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CLOSED FORM SOLUTIONS OF HOLE DISTORTION FOR USE IN DEEP-HOLE DRILLING MEASUREMENTS OF RESIDUAL STRESS IN ORTHOTROPIC PLATES

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ABSTRACT

The measurement of residual stress using the deep hole drilling method relies on the evaluation of the distortion of a hole in a plate under the action of far-field direct and shear stresses. While closed form solutions exist for isotropic materials, in previous work for orthotropic materials finite element analysis has been used to find the distortion. In this technical note, Lekhnitskii's analysis is used to find closed form solutions for the distortion of a circular hole in an orthotropic plate. The results are compared with those of finite element analysis for a range of material properties with excellent agreement.

Keywords: Residual Stress, Composites, Deep-Hole Drilling, Orthotropic Stress Analysis

Supplementary Material

The following Mathematica files have been included in the submission: *Closed form expressions for hole edge displacement.nb*, *Demonstration of zero traction at the hole edge.nb* and *Solution for tension at an angle.nb*. For each Mathematica file, a pdf version has also been included.

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1. INTRODUCTION

Deep hole drilling (DHD) is a residual stress measurement method that is particularly suitable for large, thick section components [1]. First, a hole is drilled through the thickness of the component. The diameter of the hole is measured accurately and then a cylindrical core of material around the hole is trepanned from the component, relaxing the residual stresses in the core. Finally, the diameter of the hole is re-measured and the change in diameter is used to calculate the residual stress. This calculation requires the evaluation of a set of coefficients which are obtained from the analysis of the distortion of a hole in a plate loaded by far-field direct and shear stresses. For isotropic materials these coefficients can be derived from a closed-form solution [2]. For orthotropic materials, earlier work [3] used finite element analysis (FEA) to derive the necessary coefficients but in this technical note Lekhnitskii's analysis [4] is used to obtain a closed form solution. The DHD method described here assumes the residual stresses within the orthotropic plate are of a plane stress state, thus stresses normal to the plane of the plate are taken to be negligible. We remark that if significant residual stresses normal to the plane exist, it is possible to extend the DHD method to address this case, although the necessary procedure has only been demonstrated for isotropic materials [5].

The DHD technique is closely related to the centre hole drilling technique. In this technique a hole is drilled through the plate and the resulting distortion of the plate due to the release of residual stress at the hole is measured using either surface mounted strain gauges [6-8] or full-field measurement of displacement [9-12]. This work has been based on either the solution of Smith [13] or Lekhnitskii [4] for the deformation around a hole in an orthotropic plate. The Smith solution is only valid for a limited range of material properties, but for this range it gives identical results to the Lekhnitskii solution.

2. LEKHNITSKII'S ANALYSIS

In this section we revise the equations presented by Lekhnitskii to find the stresses and displacements around an open hole in an orthotropic plate loaded away from the hole. We use the same notation as Lekhnitskii [4].

When the coordinate axes coincide with the principal material directions, the characteristic equation is

$$\mu^4 + \left(\frac{E_1}{G} - 2\nu_1 \right) \mu^2 + \frac{E_1}{E_2} = 0 \quad (1)$$

E_1 , E_2 are the Young's moduli of the material in the major and minor principal material directions, ν_1 is Poisson's ratio for loading in the major principal material direction and G the shear modulus. In general there are four roots given by

$$\mu_1 = \alpha + \beta i, \bar{\mu}_1 = \alpha - \beta i, \mu_2 = -\alpha + \delta i, \bar{\mu}_2 = -\alpha - \delta i \quad (2)$$

where α , β and δ are real, $\beta > 0$, $\delta > 0$ and

$$\mu_1, \mu_2 = \frac{1}{\sqrt{2}} \sqrt{-\frac{E_1}{G} + 2\nu_1} \pm \sqrt{\left(\frac{E_1}{G} - 2\nu_1 \right)^2 - \frac{4E_1}{E_2}} \quad (3)$$

Lekhnitskii recognised three cases. When $(E_1/G - 2\nu_1)^2 > 4E_1/E_2$ and $E_1/G > 2\nu_1$, as is usual, the four roots are imaginary ($\alpha = 0$). When $(E_1/G - 2\nu_1)^2 = 4E_1/E_2$ and $E_1/G > 2\nu_1$ there are two pairs of equal imaginary roots ($\alpha = 0$ and $\beta = \delta$). This case includes the case when the material is isotropic. Finally when $(E_1/G - 2\nu_1)^2 < 4E_1/E_2$ there are four complex roots ($\alpha \neq 0$ and $\beta = \delta$). For each of these three cases it can be shown that both the product and sum of the roots μ_1 and μ_2 are imaginary.

