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Closed-Form Recursive Estimation of MA Coefficients Using Autocorrelations and Third-Order Cumulants

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***Abstract*—We derive a simple, recursive, closed-form algorithm to estimate the parameters of an MA model of known order, using only the autocorrelation and the 1-D diagonal slice of the third-order cumulant of its response to excitation by an unobservable, non-Gaussian, i.i.d. process. The output may be corrupted by zero-mean, nonskewed white noise of unknown variance. The ARMA case is briefly discussed.**

I. Introduction

Given a zero-mean, stationary random process $x(t)$, its third-order cumulant is defined as

$$C_x(t_1, t_2) = E\{x(t)x(t + t_1)x(t + t_2)\}. \quad (1)$$

Rigorous definitions of cumulants of arbitrary order may be found in [1]. Cumulants of order $k > 2$ of a Gaussian process vanish, and as such, cumulants provide a measure of non-Gaussianity. The 1-D diagonal slice of the third-order cumulant is obtained by setting $t_1 = t_2$ in (1), i.e.,

$$c_x(\tau) = C_x(\tau, \tau) = E\{x(t)x^2(t + \tau)\}. \quad (2)$$

Rosenblatt [2] has shown that the transfer function of a finite-dimensional linear time-invariant (LTI), causal, exponentially stable system can be identified from some kth ($k > 2$)-order cumulant of the output process, without invoking the minimum phase assumption, provided the input excitation is non-Gaussian and i.i.d.

Giannakis [3] and Giannakis and Mendel [4] consider one-dimensional slices of the kth-order cumulant and develop identifiability results for ARMA models. In [3] and [4], a recursive, but rather complicated, algorithm is developed to estimate the coefficients of an MA (q) process, using both the autocorrelation and the cumulants. We develop a *simple recursive algorithm* that explicitly handles additive noise at the output.

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II. MA Parameter Estimation

The output of a finite-dimensional, LTI, MA model satisfies the equation

$$y(n) = \sum_{k=0}^{q} b(k)u(n - k). \quad (3)$$

The observed signal $z(t)$ is given by

$$z(n) = y(n) + w(n). \quad (4)$$

We assume the following.

Assumption 1: Output noise $w(n)$ is zero-mean, i.i.d., with zero third-order cumulant (possibly Gaussian), independent of $y(n)$, and has unknown variance σ_w^2.

Assumption 2: Input $u(k)$ is i.i.d., with $E\{u^2(k)\} = \sigma^2 < \infty$ and $E\{u^3(k)\} = \gamma \neq 0$; $|\gamma| < \infty$.

Assumption 3: $b(0) = 1$ and order q is known.

Assumption 4: $b(q - k) \neq b(q)[1 - b(k)]$, $k = 0, 1, \cdots, q/2$.

The autocorrelation and the diagonal slice of the third-order cumulant of $z(k)$ are given by

$$r_z(\tau) = \sigma^2 \sum_{k=0}^{q} b(k)b(k + \tau) + \sigma_w^2\delta(\tau) \quad (5)$$

$$c_z(\tau) = \gamma \sum_{k=0}^{q} b(k)b^2(k + \tau). \quad (6)$$

Both $r_z(\tau)$ and $c_z(\tau)$ have finite support over $[-q, q]$. Equation (5) defines a set of nonlinear equations in the unknown MA coefficients for which a unique solution does not generally exist. Newton–Raphson-type techniques may be used to solve (5); uniqueness is guaranteed only under a minimum phase assumption [5].

In [3] and [4], it is shown that

$$\sum_{k=0}^{q} b^2(k)r_y(\tau - k) = \frac{\sigma^2}{\gamma} \sum_{k=0}^{q} b(k)c_y(\tau - k). \quad (7)$$

Based on (7), a recursive, but rather complicated, algorithm is developed in [3], [4] to estimate the MA coefficients; based on (5) and (6), we develop a simpler recursive algorithm.

Evaluating (5) and (6) at $\tau = \pm q$ yields

$$r_z(q) = \sigma^2 b(q) + \sigma_w^2\delta(q), \quad c_z(q) = \gamma b^2(q), \quad c_z(-q) = \gamma b(q) \quad (8)$$

from which we obtain (see also [4])

$$b(q) = c_z(q)/c_z(-q); \quad \sigma^2 = r_z(q)c_z(-q)/c_z(q); \quad \gamma = c_z^2(-q)/c_z(q). \quad (9)$$

Let

$$R(\tau) = r_z(\tau)/\sigma^2 = \sum_{k=0}^{q} b(k)b(k + \tau) + \frac{\sigma_w^2}{\sigma^2}\delta(\tau) \quad (10)$$

$$C(\tau) = c_z(\tau)/\gamma = \sum_{k=0}^{q} b(k)b^2(k + \tau). \quad (11)$$

Theorem: For the MA model described in (3) and (4), operating under Assumptions 1–4, the MA parameters $b(m)$, $m = 1, 2, \cdots, q - 1$ may be recursively estimated from the output correlation and third-order cumulant as

$$b(q - m) = \frac{f(m)}{2} + \frac{b(q)h(m) - g(m)}{2[b(q) - f(m)]} \quad (12)$$

$$b(m) = \frac{f(m) - b(q - m)}{b(q)} \quad (13)$$

where

$$f(m) := R(q-m) - \sum_{k=1}^{m-1} b(k)b(k+q-m) = b(q-m) + b(m)b(q) \quad (14)$$

$$g(m) := C(q-m) - \sum_{k=1}^{m-1} b(k)b^2(k+q-m) = b^2(q-m) + b(m)b^2(q) \quad (15)$$

$$h(m) := C(m-q) - \sum_{k=1}^{m-1} b^2(k)b(k+q-m) = b(q-m) + b^2(m)b(q). \quad (16)$$

The recursion consists of evaluating (12)-(16) for $m = 1, \cdots, q/2$. Note that $b(q)$, σ^2, and γ have already been estimated via (9).

