2\,G_3^2\,I_3\,J_3\,Q_4\,Q_6+$

$6\,F_3\,G_3^2\,J_3\,Q_3\,Q_6-G_3^2\,I_3^2\,Q_3\,Q_6+2\,G_3^4\,Q_3\,Q_6+6\,F_3\,G_3\,J_3^2\,Q_2\,Q_6-$

$2\,G_3\,I_3^2\,J_3\,Q_2\,Q_6+6\,G_3^3\,J_3\,Q_2\,Q_6+2\,F_3^3\,J_3\,Q_{13}\,Q_6-F_3^2\,I_3^2\,Q_{13}\,Q_6+$

$6\,F_3^2\,G_3^2\,Q_{13}\,Q_6+6\,F_3^2\,G_3\,J_3\,Q_{12}\,Q_6-2\,F_3\,G_3\,I_3^2\,Q_{12}\,Q_6+6\,F_3\,G_3^3\,Q_{12}\,Q_6+$

$$6 F_3 G_3^2 J_3 Q_{11} Q_6 - G_3^2 I_3^2 Q_{11} Q_6 + 2 G_3^4 Q_{11} Q_6 + F_3^2 I_3 J_3 Q_{10} Q_6 +$$
$$2 F_3 G_3^2 I_3 Q_{10} Q_6 + 2 F_3 J_3^3 Q_1 Q_6 - I_3^2 J_3^2 Q_1 Q_6 + 6 G_3^2 J_3^2 Q_1 Q_6 -$$
$$F_3 G_3^3 Q_5^2 - 6 F_3 G_3^2 J_3 Q_4 Q_5 - 2 G_3^4 Q_4 Q_5 + G_3^3 I_3 Q_2 Q_5 -$$
$$3 F_3^2 G_3 I_3 Q_{12} Q_5 - 3 F_3^2 I_3 J_3 Q_{11} Q_5 + F_3 I_3^3 Q_{11} Q_5 - 6 F_3 G_3^2 I_3 Q_{11} Q_5 +$$
$$2 F_3^2 G_3^2 Q_{10} Q_5 + 3 G_3^2 I_3 J_3 Q_1 Q_5 - 3 F_3 G_3 J_3^2 Q_4^2 - 3 G_3^3 J_3 Q_4^2 +$$
$$G_3^3 I_3 Q_3 Q_4 + 3 G_3^2 I_3 J_3 Q_2 Q_4 - 3 F_3^2 G_3 I_3 Q_{13} Q_4 - 3 F_3^2 I_3 J_3 Q_{12} Q_4 +$$
$$F_3 I_3^3 Q_{12} Q_4 - 6 F_3 G_3^2 I_3 Q_{12} Q_4 - 6 F_3 G_3 I_3 J_3 Q_{11} Q_4 + G_3 I_3^3 Q_{11} Q_4 -$$
$$3 G_3^3 I_3 Q_{11} Q_4 + 4 F_3^2 G_3 J_3 Q_{10} Q_4 - F_3 G_3 I_3^2 Q_{10} Q_4 + 4 F_3 G_3^3 Q_{10} Q_4 +$$
$$3 G_3 I_3 J_3^2 Q_1 Q_4 - 2 G_3^4 Q_2 Q_3 - 2 F_3^2 G_3^2 Q_{12} Q_3 - 4 F_3^2 G_3 J_3 Q_{11} Q_3 +$$
$$4 F_3 G_3 I_3^2 Q_{11} Q_3 - 4 F_3 G_3^3 Q_{11} Q_3 - 8 G_3^3 J_3 Q_1 Q_3 - 4 G_3^3 J_3 Q_2^2 -$$
$$2 F_3^2 G_3^2 Q_{13} Q_2 - 4 F_3^2 G_3 J_3 Q_{12} Q_2 + 4 F_3 G_3 I_3^2 Q_{12} Q_2 - 4 F_3 G_3^3 Q_{12} Q_2 -$$
$$2 F_3^2 J_3^2 Q_{11} Q_2 + 4 F_3 I_3^2 J_3 Q_{11} Q_2 - 8 F_3 G_3^2 J_3 Q_{11} Q_2 - I_3^4 Q_{11} Q_2 +$$
$$4 G_3^2 I_3^2 Q_{11} Q_2 - 2 G_3^4 Q_{11} Q_2 - 3 F_3 G_3^2 I_3 Q_{10} Q_2 - 12 G_3^2 J_3^2 Q_1 Q_2 -$$
$$2 F_3^4 Q_{12} Q_{13} - 8 F_3^3 G_3 Q_{11} Q_{13} - 4 F_3^2 G_3 J_3 Q_1 Q_{13} + 4 F_3 G_3 I_3^2 Q_1 Q_{13} -$$
$$4 F_3 G_3^3 Q_1 Q_{13} - 4 F_3^3 G_3 Q_{12}^2 - 12 F_3^2 G_3^2 Q_{11} Q_{12} + F_3^3 I_3 Q_{10} Q_{12} -$$
$$2 F_3^2 J_3^2 Q_1 Q_{12} + 4 F_3 I_3^2 J_3 Q_1 Q_{12} - 8 F_3 G_3^2 J_3 Q_1 Q_{12} - I_3^4 Q_1 Q_{12} +$$
$$4 G_3^2 I_3^2 Q_1 Q_{12} - 2 G_3^4 Q_1 Q_{12} - 4 F_3 G_3^3 Q_{11}^2 + 3 F_3^2 G_3 I_3 Q_{10} Q_{11} -$$
$$4 F_3 G_3 J_3^2 Q_1 Q_{11} + 4 G_3 I_3^2 J_3 Q_1 Q_{11} - 4 G_3^3 J_3 Q_1 Q_{11} - F_3^3 G_3 Q_{10}^2 -$$
$$6 F_3 G_3 I_3 J_3 Q_1 Q_{10} + G_3 I_3^3 Q_1 Q_{10} - 3 G_3^3 I_3 Q_1 Q_{10} - 4 G_3 J_3^3 Q_1^2) \, t_1^{10} +$$
$$(-3 F_3 G_3^2 J_3 Q_9^2 - G_3^4 Q_9^2 + F_3^2 I_3 J_3 Q_8 Q_9 + 2 F_3 G_3^2 I_3 Q_8 Q_9 +$$
$$2 F_3 G_3 I_3 J_3 Q_7 Q_9 + G_3^3 I_3 Q_7 Q_9 + G_3^2 I_3 J_3 Q_6 Q_9 + 2 F_3^2 J_3^2 Q_5 Q_9 -$$
$$F_3 I_3^2 J_3 Q_5 Q_9 + 8 F_3 G_3^2 J_3 Q_5 Q_9 - G_3^2 I_3^2 Q_5 Q_9 + 2 G_3^4 Q_5 Q_9 +$$
$$4 F_3 G_3 J_3^2 Q_4 Q_9 - G_3 I_3^2 J_3 Q_4 Q_9 + 4 G_3^3 J_3 Q_4 Q_9 - 6 F_3 G_3 I_3 J_3 Q_3 Q_9 +$$
$$G_3 I_3^3 Q_3 Q_9 - 3 G_3^3 I_3 Q_3 Q_9 - 3 F_3 I_3 J_3^2 Q_2 Q_9 + I_3^3 J_3 Q_2 Q_9 -$$
$$6 G_3^2 I_3 J_3 Q_2 Q_9 + 3 F_3^2 G_3 I_3 Q_{13} Q_9 + 3 F_3 G_3^2 I_3 Q_{12} Q_9 + G_3^3 I_3 Q_{11} Q_9 -$$

$$6\,F_3{}^2\,G_3\,J_3\,Q_{10}\,Q_9 - 6\,F_3\,G_3{}^3\,Q_{10}\,Q_9 - 3\,G_3\,I_3\,J_3{}^2\,Q_1\,Q_9 - F_3{}^2\,G_3{}^2\,Q_8{}^2 -$$

