10

Closed Form Performance Distributions of a Discrete Time $GI^G/D/1/N$ Queue with Correlated Traffic

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Abstract

A novel approach is applied to the study of a queue with general correlated traffic, in that the only features of the traffic which are taken into account are the usual measures of its correlation: the traffic is modelled as a batch renewal process. The batch renewal process is a precise tool for investigation into effects of correlation because it is the least biased choice of process which is completely determined by the infinite sets of measures of the traffic's correlation (e.g. indices of dispersion, covariances or correlation functions).

The general effect of traffic correlation on waiting time, blocking probability and queue length is well known from simulation studies and numerical analysis of a variety of models. The contribution of this paper is to show that these effects are due to correlation alone (and not to any other features of the traffic or of the models used) and to show explicitly how the magnitudes of blocking, waiting time and queue length distribution are determined by the degree of correlation in the traffic.

The study focuses upon a discrete time $GI^G/D/1/N$ queue with single server, general batch renewal arrivals process, deterministic service time and finite capacity N. Closed form expressions for basic performance distributions, such as queue length and waiting time distributions and blocking probability, are derived when the batch renewal process is of the least biased form which might be expected to result from actual traffic measurements at the interior of a network or of some individual traffic source.

The effect of varying degrees of traffic correlation upon basic performance distributions is investigated and illustrative numerical results are presented. Comments on implications of the results on analysis of general discrete time queueing networks with correlated traffic are included.

Keywords

asynchronous transfer mode (ATM), batch renewal process, correlated traffic, least biased process, discrete time queue, performance distributions

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1 INTRODUCTION

ATM traffic is both bursty and correlated. Even for traffic sources described as being (bursty) renewal processes, superposition of several such sources generally yields correlated processes. The indices of dispersion have been proposed as appropriate characterisation of bursty, correlated traffic and Markov modulated processes as models of sources of bursty traffic with correlation. Sriram and Whitt (1986) described superposition of bursty sources (modelled by renewal processes) in terms of the indices of dispersion for intervals (IDI). Heffes and Lucantoni (1986) model the superposition of bursty renewal processes approximately by a 2-phase Markov modulated Poisson process (MMPP) matched on three features of the indices of dispersion for counts (IDC) and mean arrival rate. Gusella (1991) estimated indices of dispersion for measured LAN traffic and modelled the traffic approximately by a 2-phase MMPP matched on three features of the IDC and the SCV of one inter-arrival time. It should be noted that i) a 2-phase MMPP is defined by only four parameters and cannot conform entirely to all the indices of dispersion which may be used to characterise traffic of any particular source, ii) the IDI, J_n, and IDC, I_t, of a 2-phase MMPP tend to the same finite limit, i.e. $I_\infty = J_\infty$, and iii) a batch renewal process may be constructed for an arbitrary set of indices of dispersion (with finite $I_\infty = J_\infty$). If the indices of dispersion are all that is known about certain traffic (as might result from measurements of real traffic) then a batch renewal process may be constructed which incorporates all that information and no other: in that sense, batch renewal processes provide a description of the traffic which is both complete and least biased — which models, such as MMPP, with limited parameterisation do not.

Fowler and Leland (1991) have reported LAN traffic with unbounded IDC (i.e. infinite I_∞). However, it is to be expected that performance of restricted buffer systems with deterministic service (as in ATM switches) would not be affected by the magnitude of IDC for long intervals. Recently, Andrade and Martinez-Pascua (1994) have shown that queue length distribution, etc. is affected by IDC only up to a certain size of interval (determined by the buffer size) and "the value of the IDC at infinity has little importance." So it may be expected that, for practical purposes, the finite limit to IDI and IDC in batch renewal processes would not be a disadvantage in traffic models.

The general effect of traffic correlation on waiting time, blocking probability and queue length is well known from simulation studies and numerical analysis of a variety of models. The contribution of this paper is to show that these effects are due to correlation alone (and not to any other features of the traffic or of the models used) and to show explicitly how the magnitudes of blocking, waiting time and queue length distribution are determined by the degree of correlation in the traffic. The traffic is modelled as a batch renewal process which is a precise tool for investigation into effects of correlation because it is the least biased choice of process which is completely determined by the infinite sets of measures of the traffic's correlation (e.g. indices of dispersion, covariances or correlation functions).

Batch renewal processes are defined and their properties described in Section 2. In Section 3, the relationships between the component distributions of the batch renewal arrivals process and the queue length distribution, waiting time and blocking probability in a finite buffer queue with deterministic service and censored batch renewal arrivals process are presented. Such a queueing system is an appropriate model for an ATM multiplexer or partitioned buffer switch.

The analysis is specialised in Section 4 to batch renewal processes in which the component distributions are shifted generalised geometric (shifted GGeo). This form of process appears to be appropriate to measured traffic, especially where the usable data set be limited by (say) the time for which the actual process may be regarded as being stationary. Closed form expressions for queue length distribution, waiting time and blocking probability are given.

Section 5 presents analysis of the effects, on blocking probability, waiting time and queue length, of varying degrees of correlation and illustrates the results by numerical examples.

Finally, conclusions and proposals for extensions to the work, including those towards approximate analysis of general queueing networks with correlated traffic, are given in Section 6

2 BATCH RENEWAL PROCESSES

Definition A random sequence $\{\xi(t) : t = \ldots, -2, -1, 0, 1, 2, \ldots\}$ is stationary in the wide sense (equivalently, stationary in Khinchin's sense) if

- the random function $\xi(t)$ has constant finite mean $\mathbf{E}[\xi(t)] = \xi$ (which is independent of t) and
- the correlation function $\mathrm{Cov}[\xi(t), \xi(s)] \triangleq \mathbf{E}[(\xi(t) - \xi)(\xi(s) - \xi]$ is finite and depends on the lag $t - s$ only.

Observe that $\mathrm{Cov}[\xi(t), \xi(t+\ell)] = \mathrm{Cov}[\xi(t+\ell), \xi(t)]$, by symmetry of the definition, and that $\mathrm{Cov}[\xi(t+\ell), \xi(t)] = \mathrm{Cov}[\xi(t), \xi(t-\ell)]$, by change of variable t to $t-\ell$. Consequently, $\mathrm{Cov}[\xi(t), \xi(t+\ell)] = \mathrm{Cov}[\xi(t), \xi(t-\ell)]$ — only the magnitude of the lag is significant and it is therefore necessary only to consider positive lags.

Consider an arrivals process which is a two dimensional wide sense stationary sequence $\{\alpha(t), \beta(t) : t = \ldots, -2, -1, 0, 1, 2, \ldots\}$, in which realisations of $\alpha(t)$ and $\beta(t)$ are drawn from the positive integers. The $\beta(t)$ are to be interpreted as the number of arrivals (i.e. batch sizes) and the $\alpha(t)$ as the number of slots in intervals between successive batches of arrivals. From $\{\alpha(t), \beta(t)\}$ may be derived two related sequences of interest

- the numbers of arrivals $\{N(t) : t = \ldots, -2, -1, 0, 1, 2, \ldots\}$ at each epoch,
- the intervals $\{X(t) : t = \ldots, -2, -1, 0, 1, 2, \ldots\}$ between successive arrivals.

To be specific, $X(0)$ may be assigned $\alpha(0)$, $X(\beta(1))$ assigned $\alpha(1)$, $X(\beta(1)+\beta(2))$ assigned $\alpha(2)$, etc. and intermediate values, $X(1)$ through $X(\beta(1) - 1)$, etc., assigned 0. Similarly, $N(1)$ may be assigned $\beta(1)$, $N(\alpha(2)+1)$ assigned $\beta(2)$, etc. and intermediate values, $N(2)$ through $N(\alpha(2))$, etc., assigned 0.