Proof: If $q = 1$, (11) yields $C(0) = 1 + b^3(1)$; hence, $b(1) = [C(0) - 1]^{1/3}$. Hence, we assume $q > 1$ in the sequel. The equalities in (14)-(16) follow from (10) and (11). Evaluating (14)-(16) at $\tau = \pm(q-1)$ yields

$$f(1) = R(q-1) = b(q-1) + b(1)b(q) \quad (17)$$

$$g(1) = C(q-1) = b^2(q-1) + b(1)b^2(q) \quad (18)$$

$$h(1) = C(1-q) = b(q-1) + b^2(1)b(q). \quad (19)$$

Next, we determine $b(q-1)$ from (17)-(19). From (17),

$$b(1) = [R(q-1) - b(q-1)]/b(q). \quad (20)$$

Substituting for $b(1)$ from (20) into (18) and (19) yields

$$C(q-1) = b^2(q-1) + b(q)[R(q-1) - b(q-1)] \quad (21)$$

$$C(1-q) = b(q-1) + [R^2(q-1) - 2R(q-1) \cdot b(q-1) + b^2(q-1)]/b(q). \quad (22)$$

Finally, eliminating $b^2(q-1)$ from (21) and (22) and simplifying leads to

$$b(q-1) = \frac{R(q-1)}{2} + \frac{b(q)C(1-q) - C(q-1)}{2[b(q) - R(q-1)]}. \quad (23)$$

Having obtained $b(q-1)$, $b(1)$ is obtained from (20). Thus, $b(1)$ and $b(q-1)$ can be obtained from $b(q)$, $R(q-1)$, $C(q-1)$, and $C(1-q)$.

Now, assume that $b(k)$, $k = 0, 1, \cdots, m-1$ are known. From (10), it follows that $b(q-k)$, $k = 0, 1, \cdots, m-1$ are also known. Note that $f(m)$, $g(m)$, and $h(m)$ in (14)-(16) are defined in terms of known quantities. Furthermore, the very right-hand sides of (14)-(16) are in the same form as (17)-(19) (with m replacing unity). Solving (14)-(16) in the same manner as (17)-(19), for $m = 2, 3, \cdots, q/2$ [the recursion stops at $q/2$ since the coefficients are evaluated in pairs $b(m)$ and $b(q-m)$], leads to (12) and (13). □

Our recursive algorithm for estimating the MA coefficients thus consists of: 1) estimating σ^2, γ, and $b(q)$ from $r_z(q)$ and $c_z(\pm q)$ via (9); 2) normalizing $r_z(k)$ and $c_z(k)$ via (10) and (11); and 3) evaluating (12)-(16) for $m = 1, 2, \cdots, q/2$. Finally, once the MA coefficients have been obtained, the noise variance σ_w^2 may be obtained from

$$\sigma_w^2 = R_z(0) - \sigma^2 \sum_{m=0}^{q} b^2(m). \quad (24)$$

Our method is considerably simpler than the one proposed in [4]. The algorithm proposed here uses autocorrelation and cumulant lags in the range $q/2 \le |m| \le q$ only; thus, it is clear that our method will work for Gaussian or non-Gaussian colored noise if the noise can be modeled as an MA(p) process, with $p < q/2$. If the measurement noise is modeled as i.i.d., then (12)-(16) may be evaluated for $m = q/2 + 1, \cdots, q$, yielding additional estimates of $b(m)$, $m = 1, \cdots, q$.

The algorithm proposed here cannot be extended to the fourth-order cumulant because in that case, the $b(k)$'s can only be obtained as solutions of pairs of quadratic equations in $b(k)$ and $b(q-k)$. In [7], an MA parameter estimation algorithm based on correlation and an off-diagonal slice of the cumulant *of arbitrary order* is described. In [6], a closed-form solution based *only* on the kth-order cumulant is reported.

Assumption 4 is required to avoid a zero-divided-by-zero problem in (12); this assumption is also required in the algorithm in [3], [4]. If Assumption 4 does not hold, one can only obtain quadratic equations in $b(k)$ and $b(q-k)$. One could accept both solutions of the quadratic equation separately, thus obtaining two candidate MA models; the cumulant matching technique in [8] may then be used to decide between the two candidate models; note, however, that the method in [8] involves the rooting of an order q polynomial.

For an ARMA (p, q) model, the AR parameters are estimated using correlations and cumulants, and the residual AR-compensated time series is computed. The residual time series is MA; hence, our MA estimation algorithm may be applied to the residual series. Note that the additive noise is now colored; hence, the autocorrelation of the residual series must be corrected for the noise contribution.

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A Direct Algorithm for Computing 2-D AR Power Spectrum Estimates

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***Abstract*—An algorithm for computing the parameters in a 2-D AR spectral estimate without prior estimation of the correlation is described. The algorithm utilizes the multichannel form of the Burg al-**

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