$$4\,F_3{}^2\,G_3\,J_3\,Q_7\,Q_8 - 4\,F_3\,G_3{}^3\,Q_7\,Q_8 - 2\,F_3{}^2\,J_3{}^2\,Q_6\,Q_8 - 8\,F_3\,G_3{}^2\,J_3\,Q_6\,Q_8 -$$

$$2\,G_3{}^4\,Q_6\,Q_8 + F_3\,G_3{}^2\,I_3\,Q_5\,Q_8 + 2\,F_3\,G_3\,I_3\,J_3\,Q_4\,Q_8 + G_3{}^3\,I_3\,Q_4\,Q_8 +$$

$$2\,F_3\,G_3{}^3\,Q_3\,Q_8 + 6\,F_3\,G_3{}^2\,J_3\,Q_2\,Q_8 - G_3{}^2\,I_3{}^2\,Q_2\,Q_8 + 2\,G_3{}^4\,Q_2\,Q_8 +$$

$$2\,F_3{}^3\,G_3\,Q_{13}\,Q_8 + 2\,F_3{}^3\,J_3\,Q_{12}\,Q_8 - F_3{}^2\,I_3{}^2\,Q_{12}\,Q_8 + 6\,F_3{}^2\,G_3{}^2\,Q_{12}\,Q_8 +$$

$$6\,F_3{}^2\,G_3\,J_3\,Q_{11}\,Q_8 - 2\,F_3\,G_3\,I_3{}^2\,Q_{11}\,Q_8 + 6\,F_3\,G_3{}^3\,Q_{11}\,Q_8 + F_3{}^2\,G_3\,I_3\,Q_{10}\,Q_8 +$$

$$6\,F_3\,G_3\,J_3{}^2\,Q_1\,Q_8 - 2\,G_3\,I_3{}^2\,J_3\,Q_1\,Q_8 + 6\,G_3{}^3\,J_3\,Q_1\,Q_8 - F_3{}^2\,J_3{}^2\,Q_7{}^2 -$$

$$4\,F_3\,G_3{}^2\,J_3\,Q_7{}^2 - G_3{}^4\,Q_7{}^2 - 4\,F_3\,G_3\,J_3{}^2\,Q_6\,Q_7 - 4\,G_3{}^3\,J_3\,Q_6\,Q_7 +$$

$$2\,F_3\,G_3\,I_3\,J_3\,Q_5\,Q_7 + G_3{}^3\,I_3\,Q_5\,Q_7 + F_3\,I_3\,J_3{}^2\,Q_4\,Q_7 + 2\,G_3{}^2\,I_3\,J_3\,Q_4\,Q_7 +$$

$$6\,F_3\,G_3{}^2\,J_3\,Q_3\,Q_7 - G_3{}^2\,I_3{}^2\,Q_3\,Q_7 + 2\,G_3{}^4\,Q_3\,Q_7 + 6\,F_3\,G_3\,J_3{}^2\,Q_2\,Q_7 -$$

$$2\,G_3\,I_3{}^2\,J_3\,Q_2\,Q_7 + 6\,G_3{}^3\,J_3\,Q_2\,Q_7 + 2\,F_3{}^3\,J_3\,Q_{13}\,Q_7 - F_3{}^2\,I_3{}^2\,Q_{13}\,Q_7 +$$

$$6\,F_3{}^2\,G_3{}^2\,Q_{13}\,Q_7 + 6\,F_3{}^2\,G_3\,J_3\,Q_{12}\,Q_7 - 2\,F_3\,G_3\,I_3{}^2\,Q_{12}\,Q_7 + 6\,F_3\,G_3{}^3\,Q_{12}\,Q_7 +$$

$$6\,F_3\,G_3{}^2\,J_3\,Q_{11}\,Q_7 - G_3{}^2\,I_3{}^2\,Q_{11}\,Q_7 + 2\,G_3{}^4\,Q_{11}\,Q_7 + F_3{}^2\,I_3\,J_3\,Q_{10}\,Q_7 +$$

$$2\,F_3\,G_3{}^2\,I_3\,Q_{10}\,Q_7 + 2\,F_3\,J_3{}^3\,Q_1\,Q_7 - I_3{}^2\,J_3{}^2\,Q_1\,Q_7 + 6\,G_3{}^2\,J_3{}^2\,Q_1\,Q_7 -$$

$$G_3{}^2\,J_3{}^2\,Q_6{}^2 + F_3\,I_3\,J_3{}^2\,Q_5\,Q_6 + 2\,G_3{}^2\,I_3\,J_3\,Q_5\,Q_6 + G_3\,I_3\,J_3{}^2\,Q_4\,Q_6 +$$

$$6\,F_3\,G_3\,J_3{}^2\,Q_3\,Q_6 - 2\,G_3\,I_3{}^2\,J_3\,Q_3\,Q_6 + 6\,G_3{}^3\,J_3\,Q_3\,Q_6 + 2\,F_3\,J_3{}^3\,Q_2\,Q_6 -$$

$$I_3{}^2\,J_3{}^2\,Q_2\,Q_6 + 6\,G_3{}^2\,J_3{}^2\,Q_2\,Q_6 + 6\,F_3{}^2\,G_3\,J_3\,Q_{13}\,Q_6 - 2\,F_3\,G_3\,I_3{}^2\,Q_{13}\,Q_6 +$$

$$6\,F_3\,G_3{}^3\,Q_{13}\,Q_6 + 6\,F_3\,G_3{}^2\,J_3\,Q_{12}\,Q_6 - G_3{}^2\,I_3{}^2\,Q_{12}\,Q_6 + 2\,G_3{}^4\,Q_{12}\,Q_6 +$$

$$2\,G_3{}^3\,J_3\,Q_{11}\,Q_6 + 2\,F_3\,G_3\,I_3\,J_3\,Q_{10}\,Q_6 + G_3{}^3\,I_3\,Q_{10}\,Q_6 + 2\,G_3\,J_3{}^3\,Q_1\,Q_6 -$$

$$3\,F_3\,G_3{}^2\,J_3\,Q_5{}^2 - G_3{}^4\,Q_5{}^2 - 6\,F_3\,G_3\,J_3{}^2\,Q_4\,Q_5 - 6\,G_3{}^3\,J_3\,Q_4\,Q_5 +$$

$$G_3{}^3\,I_3\,Q_3\,Q_5 + 3\,G_3{}^2\,I_3\,J_3\,Q_2\,Q_5 - 3\,F_3{}^2\,G_3\,I_3\,Q_{13}\,Q_5 - 3\,F_3{}^2\,I_3\,J_3\,Q_{12}\,Q_5 +$$

$$F_3\,I_3{}^3\,Q_{12}\,Q_5 - 6\,F_3\,G_3{}^2\,I_3\,Q_{12}\,Q_5 - 6\,F_3\,G_3\,I_3\,J_3\,Q_{11}\,Q_5 + G_3\,I_3{}^3\,Q_{11}\,Q_5 -$$

$$3\,G_3{}^3\,I_3\,Q_{11}\,Q_5 + 4\,F_3{}^2\,G_3\,J_3\,Q_{10}\,Q_5 - F_3\,G_3\,I_3{}^2\,Q_{10}\,Q_5 + 4\,F_3\,G_3{}^3\,Q_{10}\,Q_5 +$$

$$3\,G_3\,I_3\,J_3{}^2\,Q_1\,Q_5 - F_3\,J_3{}^3\,Q_4{}^2 - 3\,G_3{}^2\,J_3{}^2\,Q_4{}^2 + 3\,G_3{}^2\,I_3\,J_3\,Q_3\,Q_4 +$$