It is generally true that

$$\mathbf{E}[N(t)^n] = \frac{\mathbf{E}[\beta(t)^n]}{\mathbf{E}[\alpha(t)]}$$

and, thence, generally true that

$$I_1 \triangleq \frac{\mathrm{Var}[N(t)]}{\mathbf{E}[N(t)]} = bC_b^2 + b - \frac{b}{a}$$

where $a = \mathbf{E}[\alpha]$, $b = \mathbf{E}[\beta]$ and C_b^2 is the square coefficient of variation of β and I_1 is the index of dispersion for counts (IDC) over one slot (i.e. at lag 0). Similarly,

$$J_1 \triangleq \frac{\mathrm{Var}[X(t)]}{\mathbf{E}[X(t)]^2} = bC_a^2 + b - 1$$

where C_a^2 is the square coefficient of variation of α and J_1, the index of dispersion for intervals (IDI) for one interval (i.e. at lag 0). J_1 is the square coefficient of variation of X.

To determine the correlation functions for $\{N(t)\}$ and $\{X(t)\}$ more information about $\{\alpha(t), \beta(t)\}$ is required.

Definition A batch renewal arrival process is a process in which there are batches of simultaneous arrivals such that

- the numbers of arrivals in batches are independent identically distributed random variables,
- the intervals between batches are independent identically distributed random variables.
- the batch sizes are independent of the intervals between batches.

It is shown, below, that a discrete time batch renewal arrival process may be constructed to give any degree of correlation between numbers of arrivals at different epochs and, simultaneously, any degree of correlation between interarrivals times at arbitrary lags. Indeed, there is a *one-to-one* correspondence between an arbitrary set of indices of dispersion (or, equivalently, of correlation functions or covariances) and a batch renewal process. Furthermore, the corresponding batch renewal process is the *least biased choice* given only a set of indices of dispersion or of correlation functions. (To say that a process be the "least biased choice" means that, of all possible processes which satisfy the given conditions (e.g. set of indices of dispersion), is chosen that process which involves least arbitrary additional information. For example, in the case of a 2-dimensional joint probability distribution $\mathbf{P}[X = n_1, Y = n_2]$ given only the marginal distributions $\mathbf{P}[X = n_1]$ and $\mathbf{P}[Y = n_2]$, the least biased choice for the joint distribution is $\mathbf{P}[X = n_1, Y = n_2] = \mathbf{P}[X = n_1]\mathbf{P}[Y = n_2]$. The effect is to treat X and Y as being independent. Any other choice would introduce arbitrary information in the form of the dependence between X and Y.)

2.1 Independence or Dependence at Various Lags

Consider a discrete time batch renewal process in which

- the distribution of batch size is given by the probability mass function (*pmf*) $b(n)$, $n = 1, 2, \ldots$, with mean b, square coefficient of variation (SCV) C_b^2 and probability generating function (*pgf*) $B(z) = \sum_{n=1}^{\infty} b(n)z^n$, and

- the distribution of intervals between batches is given by the *pmf* $a(t)$, $t = 1, 2, \ldots$, with mean a, SCV C_a^2 and *pgf* $A(\omega) = \sum_{t=1}^{\infty} a(t)\omega^t$.

Observe that no loss in generality ensues from the assumption that $a(0) = 0$, $b(0) = 0$.

It is readily seen that the stationary distribution of the number n of arrivals at an epoch is given by the *pmf* $\nu(n)$, $n = 0, 1, \ldots$,

$$\nu(n) = \begin{cases} 1 - \dfrac{1}{a} & n = 0 \\ \dfrac{1}{a}\, b(n) & n = 1, 2, \ldots \end{cases} \tag{1}$$

and the conditional probability $\nu_\ell(n;k)$ that there be n arrivals $(n = 0, 1, \ldots)$ at an epoch, given that there had been k arrivals $(k = 0, 1, \ldots)$ at the epoch ℓ slots earlier $(\ell = 1, 2, \ldots)$, is

$$\nu_\ell(n;k) = \begin{cases} 1 - \dfrac{1-\phi_\ell}{a-1} & n = 0,\ k = 0 \\ \dfrac{1-\phi_\ell}{a-1}\, b(n) & n = 1, 2, \ldots,\ k = 0 \\ 1 - \phi_\ell & n = 0,\ k = 1, 2, \ldots \\ \phi_\ell\, b(n) & n = 1, 2, \ldots,\ k = 1, 2, \ldots \end{cases} \tag{2}$$

where ϕ_ℓ is the probability that there be a batch at any epoch, given that there had been a batch at the epoch ℓ slots earlier. The number of arrivals at an epoch is either independent of or dependent on the number of arrivals at the epoch ℓ slots earlier according to whether $\phi_\ell = 1/a$ or not. Obviously, ϕ_ℓ satisfies the (convolution) relationship

$$\phi_\ell = \begin{cases} 1 & \ell = 0 \\ \displaystyle\sum_{t=1}^{\ell} \phi_{\ell-t}\, a(t) & \ell = 1, 2, \ldots \end{cases} \tag{3}$$

and ϕ_ℓ is generated from the *pgf*

$$\sum_{\ell=0}^{\infty} \phi_\ell\, \omega^\ell = \frac{1}{1 - A(\omega)}\,. \tag{4}$$

Note that ϕ_ℓ is determined by $a(1), \ldots, a(\ell)$ only $(\ell = 1, 2, \ldots)$ and so $a(\cdot)$ may be constructed to give independence or dependence arbitrarily at any specified lags.

The correlation functions (covariances) at lag ℓ, $\ell = 1, 2, \ldots$, are derived from equations (1) and (2) as

$$\mathrm{Cov}[N(t), N(t+\ell)] = \mathbf{E}[N(t)N(t+\ell)] - \mathbf{E}[N(t)]^2 = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} n\, k\, \nu(k)\, \nu_\ell(n;k) - \left(\frac{b}{a}\right)^2$$

$$= \left(\sum_{k=1}^{\infty} k \frac{b(k)}{a}\right) \left(\sum_{n=1}^{\infty} n\, \phi_\ell\, b(n)\right) - \left(\frac{b}{a}\right)^2$$

$$= \frac{b^2}{a}\left(\phi_\ell - \frac{1}{a}\right) \tag{5}$$

Hence, utilising equation (4), the variance and covariances are generated by

$$K(\omega) \triangleq \frac{1}{\lambda}\left(\mathrm{Var}[N] + 2\sum_{\ell=1}^{\infty} \mathrm{Cov}[N(t), N(t+\ell)]\, \omega^\ell\right) = bC_b^2 + b\frac{1+A(\omega)}{1-A(\omega)} - \lambda\frac{1+\omega}{1-\omega} \tag{6}$$

where $\lambda = b/a$. Observe that $\lambda K(z)$ is a *pgf* analog of the spectral density function for the random sequence of the number of arrivals at successive epochs. Further, the relationship between correlation functions (covariances) and indices of dispersion may be described by

$$K(\omega) = (1-\omega)^2 I'(\omega)$$

where $I(\omega)$ is the *pgf* of the indices of dispersion I_t for counts and where the prime ($'$) indicates the derivative.

In an essentially similar way, it may be shown that the batch size distribution $b(\cdot)$ may be constructed to give independence or dependence of intervals (between individual pairs of arrivals) arbitrarily at any specified lags and that the correlation functions are generated from

$$L(z) = \lambda^2\left(\mathrm{Var}[X] + 2\sum_{\ell=1}^{\infty} \mathrm{Cov}[X(t), X(t+\ell)]\, z^\ell\right) = bC_a^2 + b\frac{1+B(z)}{1-B(z)} - \frac{1+z}{1-z} \tag{7}$$

where $L(\omega)/\lambda^2$ is a *pgf* analog of the spectral density function for the random sequence of interarrival times,

$$L(z) = (1-z)^2 J'(z)$$

where $J(z)$ is the *pgf* of the indices of dispersion J_n for intervals and where the prime ($'$) indicates the derivative.

