Closed-Form Discrete Fractional and Affine Fourier Transforms

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***Abstract*—The discrete fractional Fourier transform (DFRFT) is the generalization of discrete Fourier transform. Many types of DFRFT have been derived and are useful for signal processing applications. In this paper, we will introduce a new type of DFRFT, which are unitary, reversible, and flexible; in addition, the closed-form analytic expression can be obtained. It works in performance similar to the continuous fractional Fourier transform (FRFT) and can be efficiently calculated by FFT. Since the continuous FRFT can be generalized into the continuous affine Fourier transform (AFT) (the so-called canonical transform), we also extend the DFRFT into the discrete affine Fourier transform (DAFT). We will derive two types of the DFRFT and DAFT. Type 1 will be similar to the continuous FRFT and AFT and can be used for computing the continuous FRFT and AFT. Type 2 is the improved form of type 1 and can be used for other applications of digital signal processing. Meanwhile, many important properties continuous FRFT and AFT are kept in closed-form DFRFT and DAFT, and some applications, such as the filter design and pattern recognition, will also be discussed. The closed-form DFRFT we introduce will have the lowest complexity among all current DFRFT's that are still similar to the continuous FRFT.**

***Index Terms*—Affine Fourier transform, discrete affine Fourier transform, discrete Fourier transform, discrete fractional Fourier transform, Fourier transform.**

I. Introduction

THE continuous *fractional Fourier transform (FRFT)* [1], [2], which is the generalization of Fourier transform, is defined as

$$O_F^{\alpha}(f(t)) = \sqrt{\frac{1-j\cot\alpha}{2\pi}} \cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot u^2} \cdot \int_{-\infty}^{\infty} e^{-j\cdot\csc\alpha\cdot u\cdot t^2} \cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot t^2} \cdot f(t)\cdot dt \quad (1)$$

where the phase of $\sqrt{1-j\cot\alpha}$ is constrained in the range of $(-\pi/4, \pi/4)$. It has been discussed in recent years and used in many applications such as optical system analysis, filter design, soluiton of differential equations, phase retrieval, pattern recognition, etc. The continuous FRFT satisfies the additivity property as

$$O_F^{\alpha}\left(O_F^{\beta}(f(t))\right) = O_F^{\alpha+\beta}(f(t)). \quad (2)$$

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The FRFT has been further generalized into the special affine Fourier transform (SAFT) [3] (the so-called canonical transform [4]). It is defined as

$$O_F^{(a,b,c,d)}(f(t)) = \sqrt{\frac{1}{2\pi|b|}} \cdot e^{\frac{j}{2}\cdot\frac{d}{b}\cdot u^2} \cdot \int_{-\infty}^{\infty} e^{-j\cdot\frac{u}{b}\cdot t} \cdot e^{\frac{j}{2}\cdot\frac{a}{b}\cdot t^2} \cdot f(t)\cdot dt \quad \text{when } b\neq 0 \quad (3)$$

$$O_F^{(a,0,c,d)}(f(t)) = \sqrt{d}\cdot e^{\frac{j}{2}\cdot cd\cdot u^2}\cdot f(du) \quad \text{when } b=0 \quad (4)$$

where $ad-bc=1$ must be satisfied. Special affine Fourier transform has the additive property

$$O_F^{(a_1,b_1,c_1,d_1)}\left(O_F^{(a_2,b_2,c_2,d_2)}(f(t))\right) = O_F^{(a_3,b_3,c_3,d_3)}(f(t)) \quad (5)$$

where

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}$$

and it has the reversible property

$$O_F^{(d,-b,-c,a)}\left(O_F^{(a,b,c,d)}(f(t))\right) = f(t) \quad (5a)$$

We will call this special affine Fourier transform the *affine Fourier transform (AFT)*. The affine Fourier transform can extend the utilities of FRFT and is a useful tool for the optical system analysis. The effect of the FRFT and AFT can be interpreted by the Wigner distribution function (WDF). After doing the FRFT, the WDF of $O_F^{\alpha}(f(t))$ will be the rotation of the WDF of $f(t)$ with angle α [23], and after doing the AFT, the WDF of $O_F^{(a,b,c,d)}(f(t))$ will be the twisting of the WDF of $f(t)$.

After the continuous fractional Fourier transform has been derived, many researchers have tried to derive their discrete counterpart, that is, the *discrete fractional Fourier transform (DFRFT)*. We briefly review DFRFT's below. The name for each type of DFRFT is not recalled by the original authors. We give their names for easy classification.

1) *Direct form of DFRFT*. The simplest way to derive the DFRFT is sampling the continuous FRFT and computing it directly, but when we sample the continuous FRFT directly, then the resultant discrete transform we obtain will lose many important properties. The most serious problem is the DFRFT of this type will not be unitary and reversible. Besides, lacks closed-form properties, and not additive, so its applications are very limited.
2) *Improved sampling-type DFRFT*. In [5], a way to sample the continuous FRFT properly is introduced, and then, the resultant DFRFT will have the similar transform results as

the continuous FRFT. Although, in this case, the DFRFT can work very similarly to the continuous case and has a fast algorithm, but the transform kernel will not be orthogonal and additive. Besides, many constraints, including the input signal constraint, should be satisfied.

3) *Linear combination-type DFRFT*. In [6]–[8], and [24], the discrete fractional Fourier transform is derived by using the linear combination of identity operation, DFT, time inverse operation, and IDFT. In this case, the transform matrix is orthogonal, and the additivity property and the reversibility property will satisfy for this type of DFRFT. However, the main problem is that the transform results will not match to the continuous FRFT. Besides, it will work very similarly to the original Fourier transform or the identity operation and lose the important characteristic of "fractionalization."
4) *Eigenvectors decomposition-type DFRFT*. In [9]–[11], and [16], the authors derive another type of discrete fractional Fourier transform by searching the eigenvectors and eigenvalues of the DFT matrix and then compute the fractional power of the DFT matrix. This type of DFRFT will work very similarly to the continuous FRFT and will also have the properties of orthogonal, additivity, and reversibility. In [11], they have further improved this type of DFRFT by modifying their eigenvectors more similarly to the continuous Hermite functions, which are the eigenfunctions of the FRFT. These types of DFRFT's lack the fast computation algorithm, and the eigenvectors cannot be written in a closed form.
5) *Group theory-type DFRFT*. In [13], the concept of group theory [15] is used, and the DFRFT as the multiplication of DFT and the periodic chirps are derived. The DFRFT derived will satisfy the rotation property on the Wigner distribution, and the additivity and reversible property will also be satisfied. However, this type of DFRFT can be derived only when the fractional order of the DFRFT equals some specified angles, and when the number of points N is not prime, it will be very complicated to derive.
6) *Impulse train-type DFRFT*. Recently, in [14], another type of DFRFT is derived. This type of DFRFT can be viewed as a special case of the continuous FRFT. In this case, the input function $f(t)$ is a periodic, equally spaced impulse train, and if the number of impulses in a period is N, and the period is Δ_0, then $N = \Delta_0^2$. Besides, the value of $\tan\alpha$ is limited and must be a rational number (α is the order of FRFT). Because this type of DFRFT can be viewed as a special case of continuous FRFT, many properties of the FRFT will also exist and have the fast algorithm. However, this type of DFRFT has many constraints and cannot be defined for all values of α.

Although many types of the discrete fractional Fourier transform (DFRFT) have been derived recently, no discrete affine Fourier transform (DAFT) has yet been derived.

In this paper, we will derive a new type of DFRFT, and then extend it into the discrete affine Fourier transform (DAFT). The DFRFT and DAFT we derived come from the proper sampling of the continuous FRFT and AFT. The DFRFT introduced in [5] is also derived from the sampling of the continuous FRFT. Here, however, we will sample the continuous FRFT and affine Fourier transform by some proper intervals, and therefore, the transform matrix will be orthogonal and reversible. It can be written in the closed form so that many properties can be derived, and the fast algorithms can be achieved. Our idea comes from the [12] and [22]. In these papers, when we sample the fractional Fourier transform properly, we will obtain an unitary transform. We will improve upon these ideas.

In this paper, our focus is on the *practical applications*. Thus, although our DFRFT/DAFT seem neat in concepts and sacrifice the additivity property, they are very suitable for the practical applications due to the simpler and closed form of discrete fractional convolution and correlation introduced in Section II-D and the advantages listed in Section II-E. Our DFRFT/DAFT will have the lowest complexity among all the current the DFRFT/DAFT's that still have the similar properties as the continuous FRFT/AFT.

Due to the orientation of practical usage, we will derive two types of DFRFT/DAFT. These two types of DFRFT/DAFT are essentially the same but different in parameterizations. The first type we derive has the parameters that are more directly linked to the continuous FRFT/AFT and suits the applications of computing the continuous FRFT/AFT. On the other hand, type 2 has the simpler parameters set and allows more elegant expression for the operator kernels. It is suitable for other applications of DFRFT/DAFT, such as the filter design, pattern recognition (described in Section IV), and the use for the phase retrieval discussed in [12] and [22] can also be improved by the type 2 DFRFT/DAFT proposed in this paper.

In Section II, we will give the derivation and definitions of our new types of DFRFT and DAFT. For different applications, we will use different parameterizations to define 2 types of DFRFT/DAFT. In Section III, we will discuss their properties and their transform results for some special signals. In Section IV, we discuss their applications. Finally, in Section V, we make a conclusion.

II. Derivation of Closed-Form Discrete Fractional and Affine Fourier Transforms

A. The Closed-Form Discrete Fractional Fourier Transform of Type 1

1) The Derivation: To derive the DFRFT, we first sample the input function $f(t)$ and the output function $F_\alpha(u)$ of the FRFT [see (1)] by the interval Δt,, Δu as

$$y(n) = f(n\cdot\Delta t) \quad Y_\alpha(m) = F_\alpha(m\cdot\Delta u) \tag{6}$$

where $n = -N, -N+1, \ldots, N$, and $m = -M, -M+1, \ldots, M$. Here, we do not start our sampling at $t = 0$ and $u = 0$. Instead, we try to make the DC component in the center. From (6), we can convert (1) as

$$Y_\alpha(m) = \sqrt{\frac{1-j\cdot\cot\alpha}{2\pi}}\cdot\Delta t\cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot m^2\Delta u^2} \times \sum_{n=-N}^{N} e^{-j\cdot\csc\alpha\cdot n\cdot m\Delta u\Delta t} \cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot n^2\Delta t^2}\cdot y(n). \tag{7}$$

The above equation can be written as the form of transformation matrix

$$Y_\alpha(m) = \sum_{n=-N}^{N} F_\alpha(m,n)\cdot y(n) \tag{8}$$

where

$$F_\alpha(m,n) = \sqrt{\frac{1-j\cdot\cot\alpha}{2\pi}}\cdot\Delta t\cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot m^2\Delta u^2} \cdot e^{-j\cdot\csc\alpha\cdot n\cdot m\Delta u\Delta t}\cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot n^2\Delta t^2} \tag{9}$$

in order for (8) to be reversible. We will try to make the inverse transform to be the Hermitian (conjugation and transpose) of $F_\alpha(m,n)$ when $M \geq N$, i.e.,

$$y(n) = \sum_{m=-M}^{M} F_\alpha^*(m,n)\cdot Y_\alpha(m) \quad \text{for } M \geq N. \tag{10}$$

Then, from (8) and (9)

$$\begin{aligned} y(n) &= \sum_{m=-M}^{M}\sum_{k=-N}^{N} F_\alpha^*(m,n)\cdot F_\alpha(m,k)\cdot y(k) \\ &= \frac{\Delta t^2}{2\pi|\sin\alpha|}\sum_{m=-M}^{M}\sum_{k=-N}^{N} e^{\frac{j}{2}\cdot\cot\alpha\cdot(k^2-n^2)\Delta t^2} \cdot e^{j\cdot\csc\alpha\cdot m\cdot(n-k)\cdot\Delta u\Delta t}\cdot y(k). \end{aligned} \tag{11}$$