$$3 G_3 I_3 J_3^2 Q_2 Q_4 - 3 F_3^2 I_3 J_3 Q_{13} Q_4 + F_3 I_3^3 Q_{13} Q_4 - 6 F_3 G_3^2 I_3 Q_{13} Q_4 -$$

$$6 F_3 G_3 I_3 J_3 Q_{12} Q_4 + G_3 I_3^3 Q_{12} Q_4 - 3 G_3^3 I_3 Q_{12} Q_4 - 3 G_3^2 I_3 J_3 Q_{11} Q_4 +$$

$$2 F_3^2 J_3^2 Q_{10} Q_4 - F_3 I_3^2 J_3 Q_{10} Q_4 + 8 F_3 G_3^2 J_3 Q_{10} Q_4 - G_3^2 I_3^2 Q_{10} Q_4 +$$

$$2 G_3^4 Q_{10} Q_4 + I_3 J_3^3 Q_1 Q_4 - G_3^4 Q_3^2 - 8 G_3^3 J_3 Q_2 Q_3 -$$

$$2 F_3^2 G_3^2 Q_{13} Q_3 - 4 F_3^2 G_3 J_3 Q_{12} Q_3 + 4 F_3 G_3 I_3^2 Q_{12} Q_3 - 4 F_3 G_3^3 Q_{12} Q_3 -$$

$$2 F_3^2 J_3^2 Q_{11} Q_3 + 4 F_3 I_3^2 J_3 Q_{11} Q_3 - 8 F_3 G_3^2 J_3 Q_{11} Q_3 - I_3^4 Q_{11} Q_3 +$$

$$4 G_3^2 I_3^2 Q_{11} Q_3 - 2 G_3^4 Q_{11} Q_3 - 3 F_3 G_3^2 I_3 Q_{10} Q_3 - 12 G_3^2 J_3^2 Q_1 Q_3 -$$

$$6 G_3^2 J_3^2 Q_2^2 - 4 F_3^2 G_3 J_3 Q_{13} Q_2 + 4 F_3 G_3 I_3^2 Q_{13} Q_2 - 4 F_3 G_3^3 Q_{13} Q_2 -$$

$$2 F_3^2 J_3^2 Q_{12} Q_2 + 4 F_3 I_3^2 J_3 Q_{12} Q_2 - 8 F_3 G_3^2 J_3 Q_{12} Q_2 - I_3^4 Q_{12} Q_2 +$$

$$4 G_3^2 I_3^2 Q_{12} Q_2 - 2 G_3^4 Q_{12} Q_2 - 4 F_3 G_3 J_3^2 Q_{11} Q_2 + 4 G_3 I_3^2 J_3 Q_{11} Q_2 -$$

$$4 G_3^3 J_3 Q_{11} Q_2 - 6 F_3 G_3 I_3 J_3 Q_{10} Q_2 + G_3 I_3^3 Q_{10} Q_2 - 3 G_3^3 I_3 Q_{10} Q_2 -$$

$$8 G_3 J_3^3 Q_1 Q_2 - F_3^4 Q_{13}^2 - 8 F_3^3 G_3 Q_{12} Q_{13} - 12 F_3^2 G_3^2 Q_{11} Q_{13} +$$

$$F_3^3 I_3 Q_{10} Q_{13} - 2 F_3^2 J_3^2 Q_1 Q_{13} + 4 F_3 I_3^2 J_3 Q_1 Q_{13} - 8 F_3 G_3^2 J_3 Q_1 Q_{13} -$$

$$I_3^4 Q_1 Q_{13} + 4 G_3^2 I_3^2 Q_1 Q_{13} - 2 G_3^4 Q_1 Q_{13} - 6 F_3^2 G_3^2 Q_{12}^2 -$$

$$8 F_3 G_3^3 Q_{11} Q_{12} + 3 F_3^2 G_3 I_3 Q_{10} Q_{12} - 4 F_3 G_3 J_3^2 Q_1 Q_{12} + 4 G_3 I_3^2 J_3 Q_1 Q_{12} -$$

