Closed-Form Error Analysis of the Non-Identical Nakagami-m Relay Fading Channel

Himal A. Suraweera, *Member, IEEE*, and George K. Karagiannidis, *Senior Member, IEEE*

***Abstract*— We present closed-form expressions for the average bit error probability (ABEP) of BPSK, QPSK and M-QAM of an amplify-and-forward average power scaling dual-hop relay transmission, over non-identical Nakagami-m fading channels, with integer values of m. Additionally, we evaluate in closed-form the ABEP under sufficiently large signal-to-noise ratio for the source-relay link, valid for arbitrary m. Numerical and simulation results show the validity of the proposed mathematical analysis and point out the effect of the two hops unbalanced fading conditions on the error performance.**

***Index Terms*— Wireless relays, Nakagami-m fading, amplify-and-forward, error performance.**

I. Introduction

In recent years, wireless relaying techniques have attracted a lot of research interest due to their possible exploit in cellular, ad-hoc networks and military communications [1]. In relay networks, intermediate nodes are used to relay signals between the source and the destination terminal.

Amplify-and-forward (AF) is one of the two main schemes for relaying [2]. AF relays without performing any decoding, retransmit a scaled replica of the received signal. Literature on AF relaying schemes assumes two different power constraints at the relay: fixed-gain [2] also called "average power scaling" (APS) in [3] and instantaneous power scaling [3].

The performance analysis of multihop wireless networks operating under different fading conditions has been an important field of research in the past few years. See for example, [2]–[10]. In [3], Mheidat and Uysal have investigated the impact of receive diversity on the performance of a relay-assisted network in which the relay is operating under the AF-APS constraint. In [2], [5], Hasna and Alouini have studied the average bit error probability (ABEP) of dual-hop systems with AF relaying over Rayleigh and Nakagami-m fading channels. In [4], Adinoyi and Yanikomeroglu have analyzed the error performance of a decode-and-forward (DF) based multi-antenna relay network in the presence of Nakagami-m fading. In [6] and [7] Karagiannidis *et al.* have studied the performance bounds of AF multihop transmissions over non-identically distributed Nakagami-m fading channels. In [9] Ikki and Ahmed have presented a tight lower bound for the performance of an AF multi-relay network over non-identical Nakagami-m fading channels, especially in the medium and high signal-to-noise (SNR) region.

In this letter, we present closed-form expressions for the ABEP of an AF-APS dual-hop relay link in non-identical Nakagami-m fading channels (which is the real situation in practical wireless relaying systems) with integer fading parameters. To the best of authors' knowledge, no exact closed-form ABEP expressions for the non-identical Nakagami-m AF-APS relaying are reported. Moreover, we derive a closed-form formula for the error performance under sufficiently large SNR for the source-relay link, valid for arbitrary values of m.

II. Dual-Hop Relay Model

Consider a wireless communication system, where a source terminal S communicates with a destination terminal D using a relay R [2]. Let the modulated signal transmitted by S during the first time slot denoted as x. The received signal at R is given by [2]

$$y_r = \sqrt{E_{SR}}\alpha_1 x + n_r \tag{1}$$

where α_1 is the fading amplitude of the $S-R$ link. n_r is an additive white Gaussian noise (AWGN) component with single sided power spectral density N_0. In the second time slot, the relay multiplies the received signal by a gain factor G and then retransmits to D. The received signal at D is

$$y_d = \sqrt{E_{RD}}\alpha_2 G(\sqrt{E_{SR}}\alpha_1 x + n_r) + n_d \tag{2}$$

where α_2 is the fading amplitude of the $R-D$ link and n_d is the AWGN component with power N_0 at the input of D. E_{SR} and E_{RD} represent the average energies available at R and D, taking into consideration of possibly different path loss and shadowing effects in $S-R$ and R-D links [3]. When R operates under APS constraint, $G^2 = 1/(E_{SR} + N_0)$. The instantaneous end-to-end SNR at D, γ_{eq}, is given by [2]

$$\gamma_{\text{eq}} = \frac{(E_{SR}/N_0)(E_{RD}/N_0)\alpha_1^2\alpha_2^2}{1 + E_{SR}/N_0 + (E_{RD}/N_0)\alpha_2^2} \tag{3}$$

and (3) can be reexpressed as

$$\gamma_{\text{eq}} = \frac{\gamma_1\gamma_2}{C + \gamma_2} \tag{4}$$

where $C = 1 + (E_{SR}/N_0)$ and $\gamma_1 = \alpha_1^2 E_{SR}/N_0$, $\gamma_2 = \alpha_2^2 E_{RD}/N_0$ denote the instantaneous SNRs of the S-R and R-D hops respectively. Since the hops are subject to non-identical Nakagami fading, we model α_1 and α_2 according to Nakagami-m distribution with fading severity parameters m_1 and m_2, respectively, i.e.,

$$p_{\alpha_i}(\alpha) = \frac{2m_i^{m_i}\alpha^{2m_i-1}e^{-m_i\alpha^2}}{\Gamma(m_i)} \tag{5}$$

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H. A. Suraweera is with the School of Electrical Engineering, Victoria University, PO Box 14428, Melbourne, Victoria, 8001, Australia (e-mail: himal.suraweera@vu.edu.au).