It may be shown that equations (6) and (7) together imply a *one-to-one* relationship between the set of correlation functions or covariances (equivalently, indices of dispersion) and the batch renewal process distributions of batch sizes and intervals between batches.

3 CENSORED $GI^G/D/1/N$ QUEUE UNDER DEPARTURES FIRST POLICY

Consider a $GI^G/D/1/N$ queue in discrete time in which arrivals to a full system are turned away and simply lost (i.e. censored arrivals). Events (arrivals and departures) occur at discrete points in time (epochs) only. The intervals between epochs are called *slots* and, without loss of generality, may be regarded as being of constant duration. At an epoch at which both arrivals and departures occur, the departing customers release the places, which they had been occupying, to be available to arriving customers (*departures first* memory management policy). The service time for a customer is one slot and the first customer arriving to an empty system (after any departures) receives service and departs at the end of the slot in which it arrived (*immediate service* policy). By GI^{G} arrivals process is meant the intervals between batches are independent and of general distribution and the batch size distribution is general (batch renewal process). Consider further two processes embedded at points immediately before and immediately after each batch of arrivals. Each process may be described independently by a Markov chain but the processes are mutually dependent. Let

$p_N(n)$ be the steady state probability that there be $n = 0, 1, \ldots, N$ customers in the system (either queueing or receiving service) during a slot (i.e. $\{p_N(n) : n = 0, \ldots, N\}$ is the random observer's distribution),

$p_N^A(n)$ be the steady state probability that a batch of arrivals 'see' $n = 0, \ldots, N-1$ customers in the system (i.e. $\{p_N^A(n) : n = 0, \ldots, N-1\}$ is the stationary distribution of the Markov chain embedded immediately before batch arrivals),

$p_N^D(n)$ be the steady state probability that there be $n = 1, \ldots, N$ customers in the system immediately after a batch of arrivals to the queue (i.e. $\{p_N^D(n) : n = 1, \ldots, N\}$ is the stationary distribution of the Markov chain embedded immediately after batch arrivals).

When, immediately following a batch of arrivals, the system contains k ($k = 1, \ldots, N$) customers and the interval to the next batch is t slots, then there will be one departure at the end of each of the t slots for which there remain customers in the system. If $t \leq k$ that next batch will 'see' $k - t$ customers. If $t > k$ the system will become empty before the next batch arrives. Similarly, when, immediately prior to admission of a batch of arrivals, the system contains k ($k = 0, \ldots, N-1$) customers and the batch size is r, then the buffer will become full if $r \geq N - k$. Otherwise there will be $k + r$ customers immediately after the batch arrives. Consequently, $p_N^A(\cdot)$ and $p_N^D(\cdot)$ are related by

$$p_N^A(n) = \begin{cases} \sum_{k=1}^{N} p_N^D(k) \sum_{t=k}^{\infty} a(t) & n = 0 \\ \sum_{k=n+1}^{N} p_N^D(k) a(k-n) & n = 1, \ldots, N-1 \end{cases} \tag{8}$$

and

$$p_N^D(n) = \begin{cases} \sum_{k=0}^{n-1} p_N^A(k) b(n-k) & n = 1, \ldots, N-1 \\ \sum_{k=0}^{N-1} p_N^A(k) \sum_{r=N-k}^{\infty} b(r) & n = N \end{cases} \tag{9}$$

The relationship between$p_N^D(\cdot)$ and the random observer's probability $p_N(\cdot)$ results from the following considerations.

If, immediately after the arrival of a batch, there be k customers in the system and the interval between that and the next batch be t slots (figure 1) then, during that interval of t slots,

if $t \leq k$ the system visits the states $k, \ldots, k-t+1$ for one slot each,
if $t > k$ the system visits the states $k, \ldots, 1$ for one slot each and resides in state 0 for $t-k$ slots.

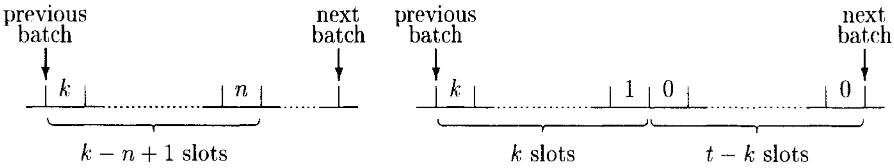

Figure 1 Ways in which state n ($n > 0$) and state 0 may be reached during an interval between batches.

Any arbitrary slot must fall in the interval between two batches. The probability that the interval be of length t is $\frac{1}{a}ta(t)$ and the probability that the slot occupy any particular position within the interval is $1/t$.

Hence,

$$p_N(n) = \begin{cases} \dfrac{1}{a}\displaystyle\sum_{k=1}^{N} p_N^D(k) \sum_{t=k+1}^{\infty} (t-k)a(t) & n = 0 \\ \dfrac{1}{a}\displaystyle\sum_{k=n}^{N} p_N^D(k) \sum_{t=k-n+1}^{\infty} a(t) & n = 1, \ldots, N \end{cases} \tag{10}$$

3.1 Blocking Probability

If an arriving batch of size $N-k+r$ see k customers in the system, then only the first $N-k$ members of the arriving batch may be admitted and r customers will be blocked. The probability that the arriving batch see k customers is $p_N^A(k)$, the probability that the batch be of size $N-k+r$ is $(N-k+r)\,b(N-k+r)/b$ and the probability of a customer being in one of the r positions, given that the batch be of size $N-k+r$, is $r/(N-k+r)$. Therefore, the blocking probability π_N^B is given by

$$\pi_N^B = \sum_{k=0}^{N-1} p_N^A(k) \sum_{r=1}^{\infty} \frac{r}{b} b(N-k+r) \tag{11}$$

3.2 Waiting Time

The waiting time of a customer is given by its position in the queue at the instant at which it arrive in the queue. Thus, given that there be k customers in the queue (including any in service) at the time of an arriving batch of size r, the customer in position $t-k$ in the batch $(1 \leq t-k \leq r \leq N-k)$ will remain in the queue for t slots.

Let $b_k(n)$, $n = 1, \ldots, k$ with mean b_k be the effective arrival distribution given that an arriving batch see k, $k = 1, \ldots, N-1$, places available in the buffer.

$$b_k(n) = \begin{cases} b(n) & 1 \leq n < k \\ \sum_{r=k}^{\infty} b(r) & n = k \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Then the mean b_k is given by

$$b_k = \sum_{n=1}^{k} n\, b_k(n) = \sum_{n=1}^{k} n\, b(n) + k \sum_{n=k+1}^{\infty} b(n) \tag{13}$$

Then waiting time is distributed as $w(t)$, $t = 1, \ldots, N$,

$$w(t) = \sum_{k=0}^{t-1} p_N^A(k) \frac{1}{b_{N-k}} \sum_{r=t-k}^{N-k} b_{N-k}(r) = \sum_{k=0}^{t-1} p_N^A(k) \frac{1}{b_{N-k}} \sum_{r=t-k}^{\infty} b(r) \tag{14}$$

4 SHIFTED GGEO DISTRIBUTIONS OF BATCH SIZE AND INTERVALS

This section presents particular forms of batch renewal arrivals process which appear to be especially appropriate to models of traffic where there are relatively few measurements from which the correlation functions (covariances) may be estimated. In such cases it is natural to plot the logarithms of covariances against lags and fit a straight line to the plot. Then, if

$$\log \mathrm{Cov}[X(t), X(t+\ell)] \simeq -C - m\ell$$

(for some constants C and m), equation (7) implies that the corresponding batch renewal process has batch size distribution of the form

$$b(n) = \begin{cases} 1-\eta & n = 1 \\ \eta\nu(1-\nu)^{n-2} & n = 2, \ldots \end{cases} \tag{15}$$

in which

$$\begin{aligned} \eta &= \frac{\lambda^2 e^{-C}}{1+\lambda^2 e^{-C}} \left(1 - e^{-m}\right) \\ \nu &= \frac{1}{1+\lambda^2 e^{-C}} \left(1 - e^{-m}\right) \end{aligned}$$

Similarly, if

$$\log \mathrm{Cov}[N(t), N(t+\ell)] \simeq -C - m\ell$$

(for some constants C and m) equation (6) implies that the corresponding batch renewal process has intervals between batches distributed as

$$a(t) = \begin{cases} 1-\sigma & t = 1 \\ \sigma\tau(1-\tau)^{t-2} & t = 2, \ldots \end{cases} \tag{16}$$

in which

$$\begin{aligned} \sigma &= \frac{e^{-C}}{\lambda^2 + e^{-C}}\left(1 - e^{-m}\right) \\ \tau &= \frac{\lambda^2}{\lambda^2 + e^{-C}}\left(1 - e^{-m}\right) \end{aligned}$$

Distributions of form (15) and (16) are known as shifted generalised geometric (shifted GGeo).