$$4 G_3^3 J_3 Q_1 Q_{12} - G_3^4 Q_{11}^2 + 3 F_3 G_3^2 I_3 Q_{10} Q_{11} - 2 G_3^2 J_3^2 Q_1 Q_{11} -$$

$$F_3^3 J_3 Q_{10}^2 - 3 F_3^2 G_3^2 Q_{10}^2 - 3 F_3 I_3 J_3^2 Q_1 Q_{10} + I_3^3 J_3 Q_1 Q_{10} -$$

$$6 G_3^2 I_3 J_3 Q_1 Q_{10} - J_3^4 Q_1^2) t_1^8 +$$

$$(-G_3^3 J_3 Q_9^2 + 2 F_3 G_3 I_3 J_3 Q_8 Q_9 + G_3^3 I_3 Q_8 Q_9 + G_3^2 I_3 J_3 Q_7 Q_9 +$$

$$4 F_3 G_3 J_3^2 Q_5 Q_9 - G_3 I_3^2 J_3 Q_5 Q_9 + 4 G_3^3 J_3 Q_5 Q_9 + 2 G_3^2 J_3^2 Q_4 Q_9 -$$

$$3 F_3 I_3 J_3^2 Q_3 Q_9 + I_3^3 J_3 Q_3 Q_9 - 6 G_3^2 I_3 J_3 Q_3 Q_9 - 3 G_3 I_3 J_3^2 Q_2 Q_9 +$$

$$3 F_3 G_3^2 I_3 Q_{13} Q_9 + G_3^3 I_3 Q_{12} Q_9 - 6 F_3 G_3^2 J_3 Q_{10} Q_9 - 2 G_3^4 Q_{10} Q_9 -$$

$$2 F_3^2 G_3 J_3 Q_8^2 - 2 F_3 G_3^3 Q_8^2 - 2 F_3^2 J_3^2 Q_7 Q_8 - 8 F_3 G_3^2 J_3 Q_7 Q_8 -$$

$$2 G_3^4 Q_7 Q_8 - 4 F_3 G_3 J_3^2 Q_6 Q_8 - 4 G_3^3 J_3 Q_6 Q_8 + 2 F_3 G_3 I_3 J_3 Q_5 Q_8 +$$

$$G_3^3 I_3 Q_5 Q_8 + F_3 I_3 J_3^2 Q_4 Q_8 + 2 G_3^2 I_3 J_3 Q_4 Q_8 + 6 F_3 G_3^2 J_3 Q_3 Q_8 -$$

$G_3^2 I_3^2 Q_3 Q_8 + 2 G_3^4 Q_3 Q_8 + 6 F_3 G_3 J_3^2 Q_2 Q_8 - 2 G_3 I_3^2 J_3 Q_2 Q_8 +$

$6 G_3^3 J_3 Q_2 Q_8 + 2 F_3^3 J_3 Q_{13} Q_8 - F_3^2 I_3^2 Q_{13} Q_8 + 6 F_3^2 G_3^2 Q_{13} Q_8 +$

$6 F_3^2 G_3 J_3 Q_{12} Q_8 - 2 F_3 G_3 I_3^2 Q_{12} Q_8 + 6 F_3 G_3^3 Q_{12} Q_8 + 6 F_3 G_3^2 J_3 Q_{11} Q_8 -$

$G_3^2 I_3^2 Q_{11} Q_8 + 2 G_3^4 Q_{11} Q_8 + F_3^2 I_3 J_3 Q_{10} Q_8 + 2 F_3 G_3^2 I_3 Q_{10} Q_8 +$

$2 F_3 J_3^3 Q_1 Q_8 - I_3^2 J_3^2 Q_1 Q_8 + 6 G_3^2 J_3^2 Q_1 Q_8 - 2 F_3 G_3 J_3^2 Q_7^2 -$

$2 G_3^3 J_3 Q_7^2 - 2 G_3^2 J_3^2 Q_6 Q_7 + F_3 I_3 J_3^2 Q_5 Q_7 + 2 G_3^2 I_3 J_3 Q_5 Q_7 +$

$G_3 I_3 J_3^2 Q_4 Q_7 + 6 F_3 G_3 J_3^2 Q_3 Q_7 - 2 G_3 I_3^2 J_3 Q_3 Q_7 + 6 G_3^3 J_3 Q_3 Q_7 +$

$2 F_3 J_3^3 Q_2 Q_7 - I_3^2 J_3^2 Q_2 Q_7 + 6 G_3^2 J_3^2 Q_2 Q_7 + 6 F_3^2 G_3 J_3 Q_{13} Q_7 -$

$2 F_3 G_3 I_3^2 Q_{13} Q_7 + 6 F_3 G_3^3 Q_{13} Q_7 + 6 F_3 G_3^2 J_3 Q_{12} Q_7 - G_3^2 I_3^2 Q_{12} Q_7 +$

$2 G_3^4 Q_{12} Q_7 + 2 G_3^3 J_3 Q_{11} Q_7 + 2 F_3 G_3 I_3 J_3 Q_{10} Q_7 + G_3^3 I_3 Q_{10} Q_7 +$

$2 G_3 J_3^3 Q_1 Q_7 + G_3 I_3 J_3^2 Q_5 Q_6 + 2 F_3 J_3^3 Q_3 Q_6 - I_3^2 J_3^2 Q_3 Q_6 +$

$6 G_3^2 J_3^2 Q_3 Q_6 + 2 G_3 J_3^3 Q_2 Q_6 + 6 F_3 G_3^2 J_3 Q_{13} Q_6 - G_3^2 I_3^2 Q_{13} Q_6 +$

$2 G_3^4 Q_{13} Q_6 + 2 G_3^3 J_3 Q_{12} Q_6 + G_3^2 I_3 J_3 Q_{10} Q_6 - 3 F_3 G_3 J_3^2 Q_5^2 -$

$3 G_3^3 J_3 Q_5^2 - 2 F_3 J_3^3 Q_4 Q_5 - 6 G_3^2 J_3^2 Q_4 Q_5 + 3 G_3^2 I_3 J_3 Q_3 Q_5 +$

$3 G_3 I_3 J_3^2 Q_2 Q_5 - 3 F_3^2 I_3 J_3 Q_{13} Q_5 + F_3 I_3^3 Q_{13} Q_5 - 6 F_3 G_3^2 I_3 Q_{13} Q_5 -$

$6 F_3 G_3 I_3 J_3 Q_{12} Q_5 + G_3 I_3^3 Q_{12} Q_5 - 3 G_3^3 I_3 Q_{12} Q_5 - 3 G_3^2 I_3 J_3 Q_{11} Q_5 +$

$2 F_3^2 J_3^2 Q_{10} Q_5 - F_3 I_3^2 J_3 Q_{10} Q_5 + 8 F_3 G_3^2 J_3 Q_{10} Q_5 - G_3^2 I_3^2 Q_{10} Q_5 +$

$2 G_3^4 Q_{10} Q_5 + I_3 J_3^3 Q_1 Q_5 - G_3 J_3^3 Q_4^2 + 3 G_3 I_3 J_3^2 Q_3 Q_4 +$

$I_3 J_3^3 Q_2 Q_4 - 6 F_3 G_3 I_3 J_3 Q_{13} Q_4 + G_3 I_3^3 Q_{13} Q_4 - 3 G_3^3 I_3 Q_{13} Q_4 -$

$3 G_3^2 I_3 J_3 Q_{12} Q_4 + 4 F_3 G_3 J_3^2 Q_{10} Q_4 - G_3 I_3^2 J_3 Q_{10} Q_4 + 4 G_3^3 J_3 Q_{10} Q_4 -$

$4 G_3^3 J_3 Q_3^2 - 12 G_3^2 J_3^2 Q_2 Q_3 - 4 F_3^2 G_3 J_3 Q_{13} Q_3 + 4 F_3 G_3 I_3^2 Q_{13} Q_3 -$

$4 F_3 G_3^3 Q_{13} Q_3 - 2 F_3^2 J_3^2 Q_{12} Q_3 + 4 F_3 I_3^2 J_3 Q_{12} Q_3 - 8 F_3 G_3^2 J_3 Q_{12} Q_3 -$

$I_3^4 Q_{12} Q_3 + 4 G_3^2 I_3^2 Q_{12} Q_3 - 2 G_3^4 Q_{12} Q_3 - 4 F_3 G_3 J_3^2 Q_{11} Q_3 +$

$4 G_3 I_3^2 J_3 Q_{11} Q_3 - 4 G_3^3 J_3 Q_{11} Q_3 - 6 F_3 G_3 I_3 J_3 Q_{10} Q_3 + G_3 I_3^3 Q_{10} Q_3 -$

$3 G_3^3 I_3 Q_{10} Q_3 - 8 G_3 J_3^3 Q_1 Q_3 - 4 G_3 J_3^3 Q_2^2 - 2 F_3^2 J_3^2 Q_{13} Q_2 +$

$4\,F_3\,I_3^{\,2}\,J_3\,Q_{13}\,Q_2 - 8\,F_3\,G_3^{\,2}\,J_3\,Q_{13}\,Q_2 - I_3^{\,4}\,Q_{13}\,Q_2 + 4\,G_3^{\,2}\,I_3^{\,2}\,Q_{13}\,Q_2 -$

$2\,G_3^{\,4}\,Q_{13}\,Q_2 - 4\,F_3\,G_3\,J_3^{\,2}\,Q_{12}\,Q_2 + 4\,G_3\,I_3^{\,2}\,J_3\,Q_{12}\,Q_2 - 4\,G_3^{\,3}\,J_3\,Q_{12}\,Q_2 -$

$2\,G_3^{\,2}\,J_3^{\,2}\,Q_{11}\,Q_2 - 3\,F_3\,I_3\,J_3^{\,2}\,Q_{10}\,Q_2 + I_3^{\,3}\,J_3\,Q_{10}\,Q_2 - 6\,G_3^{\,2}\,I_3\,J_3\,Q_{10}\,Q_2 -$

$2\,J_3^{\,4}\,Q_1\,Q_2 - 4\,F_3^{\,3}\,G_3\,Q_{13}^{\,2} - 12\,F_3^{\,2}\,G_3^{\,2}\,Q_{12}\,Q_{13} - 8\,F_3\,G_3^{\,3}\,Q_{11}\,Q_{13} +$

$3\,F_3^{\,2}\,G_3\,I_3\,Q_{10}\,Q_{13} - 4\,F_3\,G_3\,J_3^{\,2}\,Q_1\,Q_{13} + 4\,G_3\,I_3^{\,2}\,J_3\,Q_1\,Q_{13} - 4\,G_3^{\,3}\,J_3\,Q_1\,Q_{13} -$

$4\,F_3\,G_3^{\,3}\,Q_{12}^{\,2} - 2\,G_3^{\,4}\,Q_{11}\,Q_{12} + 3\,F_3\,G_3^{\,2}\,I_3\,Q_{10}\,Q_{12} - 2\,G_3^{\,2}\,J_3^{\,2}\,Q_1\,Q_{12} +$

$G_3^{\,3}\,I_3\,Q_{10}\,Q_{11} - 3\,F_3^{\,2}\,G_3\,J_3\,Q_{10}^{\,2} - 3\,F_3\,G_3^{\,3}\,Q_{10}^{\,2} - 3\,G_3\,I_3\,J_3^{\,2}\,Q_1\,Q_{10})\,t_1^{\,6} +$

$(G_3^{\,2}\,I_3\,J_3\,Q_8\,Q_9 + 2\,G_3^{\,2}\,J_3^{\,2}\,Q_5\,Q_9 - 3\,G_3\,I_3\,J_3^{\,2}\,Q_3\,Q_9 + G_3^{\,3}\,I_3\,Q_{13}\,Q_9 -$