If we want the summation for m in (11) to become $\delta(n-k)$, then

$$\Delta u\cdot\Delta t = S\cdot 2\pi\cdot\sin\alpha/(2M+1) \tag{12}$$

where $|S|$ is some integer prime to $2M+1$. In this case, (9) becomes

$$F_\alpha(m,n) = \sqrt{\frac{1-j\cdot\cot\alpha}{2\pi}}\cdot\Delta t\cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot m^2\Delta u^2} \cdot e^{-j\cdot\frac{S\cdot 2\pi\cdot n\cdot m}{2M+1}}\cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot n^2\Delta t^2} \tag{13}$$

and

$$\begin{aligned} &\sum_{m=-M}^{M}\sum_{k=-N}^{N} F_\alpha^*(m,n)\cdot F_\alpha(m,k)\cdot y(k) \\ &= \frac{2M+1}{2\pi|\sin\alpha|}\cdot\Delta t^2\cdot y(n) \\ &= \frac{2M+1}{2\pi\,\mathrm{sgn}(\sin\alpha)\cdot\sin\alpha}\cdot\Delta t^2\cdot y(n). \end{aligned} \tag{14}$$

We then normalize $F_\alpha(m,n)$ to satisfy (11) and obtain the transform matrix $F_\alpha(m,n)$ as

$$F_\alpha(m,n) = \sqrt{\frac{\mathrm{sgn}(\sin\alpha)\cdot(\sin\alpha - j\cos\alpha)}{2M+1}} \cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot m^2\Delta u^2}\cdot e^{-j\cdot\frac{S\cdot 2\pi\cdot n\cdot m}{2M+1}} \cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot n^2\Delta t^2}. \tag{15}$$

For simplicity, we can choose $S = \mathrm{sgn}(\sin\alpha) = \pm 1$, and rewrite (15) as

$$F_\alpha(m,n) = \sqrt{\frac{|\sin\alpha| - j\,\mathrm{sgn}(\sin\alpha)\cos\alpha}{2M+1}}\cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot m^2\Delta u^2} \times e^{-j\cdot\frac{\mathrm{sgn}(\sin\alpha)\cdot 2\pi\cdot n\cdot m}{2M+1}} e^{\frac{j}{2}\cdot\cot\alpha\cdot n^2\Delta t^2}. \tag{16}$$

Then, we obtain the following two formulas of discrete fractional Fourier transforms (DFRFT) for 1) $\sin\alpha > 0$ and 2) $\sin\alpha < 0$:

- **DFRFT of type 1:**

1)

$$Y_\alpha(m) = \sqrt{\frac{\sin\alpha - j\cos\alpha}{2M+1}}\cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot m^2\Delta u^2} \times \sum_{n=-N}^{N} e^{-j\cdot\frac{2\pi\cdot n\cdot m}{2M+1}}\cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot n^2\Delta t^2}\cdot y(n)$$
$$\text{when } \alpha \in 2D\pi + (0,\pi), \quad D \text{ is integer } (\text{i.e., } \sin\alpha > 0) \tag{17}$$

2)

$$Y_\alpha(m) = \sqrt{\frac{-\sin\alpha + j\cos\alpha}{2M+1}}\cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot m^2\Delta u^2} \times \sum_{n=-N}^{N} e^{j\cdot\frac{2\pi\cdot n\cdot m}{2M+1}}\cdot e^{\frac{j}{2}\cdot\cot\alpha\cdot n^2\Delta t^2}\cdot y(n)$$
$$\text{when } \alpha \in 2D\pi + (-\pi,0), \quad D \text{ is integer } (\text{i.e., } \sin\alpha < 0). \tag{18}$$

Additionally, the constraints that

$$M \geq N \quad (2N+1, 2M+1 \text{ are the number of points in the time, frequency domain})$$
$$\Delta u\cdot\Delta t = 2\pi\cdot|\sin\alpha|/(2M+1) \tag{19}$$

must also be satisfied. We note that when $M = N$ and $\alpha = \pi/2$, (17) will become the DFT, and when $\alpha = -\pi/2$, (18) will become the inverse DFT. We also note that when $\alpha = D\cdot\pi$ and D is some integer, there is no proper choice for Δu and Δt that satisfies this constraint of (15), and we cannot use (17) or (18) as the definition of DFRFT when $\alpha = D\cdot\pi$. In fact, in these cases, we can just use the following definitions:

3)

$$Y_\alpha(m) = y(m) \quad \text{when } \alpha = 2D\pi \tag{20}$$

4)

$$Y_\alpha(m) = y(-m) \quad \text{when } \alpha = (2D+1)\pi. \tag{21}$$

Equations (17)–(21) are the definition of the DFRFT.

2) Some Important Discussion About the DFRFT of Type 1: We also note, from (1) and (2), the inverse of the forward continuous FRFT with order α is just the forward continuous FRFT with order $-\alpha$. In fact, this property will also exist for the DFRFT defined as (17)–(21). Since, from (11), the inverse of $F_\alpha(m,n)$ is just its Hermitian, i.e., $F_\alpha^*(n,m)$, and if we define $F_\alpha(m,n)$ as (16), then we find

$$F_{-\alpha,\Delta u,\Delta t}(m,n) = F^*_{\alpha,\Delta t,\Delta u}(n,m). \tag{22}$$

In the above equation, we notice that the sampling interval of the input (the second subscript) and the sampling interval of the output (the third subscript) are exchanged. Then, we can conclude that

$$O_{\mathrm{DFRFT}}^{-\alpha,\Delta u,\Delta t}\left(O_{\mathrm{DFRFT}}^{\alpha,\Delta t,\Delta u}(y(n))\right) = y(n) \tag{23}$$

that is, the DFRFT of order $-\alpha$ with the sampling interval Δu in the input and Δt at the output will be the inverse of the DFRFT of order α with the sampling interval Δt in the input and Δu at the output. This is the reversible property of the DFRFT of type 1.

As the continuous FRFT, the DFRFT of type 1 also has the periodic property, that is

$$Y_\alpha(-m) = Y_{\alpha+\pi}(m) \quad Y_\alpha(m) = Y_{\alpha+2\pi}(m). \tag{24}$$

The DFRFT of type 1 will have the period of 2π as the continuous FRFT. The DFRFT's of type 1 also have the conjugation property that

$$Y_\alpha(m) = Y^*_{-\alpha}(m) \quad \text{if } y(n) \text{ is real.} \tag{25}$$

The rest important properties will be derived in the Section III.

Although the DFRFT introduced here has no additivity property, it can be convertible, that is, we can convert the DFRFT with some set of parameters into the DFRFT with another set of parameters. Suppose we use Δt, Δu_1 for the DFRFT with the parameter α_1 and use Δt, Δu_2 for the DFRFT with the parameter α_2 (The sampling interval in both time domains is the same and fixed). Then, from (16), we find

$$\begin{aligned} Y_{\alpha_2}(m) &= \sqrt{\frac{|\sin\alpha_2| - j\,\mathrm{sgn}(\sin\alpha_2)\cos\alpha_2}{2M+1}} \\ &\quad \cdot e^{\frac{j}{2}\cdot\cot\alpha_2\cdot m^2\Delta u_2^2} \cdot \sum_{n=-N}^{N} e^{-j\cdot\frac{2\pi\cdot\mathrm{sgn}(\sin\alpha_2)\cdot n\cdot m}{2M+1}} \\ &\quad \times e^{\frac{j}{2}\cdot(\cot\alpha_2-\cot\alpha_1)\cdot n^2\Delta t^2} e^{\frac{j}{2}\cdot\cot\alpha_1\cdot n^2\Delta t^2} \cdot y(n) \\ &= \sqrt{\frac{|\sin\alpha_2| - j\,\mathrm{sgn}(\sin\alpha_2)\cos\alpha_2}{2M+1}} \\ &\quad \cdot e^{\frac{j}{2}\cdot\cot\alpha_2\cdot m^2\Delta u_2^2} \cdot \sum_{r=-M}^{M} h(((m-r))_{2M+1}) \\ &\quad \cdot \sum_{n=-N}^{N} e^{-j\cdot\frac{2\pi\cdot\mathrm{sgn}(\sin\alpha_2)\cdot n\cdot r}{2M+1}} \cdot e^{\frac{j}{2}\cdot\cot\alpha_1\cdot n^2\Delta t^2} \cdot y(n) \end{aligned}$$

where $h(m)$ is the DFT or IDFT of $\exp(j(\cot\alpha_2 - \cot\alpha_1)n^2\Delta t^2/2)$

$$\begin{aligned} h(m) &= \frac{1}{2M+1}\sum_{n=-N}^{N} e^{-j\cdot\frac{2\pi\cdot\mathrm{sgn}(\sin\alpha_2)n\cdot m}{2M+1}} \\ &\quad \times e^{\frac{j}{2}\cdot(\cot\alpha_2-\cot\alpha_1)\cdot n^2\Delta t^2}. \end{aligned} \tag{26}$$

In addition

$$((n))_{2M+1} = n + D(2M+1)$$

where

$$D \text{ is some integer such that } |n + D(2M+1)| \le M. \tag{27}$$

Therefore, we obtain

$$\begin{aligned} Y_{\alpha_2}(m) &= \sqrt{\frac{|\sin\alpha_2| - j\,\mathrm{sgn}(\sin\alpha_2)\cos\alpha_2}{|\sin\alpha_1| - j\,\mathrm{sgn}(\sin\alpha_1)\cos\alpha_1}} \\ &\quad \cdot e^{\frac{j}{2}\cot\alpha_2\cdot m^2\Delta u_2^2} \cdot \sum_{r=-M}^{M} h(((m-r))_{2M+1}) \\ &\quad \cdot e^{-\frac{j}{2}\cot\alpha_1\cdot r^2\Delta u_1^2} Y_{\alpha_1}\left(\frac{\mathrm{sgn}(\sin\alpha_1)}{\mathrm{sgn}(\sin\alpha_2)}r\right). \end{aligned} \tag{28}$$

Although the DFRFT defined in (17)–(21) has no additive property, if we fix Δt, then we can convert the transform result of the DFRFT with order α_1 into order α_2 by two chirp multiplications and one convolution operation.

We note, in (19), that if $|\sin\alpha|$ is very small, then Δt and Δu must also be very small, and the number of points must increase. This will increase the computation time of the DFRFT because for the continuous FRFT

$$O_F^\alpha(f(t)) = O_F^{\alpha-\pi/2}(\mathrm{FT}(f(t)))$$

so when $|\sin\alpha|$ is very small, we can first do the forward DFT for the sampling of $f(t)$ and do the DFRFT defined as (17) or (18) with the order $\alpha - \pi/2$. Thus, we can change the DFRFT of type 1 to

$$\begin{aligned} Y_\alpha(m) &= C\cdot e^{-\frac{j}{2}\cdot\tan\alpha\cdot m^2\Delta u^2} \\ &\quad \times \sum_{r=-N}^{N}\sum_{n=-N}^{N} e^{j\cdot\frac{2\pi\cdot\mathrm{sgn}(\cos\alpha)\cdot r\cdot m}{2M+1}} \\ &\quad \times e^{-\frac{j}{2}\cdot\tan\alpha\cdot r^2\Delta f^2} e^{-j\cdot\frac{2\pi\cdot n\cdot r}{2N+1}} \cdot y(n) \end{aligned} \tag{29}$$

where

$$\begin{aligned} \Delta u\cdot\Delta f &= 2\pi\cdot|\cos\alpha|/(2M+1) \\ C &= \sqrt{\frac{|\cos\alpha| + j\,\mathrm{sgn}(\cos\alpha)\sin\alpha}{(2M+1)(2N+1)}}. \end{aligned} \tag{29a}$$

Then, because

$$\Delta t\cdot\Delta f = 2\pi/(2N+1), \quad \Delta f = 2\pi/(\Delta t(2N+1))$$

for the case that $|\sin\alpha|$ is small, we can define the DFRFT as follows.

- **Modification form of the DFRFT of type 1 when $|\sin\alpha| \approx 0$**

$$\begin{aligned} Y_\alpha(m) &= C\cdot e^{-\frac{j}{2}\cdot\tan\alpha\cdot m^2\Delta u^2} \\ &\quad \times \sum_{r=-N}^{N}\sum_{n=-N}^{N} e^{j\cdot\frac{2\pi\cdot\mathrm{sgn}(\cos\alpha)\cdot r\cdot m}{2M+1}} \\ &\quad \times e^{-\frac{j\cdot 2\pi^2\tan\alpha\cdot r^2}{(2N+1)^2\Delta t^2}} e^{-j\cdot\frac{2\pi\cdot n\cdot r}{2N+1}} \cdot y(n) \end{aligned} \tag{30}$$

where C is the same as (29a), and the constraint for Δt becomes

$$\Delta u = (2N+1)\cdot|\cos\alpha|\cdot\Delta t/(2M+1). \tag{31}$$

When $|\sin\alpha|$ is small, we can use (30) as the DFRFT.