This section first discusses the solution of GI^G/D/1/N queues in which the intervals between batches are distributed as a shifted GGeo. (A similar solution method may be applied to GI^G/D/1/N queues in which the batch sizes are distributed as a shifted GGeo.) Then closed form expressions for queue length distribution, waiting time distribution and blocking probability are derived for the interesting case when both batch sizes and intervals are distributed as shifted GGeo.

4.1 GGeo^G/D/1/N Queues

When the distribution of intervals between batches is shifted generalised geometric with parameters σ and τ the correlation functions (covariances) for the numbers of arrivals per slot are

$$\mathrm{Cov}[N(t), N(t+\ell)] = \frac{b^2}{a}\phi_\ell - \left(\frac{b}{a}\right)^2 = \lambda^2\frac{\sigma}{\tau}\left(1-\sigma-\tau\right)^\ell = \lambda^2\frac{a-1}{a}{\beta_a}^\ell$$

which shows, for given interval a between batches, the significance of the "correlation factor" $\beta_a \triangleq 1 - \sigma - \tau$.

Equation (8) becomes

$$p_N^A(n) = \begin{cases} p_N^D(1) + \sigma\sum\limits_{k=2}^{N} p_N^D(k)(1-\tau)^{k-2} & n = 0 \\ (1-\sigma)p_N^D(n+1) + \sigma\tau\sum\limits_{k=n+2}^{N} p_N^D(k)(1-\tau)^{k-n-2} & n = 1, \ldots, N-2 \\ (1-\sigma)p_N^D(N) & n = N-1 \end{cases} \tag{17}$$

Consideration of the differences between the $p_N^A(n)$ for successive values of n leads to the difference relations below. For $n = 1, \ldots, N-3$

$$\begin{aligned}(1-\tau)p_N^A(n+1) - p_N^A(n) &= (1-\sigma-\tau)p_N^D(n+2) - (1-\sigma)p_N^D(n+1) \\ &= (1-\sigma-\tau)\sum_{k=0}^{n+1} p_N^A(k)b(n-k+2) \\ &\quad -(1-\sigma)\sum_{k=0}^{n} p_N^A(k)b(n-k+1)\end{aligned} \tag{18}$$

and for $n = 0$

$$\begin{aligned}(1-\tau)p_N^A(1) - p_N^A(0) &= (1-\sigma-\tau)p_N^D(2) - \tau p_N^D(1) \\ &= (1-\sigma-\tau)\left[p_N^A(0)b(2) + p_N^A(1)b(1)\right] - \tau p_N^A(0)b(1)\end{aligned} \tag{19}$$

The system of linear equations (18) and (19) establish ratios between $p_N^A(n)$ and $p_N^A(0)$ (for $n = 1, \ldots, N-2$) which are independent of N and are the same as in the corresponding unrestricted buffer $\text{GGeo}^{\text{G}}/\text{D}/1$ system. Therefore, writing $p^A(n)$ for the steady state probability that a batch of arrivals to the unrestricted queue 'see' n in the system,

$$p_N^A(n) = \tfrac{1}{Z}p^A(n) \quad n = 0, \ldots, N-2 \tag{20}$$

for some normalising constant Z, and so, writing

$$P^A(z) = \sum_{n=0}^{\infty} p_N^A(n) z^n$$

for the generating function of $p^A(n)$, gives

$$P^A(z) = \frac{\sigma+\tau-b}{1-(1-\sigma-\tau)\dfrac{B(z)}{z} - \tau\dfrac{1-B(z)}{1-z}} \tag{21}$$

Given the distribution $b(\cdot)$ explicitly, equation (21) may (in principle) be solved, leading (via equation (20)) to the solution of equation (17). Thence, relations (9), (10), (14) and (11) give queue length distribution, waiting time distribution and blocking probability. This method is shown, in the next subsection, when the batch size is distributed as GGeo.

4.2 $\text{GGeo}^{\text{GGeo}}/\text{D}/1/\text{N}$ Queues

When both the intervals between batches and the batches are distributed as shifted Generalised Geometric equation (21) becomes

$$P^A(z) = \frac{\sigma - \tau\dfrac{\eta}{\nu}}{1-\tau\left(1+\dfrac{\eta z}{1-(1-\nu)z}\right) - (1-\sigma-\tau)\left(1-\dfrac{\eta(1-z)}{1-(1-\nu)z}\right)}$$

$$= \frac{1}{\nu}\frac{(\sigma\nu-\tau\eta)(1-(1-\nu)z)}{(\sigma+(1-\sigma-\tau)\eta)-(\sigma(1-\eta-\nu)+\eta)z} \tag{22}$$

It follows immediately that

$$p^A(n) = \begin{cases} \frac{1}{\nu}(1-x) & n=0 \\ \frac{\eta}{\nu}\frac{\tau+(1-\sigma-\tau)\nu}{\sigma+(1-\sigma-\tau)\eta}(1-x)x^{n-1} & n=1,2,\ldots \end{cases} \tag{23}$$

where the geometric term x is

$$x = \frac{\sigma(1-\eta-\nu)+\eta}{\sigma+(1-\sigma-\tau)\eta}\ .$$

Then, from equation (17) for $n=N$, equation (9) and the distribution of batch sizes,

$$\begin{aligned} p_N^A(N-1) &= (1-\sigma)p_N^D = (1-\sigma)\sum_{k=0}^{N-1} p_N^A(k)\sum_{r=N-k}^{\infty} b(r) \\ &= \frac{1}{Z}\frac{1-\sigma}{\sigma}\frac{\eta}{\nu}(1-x)x^{N-2} \end{aligned} \tag{24}$$

and so the normalising constant Z is seen to be

$$Z = 1-\frac{\eta}{\nu}\frac{\tau}{\sigma}x^{N-1} \tag{25}$$

Combining equations (20), (23) and (24) yields

$$p_N^A(n) = \begin{cases} \frac{1}{Z}\frac{1}{\nu}(1-x) & n=0 \\ \frac{1}{Z}\frac{\eta}{\nu}\frac{\tau+(1-\sigma-\tau)\nu}{\sigma+(1-\sigma-\tau)\eta}(1-x)x^{n-1} & n=1,\ldots,N-2 \\ \frac{1}{Z}\frac{\eta}{\nu}\frac{1-\sigma}{\sigma}(1-x)x^{N-2} & n=N-1 \end{cases} \tag{26}$$

Applying equation (9) to (26) yields

$$p_N^D(n) = \begin{cases} \frac{1}{Z}\frac{1-\eta}{\nu}(1-x) & n=1 \\ \frac{1}{Z}\frac{\eta}{\nu}\frac{\tau(1-\eta-\nu)+\nu}{\sigma+(1-\sigma-\tau)\eta}(1-x)x^{n-2} & n=2,\ldots,N-1 \\ \frac{1}{Z}\frac{\eta}{\nu}\frac{1}{\sigma}(1-x)x^{N-2} & n=N \end{cases} \tag{27}$$