$2\,G_3^{\,3}\,J_3\,Q_{10}\,Q_9 - F_3^{\,2}\,J_3^{\,2}\,Q_8^{\,2} - 4\,F_3\,G_3^{\,2}\,J_3\,Q_8^{\,2} - G_3^{\,4}\,Q_8^{\,2} -$

$4\,F_3\,G_3\,J_3^{\,2}\,Q_7\,Q_8 - 4\,G_3^{\,3}\,J_3\,Q_7\,Q_8 - 2\,G_3^{\,2}\,J_3^{\,2}\,Q_6\,Q_8 + F_3\,I_3\,J_3^{\,2}\,Q_5\,Q_8 +$

$2\,G_3^{\,2}\,I_3\,J_3\,Q_5\,Q_8 + G_3\,I_3\,J_3^{\,2}\,Q_4\,Q_8 + 6\,F_3\,G_3\,J_3^{\,2}\,Q_3\,Q_8 - 2\,G_3\,I_3^{\,2}\,J_3\,Q_3\,Q_8 +$

$6\,G_3^{\,3}\,J_3\,Q_3\,Q_8 + 2\,F_3\,J_3^{\,3}\,Q_2\,Q_8 - I_3^{\,2}\,J_3^{\,2}\,Q_2\,Q_8 + 6\,G_3^{\,2}\,J_3^{\,2}\,Q_2\,Q_8 +$

$6\,F_3^{\,2}\,G_3\,J_3\,Q_{13}\,Q_8 - 2\,F_3\,G_3\,I_3^{\,2}\,Q_{13}\,Q_8 + 6\,F_3\,G_3^{\,3}\,Q_{13}\,Q_8 + 6\,F_3\,G_3^{\,2}\,J_3\,Q_{12}\,Q_8 -$

$G_3^{\,2}\,I_3^{\,2}\,Q_{12}\,Q_8 + 2\,G_3^{\,4}\,Q_{12}\,Q_8 + 2\,G_3^{\,3}\,J_3\,Q_{11}\,Q_8 + 2\,F_3\,G_3\,I_3\,J_3\,Q_{10}\,Q_8 +$

$G_3^{\,3}\,I_3\,Q_{10}\,Q_8 + 2\,G_3\,J_3^{\,3}\,Q_1\,Q_8 - G_3^{\,2}\,J_3^{\,2}\,Q_7^{\,2} + G_3\,I_3\,J_3^{\,2}\,Q_5\,Q_7 +$

$2\,F_3\,J_3^{\,3}\,Q_3\,Q_7 - I_3^{\,2}\,J_3^{\,2}\,Q_3\,Q_7 + 6\,G_3^{\,2}\,J_3^{\,2}\,Q_3\,Q_7 + 2\,G_3\,J_3^{\,3}\,Q_2\,Q_7 +$

$6\,F_3\,G_3^{\,2}\,J_3\,Q_{13}\,Q_7 - G_3^{\,2}\,I_3^{\,2}\,Q_{13}\,Q_7 + 2\,G_3^{\,4}\,Q_{13}\,Q_7 + 2\,G_3^{\,3}\,J_3\,Q_{12}\,Q_7 +$

$G_3^{\,2}\,I_3\,J_3\,Q_{10}\,Q_7 + 2\,G_3\,J_3^{\,3}\,Q_3\,Q_6 + 2\,G_3^{\,3}\,J_3\,Q_{13}\,Q_6 - F_3\,J_3^{\,3}\,Q_5^{\,2} -$

$3\,G_3^{\,2}\,J_3^{\,2}\,Q_5^{\,2} - 2\,G_3\,J_3^{\,3}\,Q_4\,Q_5 + 3\,G_3\,I_3\,J_3^{\,2}\,Q_3\,Q_5 + I_3\,J_3^{\,3}\,Q_2\,Q_5 -$

$6\,F_3\,G_3\,I_3\,J_3\,Q_{13}\,Q_5 + G_3\,I_3^{\,3}\,Q_{13}\,Q_5 - 3\,G_3^{\,3}\,I_3\,Q_{13}\,Q_5 - 3\,G_3^{\,2}\,I_3\,J_3\,Q_{12}\,Q_5 +$

$4\,F_3\,G_3\,J_3^{\,2}\,Q_{10}\,Q_5 - G_3\,I_3^{\,2}\,J_3\,Q_{10}\,Q_5 + 4\,G_3^{\,3}\,J_3\,Q_{10}\,Q_5 + I_3\,J_3^{\,3}\,Q_3\,Q_4 -$

$3\,G_3^{\,2}\,I_3\,J_3\,Q_{13}\,Q_4 + 2\,G_3^{\,2}\,J_3^{\,2}\,Q_{10}\,Q_4 - 6\,G_3^{\,2}\,J_3^{\,2}\,Q_3^{\,2} - 8\,G_3\,J_3^{\,3}\,Q_2\,Q_3 -$

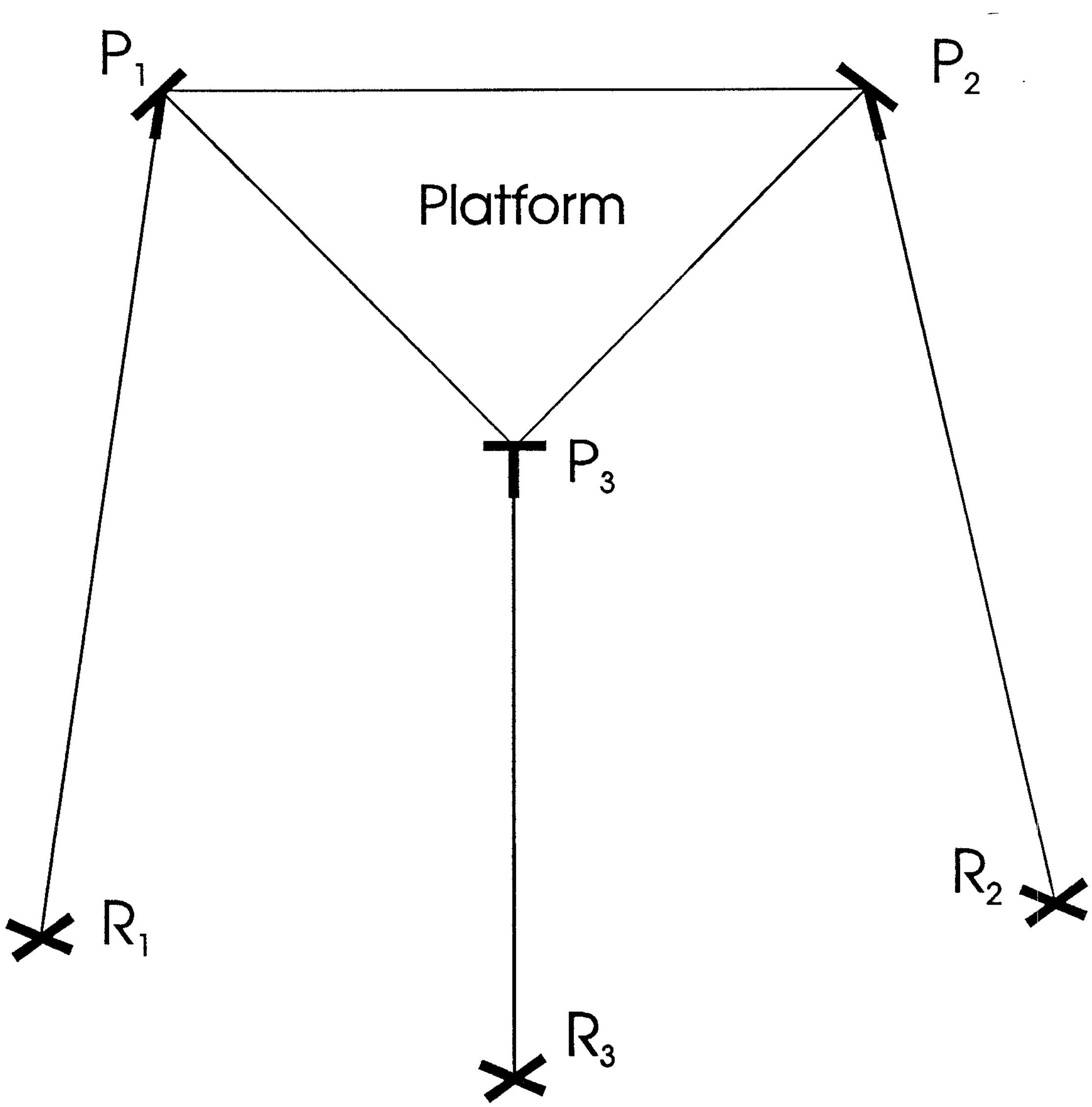
$2\,F_3^{\,2}\,J_3^{\,2}\,Q_{13}\,Q_3 + 4\,F_3\,I_3^{\,2}\,J_3\,Q_{13}\,Q_3 - 8\,F_3\,G_3^{\,2}\,J_3\,Q_{13}\,Q_3 - I_3^{\,4}\,Q_{13}\,Q_3 +$