The DFRFT of type 1 has a very important advantage, that is, it is *efficient to calculate and implement*. Because there are two chirp multiplications and one FFT, the total number of the multiplication operations required is $2P + (P/2)\cdot\log_2 P$, where $P = 2M+1$ is the length of the output. Among all types of DFRFT, the linear combination type DFRFT [6]–[8], [24] will have the least complexity and only require $(P/2)\cdot\log_2 P$ multiplication operations. However, it does not match the continuous FRFT and lacks many of the characteristics of the continuous FRFT. For example, it is hard to filter out the chirp noise with this type of DFRFT. For most of the other types of DFRFT, such as the improved directly sampling-type DFRFT [5], we need $2P + P\cdot\log_2 P$ multiplication operations because there is one convolution operation and two chirp multiplication operations required. The DFRFT we introduce will have the lowest complexity among all types of DFRFT that still work similarly to the continuous FRFT.

3) Applications for Calculating the Continuous FRFT: We can use the DFRFT of type 1 to calculate the continuous FRFT. When using the DFRFT for this application, we first sample the input continuous function into a discrete sequence, do the forward DFRFT, and get the output of DFRFT as the sampling of the transform results of continuous FRFT. We note that because when we derive the DFRFT of type 1, we have normalized the

unitary [from (14) to (15)]. Thus, when using the DFRFT of type 1 to implement the continuous FRFT, we must consider this normalization factor, that is, if

$$Y_\alpha(f) = O_F^\alpha(y(t)), \quad \tilde{y}(n) = y(n\Delta t)$$
$$\tilde{Y}_\alpha(m) = O_{\mathrm{DFRFT}}^\alpha(\tilde{y}(n)) \tag{32}$$

then

$$Y_\alpha(f)|_{f=m\Delta u} = Y_\alpha(m\Delta u) \approx \sqrt{\frac{2M+1}{2\pi|\sin\alpha|}} \cdot \Delta t \cdot \tilde{Y}_\alpha(m). \tag{33}$$

We will use some examples to discuss this.

In Figs. 1 and 2, we will give some examples to illustrate the application of DFRFT for calculating the continuous FRFT. The original continuous input functions are:

Fig. 1: $f(t) = \Pi(t/4.5)$ (rectangular)

Fig. 2: $f(t) = \Lambda(t/2.5)$ (triangular).

Then, we sample the input function with the sampling interval $\Delta t = 0.02$

Fig. 1: $x(n) = \Pi(n/225)$ (rectangular)

Fig. 2: $x(n) = \Lambda(n/125)$

and use the DFRFT of order $\alpha = 0.05, 0.2, 0.4, \pi/4$, and $N = 150$. The value of M is chosen as

for $\alpha = 0.05, 0.2$: $M = 150$

for $\alpha = 0.4$: $M = 225$

for $\alpha = \pi/4$: $M = 300$.

We can compare the transform results for the rectangular function with the results of the continuous FRFT in [2]. We find that the transform results of the DFRFT are similar to the ones of the continuous FRFT in these two examples. The closed form of the continuous FRFT of rectangular function is derived in [24], and the continuous FRFT of the triangular function can be calculated from the numerical method. We use these results to calculate our errors of the transform results in Figs. 1 and 2. The error is calculated from

$$\mathrm{err} = \sum_{m=-Ma}^{Ma} \left| \sqrt{\frac{2M+1}{2\pi|\sin\alpha|}} \cdot \Delta t \cdot \tilde{Y}_\alpha(m) - Y_\alpha(m\Delta u) \right|^2 \Bigg/ \sum_{m=-Ma}^{Ma} |Y_\alpha(m\Delta u)|^2 \tag{34}$$

where $Y_\alpha(m)$ and $\tilde{Y}_\alpha(m)$ are defined as (32), and Ma is the largest integer such that $Ma \cdot \Delta u < 7$ (we just consider the interval that $m\Delta t \in [-7, 7]$ for simplification). Then, we obtain the error as in Table I. In fact, when choosing the same value of sampling interval in the time and frequency domains and the same number of points, the error of our DFRFT when used to calculate the continuous FRFT will be the same as the DFRFT introduced in [5]. However, our DFRFT will only require about half of the computation of the DFRFT introduced in [5], and many of the constraints in [5], such as the original signal, must be bandlimited in all the domains ($Y_\alpha(u) \approx 0$ when $|u| < B_\alpha$ for all the value of α) will be not required here.

Not all the input functions of the FRFT will be time limited, as is the above experiment. If the input function has very long time duration, we will modify the above process a little. We will cut the input function into several subsections with short time duration and sample them

$$y_h(t) = y(t - h \cdot L) \cdot \Pi(t/L), \quad h = -\infty, \ldots, -1, 0, 1, \ldots, \infty \tag{35a}$$
$$\tilde{y}_h(m \cdot \Delta t) = y_h(m \cdot \Delta t) \quad \text{where } m \in [-N, N], \quad (2N+1)\Delta t = L \tag{35b}$$

and input them into DFRFT. We will use the shifting property for the continuous FRFT [2]

$$O_F^\alpha(f(t-\tau)) = e^{j\frac{\tau^2}{2}\sin\alpha\cos\alpha} e^{-ju\tau\sin\alpha} \cdot F_\alpha(u - \tau\cos\alpha). \tag{36}$$

Thus, we require that $\tau\cos\alpha$ ($\tau = L$) must be the multiple integers of Δu

$$L \cdot \cos\alpha = C \cdot \Delta u \quad \text{where } C \text{ is some integer.}$$

Then, from $(2N+1)\Delta t = L$ and the relation of (19), we see that Δt must satisfy

$$\Delta t = \sqrt{\frac{2\pi}{(2M+1)(2N+1)} \frac{|\sin\alpha|}{\cos\alpha} \cdot C} \quad \text{where } \mathrm{sgn}(C) = \mathrm{sgn}(\cos\alpha). \tag{37}$$

Thus, if we choose Δt as(37), then together with the shift property, we obtain

$$\tilde{Y}_{\alpha,h}(m) = O_{\mathrm{DFRFT}}^\alpha(\tilde{y}_h(n)) \tag{38a}$$
$$\tilde{Y}_\alpha(m) = \sum_{h=-\infty}^{\infty} e^{j\left(\frac{h\cdot C}{2} - m\right)h\cdot C\cdot\tan\alpha\cdot\Delta u^2} \cdot \tilde{Y}_{\alpha,h}(m - h \cdot C) \tag{38b}$$

and we can obtain the approximated value of $Y_\alpha(f)$ (the continuous FRFT of $y(t)$) from

$$Y_\alpha(f)|_{f=m\Delta u} = Y_\alpha(m\Delta u) \approx \sqrt{\frac{2M+1}{2\pi|\sin\alpha|}} \cdot \Delta t \cdot \tilde{Y}_\alpha(m). \tag{38c}$$

Therefore, for very long input, we can also use the DFRFT of type 1 to compute the transform results of the continuous FRFT [but the sampling interval Δt must be chosen as (37)]. In Fig. 3, we will show an example. Here, we use

$$f(t) = e^{-(t-\tau)^2\sigma/2} \tag{39}$$

as the input of continuous FRFT. The transform result of $f(t)$ for continuous FRFT is

$$F_\alpha(u) = \sqrt{\frac{1 - j\cot\alpha}{c - j\cot\alpha}} \cdot e^{\frac{j}{2}\tau^2\sin\alpha\cos\alpha - ju\tau\cos\alpha} \cdot e^{\frac{j(\sigma^2-1)\cot\alpha - \sigma\csc^2\alpha}{2(\sigma^2+\cot^2\alpha)}\cdot(u-\tau\cos\alpha)^2}. \tag{39a}$$

We will choose $\tau = 1.6025$, $\sigma = 0.1$, and $\alpha = \pi/6$. In (37), we choose

$$C = 20, \quad M = 250, \quad N = 70$$

and then, $\Delta t = 0.032\,05$, $\Delta u = 0.1957$. We then use the DFRFT to compute the transform result of $f(t)$ for continuous FRFT by the method from (35a)–(38c), and in (35a), we choose $h = -2{\sim}2$ and $-5{\sim}5$. We plot the result in the Fig. 3. Then, we also use (34) to compute the error and obtain

$$\mathrm{err} = 1.463 \cdot 10^{-2} \quad \text{for } h = -2{\sim}2$$
$$\mathrm{err} = 7.1563 \cdot 10^{-5} \quad \text{for } h = -5{\sim}5.$$

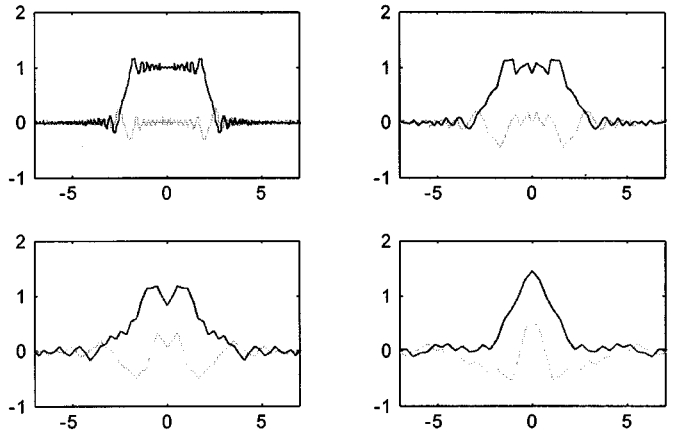

Fig. 1. DFRFT for the rectangular function $x(n) = \Pi(n/225)$, i.e., $f(t) = \Pi(t/4.5)$. Upper left: $\alpha = 0.05$. Upper right: $\alpha = 0.2$. Lower left: $\alpha = 0.4$, Lower right: $\alpha = \pi/4$.

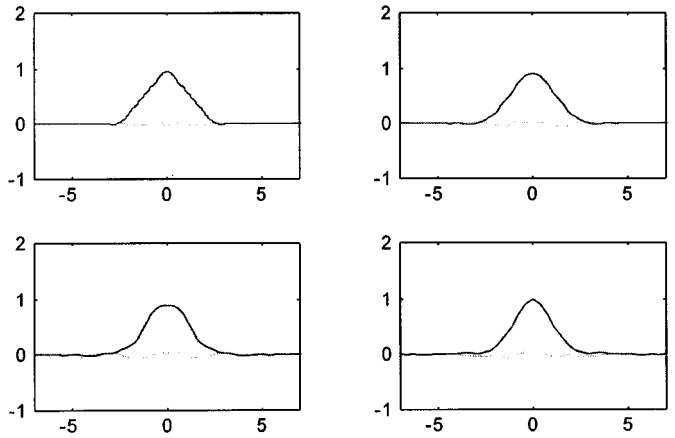

Fig. 2. DFRFT for the triangular function $x(n) = \Lambda(n/125)$, i.e., $f(t) = \Lambda(t/2.5)$). Upper left: $\alpha = 0.05$. Upper right: $\alpha = 0.2$. Lower left: $\alpha = 0.4$. Lower right: $\alpha = \pi/4$.

When we use the DFRFT to calculate the continuous FRFT, we should consider its precision. There are two main constraints that must be satisfied for precision. First, the value of $x(t)$ can be ignored outside $[-L, L]$

$$\text{(constraint 1):} \int_{-L}^{L} |x(t)| \cdot dt \Big/ \int_{-\infty}^{\infty} |x(t)| \cdot dt \approx 1$$

$$\text{where } L = N\Delta t = \frac{2\pi N \sin\alpha}{(2M+1)\Delta u}.$$

Second, we must consider the aliasing effect of the sampling. Consider first the bandwidth of the term

$$\exp(j \cot\alpha \cdot t^2/2) \cdot x(t). \tag{40}$$

Because

$$\text{Max}\left(\left|\frac{d}{dt}\frac{1}{2}\cot\alpha \cdot t^2\right|\right) = \text{Max}(|\cot\alpha \cdot t|) = |\cot\alpha| \cdot L = |\cot\alpha| \cdot N\Delta t$$

if the bandwidth of $x(t)$ is W, then the bandwidth of (40) is $W + |\cot\alpha| \cdot N\Delta t$. Then, from the sampling theory

$$\text{(constraint 2): } (\Delta t)^{-1} > 2W + |\cot\alpha| \cdot 2N\Delta t = 2W + |\cot\alpha| \cdot 2L.$$

There are some remarks about the above two constraints.