Applying equation (10) to (27) yields

$$p_N(n) = \begin{cases} \frac{1}{Z}\frac{1}{\nu}\frac{1}{\sigma+\tau}(\sigma\nu - \tau\eta) = \frac{1}{Z}(1-\lambda) & n = 0 \\ \frac{1}{Z}\frac{1}{\nu}\frac{\tau}{\sigma+\tau}(1-x) & n = 1 \\ \frac{1}{Z}\frac{\eta}{\nu}\frac{\tau}{\sigma+\tau}\frac{1-(1-\sigma-\tau)(1-\eta-\nu)}{\sigma+(1-\sigma-\tau)\eta}(1-x)x^{n-2} & n = 2,\ldots,N-1 \\ \frac{1}{Z}\frac{\eta}{\nu}\frac{\tau}{\sigma+\tau}\frac{1}{\sigma}(1-x)x^{N-2} & n = N \end{cases} \tag{28}$$

Hence, mean queue length L_N is

$$\begin{aligned} L_N &= \frac{1}{Z}\frac{\tau}{\sigma+\tau}\frac{1}{\nu}\Bigg((\eta+\nu) + \eta\frac{1-(1-\sigma-\tau)(1-\eta-\nu)}{\sigma\nu - \tau\eta} \\ &\qquad - \eta\,\frac{1-(1-\sigma-\tau)(1-\eta-\nu)}{\sigma\nu-\tau\eta}x^{N-1} \\ &\qquad + N\frac{\eta}{\sigma}(1-\sigma-\tau)x^{N-1}\Bigg) \end{aligned} \tag{29}$$

From equations (11) and (26) the blocking probability π_N^B is

$$\pi_N^B = \sum_{k=0}^{N-1} p_N^A(k)\sum_{r=1}^{\infty}\frac{r}{b}b(N-k+r) = \frac{1}{Z}\frac{\eta}{\eta+\nu}\frac{\sigma\nu-\tau\eta}{\sigma\nu}x^{N-1} = \frac{1-Z}{Z}\frac{1-\lambda}{\lambda} \tag{30}$$

Hence

$$\frac{\pi_{N+1}^B}{\pi_N^B} \to x = \frac{\sigma(1-\eta-\nu)+\eta}{\sigma+(1-\sigma-\tau)\eta} \quad \text{as} \quad N\to\infty$$

which illustrates the typical log-linear relationship between blocking probability $\pi_N^B$and buffer size N.

From equation (13), the mean effective batch size given k buffer places available to arrivals is given by

$$b_k = (1-\eta) + \eta\nu\sum_{n=2}^{k-1} n(1-\nu)^{n-2} + k\eta\nu\sum_{n=k}^{\infty}(1-\nu)^{n-2} = 1 + \frac{\eta}{\nu}\left(1-(1-\nu)^{k-1}\right) \tag{31}$$

Hence, waiting time is distributed as

$$\begin{aligned} w(t) &= \sum_{k=0}^{t-1} p_N^A(k)\frac{1}{b_{N-k}}\sum_{r=t-k}^{\infty} b(r) \\ &= (t>1)\sum_{k=0}^{t-2} p_N^A(k)\frac{\eta(1-\nu)^{t-k-2}}{1+\frac{\eta}{\nu}\left(1-(1-\nu)^{N-k-1}\right)} + p_N^A(t-1)\frac{1}{1+\frac{\eta}{\nu}\left(1-(1-\nu)^{N-t}\right)} \end{aligned} \tag{32}$$

4.3 Infinite Buffer

In the limit as the buffer size $N \to \infty$, the expressions for mean queue length and waiting time reduce to

$$L = \frac{\tau}{\sigma+\tau}\frac{1}{\nu}\left((\eta+\nu)+\eta\frac{1-(1-\sigma-\tau)(1-\eta-\nu)}{\sigma\nu-\tau\eta}\right) \tag{33}$$

and

$$w(t) = \begin{cases} \frac{1}{\eta+\nu}(1-x) & t=1 \\ \frac{\eta}{\eta+\nu}\frac{1-(1-\sigma-\tau)(1-\eta-\nu)}{\sigma+(1-\sigma-\tau)\eta}(1-x)x^{t-2} & t=2,3,\ldots \end{cases} \tag{34}$$

Hence, the mean waiting time becomes

$$W = 1+\frac{\eta}{\eta+\nu}\frac{1-(1-\sigma-\tau)(1-\eta-\nu)}{\sigma\nu-\tau\eta} = \frac{L}{\lambda} \tag{35}$$

5 EFFECTS OF CORRELATION

In view of the significance of the following terms in the analysis, it is convenient to introduce symbols for them.

$\beta_a \triangleq (1-\sigma-\tau)$ as the geometric factor in the correlation function for numbers of arrivals per slot (and in the IDC),

$\beta_b \triangleq (1-\eta-\nu)$ as the geometric factor in the correlation function for intervals between individual arrivals (and in the IDI),

$x \triangleq \frac{\sigma(1-\eta-\nu)+\eta}{\sigma+(1-\sigma-\tau)\eta}$ as the geometric factor in the queue length distibution, asymptotic blocking probability, etc.

Further, it is convenient to investigation of queue behaviour when β_a or β_b be close to 1 to define additional symbols $\kappa_a \triangleq (1-\beta_a)^{-1}$ and $\kappa_b \triangleq (1-\beta_b)^{-1}$.

5.1 Choice of Reference System

The factors β_a and β_b appear to be good indicators of the type of correlation in the $\mathrm{GGeo}^{\mathrm{GGeo}}$ batch renewal process.

- A β value of 0 implies no correlation (in the *number* or *time* dimension, as appropriate). If $\beta_a = 0$, the process is Batch Bernoulli and there is no correlation between numbers of events at different epochs. If $\beta_b = 0$, the process is renewal and there is no correlation between intervals (i.e. between the interval between one pair of successive events and the interval between another pair of successive events).

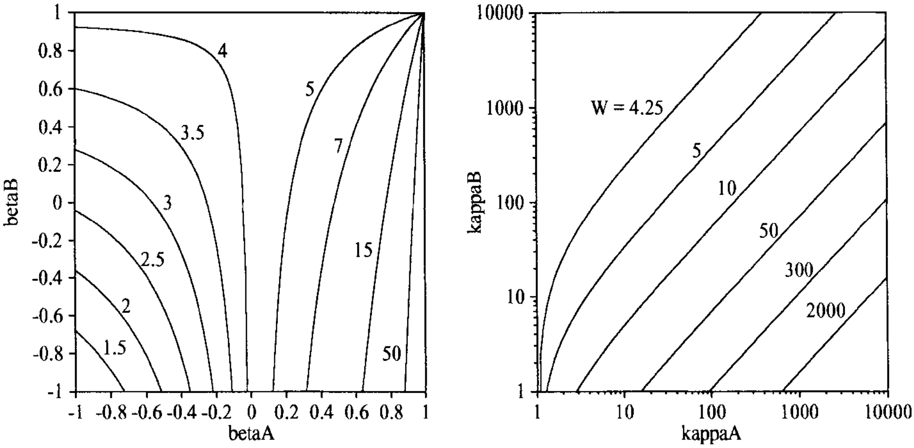

Figure 2 Waiting time (in slots) against correlation factors β_b and $\kappa_b = (1 - \beta_b)^{-1}$ (vertical scale) of batch size and β_a and $\kappa_a = (1 - \beta_a)^{-1}$ (horizontal scale) of intervals between batches for SCV of individual interarrival times $J_1 = 6.25 \times (1 - \lambda)$.

- A positive (negative) value for the β implies positive (negative) correlation in the appropriate (*number* or *time*) dimension. A greater magnitude of the β value implies stronger (positive or negative) correlation in that dimension.
- Only if both $\beta_a = 0$ and $\beta_b = 0$ is the process completely free of correlation.