$4\,G_3^{\,2}\,I_3^{\,2}\,Q_{13}\,Q_3 - 2\,G_3^{\,4}\,Q_{13}\,Q_3 - 4\,F_3\,G_3\,J_3^{\,2}\,Q_{12}\,Q_3 + 4\,G_3\,I_3^{\,2}\,J_3\,Q_{12}\,Q_3 -$

$4\,G_3^{\,3}\,J_3\,Q_{12}\,Q_3 - 2\,G_3^{\,2}\,J_3^{\,2}\,Q_{11}\,Q_3 - 3\,F_3\,I_3\,J_3^{\,2}\,Q_{10}\,Q_3 + I_3^{\,3}\,J_3\,Q_{10}\,Q_3 -$

$$
\begin{aligned}
&6\,G_3{}^2\,I_3\,J_3\,Q_{10}\,Q_3 - 2\,J_3{}^4\,Q_1\,Q_3 - J_3{}^4\,Q_2{}^2 - 4\,F_3\,G_3\,J_3{}^2\,Q_{13}\,Q_2 + \\
&4\,G_3\,I_3{}^2\,J_3\,Q_{13}\,Q_2 - 4\,G_3{}^3\,J_3\,Q_{13}\,Q_2 - 2\,G_3{}^2\,J_3{}^2\,Q_{12}\,Q_2 - 3\,G_3\,I_3\,J_3{}^2\,Q_{10}\,Q_2 - \\
&6\,F_3{}^2\,G_3{}^2\,Q_{13}{}^2 - 8\,F_3\,G_3{}^3\,Q_{12}\,Q_{13} - 2\,G_3{}^4\,Q_{11}\,Q_{13} + 3\,F_3\,G_3{}^2\,I_3\,Q_{10}\,Q_{13} - \\
&2\,G_3{}^2\,J_3{}^2\,Q_1\,Q_{13} - G_3{}^4\,Q_{12}{}^2 + G_3{}^3\,I_3\,Q_{10}\,Q_{12} - 3\,F_3\,G_3{}^2\,J_3\,Q_{10}{}^2 - \\
&G_3{}^4\,Q_{10}{}^2)\,t_1{}^4 + \\
&(-2\,F_3\,G_3\,J_3{}^2\,Q_8{}^2 - 2\,G_3{}^3\,J_3\,Q_8{}^2 - 2\,G_3{}^2\,J_3{}^2\,Q_7\,Q_8 + G_3\,I_3\,J_3{}^2\,Q_5\,Q_8 + \\
&2\,F_3\,J_3{}^3\,Q_3\,Q_8 - I_3{}^2\,J_3{}^2\,Q_3\,Q_8 + 6\,G_3{}^2\,J_3{}^2\,Q_3\,Q_8 + 2\,G_3\,J_3{}^3\,Q_2\,Q_8 + \\
&6\,F_3\,G_3{}^2\,J_3\,Q_{13}\,Q_8 - G_3{}^2\,I_3{}^2\,Q_{13}\,Q_8 + 2\,G_3{}^4\,Q_{13}\,Q_8 + 2\,G_3{}^3\,J_3\,Q_{12}\,Q_8 + \\
&G_3{}^2\,I_3\,J_3\,Q_{10}\,Q_8 + 2\,G_3\,J_3{}^3\,Q_3\,Q_7 + 2\,G_3{}^3\,J_3\,Q_{13}\,Q_7 - G_3\,J_3{}^3\,Q_5{}^2 + \\
&I_3\,J_3{}^3\,Q_3\,Q_5 - 3\,G_3{}^2\,I_3\,J_3\,Q_{13}\,Q_5 + 2\,G_3{}^2\,J_3{}^2\,Q_{10}\,Q_5 - 4\,G_3\,J_3{}^3\,Q_3{}^2 - \\
&2\,J_3{}^4\,Q_2\,Q_3 - 4\,F_3\,G_3\,J_3{}^2\,Q_{13}\,Q_3 + 4\,G_3\,I_3{}^2\,J_3\,Q_{13}\,Q_3 - 4\,G_3{}^3\,J_3\,Q_{13}\,Q_3 - \\
&2\,G_3{}^2\,J_3{}^2\,Q_{12}\,Q_3 - 3\,G_3\,I_3\,J_3{}^2\,Q_{10}\,Q_3 - 2\,G_3{}^2\,J_3{}^2\,Q_{13}\,Q_2 - 4\,F_3\,G_3{}^3\,Q_{13}{}^2 - \\
&2\,G_3{}^4\,Q_{12}\,Q_{13} + G_3{}^3\,I_3\,Q_{10}\,Q_{13} - G_3{}^3\,J_3\,Q_{10}{}^2)\,t_1{}^2 - \\
&G_3{}^2\,J_3{}^2\,Q_8{}^2 + 2\,G_3\,J_3{}^3\,Q_3\,Q_8 + 2\,G_3{}^3\,J_3\,Q_{13}\,Q_8 - J_3{}^4\,Q_3{}^2 - \\
&2\,G_3{}^2\,J_3{}^2\,Q_{13}\,Q_3 - G_3{}^4\,Q_{13}{}^2 = 0 \qquad (54)
\end{aligned}
$$

The above method is introduced by Salmon (1964).



R_1, R_2, and R_3 are connected to drivers.