Lekhnitskii provides a solution for the stresses and displacements around an open elliptic hole in a plate loaded by a tension p far away from the hole at an angle φ to the major axis of the ellipse, where the axes of the ellipse coincide with the principal material directions. The solution is constructed by summing the stresses in a uniform plate loaded in tension at an angle φ with those applied to the edge of the elliptic hole in such a way that the sum of the normal and shear stresses in the direction normal to the hole edge are zero. For simplicity the equations are specialised for a circular hole of radius a .

The stresses around the hole in the principal material directions are calculated by

$$\begin{aligned}\sigma_x &= p \cos^2 \varphi + 2 \operatorname{Re} \left[\mu_1^2 \Phi_1'(z_1) + \mu_2^2 \Phi_2'(z_2) \right] \\ \sigma_y &= p \sin^2 \varphi + 2 \operatorname{Re} \left[\Phi_1'(z_1) + \Phi_2'(z_2) \right] \\ \tau_{xy} &= p \sin \varphi \cos \varphi - 2 \operatorname{Re} \left[\mu_1 \Phi_1'(z_1) + \mu_2 \Phi_2'(z_2) \right]\end{aligned}\quad (4)$$

The first term in each of the equations is that due to the stress in a uniform plate while the second term is that due to the loading applied to the hole edge.

In Eq. (4)

$$\Phi_1'(z_1) = -\frac{1}{\sqrt{z_1^2 - a^2(1 + \mu_1^2)}} \frac{\bar{\beta}_1 - \mu_2 \bar{\alpha}_1}{\xi_1(\mu_1 - \mu_2)}, \quad \Phi_2'(z_2) = -\frac{1}{\sqrt{z_2^2 - a^2(1 + \mu_2^2)}} \frac{\bar{\beta}_1 - \mu_1 \bar{\alpha}_1}{\xi_2(\mu_1 - \mu_2)} \quad (5)$$

$$z_1 = x + \mu_1 y, \quad z_2 = x + \mu_2 y \quad (6)$$

$$\bar{\alpha}_1 = -\frac{pa \sin \varphi}{2} (\sin \varphi - i \cos \varphi), \quad \bar{\beta}_1 = \frac{pa \cos \varphi}{2} (\sin \varphi - i \cos \varphi) \quad (7)$$

$$\xi_1 = \frac{z_1 + \sqrt{z_1^2 - a^2(1 + \mu_1^2)}}{a(1 - i\mu_1)}, \quad \xi_2 = \frac{z_2 + \sqrt{z_2^2 - a^2(1 + \mu_2^2)}}{a(1 - i\mu_2)} \quad (8)$$

In general, $\Phi'_1(z_1)$ and $\Phi'_2(z_2)$ are represented by a Fourier series, but for the case considered here a solution is achieved using only the first term in the series.

At the hole edge, the direct and shear stresses σ_n and τ_n may be calculated from Eq. (4) as

$$\begin{aligned}\sigma_n &= \frac{p}{2}(1 + \cos(2\theta - 2\varphi)) + 2 \operatorname{Re} \left[\begin{aligned} &\Phi'_1(z_1)(\mu_1^2 \cos^2 \theta + \sin^2 \theta - 2\mu_1 \sin \theta \cos \theta) + \\ &\Phi'_2(z_2)(\mu_2^2 \cos^2 \theta + \sin^2 \theta - 2\mu_2 \sin \theta \cos \theta) \end{aligned} \right] \\ \tau_n &= -\frac{p}{2} \sin(2\theta - 2\varphi) + 2 \operatorname{Re} \left[\begin{aligned} &\Phi'_1(z_1)((1 - \mu_1^2) \sin \theta \cos \theta - \mu_1(\cos^2 \theta - \sin^2 \theta)) + \\ &\Phi'_2(z_2)((1 - \mu_2^2) \sin \theta \cos \theta - \mu_2(\cos^2 \theta - \sin^2 \theta)) \end{aligned} \right]\end{aligned}\quad (9)$$

where the position around the hole from the major principal material direction is defined by the angle θ .