1) From (40), we find that if the value of L (the effective width of the input signal) increases, then Δt must decrease, that is, the sampling interval in the time domain will also depend on the effective width of the input signal.
2) When W (the bandwidth of $f(t)$) increases and L is fixed, then Δt must be decreased, and N should be increased.
3) When $|\cot\alpha|$ increases, i.e., $\alpha \to 0, \pi$, and L is fixed, then Δt must be decreased, and N will be increased.

B. Closed-Form Discrete Affine Fourier Transform of Type 1

We can use a similar way to derive the DAFT that is analogous to the continuous case in (3). Similar to the process to derive the DFRFT, we find that if Δt and Δu satisfy

$$\Delta u \cdot \Delta t = 2\pi \cdot |b|/(2M+1) \tag{41}$$

and $M \geq N$, then the transform matrix $F_{(a,b,c,d)}(m,n)$ will be reversible, where

$$F_{(a,b,c,d)}(m,n) = \sqrt{\frac{1}{2M+1}} \cdot e^{j\frac{d}{2b}\cdot m^2\Delta u^2} \cdot e^{-j\frac{2\pi\cdot\text{sgn}(b)\cdot n\cdot m}{2M+1}} \cdot e^{j\frac{a}{2b}\cdot n^2\Delta t^2} \tag{42}$$

and $m \in [-M, M]$, $n \in [-N, N]$. Thus, the DAFT of type 1 is as follows.

- **DAFT of type 1**

1)

$$Y_{(a,b,c,d)}(m) = \sqrt{\frac{1}{2M+1}} \cdot e^{j\frac{d}{2b}\cdot m^2\Delta u^2} \sum_{n=-N}^{N} e^{-j\cdot\frac{2\pi\cdot n\cdot m}{2M+1}} \times e^{j\frac{a}{2b}\cdot n^2\Delta t^2} \cdot y(n) \quad \text{when } b > 0 \tag{43}$$

2)

$$Y_{(a,b,c,d)}(m) = \sqrt{\frac{1}{2M+1}} \cdot e^{j\frac{d}{2b}\cdot m^2\Delta u^2} \sum_{n=-N}^{N} e^{j\cdot\frac{2\pi\cdot n\cdot m}{2M+1}} \times e^{j\frac{a}{2b}\cdot n^2\Delta t^2} \cdot y(n) \quad \text{when } b < 0. \tag{44}$$

The above DFRFT defined as (17), and (18) is a special case of the DAFT wherein $\{a, b, c, d\} = \{\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha\}$. The reversible property for the DAFT of type 1 is

$$O_{\text{DAFT}}^{(d,-b,-c,a),\Delta u,\Delta t}\left(O_{\text{DAFT}}^{(a,b,c,d),\Delta t,\Delta u}(y(n))\right) \tag{45}$$

where $O_{\text{DAFT}}^{(a,b,c,d),\Delta t,\Delta u}(y(n))$ is the DAFT of $y(n)$ for parameters $\{a, b, c, d\}$ and where the sampling interval Δt is the input, and Δu is the output. This reversible property is the same as the continuous AFT.

As for the case of DFRFT, when $b = 0$, we also find that the DAFT fails to be defined from (43) and (44) because the right side of (41) will be 0. However, for $b = 0$, the continuous affine Fourier transform results will be

$$O_F^{(a,0,c,d)}(f(t)) = \sqrt{|d|} \cdot e^{\frac{j}{2}\cdot cd\cdot u^2} \cdot f(du) \tag{46}$$

or

$$O_F^{(a,0,c,d)}(f(t)) = O_F^{(0,-a,d,-c)}(F(w)) \tag{47}$$

where $F(w)$ is the Fourier transform of $f(t)$. Therefore, we can define the DAFT, when $b = 0$, as follows.

1)

$$Y_{(a,0,c,d)}(m) = \sqrt{d} \cdot e^{\frac{j}{2}\cdot cd\cdot m^2\Delta u^2} \cdot y(d\cdot m) \quad \text{when } b = 0, \quad d \text{ is an integer.} \tag{48}$$

TABLE I
ERRORS FOR USING DFRFT OF TYPE 1 TO COMPUTE THE CONTINUOUS DFRFT

	$\alpha = 0.05$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = \pi/4$
Rectangle	1.191×10^{-4}	8.793×10^{-6}	3.985×10^{-6}	2.172×10^{-6}
Triangle	4.304×10^{-8}	7.111×10^{-9}	3.600×10^{-9}	1.825×10^{-9}

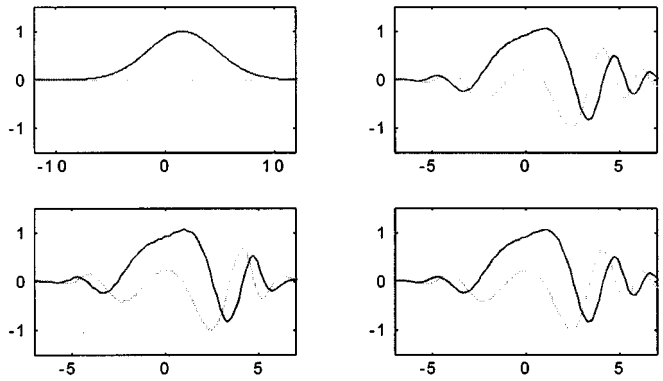

Fig. 3. Experiment for using DFRFT to compute the transform result of continuous FRFT for the signal with nonfinite time duration. Upper-left: Input [see (39)]. Upper-right: Exact transform result [see (39a)]. Lower left: Transform results calculated from DFRFT with $h = -2\sim2$. Lower right: Transform results calculated from DFRFT with $h = -5\sim5$.

2)

$$Y_{(a,0,c,d)}(m) = \sqrt{\frac{1}{R}} \cdot e^{j\frac{c}{2a}\cdot m^2\Delta u^2} \cdot \sum_{n=-N}^{N}\sum_{k=-N}^{N} e^{j\cdot\frac{2\pi\cdot\mathrm{sgn}(a)\cdot k\cdot m}{2M+1}} \cdot e^{-j\cdot\frac{2\pi\cdot n\cdot k}{2N+1}} \cdot y(n) \quad \text{when } b=0, \quad d \text{ is not an integer.} \tag{49}$$

In (49), $R = (2M+1)\cdot(2N+1)$, and

$$\Delta u = (2N+1)\cdot|a|\cdot\Delta t/(2M+1). \tag{50}$$

We note that it is no problem for (48) to be reversible. Besides, (49) is also reversible

$$y(n) = \sqrt{\frac{1}{R}} \cdot \sum_{n=-N}^{N}\sum_{k=-M}^{M} e^{j\cdot\frac{2\pi\cdot n\cdot k}{2N+1}} \cdot e^{j\cdot\frac{2\pi\cdot k\cdot m}{2M+1}} \cdot e^{-j\frac{c}{2a}\cdot m^2\Delta u^2} \cdot Y_{(a,0,c,d)}(m). \tag{51}$$

From the constraint of (41), we find that (43) and (44) can also be written as

$$Y_{(a,b,c,d)}(m) = \sqrt{\frac{1}{2M+1}} \cdot e^{j\frac{2\pi^2\cdot d\cdot b}{(2M+1)^2\Delta t^2}\cdot m^2} \sum_{n=-N}^{N} \times e^{-j\cdot\frac{2\pi\cdot\mathrm{sgn}(b)\cdot n\cdot m}{2M+1}} e^{j\frac{a}{2b}\cdot n^2\Delta t^2} \cdot y(n) \ (b\neq 0). \tag{52}$$

Thus, if we fix Δt and M, then only the values of a/b, $\mathrm{sgn}(b)$, $b\cdot d$ will affect the transform result. Some important equality relations for the DAFT of type 1 are

$$Y_{(a,b,c,d)}(m) = Y_{(-a,-b,-c,-d)}(-m) \tag{53}$$

$$Y^*_{(a,b,c,d)}(m) = Y_{(a,-b,-c,d)}(m). \tag{54}$$

Here we have assumed that the values of Δt and M are fixed.

The number of computations for the DAFT will also be proportional to $2P + (P/2)\cdot\log_2 P$, where $P = 2M+1$. This is the same as the case of the DFRFT.

The DAFT has no additivity property, but it is convertible. That is, we can convert the DAFT with some set of parameters into another set of parameters. Suppose we use Δt, Δu_1 for the DAFT with the parameter $\{a_1, b_1, c_1, d_1,\}$ and use Δt, Δu_2 for the DAFT with the parameter $\{a_2, b_2, c_2, d_2,\}$ (Δt and M are fixed). Then, as for the case of DFRFT, we find that the following relation can be satisfied:

$$Y_{a_2,b_2,c_2,d_2}(m) = e^{\frac{j\cdot d_2}{2b_2}\cdot m^2\Delta u_2^2} \sum_{r=-M}^{M} h(((m-r))_{2M+1}) \cdot e^{-\frac{j\cdot d_1}{2b_1}\cdot r^2\Delta u_1^2} Y_{a_1,b_1,c_1,d_1}\left(\frac{\mathrm{sgn}(b_2)\cdot r}{\mathrm{sgn}(b_1)}\right) \tag{55}$$

where $((n))_{2M+1}$ is defined as (27), and $h(m)$ can be obtained from the DFT of a chirp

$$h(m) = \frac{1}{2M+1}\sum_{n=-N}^{N} e^{j\cdot\frac{2\pi\cdot\mathrm{sgn}(b_2)\cdot n\cdot m}{2M+1}} e^{\frac{j}{2}\cdot\left(\frac{a_2}{b_2}-\frac{a_1}{b_1}\right)\cdot n^2\Delta t^2}. \tag{56}$$

In addition, we can use the DAFT to compute the continuous affine Fourier transform, and then, as in the case of the DFRFT, the following two constraints must be satisfied:

$$\text{constraint 1: } \int_{-L}^{L}|x(t)|\cdot dt \Big/ \int_{-\infty}^{\infty}|x(t)|\cdot dt \approx 1 \quad \text{where } L = N\Delta t = \frac{2\pi Nb}{(2M+1)\Delta u} \tag{57}$$

$$\text{constraint 2: } \frac{1}{\Delta t} > 2W + \left|\frac{a}{b}\right|\cdot 2N\Delta t = 2W + \left|\frac{a}{b}\right|\cdot 2L. \tag{58}$$

The above three remarks for using DFRFT to calculate the continuous FRFT listed in Section II-A will also be applied here.

C. Closed-Form Fractional and Affine Fourier Transform of Type 2

In general, we can use the discrete transform to do the following: 1) Compute the continuous transform for spectral analysis, and 2) suit for processing the discrete data signals. For the former case, the input of the discrete transform is the sampling of some continuous function. For the later case, the input is just a pure discrete sequence. For example, we can use DFT for computing the continuous Fourier transform, and the input is the sampling of some continuous function. Meanwhile, we can also use DFT for other digital signal processing applications, and in this case, the input is inherently a discrete sequence itself, such as the daily stock market or checking account, etc. Similarly, except for computing the continuous FRFT/AFT, we can also use DFRFT/DAFT for some other applications and just use them as the discrete data transforms.

When we use DFRFT and DAFT to compute the continuous FRFT/AFT, the mathematical form of the DFRFT and DAFT

should be almost the same as the continuous FRFT and AFT. Thus, in Sections II-A and II-B, we derive the DFRFT and DAFT from the sampling of kernels of the continuous FRFT and AFT. However, when we use DFRFT and DAFT for some other applications, the above requirement, such as phase alignment, is not necessary. We make DFRFT and DAFT remain the simple basic structures of FRFT and AFT, but they have the same abilities and are easier to compute and design.