In order to determine effects on queueing behaviour of correlation arising from a $\mathrm{GGeo}^{\mathrm{GGeo}}$ batch renewal arrivals process, the performance distributions and statistics for the queue must be compared with those of a reference process which is free of correlation but invariant in other significant characteristics. The $\mathrm{GGeo}^{\mathrm{GGeo}}$ process is determined by 4 parameters and, since two degrees of freedom are determined by the choice of the factors β_a and β_b, there remain 2 characteristics to be chosen to be invariant. An obvious requirement is that the intensity λ be invariant.

For the last remaining choice of invariant, it would appear natural, in view of 'traditional' traffic characterisation, to chose J_1, the SCV of intervals between successive arrivals. Figure 2 shows that, for $\beta_a < 0$, mean waiting time and (by Little's Law) mean queue length increase with β_a and with β_b, as would be expected. However, for $\beta_a > 0$, mean waiting time and mean queue length increase with β_a but reduce as β_b increases.

Similar difficulties arise with other obvious choices of other statistics to be invariant: the limiting values of the indices of dispersion $I_\infty = J_\infty$; the mean queue length or mean waiting time in an infinite buffer.

The best choice was found to be when both the mean batch size b and the mean interval a between batches were invariant. This choice is intuitively appealing because the factor β_b (equivalently κ_b) is closely related to the variability in batch sizes and the factor β_a

(equivalently κ_a) is closely related to the variability in both intervals between batches and the individual interarrival times.

$$C_a^2 = \frac{a-1}{a}\left(\frac{2}{1-\beta_a} - 1\right) = \frac{a-1}{a}(2\kappa_a - 1)$$

$$C_b^2 = \frac{b-1}{b}\left(\frac{2}{1-\beta_b} - 1\right) = \frac{b-1}{b}(2\kappa_b - 1)$$

Finally the reference system was chosen to be that with the same mean batch size and same mean interval between batches. Compare figure 3 with figure 2.

5.2 Results

Measures of interest are recast, below, in terms of the parameters a, b, κ_a and κ_b.

The geometric term x in queue length distribution, etc.

$$x = 1 - \frac{a-b}{a\,(b-1)(\kappa_a - 1) + (a-1)\,b\,\kappa_b}$$

The normalising factor Z in queue length distribution, etc. (cf equation (25))

$$Z = 1 - \frac{b-1}{a-1}\,x^{N-1}$$

The blocking probability π_N^B (cf equation (30))

$$\pi_N^B = \frac{1-Z}{Z}\frac{1-\lambda}{\lambda} = \frac{1}{Z}\frac{a-b}{a}\frac{b-1}{a-1}\,x^{N-1}$$

Mean queue length (cf equation (29))

$$L_N = \frac{1}{Z}\left(\frac{b}{a} + \frac{b(b-1)}{a-b}(\kappa_a + \kappa_b - 1)(1 - x^{N-1}) + N\frac{b-1}{a-1}(\kappa_a - 1)x^{N-1}\right)$$

Mean waiting time W in the infinite buffer queue (cf equation (35))

$$W = 1 + \frac{1}{1-x} - \kappa_b = 1 + \frac{a(b-1)}{a-b}(\kappa_a + \kappa_b - 1)$$

Figures 3, 4, 5, 6 and 7 illustrate the effects of varying correlation on mean waiting time in the infinite buffer queue, the factor x which appears as a geometric term in queue length distribution, etc., blocking probability against buffer size and mean queue length in a finite buffer. All the illustrations are for an intensity $\lambda = 0.2$. From the relations given at the begining of this sub-section, in can be appreciated that results for other intensities show similar forms.

Figure 3 shows the impact on waiting time in the infinite buffer queue. The numbers

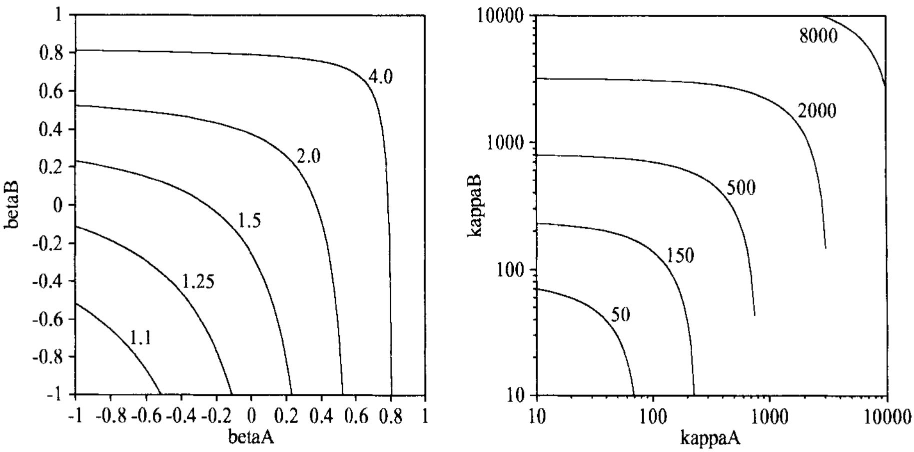

Figure 3 Waiting time (in slots) against correlation factors β_b and $\kappa_b = (1 - \beta_b)^{-1}$ (vertical scale) of batch size and β_a and $\kappa_a = (1 - \beta_a)^{-1}$ (horizontal scale) of intervals between batches for mean batch size $b = 1.5$, mean interval $a = 7.5$ slots between batches, intensity $\lambda = 0.2$.

on the contours give the waiting time as a number of slots. The right hand of the pair of charts gives an expanded view of the upper right hand corner of the left hand chart ($\beta_a \geq 0.9$, $\beta_b \geq 0.9$). The charts show that waiting time increases increasingly rapidly (and without limit) as either β_a or β_b approach unity.

Figure 4 shows the impact on the geometric term x of various degrees of correlation. The numbers on the contours give the value of x. It is seen that, as either κ_a or κ_b increases (β_a or β_b approaches 1), the value of x increases increasingly rapidly towards unity. For relatively low intensity ($\lambda = 0.2$ in the examples), the effect of variablity in batch size (given by κ_b or β_b) is stronger than that of variability in interval between batches (given by κ_a or β_a). This distinction is more pronounced when the mean batch size b is close to unity, as comparison of the two charts of figure 4 shows.

Blocking probabilty is also markedly effected by correlation in either the *time* dimension or in the *number* dimension. The two charts of figure 5 give blocking probability against buffer size for various values of β_b. The legend on each line is the value of β_b. The charts show that blocking probability increases rapidly with correlation.

The effects of correlation on mean queue length in finite buffers is shown figures 6 and 7. Each figure comprises two charts of mean queue length against buffer size for various values of κ_b, the upper chart for $\kappa_a = 1$ ($\beta_a = 0$, no correlation between interarrival times) and the lower for $\kappa_a = 5$ ($\beta_b = 0.5$, moderate correlation between interarrival times). The effects of positive correlation are marked. However, comparison of figures 6 and 7 shows that the impact of positive correlation in interarrival times is greater when the mean batch size is closer to unity.