Note that at the hole edge

$$z_1 = a(\cos \theta + \mu_1 \sin \theta), \quad z_2 = a(\cos \theta + \mu_2 \sin \theta) \quad (10)$$

and

$$\xi_1 = \xi_2 = \cos \theta + i \sin \theta \quad (11)$$

The Mathematica file *Demonstration of zero traction at the hole edge.nb* included in the supplementary material shows that both σ_n and τ_n in Eq. (9) are zero for any angle φ and for any angle θ . Therefore the expressions derived by Lekhnitskii are precise.

The displacements around the hole are evaluated by

$$\begin{aligned}u &= pa \left(\frac{\cos \theta \cos^2 \varphi}{E_1} - \frac{\nu_2 \cos \theta \sin^2 \varphi}{E_2} + \frac{\sin \theta \sin \varphi \cos \varphi}{2G} \right) + 2 \operatorname{Re} [p_1 \Phi_1(z_1) + p_2 \Phi_2(z_2)] \\ v &= pa \left(-\frac{\nu_1 \sin \theta \cos^2 \varphi}{E_1} + \frac{\sin \theta \sin^2 \varphi}{E_2} + \frac{\cos \theta \sin \varphi \cos \varphi}{2G} \right) + 2 \operatorname{Re} [q_1 \Phi_1(z_1) + q_2 \Phi_2(z_2)]\end{aligned}\quad (12)$$

Again, the first term in each of the equations corresponds to the displacements in a uniform plate while the second term corresponds to those due to the loading applied to the hole edge. These equations take $u = v = 0$ at the centre of hole and assume zero rigid body rotation.

In Eq. (12)

$$\Phi_1(z_1) = \frac{\bar{\beta}_1 - \mu_2 \bar{\alpha}_1}{\xi_1(\mu_1 - \mu_2)}, \quad \Phi_2(z_2) = \frac{\bar{\beta}_1 - \mu_1 \bar{\alpha}_1}{\xi_2(\mu_1 - \mu_2)} \quad (13)$$

$$p_1 = a_{11}\mu_1^2 + a_{12}, \quad p_2 = a_{11}\mu_2^2 + a_{12}, \quad q_1 = a_{12}\mu_1 + \frac{a_{22}}{\mu_1}, \quad q_2 = a_{12}\mu_2 + \frac{a_{22}}{\mu_2} \quad (14)$$

and

$$a_{11} = \frac{1}{E_1}, \quad a_{12} = -\frac{\nu_1}{E_1}, \quad a_{22} = \frac{1}{E_2} \quad \text{and} \quad \nu_2 = \nu_1 \frac{E_2}{E_1} \quad (15)$$

The Mathematica file *Solution for tension at an angle.nb* allows the calculation of the stresses and displacements at any point in a plate with a hole loaded in tension at an angle to the principal material direction.

3. DISTORTION OF THE HOLE EDGE

In this section the results of the previous description regarding Lekhnitskii's solution are specialised to the radial distortion of the hole edge. In this form they are directly applicable to the deep-hole drilling (DHD) method for measuring the residual stress in an orthotropic composite laminate:

$$\frac{u_r}{a} = \frac{1}{E_1} (f_\theta \sigma_{11}^0 + g_\theta \sigma_{22}^0 + h_\theta \sigma_{12}^0) \quad (16)$$

where u_r is the radial displacement at the hole edge at angle θ to the major principal material direction, a is the radius of the hole, σ_{11}^0 , σ_{22}^0 and σ_{12}^0 are far-field applied stresses, and f_θ , g_θ and h_θ are

coefficients that are used in the DHD calculations of residual stress. These coefficients are dimensionless functions of θ that depend on the orthotropic material constants.