G. K. Karagiannidis is with the Department of Electrical and Computer Engineering, Aristotle University of Thessaloniki, GR-54124, Greece (e-mail: geokarag@auth.gr).

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where $i = 1, 2$ and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the gamma function. In the probability density functions (pdfs) of α_1 and α_2, without loss of generality, we have set $E\{\alpha_1^2\}$ and $E\{\alpha_2^2\}$ to unity. Since α_i is modeled as a Nakagami-m random variable (RV), the instantaneous SNR γ_i is a gamma distributed RV with pdf given by

$$p_{\gamma_i}(\gamma) = \frac{m_i^{m_i} \gamma^{m_i - 1} e^{-m_i \gamma/\Omega_i}}{\Omega_i^{m_i} \Gamma(m_i)} \tag{6}$$

where $\Omega_1 = E_{SR}/N_0$ and $\Omega_2 = E_{RD}/N_0$.

III. Error Analysis

Traditionally the ABEP is computed by determining the pdf of γ_{eq} and then averaging the conditional BEP in AWGN, $P_b(e|\gamma)$, over this pdf. Mathematically, $P_b(e)$ is given by

$$P_b(e) = \int_0^\infty p(e|\gamma) p_{\gamma_{\text{eq}}}(\gamma) d\gamma \tag{7}$$

Note that for several Gray bit-mapped constellations employed in practical systems, $P_b(e|\gamma)$ is in the form of $Q\left(\sqrt{\beta\gamma}\right)$ with $Q(x)$ being the Gaussian Q-function defined as $Q(x) = (1/\sqrt{2\pi}) \int_x^\infty e^{-t^2/2} dt$ and β is a constant (BPSK: $P_b(e|\gamma) = Q(\sqrt{2\gamma})$, QPSK: $P_b(e|\gamma) = Q(\sqrt{\gamma})$ and in the case of square/rectangular M-QAM, $P_b(e|\gamma)$ can be written as a finite weighted sum of $Q\left(\sqrt{\beta\gamma}\right)$ terms [11]).

To evaluate the integral in (7), we invoke the technique described in [10]. That is, after introducing a new RV with standard Normal distribution, $P_b(e) = \int_0^\infty Q(\sqrt{\beta\gamma}) p_{\gamma_{\text{eq}}}(\gamma) d\gamma$ can be reexpressed as

$$P_b(e) = \frac{1}{\sqrt{2\pi}} \int_0^\infty F_{\gamma_{\text{eq}}}\left(t^2/\beta\right) e^{-t^2/2} dt \tag{8}$$

Fortunately, Tsiftsis *et al.* in [8] derived the cumulative distribution function of γ_{eq}, $F_{\gamma_{\text{eq}}}(\gamma)$, valid for integer m_1 and m_2. Using [8, eq. 18] $F_{\gamma_{\text{eq}}}(\gamma)$ can be written as[1]

$$F_{\gamma_{\text{eq}}}(\gamma) = 1 - \sum_{i=0}^{m_1 - 1} \sum_{j=0}^{i} \Upsilon(i, j) e^{-m_1 \gamma/\Omega_1} \tag{9}$$
$$\gamma^{\frac{2i + m_2 - j}{2}} K_{m_2 - j}\left(2\sqrt{\frac{m_1 m_2 C \gamma}{\Omega_1 \Omega_2}}\right)$$

and

$$\Upsilon(i, j) = \frac{2\binom{i}{j}}{\Gamma(m_2) i!} \left(\frac{m_1}{\Omega_1}\right)^{\frac{2i + m_2 - j}{2}} \left(\frac{C m_2}{\Omega_2}\right)^{\frac{m_2 + j}{2}} \tag{10}$$

In (9) $K_\nu(\cdot)$ is the ν-th order modified Bessel function of the second kind. Substituting (9) into (8) and using [12, eq. 2.16.8.4] $P_b(e)$ can be computed in closed-form as

$$P_b(e) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{m_1 - 1} \sum_{j=0}^{i} \frac{0.25 \Upsilon(i, j) \vartheta^{\lambda_1} \Gamma\left(i + \frac{1}{2}\right)}{\left(\beta\left(\frac{1}{2} + \frac{m_1}{\Omega_1 \beta}\right)\right)^{\lambda_2}} \tag{11}$$
$$\cdot \frac{1}{\left(\frac{m_1 m_2 C}{\Omega_1 \Omega_2 \beta}\right)^{\frac{1}{2}}} \Gamma\left(2\lambda_2 - i + \frac{1}{2}\right) \Psi\left(2\lambda_2 - i + \frac{1}{2}, 2\lambda_1; \vartheta\right)$$