We will derive the DAFT of type 2 from the transform matrix of the DAFT of type 1 and then simplify it into the DFRFT of type 2. The DAFT of type 1 defined as (43) and (44) has too many parameters. We can try to simplify it and set $p = (d/b)\cdot\Delta u^2$, $q = (a/b)\cdot\Delta t^2$. Then

$$F_{(p,q)}(m,n) = \sqrt{\frac{1}{2M+1}}\cdot e^{\frac{j}{2}\cdot p\cdot m^2} \cdot e^{j\cdot\frac{2\pi\cdot\mathrm{sgn}(b)\cdot n\cdot m}{2M+1}}\cdot e^{\frac{j}{2}\cdot q\cdot n^2}. \tag{59}$$

Then, from $\Delta t\cdot\Delta u = 2\pi|b|/(2M+1)$, we find

$$p\cdot q = (2\pi/(2M+1)^2)\cdot ad. \tag{60}$$

Because a, d can be any real value, there will be no constraint for p, q, and p, q can be any real value. Thus, the DFRFT matrix defined as (59) will have three parameters p, q, b without any constraint and has the free dimension of 3.

We note that the continuous affine Fourier transform has four parameters $\{a, b, c, d\}$ plus one constraint and has the free dimension of 3 in total. Although, in (59), the free dimension is also 3, but the value of $\mathrm{sgn}(b)$ can only be ±1, in fact, the free dimension is near 2. Thus, as in (12), which has a parameter s in the Fourier term, we can also put a parameter s into the Fourier transform term of (59)

$$F_{(p,q,s)}(m,n) = \sqrt{\frac{1}{2M+1}}\cdot e^{\frac{j}{2}\cdot p\cdot m^2} \cdot e^{-j\frac{2\pi\cdot s\cdot n\cdot m}{2M+1}}\cdot e^{\frac{j}{2}\cdot q\cdot n^2} \tag{61}$$

where s is prime to $2M+1$. Then, we find that the reversible property will be satisfied:

$$y(n) = \sum_{m=-M}^{M}\sum_{k=-N}^{N} F^*_{(p,q,s)}(m,n)\cdot F_{(p,q,s)}(m,k)\cdot y(k). \tag{62}$$

Therefore, we can define the discrete affine Fourier transform (DAFT) as follows.

- **DAFT of type 2**

$$Y_{(p,q,s)}(m) = \sqrt{\frac{1}{2M+1}}\cdot e^{\frac{j}{2}\cdot m^2\cdot p}\cdot\sum_{n=-N}^{N} e^{-j\cdot\frac{2\pi\cdot s\cdot n\cdot m}{2M+1}} \cdot e^{\frac{j}{2}\cdot n^2\cdot q}\cdot y(n) \tag{63}$$

where

$$M \geq N\ (2N+1, 2M+1 \text{ are the number of points in the time, frequency domain})$$
$$s \text{ is prime to } M \tag{64}$$

and its inverse transform is the following.

- **Inverse DAFT of type 2**

$$y(n) = \sqrt{\frac{1}{2M+1}}\cdot e^{-\frac{j}{2}\cdot n^2\cdot q}\cdot\sum_{m=-M}^{M} e^{j\cdot\frac{2\pi\cdot s\cdot n\cdot m}{2M+1}} \cdot e^{-\frac{j}{2}\cdot m^2\cdot p}\cdot Y_{(p,q,s)}(m). \tag{65}$$

We note that when $M = N$, the inverse transform with the parameters $\{p, q, s\}$ is just the same as forward transform with the parameters $\{-q, -p, -s\}$. Thus, when $M = N$, the DAFT with the parameters $\{-q, -p, -s\}$ is the inverse of the DAFT with the parameters $\{p, q, s\}$

$$O_{\mathrm{DAFT}}^{(-q,-p,-s)}\left(O_{\mathrm{DAFT}}^{(p,q,s)}(y(n))\right) = y(n) \quad \text{when } M = N. \tag{66}$$

In this paper, we use $O_{\mathrm{DAFT}}^{(p,q,s)}$ to denote the DAFT of type 2

$$O_{\mathrm{DAFT}}^{(p,q,s)}(y(n)) = Y_{(p,q,s)}(m) = \sum_{n=-N}^{N} F_{(p,q,s)}(m,n)\cdot y(n). \tag{67}$$

We note, from (1) and (3), that the continuous FRFT is a special case of the continuous AFT in that the inner and outer chirp terms have the same parameters ($a/b = d/b = \cot\alpha$). In the similar way, we can define the DFRFT from the DAFT by setting $p = q$ and $s = \pm1$ and obtain the following.

- **DFRFT of type 2**

$$Y_{(p)}(M) = \sqrt{\frac{1}{2M+1}}\cdot e^{\frac{j}{2}\cdot m^2\cdot p}\cdot\sum_{n=-N}^{N} e^{\pm j\cdot\frac{2\pi\cdot n\cdot m}{2M+1}} \cdot e^{\frac{j}{2}\cdot n^2\cdot p}\cdot y(n) \quad \text{where } M \geq N. \tag{68}$$

We call the above DFRFT/DAFT defined in Sections II-A and B the DFRFT/DAFT of type 1. They are suitable for calculating the continuous FRFT/AFT. In addition, we call the DFRFT and DAFT defined in (68) and (63) the DFRFT/DAFT of type 2. The DFRFT/DAFT of type 2 are simple and suitable for other digital signal processing applications.

For the DAFT of type 2, we can use p and q to control the variation of the chirp in the frequency and time domains. When $p \to 0$, the DAFT defined as (63) will be similar to a chirp multiplication operation followed by a DFT. In addition, when $q \to 0$, the DAFT will be similar to the DFT followed by a chirp multiplication operation. When $p = q$, then the transform matrix $F_{(p,p,s)}(m,n)$ will be a symmetry matrix, i.e., $F_{(p,p,s)}(m,n) = F_{(p,p,s)}(n,m)$, and the transform matrix for the inverse DAFT is just the conjugate of $F_{(p,p,s)}(m,n)$.

The DAFT/DFRFT of type 2 also need $2P + (P/2)\cdot\log_2 P$ multiplication operations. They also have no additive property, but they are convertible. We can calculate the DAFT with the parameters $\{p_2, q_2, s_2\}$ from the DAFT with the parameters $\{p_1, q_1, s_1\}$ from

$$Y_{p_2,q_2,s_2}(m) = e^{\frac{j}{2}p_2\cdot m^2}\sum_{r=-M}^{M} h(((s_2\cdot m - r))_{2M+1}) \cdot e^{-\frac{j}{2}p_1\cdot\left((s_1^{-1})_{2M+1}r\right)^2} Y_{p_1,q_1,s_1}\left((s_1^{-1})_{2M+1}\, r\right) \tag{69}$$

where $((\,))_{2M+1}$ is the modulo symbol defined as (27), and $(s_1^{-1})_{2M+1}$ is defined as

$$\left(\left(s_1\cdot(s_1^{-1})_{2M+1}\right)\right)_{2M+1} = 1. \tag{70}$$

$h(m)$ can also be calculated from the DFT of a chirp

$$h(m)=\frac{1}{2M+1}\sum_{n=-N}^{N}e^{-j\cdot\frac{2\pi\cdot n\cdot m}{2M+1}}\cdot e^{\frac{j}{2}\cdot(q_2-q_1)\cdot n^2}. \quad (71)$$

After two chirp multiplications and one convolution, we can convert the DAFT with some parameters into the DAFT with other parameters. In the case where $q_1=q_2$

$$Y_{p_2,q_1,s_2}(m)=e^{\frac{j}{2}p_2\cdot m^2}e^{-\frac{j}{2}p_1\cdot\left(\left(\left(s_1^{-1}\right)_{2M+1}s_2\cdot m\right)\right)_{2M+1}\Big)^2}\times Y_{p_1,q_1,s_1}\left(\left(\left(\left(s_1^{-1}\right)_{2M+1}s_2\cdot m\right)\right)_{2M+1}\right). \quad (72)$$

In this case, we can even save the convolution operation. Other important properties of the DFRFT/DAFT of type 2 are introduced in Section III.

We show the relations between DAFT of type 2 and its special cases in Table II.

TABLE II
RELATIONS BETWEEN THE DAFT OF TYPE 2 AND ITS SPECIAL CASES

DFRFT of type 2	$p=q, s=\pm1$
DFRFT of type 1	$p=\cot\alpha\cdot\Delta u^2$, $q=\cot\alpha\cdot\Delta t^2$, $s=\mathrm{sgn}(\sin\alpha)$
DAFT of type 1	$p=d/b\cdot\Delta u^2$, $q=a/b\cdot\Delta t^2$, $s=\mathrm{sgn}(b)$
DFT, IDFT	$p=q=0$, $s=1$ for DFT, $s=-1$ for IDFT

D. Discrete Fractional/Affine Convolutions and Correlations

Since the discrete fractional and affine Fourier transforms have been defined, we can use them to define the discrete fractional and affine convolutions and correlations. We only discuss the affine case, and the rest of the discrete fractional convolution and convolution can be obtained by substituting $\{p,q,s\}$ as $\{p,p,\pm1\}$.

The discrete affine convolution can be defined as follows:

- **Discrete affine convolution**

$$f(n)\underset{p,q,s}{*}g(n)=O_{\mathrm{DAFT}}^{(-q,-p,-s)}\left(O_{\mathrm{DAFT}}^{(p,q,s)}(f(n))\cdot O_{\mathrm{DAFT}}^{(p,q,s)}(g(n))\right). \quad (73)$$

We must remember that the DAFT with the parameters $\{-q,-p,s\}$ is the inverse of the DAFT with the parameters $\{p,q,s\}$. If $G_{p,q,s}(m)=O_{\mathrm{DAFT}}^{(p,q,s)}(g(n))$, then (73) can be rewritten as

$$f(n)\underset{p,q,s}{*}g(n)=\frac{1}{2M+1}e^{-\frac{j}{2}\cdot q\cdot n^2}\sum_{m=-M}^{M}e^{j\frac{2\pi\cdot s\cdot m\cdot e}{2M+1}}\times G_{p,q,s}(m)\sum_{r=-N}^{N}e^{-j\frac{2\pi\cdot s\cdot m\cdot r}{2M+1}}e^{\frac{j}{2}\cdot q\cdot r^2}f(r). \quad (74)$$

We note that the term $\exp(j\cdot p\cdot m^2/2)$ has been cancelled. The above equation can also be written as

$$f(n)\underset{p,q,s}{*}g(n)=\frac{1}{\sqrt{2M+1}}\cdot e^{-\frac{j}{2}\cdot q\cdot n^2}\times\sum_{r=-N}^{N}\tilde{g}_{p,q,s}(((n-r))_{2M+1})\cdot e^{\frac{j}{2}\cdot q\cdot r^2}\cdot f(r) \quad (75)$$

where

$$\tilde{g}_{p,q,s}(n)=\frac{1}{\sqrt{2M+1}}\sum_{m=-M}^{M}e^{j\cdot\frac{2\pi\cdot s\cdot n\cdot m}{2M+1}}\cdot G_{p,q,s}(m). \quad (76)$$

In the case where $p=0$

$$\tilde{g}_{0,q,s}(n)=\frac{1}{\sqrt{2M+1}}\sum_{m=-M}^{M}e^{j\cdot\frac{2\pi\cdot s\cdot n\cdot m}{2M+1}}\sum_{k=-N}^{N}e^{-j\cdot\frac{2\pi\cdot s\cdot m\cdot k}{2M+1}}\cdot e^{\frac{j}{2}\cdot p\cdot k^2}\cdot g(k)=\sqrt{2M+1}\cdot e^{j\cdot q\cdot n^2/2}\cdot g(n).$$

Thus, when $p=0$, the discrete affine convolution can be written as follows.

- **Simplification form of the discrete affine convolution** $(p=0)$

$$f(n)\underset{0,q,s}{*}g(n)=\cdot e^{-\frac{j}{2}\cdot q\cdot n^2}\sum_{r=-N}^{N}e^{\frac{j}{2}\cdot q((n-r))_{2M+1}}\cdot g(((n-r))_{2M+1})\cdot e^{\frac{j}{2}\cdot q\cdot r^2}\cdot f(r). \quad (77)$$

This is just the conventional discrete fractional convolution of $\exp(j\cdot q\cdot n^2/2)\cdot f(n)$ and $\exp(j\cdot q\cdot n^2/2)\cdot g(n)$ and with the extra multiplication of $\exp(-j\cdot q\cdot n^2/2)$. We note that in this case, s will have no effect. Thus, for the simplification, when using the discrete affine convolution, we often set $p=0$. In fact, for practical applications, control of the parameter q is quite sufficient to control performance. The discrete affine convolution can be used in digital filter design, discrete fractional Hilbert transforms, etc.

From the definition of the continuous fractional correlation [18], we can use the similar way to define the discrete affine correlation. If $z(n)$ is the fractional correlation of $x(n)$ and $g(n)$, then we have the following.