For applied stresses at the edge of the plate in the major principal material direction, that is for $\varphi = 0$ and $p = \sigma_{11}^0$, Lekhnitskii presents a closed-form expression for the hole edge displacement at positions on the hole edge defined by $\theta = 0$ and $\theta = \pi/2$. The Mathematica file *Closed form expressions for hole edge displacement.nb* included in the supplementary material shows that the hole edge displacement for intermediate angles of θ may be calculated by

$$f_{\theta} = \frac{1}{2} \left[1 + n - k + (1 + n + k) \cos 2\theta \right] \quad (17)$$

In Eq. (17) and in following equations

$$k = -\mu_1 \mu_2 = \sqrt{\frac{E_1}{E_2}}, \quad n = -i(\mu_1 + \mu_2) = \sqrt{2(k - \nu_1) + \frac{E_1}{G}} \quad (18)$$

Note that $f_0 = 1 + n$ and $f_{\pi/2} = -k$

Lekhnitskii does not provide expressions for the cases of applied stress normal to the major principal material direction, or for applied shear. For the case of applied stress normal to the major principal material direction the Mathematica file *Closed form expressions for hole edge displacement.nb* shows that for $\varphi = \pi/2$ and $p = \sigma_{22}^0$

$$g_{\theta} = \frac{1}{2} \left[k^2 + nk - k - (k^2 + nk + k) \cos 2\theta \right] \quad (19)$$

Again, note that $g_0 = -k$ and $g_{\pi/2} = k^2 + nk$

Finally, for applied shear stress the solution for the hole edge displacement is obtained by superposition of two cases where $\varphi = \pi/4$ and $p = \sigma_{12}^0$, and where $\varphi = -\pi/4$ and $p = -\sigma_{12}^0$ giving

$$h_{\theta} = \frac{1}{2}(n^2 + nk + n)\sin 2\theta \quad (20)$$

The relationships presented in this section apply to all three cases of the solution to the characteristic equation (Eq. (1)) since the relationship between the parameters k , n and the complex parameters μ_1 , μ_2 in Eq. (18) are always valid.

4. COMPARISON WITH FINITE ELEMENT ANALYSIS

Finite element analysis was carried out using ABAQUS/CAE version 6.12 [14] to provide validation of the correctness of the equations for the distortion of the hole presented in the previous section. Three separate two-dimensional finite element analyses were performed as shown in Figure 1 to determine the distortion of the hole for far-field applied stress in the major principal material direction, for applied stress in the direction normal to the major principal material direction and for far-field shear. The material properties used for these analyses are for unidirectional carbon/epoxy (AS4/8552): $E_1 = 135$ GPa, $E_2 = 9.6$ GPa, $G = 5.2$ GPa and $\nu_1 = 0.3$. The dimensions of the square model are 100 mm by 100 mm and the radius of the reference hole is 2 mm, small enough compared to the size of the square model that the results are considered to be close to those for an infinite plate.