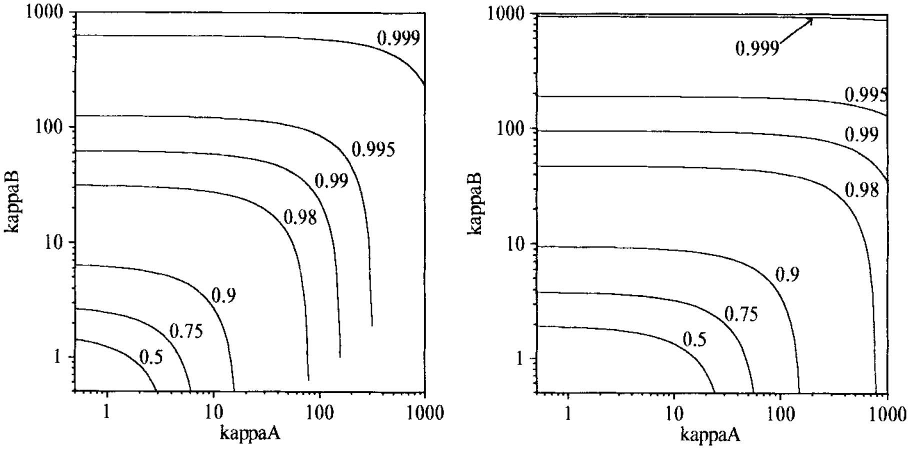

Figure 4 Geometric term x against correlation factors $\kappa_b = (1 - \beta_b)^{-1}$ (vertical scale) of batch size and $\kappa_a = (1 - \beta_a)^{-1}$ (horizontal scale) of intervals between batches for intensity $\lambda = 0.2$ and, in the left hand chart, mean batch size $b = 1.5$, mean interval $a = 7.5$ slots between batches and, in the right hand chart, mean batch size $b = 1.05$, mean interval $a = 5.25$ slots between batches.

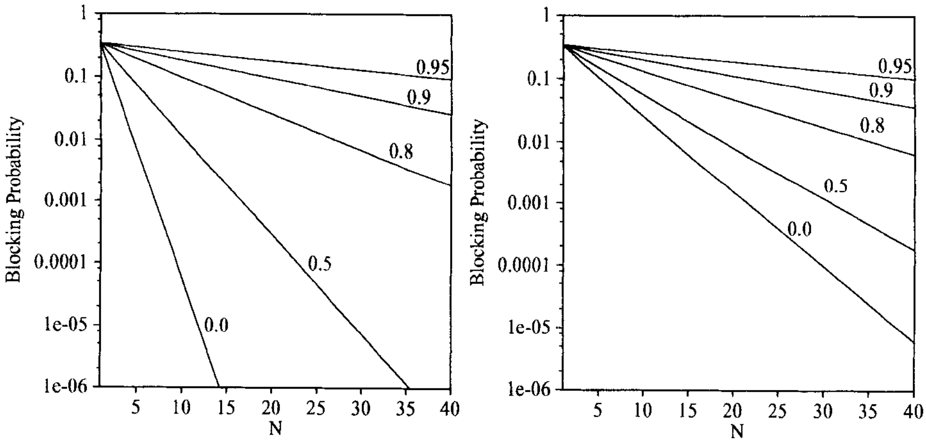

Figure 5 Blocking probability against buffer size for mean batch size $b = 1.5$, mean interval $a = 7.5$ slots between batches, intensity $\lambda = 0.2$ with $\beta_b = 0, 0.5, 0.8, 0.9, 0.95$ ($\kappa_b = 1, 2, 5, 10, 20$) and, in the left hand chart, $\beta_a = 0$ ($\kappa_a = 1$) and, in the right hand chart, $\beta_a = 0.8$ ($\kappa_a = 5$).

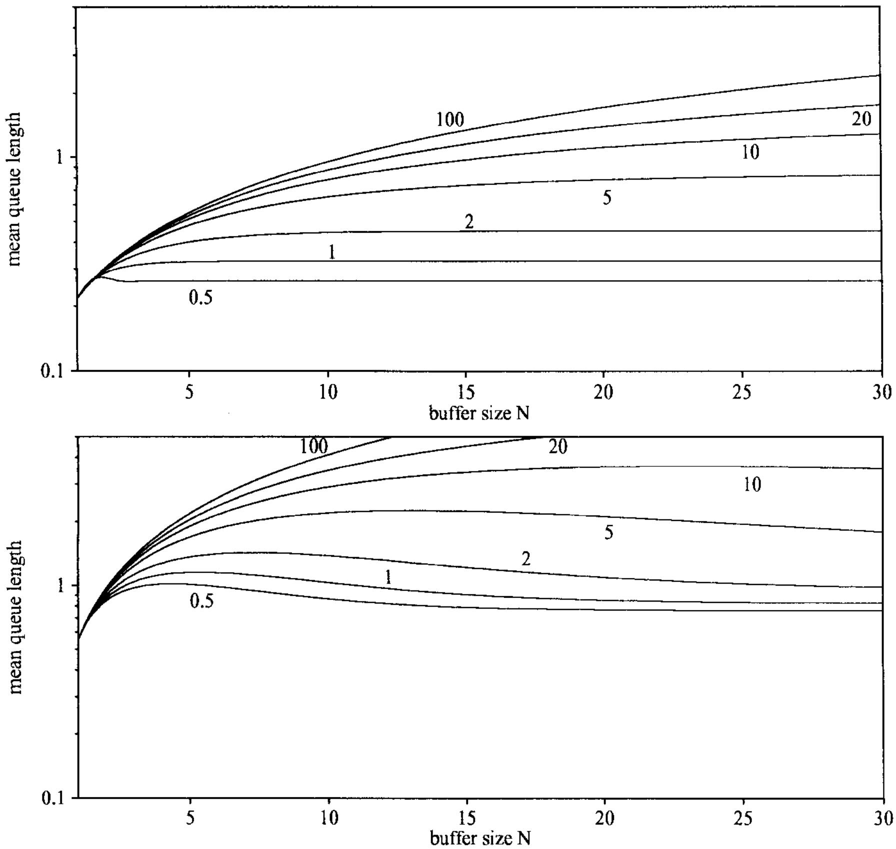

Figure 6 Mean queue length against buffer size N for mean batch size $b = 1.5$, mean interval $a = 7.5$ slots between batches, intensity $\lambda = 0.2$ and various values of κ_b $(1, 2, 5, 10, 20, 100$ and, in the upper chart, $\kappa_a = 1$ and, in the lower chart, $\kappa_a = 5$.

6 CONCLUSIONS AND PROPOSALS FOR FURTHER WORK

A discrete time $\mathrm{GI^G/D/1/N}$ queue with single server, general batch renewal arrivals process, deterministic service time and finite capacity N is analysed. Closed form expressions for basic performance distributions, such as queue length and waiting time distributions and blocking probability, are derived when the batch renewal process is of the form which might be expected to result from actual traffic measurements. Those closed form expressions are used to show the effect of varying degrees of traffic correlation upon the basic performance distributions and the results are illustrated by numerical examples.

It is seen that positive correlation has markedly adverse impact on crucial quality of