- **Discrete affine correlation**

$$z(n)=O_{\mathrm{DAFT}}^{(p3,q3,s3)}\left(O_{\mathrm{DAFT}}^{(p1,q1,s1)}(x(n))\cdot\mathrm{conj}\left(O_{\mathrm{DAFT}}^{(p2,q2,s2)}(g(n))\right)\right). \quad (78)$$

We use conj( ) to denote the conjugation operation. We will denote it as

$$z(n)=x(n)\otimes_{\{(p1,q1,s1),(p2,q2,s2),(p3,q3,s3)\}}g(n). \quad (79)$$

The original discrete correlation is the special case of (78) that $\{p1,q1,s1\}=\{p2,q2,s2\}=\{0,0,1\},\{p3,q3,s3\}=\{0,0,-1\}$. Equation (78) can also be written as

$$z(n)=(2M+1)^{-3/2}\cdot e^{\frac{j}{2}p3\cdot n^2}\times\sum_{m=-M}^{M}e^{-j\frac{2\pi\cdot s3\cdot n\cdot m}{2M+1}}\cdot e^{\frac{j}{2}\cdot(q3+p1-p2)\cdot m^2}\cdot\sum_{n1=-M}^{M}\sum_{n2=-M}^{M}e^{-j\cdot\frac{2\pi\cdot s1\cdot n1\cdot m}{2M+1}}\cdot e^{j\cdot\frac{2\pi\cdot s2\cdot n2\cdot m}{2M+1}}\cdot e^{\frac{j}{2}\cdot q1\cdot n1^2}\cdot e^{-\frac{j}{2}\cdot q2\cdot n2^2}\cdot x(n1)\cdot g^*(n2). \quad (80)$$

For simplification, we set $p1 = p2 = q3 = p3 = 0$, and $s1 = s2$. Then, the above equation becomes

$$z(n) = (2M+1)^{-\frac{3}{2}} \sum_{m=-M}^{M} e^{-j\cdot\frac{2\pi\cdot s3\cdot n\cdot m}{2M+1}} \times \sum_{n1=-M}^{M}\sum_{n2=-M}^{M} e^{-j\cdot\frac{2\pi\cdot s1\cdot(n1-n2)\cdot m}{2M+1}} \times e^{\frac{j}{2}(q1\cdot n1^2 - q^2\cdot n2^2)}\cdot x(n1)\cdot g^*(n2). \quad (81)$$

We obtain the following.

- **Simplification form of the discrete affine correlation**

$$\begin{aligned} z(n) &= x(n) \otimes_{\{(0,q1,s1),(0,q2,s1),(0,q3,0)\}} g(n) \\ &= (2M+1)^{-1/2} \sum_{n1=-M}^{M} e^{\frac{j}{2}\cdot q1\cdot n1^2} \\ &\quad \times e^{-\frac{j}{2}\cdot q2\cdot(((s1^{-1})_{2M+1}s3\cdot n+n1))^2_{2M+1}} \\ &\quad \cdot x(n1)\cdot g^*((((s1^{-1})_{2M+1}s3\cdot n+n1))_{2M+1}) \end{aligned} \quad (82)$$

where $((s))_{2M+1}$, $(s^{-1})_{2M+1}$ are defined in (27) and (70). Equation (81) will be much simpler than (80). Thus, for the simplification, we can set $p1 = p2 = q3 = p3 = 0$ and $s1 = s2$ for discrete affine correlation. The discrete affine correlation can be used for pattern recognition. In Section IV-B, we will illustrate this.

The simplification form of discrete affine convolution/correlation defined as (77) and (81) only require one conventional discrete convolution, and the relations between its input and output are very clear. Therefore, the discrete affine convolution/correlation is easier to implement and analyze. It further enhances the proposed DAFT/DFRFT as a useful tool for digital signal processing.

E. Comparison of Closed-Form DFRFT and DAFT with Other Types of DFRFT

At the end of this section, we will compare the DFRFT and the DAFT introduced in this paper with other types of DFRFT.

The name for each type of DFRFT is as follows.

- *Direct*: direct form of DFRFT;
- *Improved*: improved sampling type DFRFT [5];
- *Linear*: linear combination type DFRFT [6]–[8], [24];
- *Eigenfxs.*: eigenvectors decomposition-type DFRFT [9]–[11], [16];
- *Group*: group theory-type DFRFT [13];
- *Impulse*: impulse train-type DFRFT [14];
- *Proposed*: the DFRFT/DAFT we derive in this paper, i.e., the closed-form DFRFT/DAFT.

Each term of the comparison means the following.

- *Reversible*: whether the DFRFT is reversible;
- *Similarity*: whether the DFRFT is similar to its continuous counterparts;
- *Closed form*: whether the DFRFT can be written in the closed form;
- *Complexity*: the number of multiplication operations required (P is the number of points);
- *FFT*: whether the DFRFT can be implemented by FFT and the number of FFT's required;
- *Constraints*: the number of constraints used for calculating the continuous FRFT;
- *All orders*: whether the DFRFT can be defined for all the order α without constraints;
- *Properties*: the number of properties that can be derived;
- *Addv./Convt.*: whether the DFRFT is additive or convertible (the convertibility means that we can convert the DFRFT with some parameters into the DFRFT with other parameters);
- *DSP*: whether the DFRFT is suitable for the digital signal processing applications.

From Table III, we have seen that there are many advantages for the DFRFT/DAFT defined in this paper. The only disadvantage is that the additivity property is not satisfied. However, this disadvantage will have only a small affect on the practical usage of the DFRFT/DAFT. Besides, we can use the chirp multiplication and convolution to convert the DFRFT/DAFT with some parameters into another set of parameters, i.e., convertible. We give the proposed DFRFT/DAFT "OO" for last the term. This is because of their advantages of reversibility, less complexity, and the characteristic of "fractionalization;" in addition, many properties can be derived. Meanwhile, the DFRFT/DAFT defined here is very simple, and each of the parameters p, q, and s will have the clear roles; thus, the design and the analysis will be very easy for practical applications.

III. Properties of the Discrete Fractional and Affine Fourier Transforms

A. Properties of the DFRFT and DAFT

Because the DFRFT/DAFT we derived are reversible, simple, and can be written in the closed form, their properties can be easily derived. We will discuss the properties of the DFRFT and DAFT in this subsection. Only the properties of the DAFT of type 2 are listed in Table IV and discussed here. The properties of the DFRFT/DAFT of type 1 and the properties of the DFRFT of type 2 can be obtained by the parameters substituting listed in Table I. We will use $F_{(p,q,s)}(m,n)$ to represent the transform matrices of the DAFT, use $O_{\text{DAFT}}^{(p,q,s)}$ to represent the transform operation, use $y(n)$ to represent the input, and use $Y_{(p,q,s)}(m)$ to represent the transform results of $y(n)$.

Some of the properties are proved as follows.

a) Proof of Property 7:

$$\begin{aligned} &O_{\text{DAFT}}^{(p,q,s)}\left(e^{-j\frac{2\pi k}{2M+1}n}y(n)\right) \\ &\quad = \sum_{n=-N}^{N} F_{(p,q,s)}(m,n)\cdot e^{-j\frac{2\pi k}{2M+1}n}y(n) \\ &\quad = e^{\frac{j}{2}\cdot p(-2mk-k^2)}\sum_{n=-N}^{N} F_{(p,q,s)}(m+k,n)\cdot y(n) \\ &\quad = e^{-\frac{j}{2}\cdot p(2mk+k^2)}Y_{(p,q,s)}(m+k). \end{aligned}$$

□

TABLE III
COMPARISONS FOR DIFFERENT TYPES OF DFRFT/DAFT

	Directly	Improved	Linear	Eigenfxs.	Group	Impulse	Proposed
Reversible	×	O *	O	O	O	O	O
Closed form	O	O	O	×	O	O	O
Similarity	O	O	×	O	Δ	O	O
Complexity	P^2	$P\cdot\log_2 P+2P$	$(P/2)\cdot\log_2 P$	$P^2/2$	$P\cdot\log_2 P+2P$	$P\cdot\log_2 P+2P$	$(P/2)\cdot\log_2 P+2P$
FFT	×	2 FFT	1 FFT	×	2 FFT	2 FFT	1 FFT
Constraints	Less	Middle	Unable	Less	Much	Much	Less
All orders	O	O	O	O	×	×	O
Properties	Less	Middle	Middle	Less	Many	Many	Many
Addv./Convt.	No	Convt.	Additive	Additive	Additive	Additive	Convt.
DSP	×	O	O	O	O	O	OO

* Although the improved sampling type of DFRFT is reversible, but the reverse operations would require 3 FFTs computation; The proposed DFRFT needs only 1 FFT computation.

b) Proof of Property 10:

$$\begin{aligned}\sum_{m=-M}^{M}|Y_{(p,q,s)}(m)|^2 &= \sum_{m=-M}^{M} Y_{(p,q,s)}(m)\cdot Y^*_{(p,q,s)}(m)\\ &= \sum_{m=-M}^{M}\sum_{n=-N}^{N}\sum_{r=-N}^{N} F_{(p,q,s)}(m)\\ &\quad\cdot F^*_{(p,q,s)}(m)\cdot y(n)\cdot y^*(r)\\ &= \sum_{n=-N}^{N}\sum_{r=-N}^{N}\delta(n-r)\cdot y(n)\cdot y^*(r)\\ &= \sum_{n=-N}^{N}|y(n)|^2.\end{aligned}$$

From the conjugation property for the transform matrix, we find that the inverse transform

$$\sum_{m=-M}^{M} F^*_{(p,q,s)}(m,n)\cdot Y_{(p,q,s)}(m) = y(n) \tag{83}$$

can be rewritten as

$$\sum_{m=-M}^{M} F_{(-p,-q,s)}(-m,n)\cdot Y_{(p,q,s)}(m) = y(n)$$

$$\sum_{m=-M}^{M} F_{(-p,-q,s)}(m,n)\cdot Y_{(p,q,s)}(-m) = y(n).$$

Then applying property 1, we find that

$$e^{\frac{j}{2}(p-q)\cdot n^2} O_{\mathrm{DAFT}}^{(-p,-q,s)}\left(e^{\frac{j}{2}(q-p)\cdot m^2} Y_{(p,q,s)}(-m)\right) = y(n). \tag{84}$$

Then using the conjugation property for the transform matrix again, we obtain

$$e^{\frac{j}{2}(q-p)\cdot n^2} O_{\mathrm{DAFT}}^{(p,q,s)}\left(e^{\frac{j}{2}(p-q)\cdot m^2} Y^*_{(p,q,s)}(m)\right) = y^*(n). \tag{85}$$

From (85), we can calculate the inverse DAFT from the forward DAFT with the same parameters. (Remember that we can calculate the DFT from the IDFT).

From the modulation property, we find, as the continuous FRFT, after DAFT, that the modulation will partially remain as the modulation and partially become the shifting operation. Similarly, from the time-shifting property, we find that after the DAFT, the shifting operation will partially remain as the shifting operation and partially become the modulation operation. We note that for the original DFT and the IDFT, the shifting operation will totally become the modulation operation, and the modulation operation will totally become the shifting operation.

Thus, from the above discussion, we can say that many of the properties of continuous FRFR/AFT in [2] are also kept in our closed-form DFRFT/DAFT.

We also discuss the scaling property that only exists for the DFRFT and DAFT of type 1. In these cases, the input is the sampling of some continuous function.