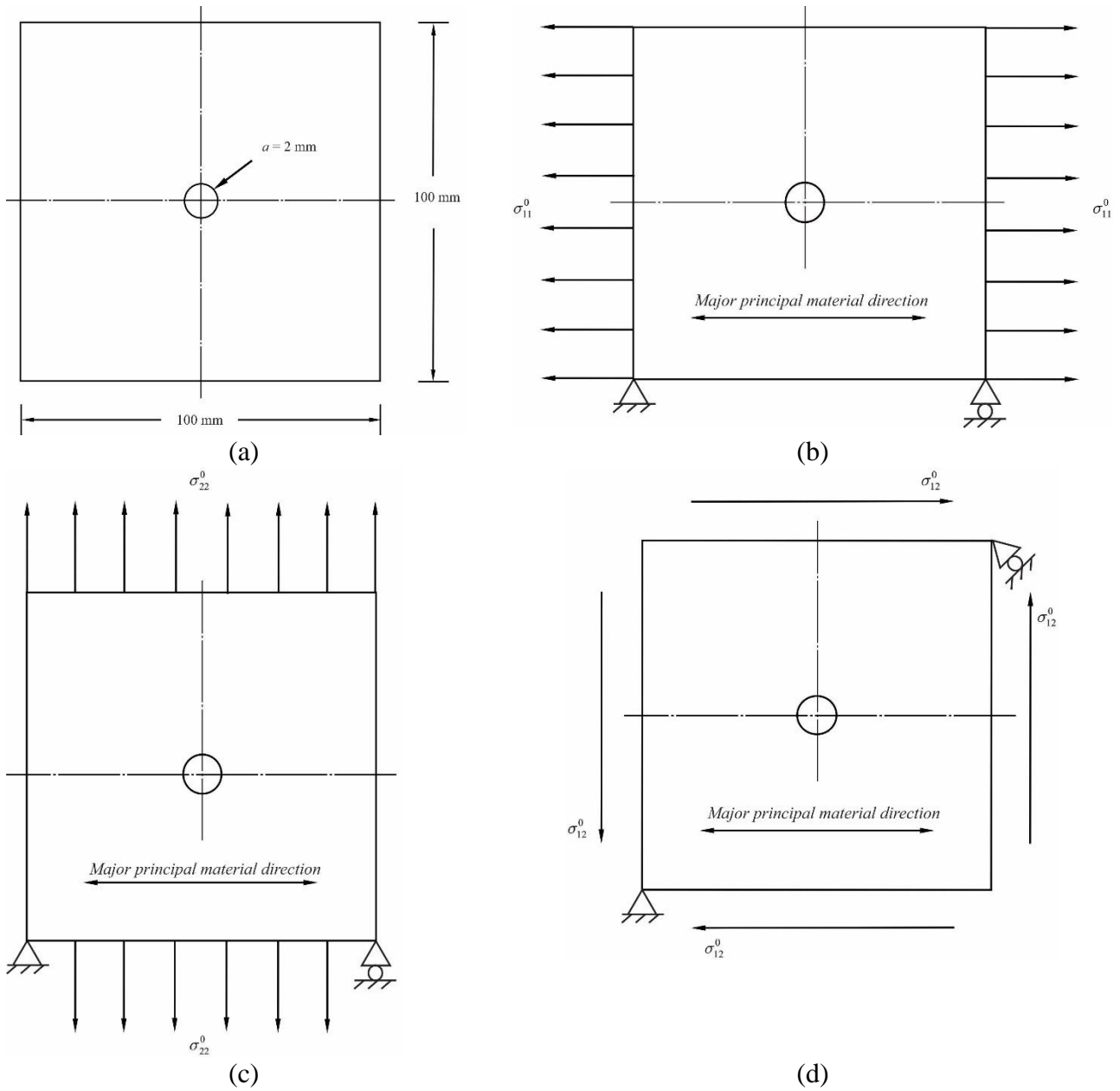


Figure 1. (a) Dimensions of the finite element model, (b) far-field loading applied to the model in the major principal material, (c) far-field loading normal to the major principal material direction and (d) far-field shear loading.

The finite element mesh used to perform the analyses consisted in 8-node biquadratic elements (CPS8R) with reduced integration and plane stress conditions. There were 18 elements for every 90 degrees around the edge of the hole and 20 elements radially, between the edge of the hole and the edge of the plate, see Figure 2.