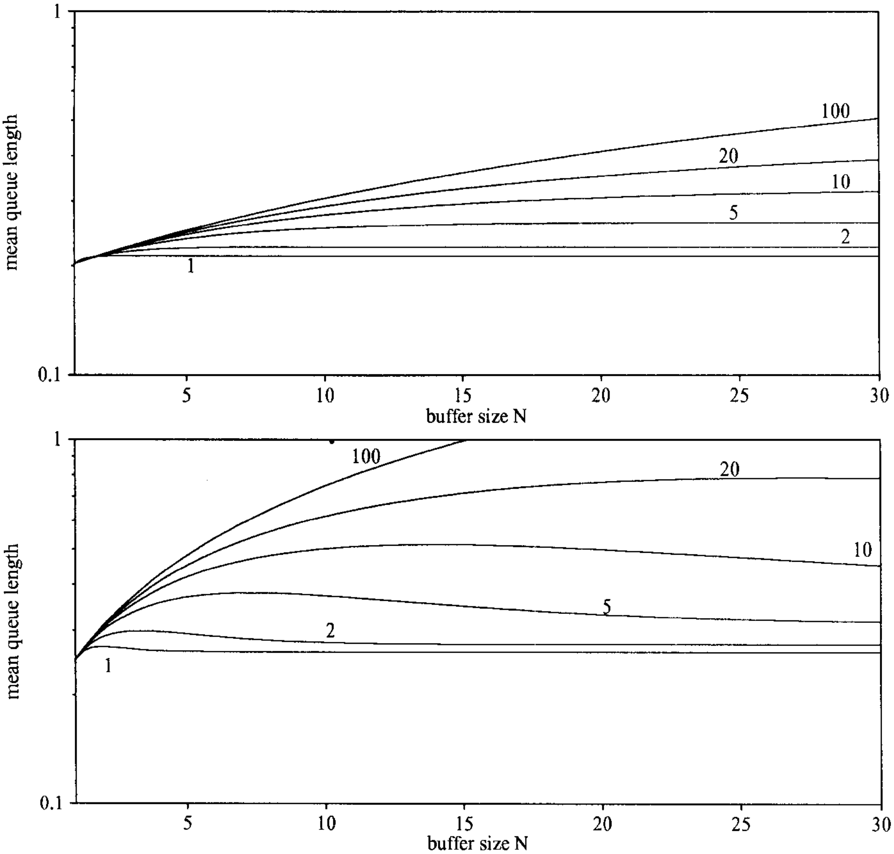

Figure 7 Mean queue length against buffer size N for mean batch size $b = 1.05$, mean interval $a = 5.25$ slots between batches, intensity $\lambda = 0.2$ and various values of κ_b $(1, 2, 5, 10, 20, 100$ and, in the upper chart, $\kappa_a = 1$ and, in the lower chart, $\kappa_a = 5$.

service (QoS) measures such as blocking probability and waiting time. Both correlation of interarrival times and correlation of counts have similar impact.

The importance of the analysis is that it shows explicitly how the magnitudes of blocking, waiting time and queue length distribution are determined by the degree of correlation in the traffic.

Characterisation of the departure process from a $GI^G/D/1/N$ queue is required in order to investigate the transmission of correlation in traffic though a multiplexer or partitioned buffer switch. Further research is required into effects of correlated traffic on the behaviour of queueing networks and, particularly, into propagation of correlation across networks of ATM switches (shared buffer, space division e.g. banyan interconnection networks). These are subjects of current study.

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APPENDIX 1 LEAST BIASED CHOICE OF PROCESS

In the notation of Section 2, the objective is to find the least biased choice for the wide sense stationary process $\{(\alpha(t), \beta(t)) : t = \ldots, -1, 0, 1, 2, \ldots\}$ given only $\mathbf{E}[N(t)N(t+\ell)]$ and $\mathbf{E}[X(t)X(t+\ell)]$ for all ℓ.

It is shown, by the outline proof below, that the least biased choice is that the $\alpha(t)$ and the $\beta(t)$ each be both stationary (in the strict sense) and independent.

First, introduce additional notation. Let

$a(n,t)$ be $\mathbf{P}[\alpha(t) = n]$ $(n = 1, 2, \ldots)$ with mean $\mathbf{E}[\alpha(t)] = a$,
$b(n,t)$ be $\mathbf{P}[\beta(t) = n]$ with mean $\mathbf{E}[\beta(t)] = b$,
$\phi_\ell(t)$ be $\mathbf{P}[N(t+\ell) \geq 1 | N(t) \geq 1]$ and
$\psi_\ell(t)$ be $\mathbf{P}[X(t+\ell) \geq 1 | X(t) \geq 1]$.

The method is first to show that $\phi_\ell(t)$ be stationary and that the $\beta(\cdot)$ be independent of each other: by similar reasoning, that $\psi_\ell(t)$ be stationary and that the $\alpha(\cdot)$ be independent of each other. Then, using the independence of the $\alpha(\cdot)$ and the stationarity of $\phi_\ell(t)$, it is readily seen that the $\alpha(\cdot)$ be stationary also: by similar reasoning, that the $\beta(\cdot)$ be stationary also.

Now

$$\mathbf{E}[N(t)N(t+\ell)] = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} n\,k\,\mathbf{P}[N(t) = k, N(t+\ell) = n]$$

Observe that only terms with $n \geq 1$ and $k \geq 1$ contribute to the sums and so it is sufficient to consider

$$\begin{aligned}\mathbf{P}[N(t) = k \geq 1, N(t+\ell) = n \geq 1] &= \mathbf{P}[N(t) = k, N(t+\ell) = n | N(t) \geq 1, N(t+\ell) \geq 1] \\ &\quad \times \mathbf{P}[N(t) \geq 1 | N(t+\ell) \geq 1] \mathbf{P}[N(t) \geq 1] .\end{aligned}$$

Now, $\mathbf{P}[N(t) \geq 1]$ is simply the probability of there being a batch at epoch t, i.e. $\mathbf{P}[N(t) \geq 1] = 1/a$, and $\mathbf{P}[N(t) \geq 1 | N(t+\ell) \geq 1]$ is $\phi_\ell(t)$, by definition, and

$$\mathbf{P}[N(t) = k, N(t+\ell) = n \,|\, N(t) \geq 1, N(t+\ell) \geq 1] = \mathbf{P}[\beta(t_1) = k, \beta(t_2) = n]$$

where t_1 is the index in the sequence $\beta(\cdot)$ which corresponds to the same batch as that indexed by t in the sequence $N(\cdot)$ and where t_2 is the index in the sequence $\beta(\cdot)$ which corresponds to the same batch as that indexed by $t + \ell$ in the sequence $N(\cdot)$.

A well known consequence of the Principle of Maximum Entropy is that, given only the marginal distributions, the least biased choice for the joint distribution is the product of the marginals. Thus, the least biased choice for the distribution $\mathbf{P}[\beta(t_1) = k, \beta(t_2) = n]$ is

$$\mathbf{P}[\beta(t_1) = k, \beta(t_2) = n] = \mathbf{P}[\beta(t_1) = k]\,\mathbf{P}[\beta(t_2 = n] = b(k, t_1)\,b(n, t_2)$$

i.e. that $\beta(t_1)$ and $\beta(t_2)$ are independent. Hence, the least biased choice for process $\{\,\alpha(t), \beta(t)\,\}$ requires that $\mathbf{E}[N(t)N(t+\ell)]$ satisfy

$$\mathbf{E}[N(t)N(t+\ell)] = \frac{1}{a}\,\phi_\ell(t) \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} n\,k\,b(k,t_1)\,b(n,t_2) = \frac{1}{a}\,\phi_\ell(t)\,\mathbf{E}[\beta(t)]^2 = \frac{1}{a}\,\phi_\ell(t)\,b^2$$

But, because the process $N(t)$ is wide sense stationary, $\mathbf{E}[N(t)N(t+\ell)]$ must be independent of t. Consequently, $\phi_\ell(t)$ is independent of t: $\phi_\ell(t)$ is stationary and may be written $\phi_\ell(t) = \phi_\ell$.

Similarly, by consideration of $\mathbf{E}[X(t)X(t+\ell)]$, the $\alpha(\cdot)$ are independent and $\psi_\ell(t)$ is stationary.

Thence, using the independence of $\alpha(t)$, $\alpha(t+\ell)$ and the stationarity of $\phi_\ell(t) = \phi_\ell$

$$\phi_1 = \mathbf{P}[\alpha(t) = 1] = a(1,t)$$

so $a(1,t)$ is independent of t,

$$\phi_2 = \mathbf{P}[\alpha(t) = 2] + \mathbf{P}[\alpha(t) = 1, \alpha(t+1) = 1] = a(2,t) + a(1,t)^2$$

so $a(2,t)$ is independent of t, etc.

Thus, $\alpha(t)$ is stationary.

By similar argument on the independence of the $\beta(t)$ and the stationarity of $\psi_\ell(t)$, it may be seen that $\beta(t)$ is stationary.

Finally, because in process $\{(\,\alpha(t), \beta(t)\,) \,:\, t = \ldots, -1, 0, 1, 2, \ldots\}$ the sequences $\alpha(t)$ and $\beta(t)$ are stationary and mutually independent, the process $\{\,\alpha(t), \beta(t)\,\}$ is a batch renewal process by definition.