• Scaling property for the DFRFT and DAFT of type 1

Suppose $y(n)$ is the sampling of the continuous function $x(t)$ with the interval of Δt

$$y(n) = x(n\cdot\Delta t) \tag{86}$$

and $s(t)$ is the scaling of $x(t)$

$$s(t) = x(\sigma\cdot t). \tag{87}$$

Now, if we sample $s(t)$ with the interval of $\Delta t/\sigma$

$$g(n) = s(n\cdot\Delta t/\sigma) = x(\sigma\cdot n\cdot\Delta t/\sigma) = x(n\cdot\Delta t) \tag{88}$$

then we will find $g(n) = y(n)$. Because the constraint of (19) or (41) must be satisfied, the sampling interval in the fractional

TABLE IV
PROPERTIES OF DAFT

(1) Symmetry property of the transform matrix	$F_{(p,q,s)}(m,n)=e^{\frac{j}{2}(p-q)(m^2-n^2)}\cdot F_{(p,q,s)}(n,m)$ Specially, when $p=q$ (including the case of DFRFT) $F_{(p,q,s)}(m,n)=F_{(p,q,s)}(n,m)$
(2) Conjugation property for the transform matrix	$F^*_{(p,q,s)}(m,n)=F_{(-p,-q,-s)}(m,n)=F_{(-p,-q,s)}(-m,n)$ $=F_{(-p,-q,s)}(m,-n)$
(3) Conjugation property for the transform result	If $Y1_{(p,q,s)}(m)=O^{(p,q,s)}_{DAFT}(y^*(n))$, $Y_{(p,q,s)}(m)=O^{(p,q,s)}_{DAFT}(y(n))$ then $Y^*_{(p,q,s)}(m)=Y1_{(-p,-q,-s)}(m)=Y1_{(-p,-q,s)}(-m)$ When $y(n)$ is pure real: $Y^*_{(p,q,s)}(m)=Y_{(-p,-q,-s)}(m)=Y_{(-p,-q,s)}(-m)$ When $y(n)$ is pure imaginary: $Y^*_{(p,q,s)}(m)=-Y_{(-p,-q,-s)}(m)=-Y_{(-p,-q,s)}(-m)$
(4) Time-reverse property	(a) $O^{(p,q,s)}_{DAFT}(y(-n))=Y_{(p,q,s)}(-m)$ (b) $Y_{(p,q,s)}(-m)=Y_{(p,q,-s)}(m)$
(5) Transform for even and odd functions	If $y(n)$ is even, then the transform result is also even: $Y_{(p,q,s)}(m)=Y_{(p,q,s)}(-m)$ If $y(n)$ is odd, then the transform result is also odd: $Y_{(p,q,s)}(m)=-Y_{(p,q,s)}(-m)$
(6) DC property	$Y_{(p,q,s)}(0)=\sqrt{\frac{1}{2M+1}}\cdot\sum_{n=-N}^{N}e^{\frac{j}{2}n^2\cdot q}\cdot y(n)$ If $q=0$, $Y_{(p,q,s)}(0)=\sqrt{\frac{1}{2M+1}}\cdot\sum_{n=-N}^{N}y(n)$
(7) Modulation property	If $\widetilde{Y}_{(p,q,s)}(m)=O^{(p,q,s)}_{DAFT}(e^{-j\frac{2\pi k}{2M+1}n}y(n))$, then $\widetilde{Y}_{(p,q,s)}(m)=e^{-\frac{j}{2}p\left(2m(s^{-1})_{2M+1}k+((s^{-1})_{2M+1}k)^2\right)}\cdot Y_{(p,q,s)}\left(m+(s^{-1})_{2M+1}k\right)$ where $(s^{-1})_{2M+1}$ is defined as Eq. (70). In other world, $\left\|\widetilde{Y}_{(p,q,s)}(m)\right\|=\left\|Y_{(p,q,s)}\left((m+(s^{-1})_{2M+1}k)_{2M+1}\right)\right\|$

domain now is $\sigma\cdot\Delta u$, where Δu is the original sampling interval in the fractional domain. Thus

$$\begin{aligned}G_{(a,b,c,d)}(m)&=\sqrt{\frac{1}{2M+1}}\cdot e^{\frac{j}{2}\cdot\frac{d}{b}\cdot m^2(\sigma\cdot\Delta u)^2}\\&\quad\cdot\sum_{n=-N}^{N}e^{-j\cdot\frac{2\pi\cdot\mathrm{sgn}(b)\cdot n\cdot m}{2M+1}}\cdot e^{\frac{j}{2}\cdot\frac{a}{b}\cdot n^2(\Delta t/\sigma)^2}\cdot g(n)\\&=\sqrt{\frac{1}{2M+1}}\cdot e^{\frac{j}{2}\cdot\frac{\sigma^2\cdot d}{b}\cdot m^2\cdot\Delta u^2}\\&\quad\cdot\sum_{n=-N}^{N}e^{-j\cdot\frac{2\pi\cdot\mathrm{sgn}(b)\cdot n\cdot m}{2M+1}}\cdot e^{\frac{j}{2}\cdot\frac{a}{\sigma^2 b}\cdot n^2\cdot\Delta t^2}\cdot y(n).\end{aligned}\tag{89}$$

We can conclude that if

$$Y_{(a,b,c,d)}(m)=O^{(a,b,c,d)}_{\mathrm{DAFT}}(y(n))=O^{(a,b,c,d)}_{\mathrm{DAFT}}(S_{\Delta t}(x(t)))\tag{90}$$

$$\begin{aligned}&G_{(a1,b1,c1,d1)}(m)\\&=O^{(a1,b1,c1,d1)}_{\mathrm{DAFT}}(g(n))=O^{(a1,b1,c1,d1)}_{\mathrm{DAFT}}(S_{\Delta t/\sigma}(x(\sigma t)))\end{aligned}\tag{91}$$

where we use $S_{\Delta t}(x(t))$ to denote sampling $x(t)$ with the interval of Δt, then from (43) and (44)

$$\begin{aligned}G_{(a,b,c,d)}(m)&=e^{\frac{j}{2}\frac{d}{b}(\sigma^2-\sigma^{-2})\cdot m^2\Delta u^2}Y_{(a,\sigma^2 b,\sigma^{-2}c,d)}(m)\\&=Y_{(\sigma^{-2}a,b,c,\sigma^2 d)}(m).\end{aligned}\tag{92}$$

TABLE IV
PROPERTIES OF DAFT (*Continued*)

(Cont.)

(8) Shifting property	Suppose (a) $y(n)=0$ for $n > N$ and $n < -N+k$ when $k > 0$ $y(n)=0$ for $n > N+k$ and $n < -N$ when $k < 0$ (b) $q \cdot k = 2\pi r/(2M+1)$, r is some integer then $O_{DAFT}^{(p,q,s)}\left(y(n+k)\right) = e^{j\frac{\pi(2m+\tau)k}{2M+1}} e^{\frac{j}{2}(2m-\tau)\cdot p\tau} \cdot Y_{(p,q,s)}(m+\tau)$ where $\tau = (s^{-1})_{2M+1} \cdot r$. If the shifting is changed as the circular shifting, i.e., $y(n+k)$ is modified as $y(((n+k))_{2M+1})$, then the above equation will still be satisfied, and the constraint (a) will be not required.
(9) Scaling property	Suppose (a) σ is prime to $2M+1$ (b) $q = 4\pi r/(2M+1)$, r is some integer then $O_{DAFT}^{(p,q,s)}\left(y(((\sigma \cdot n))_{2M+1})\right) = Y_{(p,\rho^2 \cdot q, \rho \cdot s)}(n)$ where $\rho = (\sigma^{-1})_{2M+1}$ (i.e., $((\sigma \cdot \rho))_{2M+1} = 1$)
(10) Parseval's theorem (Energy conservation)	$\sum_{m=-M}^{M} \left\|Y_{(p,q,s)}(m)\right\|^2 = \sum_{n=-N}^{N} \left\|y(n)\right\|^2$
(11) Generalized Parseval's theorem (Generalized energy conservation)	$\sum_{m=-M}^{M} Y1_{(p,q,s)}(m) \cdot Y2^*_{(p,q,s)}(m) = \sum_{n=-N}^{N} y1(n) \cdot y2^*(n)$

For the special case of DFRFT ($a = d = \cos\alpha, b = -c = \sin\alpha$)

$$G_\alpha(m) = e^{\frac{j}{2}\cot\alpha \cdot \sigma^2 \cdot \left(1 - \frac{\cot^2\beta}{\cot^2\alpha}\right) \cdot m^2 \Delta u^2} Y_\beta(m) \quad \text{where } \cot\beta = \cot\alpha/\sigma^2. \tag{93}$$

This scaling property is very similar to the continuous FRFT case in [2]. The scaling property of the continuous FRFT is

$$O_F^\alpha\left(f\left(\frac{t}{\sigma}\right)\right) = \sqrt{\frac{1 - j\cot\alpha}{\sigma^2 - j\cot\alpha}} \cdot e^{j \cdot \frac{u^2}{2}\cot\alpha\left(1 - \frac{\cos^2\beta}{\cos^2\alpha}\right)} \cdot F_\beta\left(u \cdot \frac{\sin\beta}{\sigma\sin\alpha}\right) \tag{93a}$$

where $\cot\beta = \cot\alpha/\sigma^2$.

B. Transform Results for Some Special Signals

In Table V, we just list the transform results of some special signals for the DAFT of type 2. The transform result for $\exp(-jqm^2/2) \cdot \exp(j2\pi r \cdot m/(2M+1))$ comes from the first transform result and (85).

IV. APPLICATIONS OF THE DFRFT AND DAFT

Because of the advantages of the DFRFT/DAFT listed in Section II-E, there are many signal processing applications for the DFRFT/DAFT. The main application for the DFRAT/DAFT of type 1 is useful for computing the continuous FRF/AFT. In addition, for the DFRAT/DAFT of type 2, there are also some practical applications. We will introduce two examples, that is, the filter design (a special case of discrete fractional convolution) and the pattern recognition (use discrete fractional correlation).

In fact, except for these applications, there are also some potential applications for the DFRFT/DAFT. For example, because the DFRFT/DAFT are the unitary transform constructed from the orthogonal chirp basis [see (94)] if a function is similar to the combination of several chirps, then it is convenient to use the DFRFT/DAFT to expand this function. It is also possible to use the DFRFT/DAFT for the phase retrieval [12], discrete fractional Hilbert transform, and beam shaping, etc.

According to our experiment, among the three parameters of DAFT, q is the most important, next is s, and p will have the least importance. Besides, from the discussion in Section II-D, we find that when using the discrete affine convolution and correlation, it is convenient to set $p = 0$ for the simplification. Thus, for digital signal processing applications, we have the following.

1) We usually use the parameter q to control the performance, and set $s = 1$ and $p = 0$.
2) Sometimes (the applications about scaling), we also adjust s and still set $p = 0$.
3) In lesser conditions (such as the phase retrieval), we also adjust the value of p.

The design method of the digital signal processing applications when using DAFT can be simplified by the above principles.

TABLE V
TRANSFORM RESULTS OF DAFT FOR SOME SPECIAL FUNCTIONS

Input function	Transform result (s is any integer relatively prime to $2M+1$)
(1) $\delta(n-r)$	$\sqrt{\frac{1}{2M+1}} \cdot e^{\frac{j}{2}\cdot p\cdot m^2} \cdot e^{-j\cdot\frac{2\pi\cdot s\cdot r\cdot m}{M}} \cdot e^{\frac{j}{2}\cdot q\cdot r^2}$
(2) $e^{-\frac{j}{2}\cdot q\cdot n^2} e^{j\cdot\frac{2\pi\cdot r\cdot n}{2M+1}}$	$\sqrt{2M+1}\cdot e^{\frac{j}{2}p\cdot n^2}\,\delta(sn-r)$ where $\delta(sn-r)=0$ when $((sn))_{2M+1}\neq r$ $\delta(sn-r)=1$ when $((sn))_{2M+1}=r$
(3) $e^{-\frac{j}{2}\cdot q\cdot m^2}\cos\left(\frac{2\pi rm}{2M+1}\right)$	$\sqrt{2M+1}\cdot e^{\frac{j}{2}p\cdot n^2}\left(\delta(sn+r)+\delta(sn-r)\right)/2$
(4) $e^{-\frac{j}{2}\cdot q\cdot m^2}\sin\left(\frac{2\pi rm}{2M+1}\right)$	$j\sqrt{2M+1}\cdot e^{\frac{j}{2}p\cdot n^2}\left(\delta(sn+r)-\delta(sn-r)\right)/2$

A. Filter Design

For the inverse formula of DAFT of (63), we find

$$y(n)=\sum_{m=-M}^{M} Y_{(p,q,s)}(m)\cdot e^{-\frac{j}{2}p\cdot m^2}\cdot b_m(n) \tag{94}$$

where

$$b_m(n)=\sqrt{1/M}\cdot e^{-\frac{j}{2}\cdot q\cdot n^2}\cdot e^{j\cdot\frac{2\pi\cdot s\cdot n\cdot m}{2M+1}},\quad m\in[-M,M]. \tag{95}$$