Figure 1 - Represntation of a Minimanipulator

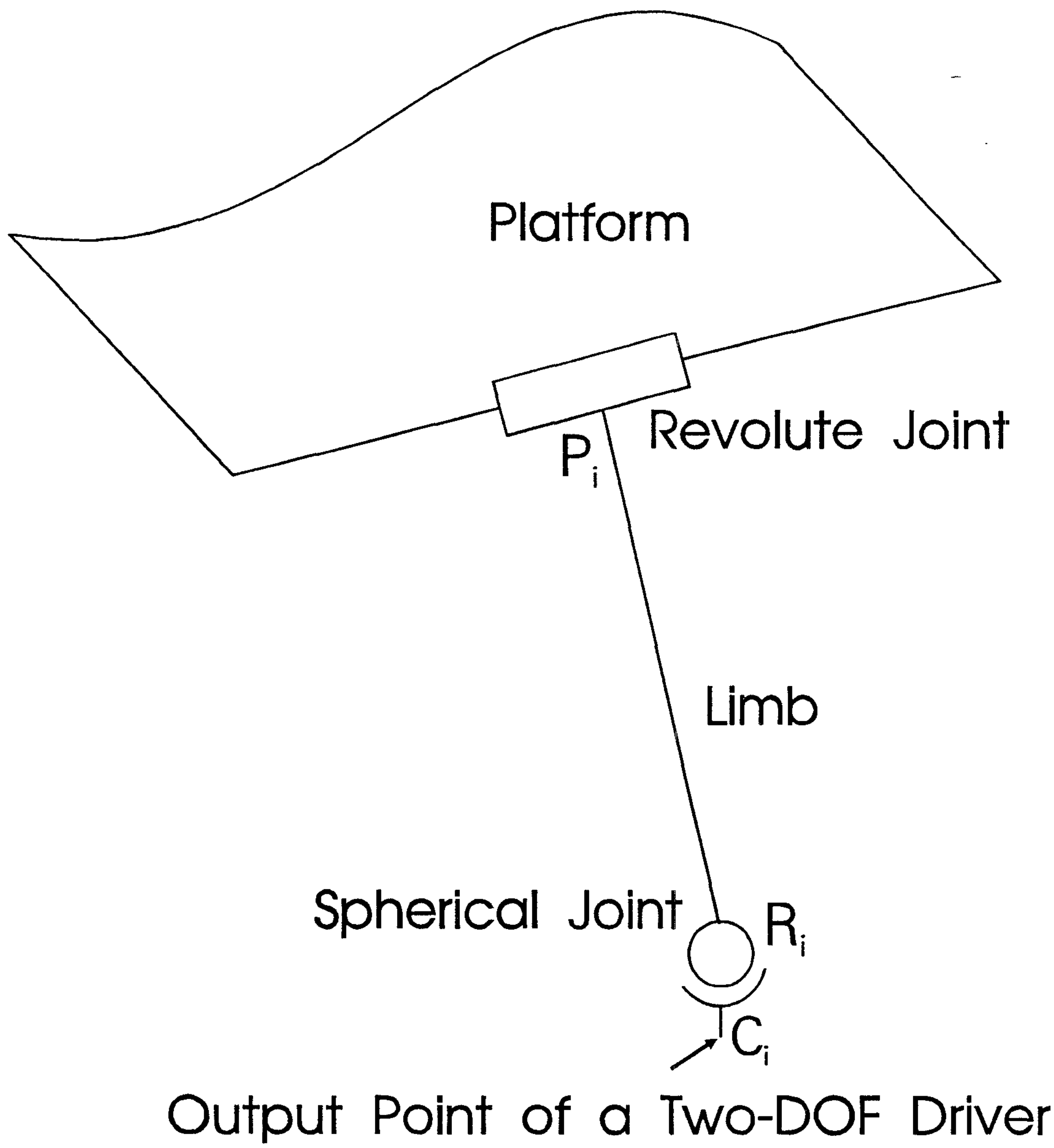


Figure 2 - Kinematic Equivalent of a Limb

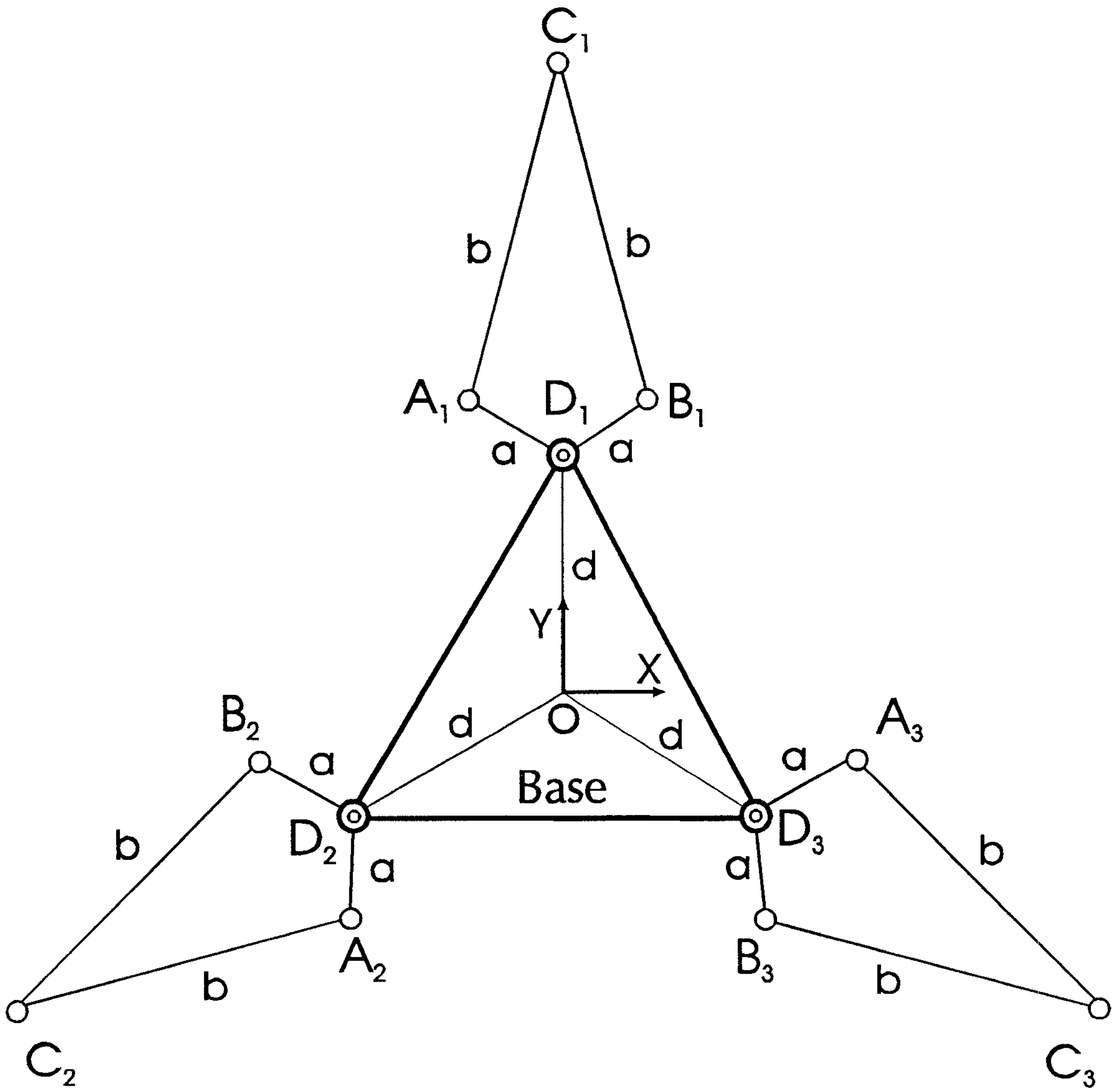
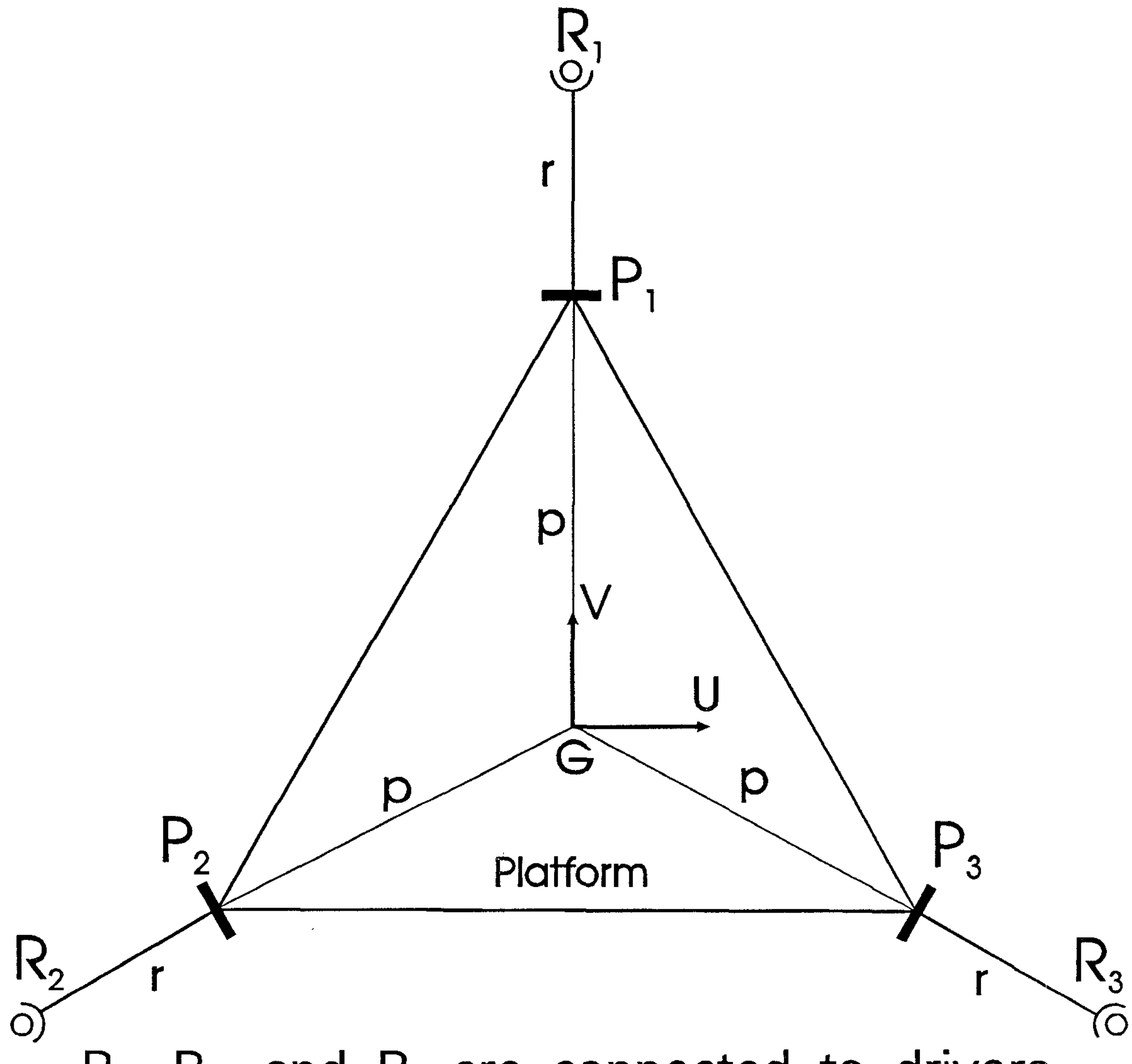