[1] It is noted that Eqs. 18 and 19 in [8] include typos which we have corrected in (9) and (10).

where $\lambda_1 = \frac{m_2 - j + 1}{2}$, $\lambda_2 = \frac{2i + m_2 - j}{2}$, $\vartheta = \frac{2 m_1 m_2 C}{(2 m_1 + \Omega_1 \beta) \Omega_2}$ and $\Psi(a, b; z)$ is the Tricomi confluent hypergeometric function [13, p. 504]. Note that to arrive at (11) we have employed the well known relationship between the Whittaker function and $\Psi(a, b; z)$ [13, p. 505].

In the special case of Rayleigh fading, a closed-form for $P_b(e)$ can be obtained setting $m_1 = m_2 = 1$, in (11) and after some manipulations as

$$P_b(e) = \frac{1}{2} - \xi \int_0^\infty t\, e^{-\left(\frac{2 + \Omega_1}{2\Omega_1}\right) t^2} K_1\left(2\sqrt{\frac{C}{\Omega_1 \Omega_2}} t\right) dt \tag{12}$$
$$= \frac{1}{2}\left(1 - \frac{\ell}{\sqrt{1 + (2/\Omega_1 \beta)}}\, e^{\ell} \left[K_1(\ell) - K_0(\ell)\right]\right)$$

where $\xi = \sqrt{\frac{2C}{\pi \Omega_1 \Omega_2 \beta}}$ and $\ell = \frac{C}{(2 + \Omega_1 \beta) \Omega_2}$. Note, that (12) can be also derived from (7) and [2, eq. 9], pointing out the validity and the generality of our approach.

A. ABEP under Sufficiently Large SNR for S-R Link and Arbitrary m

In order to derive the ABEP for arbitrary m, we assume sufficiently large SNR for the S-R hop [3], i.e., $E_{SR}/N_0 > E_{RD}/N_0$. Under this assumption, the end-to-end SNR is [3]

$$\omega_{\text{eq}} = \frac{E_{RD}}{N_0} \alpha_1^2 \alpha_2^2 \tag{13}$$

A squared Nakagami RV is gamma distributed. Let two independent RVs X and Y be gamma distributed, i.e., $X \sim \mathcal{G}(a_X, b_X)$ and $Y \sim \mathcal{G}(a_Y, b_Y)$. The pdf of the product Z of X and Y is given by

$$p_Z(z) = \int_0^\infty \frac{1}{t} p_X(t) p_Y\left(\frac{z}{t}\right) dt \tag{14}$$
$$= \frac{2}{\Gamma(a_X)\Gamma(a_Y)} (b_X b_Y)^{-\frac{a_X + a_Y}{2}} z^{\left(\frac{a_X + a_Y}{2} - 1\right)}$$
$$\times K_{a_Y - a_X}\left(2\sqrt{\frac{z}{b_X b_Y}}\right)$$

Therefore, the pdf of ω_{eq} can be expressed as

$$p_{\omega_{\text{eq}}}(\omega) = \frac{2}{\Gamma(m_1)\Gamma(m_2)} \left(\frac{m_1 m_2}{\Omega_2}\right)^{\frac{m_1 + m_2}{2}} \omega^{\frac{m_1 + m_2}{2} - 1} \tag{15}$$
$$\times K_{m_2 - m_1}\left(2\sqrt{\frac{m_1 m_2}{\Omega_2} \omega}\right)$$

Making the substitution $t^2 = \omega$ and using the formula $Q(x) = 0.5\, \text{erfc}(x/\sqrt{2})$, the ABEP is given by

$$P_b(e) = \frac{2\left(\frac{m_1 m_2}{\Omega_2}\right)^{\frac{m_1 + m_2}{2}}}{\Gamma(m_1)\Gamma(m_2)} \int_0^\infty t^{m_1 + m_2 - 1} \tag{16}$$
$$\times\ \text{erfc}\left(\sqrt{\frac{\beta}{2}} t\right) K_{m_2 - m_1}\left(2\sqrt{\frac{m_1 m_2}{\Omega_2}} t\right)$$

The involved integral in (16) can be evaluated using [12, 2.16.59.1] and the ABEP is expressed as shown in (17) where ${}_2F_2(a, b; c, d; z)$ is a generalized hypergeometric function [12]. Note that $\Psi(\cdot, \cdot; z)$ and ${}_2F_2(a, b; c, d; z)$ can be evaluated using popular symbolic software such as MAPLE, MATHEMATICA and MATLAB.