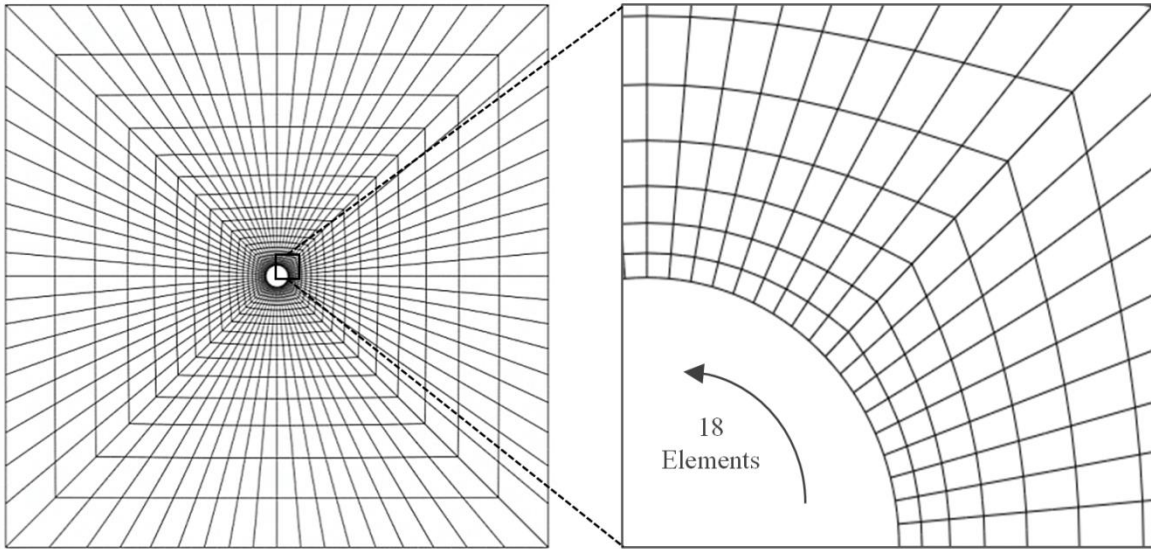


Figure 2. Finite element mesh used for calculation of the hole distortion

Figure 3 shows a comparison of the values of the coefficients obtained by finite element analysis with the equations presented in this technical note (Eqs. (17), (19) and (20)). Note that g_q and h_q have been divided by k so that the ranges of values are similar.

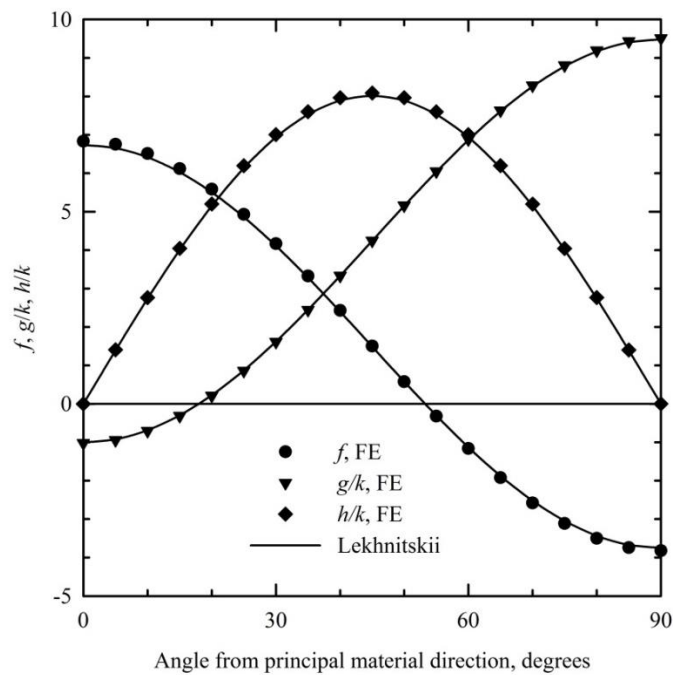


Figure 3. Comparison of coefficients calculated by finite element analysis with the equations presented in this technical note based on the analysis of Lekhnitskii.

Two additional finite element studies were carried out for the ranges of material properties shown in Table 1. For the first study the value of E_2 was varied while the other material properties were held constant. For the second study the value of G was varied. A further study was carried out where n_1 was varied but it was found that the coefficients did not change significantly and therefore the results of this simulation will not be reported here.

Material property	Study 1	Study 2
E_1	100 GPa	100 GPa
E_2	10, 20, 50, 80 GPa	10 GPa
n_1	0.3	0.3
G	5 GPa	5, 10, 15, 20 GPa

Table 1. Mechanical properties used in the models for study 1 and study 2.

The results of the finite element studies are compared to those of the equations presented here in Figure 4(a) where E_2 was varied and in Figure 4(b) where G was varied. The figure shows the value of the f_q coefficient for angles of $q=0$ and $q=p/2$, the value of the g_q coefficient for angles of $q=0$ and $q=p/2$ and the value of the h_q coefficient for an angle of $q=p/4$. Some of the coefficients have been divided by k^2 to allow the values of the coefficients to be presented easily on the same graph. The figures show close agreement for a wide range of material properties. The vertical dashed line in Figure 4(b) at $G/E_1 \gg 0.144$ shows the point when $(E_1/G - 2\nu_1)^2 = 4E_1/E_2$ which divides the cases in the solution of the characteristic equation. It can be seen that the equations presented here are valid for all cases.