Figure 3 - Simplified Five-Bar Linkage Drivers



R_1, R_2, and R_3 are connected to drivers.

Figure 4 - Kinematic Equivalent of a Minimanipulator

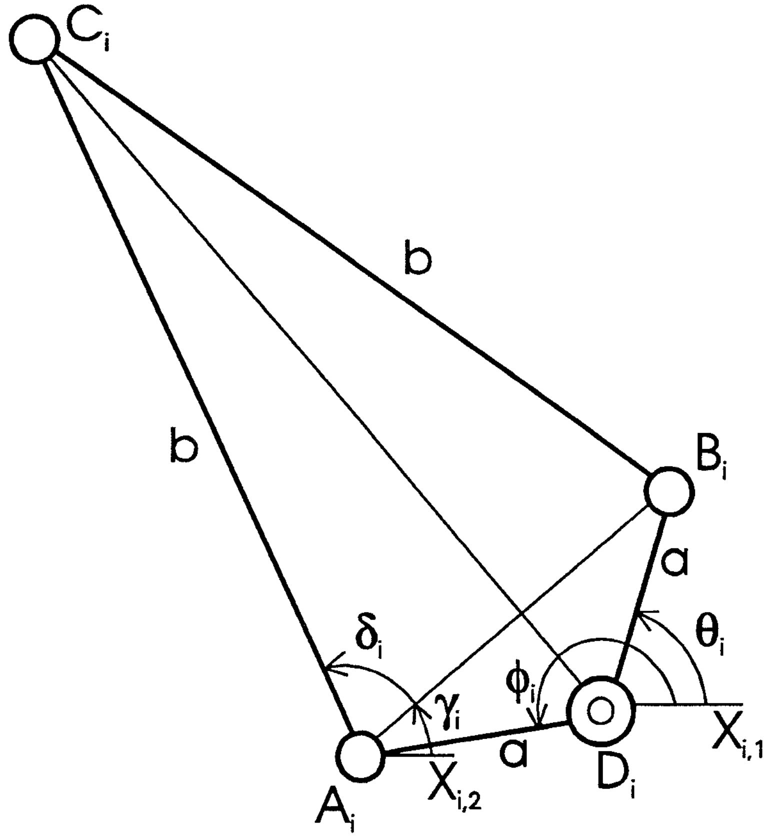


Figure 5 - Depiction of angles θ_i, ϕ_i, γ_i, and δ_i

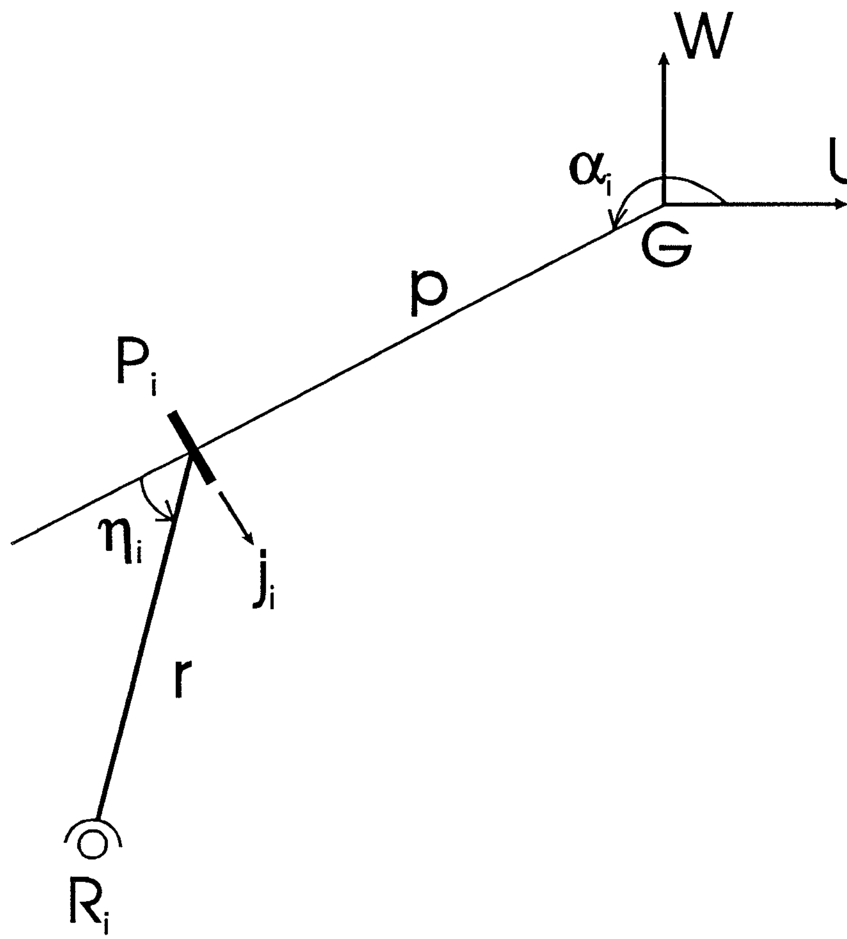


Figure 6 - Depiction of angles α_i and η_i