$$P_b(e) = \frac{1}{\sqrt{\pi}\,\Gamma(m_1)\Gamma(m_2)}\left[\frac{2^{m_1-1}\Gamma(m_2-m_1)\Gamma(0.5+m_1)}{m_1(m_1m_2/\Omega_2\beta)^{-m_1}}\,{}_2F_2\left(m_1, 0.5+m_1; 1+m_1, 1+m_1-m_2; \frac{2m_1m_2}{\Omega_2\beta}\right)\right. \\ \left. + \frac{2^{m_2-1}\Gamma(m_1-m_2)\Gamma(0.5+m_2)}{m_2(m_1m_2/\Omega_2\beta)^{-m_2}}\,{}_2F_2\left(m_2, 0.5+m_2; 1+m_2, 1+m_2-m_1; \frac{2m_1m_2}{\Omega_2\beta}\right)\right] \quad (17)$$

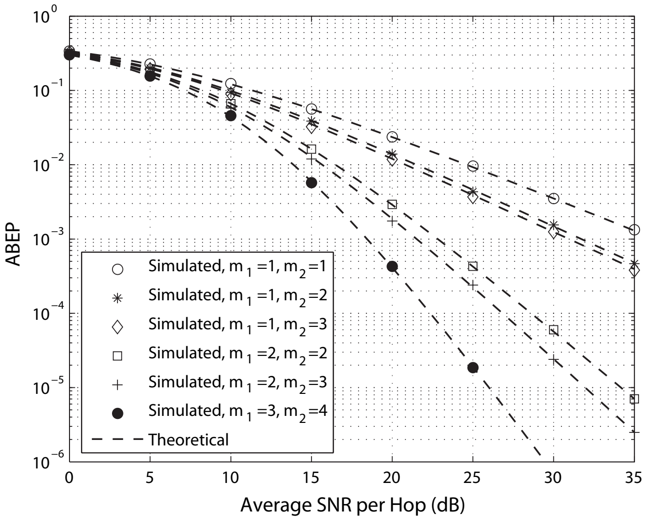

Fig. 1. Simulated and theoretical ABEP of the dual-hop relay link.

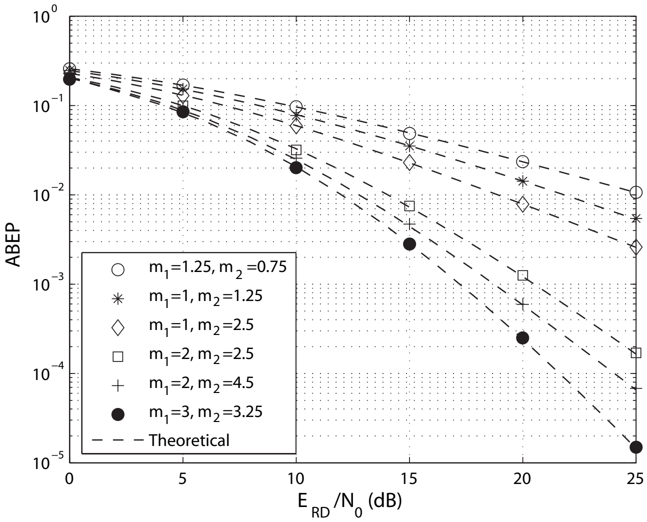

Fig. 2. Comparison of simulated and theoretical ABEP. $\Omega_1 = 35$ dB.

IV. Numerical and Simulation Results

Fig. 1 shows the ABEP of 4-QAM against the average SNR per hop, different values of m and $E_{SR}/N_0 = E_{RD}/N_0$. It should be noted, that although AF relaying will decrease the complexity at the relay, the destination needs to have channel state information knowledge of both the S-R and the R-D links. For comparison, the Rayleigh faded relay performance is also plotted. Observe that all the numerical results (the curves) are in exact agreement with the simulated ABEP results. With $S-R$ link subject to Rayleigh fading, no significant improvement in ABEP can be obtained for $m > 3$ in the R-D link. However, improved fading severity conditions in both links lower the ABEP significantly and the achieved diversity order is increased. As noticed from Fig. 1 for an ABEP equal to 10^{-5}, when fading severity changes from $m_1 = m_2 = 2$ to $m_1 = 3, m_2 = 4$, a SNR gain of 8 dB can be achieved. Fig. 2 illustrates the ABEP of the dual-hop relay link when the average SNR of the S-R link is 35 dB [3]. Again as expected, the error performance improves for large m values in both hops. Finally, simulations were performed to check the validity of (17). As it is evident from Fig. 2 they perfectly match with the analytical ABEP.

V. Conclusion

We have derived closed-form ABEP expressions for several modulation schemes of an AF-APS dual-hop relay link operating in independent and non-identical Nakagami-m fading channels. This analysis is useful to investigate the performance of AF-APS relaying subject to different fading conditions both for source to relay and relay to destination links.

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