Therefore, the DAFT is an unitary transform with the orthonormal chirp basis of $b_m(n)$ as above. Thus, if a discrete function is a chirp or the combination of several chirps, then it is convenient to use the DAFT to analyze so that DAFT can be used for filter design to remove the chirp noise. If the chirp noise has the form

$$c(n)=A\cdot e^{-\frac{j}{2}\cdot\eta\cdot(n-\tau)^2} \tag{96}$$

then we can use the DAFT to filter out this chirp

$$x(n)\cong O_{\mathrm{IDAFT}}^{(p,q,s)}\left(\left[1-\Pi\left(\frac{m-m_0}{K}\right)\right]\cdot O_{\mathrm{DAFT}}^{(p,q,s)}(x(n)+c(n))\right) \tag{97}$$

where $O_{\mathrm{IDAFT}}^{(p,q,s)}$ represents the inverse of the DAFT with parameters $\{p,q,s\}$, K represents the width of bandstop filter, and

$$q=\eta\quad m_0\cong(2M+1)\eta\tau/2\pi s. \tag{98}$$

We will give an example in the following. In this example, the number of points for the function is $N=90$. We use the Gaussian function as the input

$$x(n)=\exp(-0.005\cdot n^2). \tag{99}$$

Suppose it is interfered by the chirp noise so that the received signal becomes

$$t(n)=\exp(-0.005\cdot n^2)+0.3\cdot\exp(-i\cdot 0.025\cdot(n-50)^2/2). \tag{100}$$

Then, from (96) and (98), we find that

$$q=0.025. \tag{101}$$

Then, we set $s=1$ and $p=0$ and obtain the center of the bandstop filter as

$$m_0=181*0.025*50/2/\pi/1\approx 36. \tag{102}$$

We also choose the width of the bandstop filter K as 9. Therefore, the filter in the fractional domain is

$$f(m)=1-\Pi((m-36)/9). \tag{103}$$

Then we substitute them into (97) to obtain the recovered signal. In Fig. 4, we will show the results, and the recovered signal will be very similar to the original signal $x(n)$. We also show the conventional DFT of $t(n)$ defined in (100) and plot the result in the middle-left of Fig. 4. It is clear that the conventional DFT cannot separate the signal from the chirp interference.

Except for filtering out the chirp noise, since the DAFT system is time (space) variant, we can also use the DAFT to design the time-varying filter to remove some noise or distortions with spectrum varying with time (space). The general formula for the filter design by DAFT is

$$r(n)=O_{\mathrm{IDAFT}}^{(p,q,s)}\cdot\left(f(m)\cdot O_{\mathrm{DAFT}}^{(p,q,s)}(t(n))\right). \tag{104}$$

For simplification of computation, when we use the DAFT for the application of filter design, it is convenient to set $p=0$ and $s=1$, as in the above case. It is even simpler than using the DFRFT ($p=q$ and $s=\pm 1$) because we save two chirp multiplications (one for the forward DAFT and another one for the inverse).

B. Pattern Recognition

Because of the space-variant properties of the DFRFT and DAFT, we can use them for the pattern recognition to determine the same object located in a different place.

We will use the discrete fractional correlation described in Section II-D. We will apply the result of (81), that is, $z(n)$ is the fractional correlation of $x(n)$ and $g(n)$, and $p1=p2=q3=p3=0$, $s1=s2$. We will further simplify it by setting $q1=q2$, $s1=1$, and $s3=1$, i.e.,

$$z(n)=\mathrm{DFT}\left(X_{(0,q1,1)}(n)\cdot G^{*}_{(0,q1,1)}(n)\right). \tag{105}$$

Then

$$z(n)=(2M+1)^{-1/2}\sum_{n1=-M}^{M} e^{\frac{j}{2}\cdot q1\cdot n1^2}e^{-\frac{j}{2}\cdot q1\cdot((n1+n))_{2M+1}^2}\cdot x(n1)\cdot g^{*}(((n+n1))_{2M+1}). \tag{106}$$

Suppose the input signal is time limited as

$$x(n1)=0\quad\text{for } |n1|>B \tag{107}$$

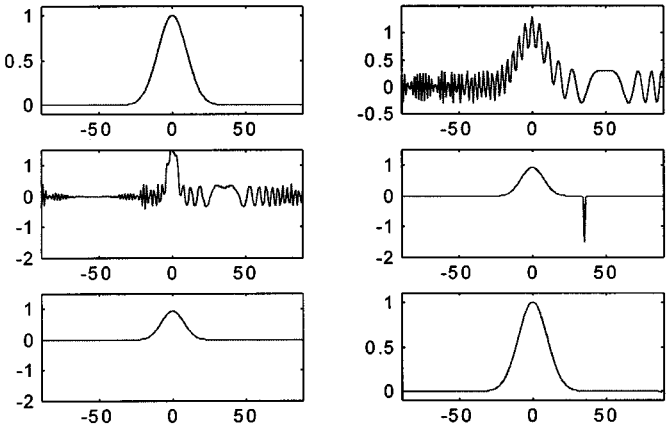

Fig. 4. Example of filter design by DAFT in Section IV-A. Upper left: Original signal. Upper right: Interfered signal (with chirp noise). Middle left: DFT of the interfered signal. Middle right: DAFT ($p = 0, q = 0.025, s = 1$) of the interfered signal. Lower left: Noise part filtered out. Lower right: Recovered signal.

and we only consider $z(n)$ in the range that $|n| \leq M - B$ so that (106) can be written as

$$z(n) = \sqrt{\frac{1}{2M+1}} \sum_{n1=-B}^{B} e^{\frac{j}{2}\cdot q1\cdot n1^2} e^{-\frac{j}{2}\cdot q1\cdot(n1+n)^2} \cdot x(n1)\cdot g^*(n+n1). \tag{108}$$

If the magnitude response is only considered, then

$$|z(n)| = \sqrt{\frac{1}{2M+1}} \times \left| \sum_{n1=-B}^{B} e^{-j\cdot q1\cdot n1\cdot n} \cdot x(n1)\cdot g^*(n+n1) \right|. \tag{109}$$

Now, we use $x(n)$ as the reference template $r(n)$ and suppose that $g(n)$ is a space-shifted version of the reference pattern, i.e.,

$$x(n) = r(n) \quad g(n) = r(n - n_0) \tag{110}$$

and suppose $r(n)$ is real. Then

$$|z(n)| = \sqrt{\frac{1}{2M+1}} \times \left| \sum_{n1=-B}^{B} e^{-j\cdot q1\cdot n1\cdot n} \cdot r(n1)\cdot r(n+n1-n_0) \right| \tag{111}$$

then the peak of $|z(n)|$ is located at $n = n_0$, and its value is

$$\text{peak of } |z(n)|\text{: } |z(n_0)| = \sqrt{\frac{1}{2M+1}} \times \left| \sum_{n1=-B}^{B} e^{-j\cdot q1\cdot n1\cdot n_0} \cdot r^2(n1) \right|. \tag{112}$$

This is an ideal peak at $n_0 = 0$. Because the width of $r(n)$ is B, the peak will only distort a little when the phase of the exponential term is in the range of $[-\pi, \pi]$ when $|n| < B$, i.e.,

$$|n_0| < \left| \frac{\pi}{B\cdot q1} \right|. \tag{113}$$

If n_0 is outside the above range, then the peak will distort seriously, and we can identify that the object is out of some region.

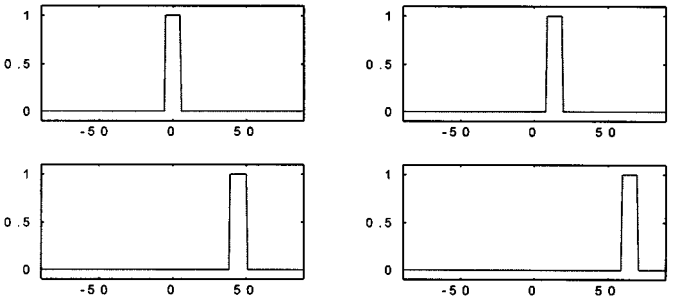

Fig. 5. Reference and the shifted object of the example in Section IV-B. Upper left: Reference. Upper right, Lower left, and Lower right: Shifted object with $n_0 = 14, 45, 65$, respectively.

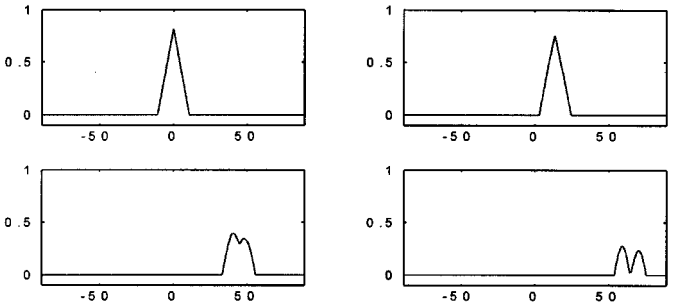

Fig. 6. Discrete fractional correlation between the shifted object and reference in the example in Secion IV-B. Upper left to lower right: $n_0 = 0, 14, 45$, and 65, respectively.

Thus, the discrete fractional correlation and, hence, the DAFT, can be used for the space variant pattern recognition to detect a pattern in a certain region.

We will give an example in the following. The total number of points is $M = 90$. The reference here is a rectangular function $r(n)$

$$x(n) = r(n) = \Pi(n/11) \tag{114}$$

and the object $g(n)$ is the space-shifted version of $r(n)$

$$g(n) = \Pi((n - n_0)/11). \tag{115}$$

We try four cases for $n_0 = 0$, 14, 45, and 65. The fractional order is $q1 = 0.009$, and from the criterion of (113), $|n_0| \leq \pi/(11\times 0.009) = 31.73$. Thus, the criterion of (113) is satisfied when $n_0 = 0, 14$, but for $n_0 = 45$ and 65, (113) is violated. We find that for $n_0 = 0$, 14, the peak of $|z(n)|$ will almost have no attenuation, and for $n_0 = 45$, 65, the peak of $|z(n)|$ will attenuate a lot. We show the results in Figs. 5 and 6. We also calculate the conventional discrete correlation of $x(n)$ and $g(n)$ for comparison

$$z(n) = \text{DFT}(X(m)G^*(m))$$

where

$$X(m) = \text{DFT}(x(n)), \quad G(m) = \text{DFT}(g(n)). \tag{116}$$

We change the last operation to be DFT instead of IDFT to avoid the time reverse. The results are shown in Fig. 7. We find, for the original discrete correlation, that the peak will never distort, no matter how much displacement there is. The conventional discrete correlation can be used for space-invariant pattern recognition, and the discrete fractional correlation can be used for space-variant pattern recognition.

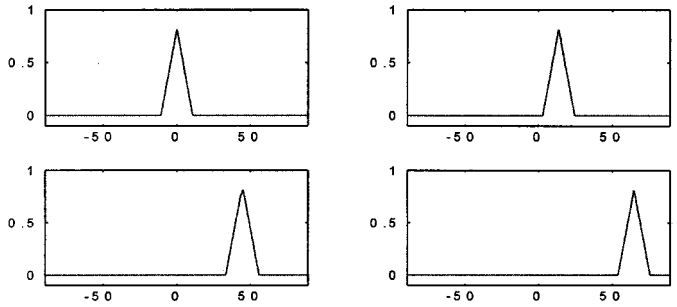

Fig. 7. Conventional discrete correlation between the shifted object and reference in the example in Section IV-B. Upper left to lower right: $n_0 = 0, 14, 45,$ and 65, respectively.

V. Conclusion

In this paper, we have introduced new types of the discrete fractional Fourier transform (DFRFT) and the discrete affine Fourier transform (DAFT). The first type comes from sampling the continuous transforms directly, and the second type is the simplification of the former. We also discuss their applications for computing the continuous FRFT and affine Fourier transform, discrete filter design, and pattern recognition.

The DFRFT and DAFT we derive in this paper keep many of the important properties that the continuous FRFT and affine Fourier transform have. For example, they have similar formulas, are all unitary, reversible, partial time variant, use the chirp functions as their transform basis (see Sections III and IV-A), and they all transform a signal into the intermediate domain between time and frequency. Thus, as the continuous fractional and affine Fourier transforms are useful tools for continuous signal processing, we think the DFRFT and DAFT will also be very useful for digital signal processing. Since the DFRFT and DAFT can be efficiently implemented by FFT, and many important properties of FRFT and AFT can be kept, we believe they will have many signal processing applications in the future.

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