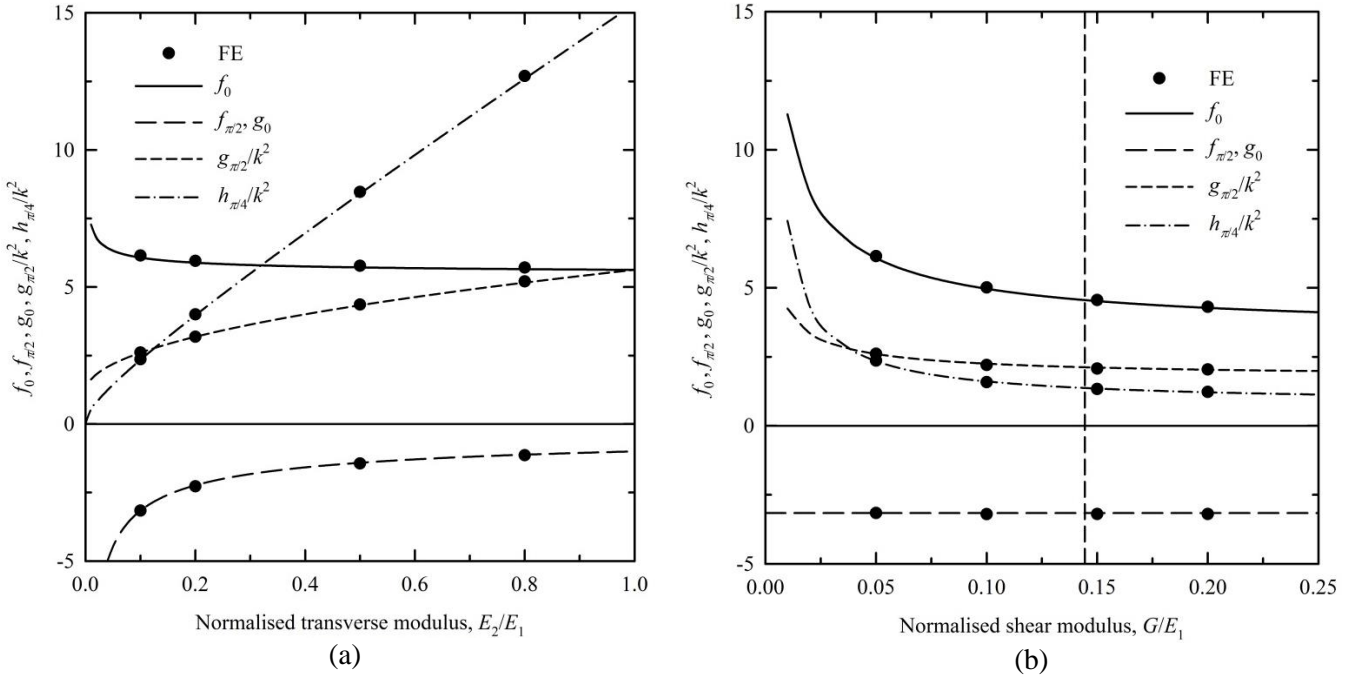


Figure 4. Comparison between coefficients calculated by Lekhnitskii's solution and by finite element analysis using the models of (a) study 1 and (b) study 2.

5. CONCLUSIONS

A set of closed form solutions have been presented for the distortion of a circular hole due to far-field applied direct stress and shear stress in an orthotropic plate based on Lekhnitskii's analysis. Excellent agreement with the results of finite element analysis has been demonstrated for a range of material properties.

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REFERENCES

1. Leggatt RH, Smith DJ, Smith SD, Faure F. Development and experimental validation of the deep-hole method for residual stress measurements. *Journal of Strain Analysis*, 31, 177-86, 1996.
2. Timoshenko S, Goodier JN. *Theory of elasticity*, McGraw-Hill, 1951.
3. Bateman MG, Miller OH, Palmer TJ, Breen CEP, Kingston EJ, Smith DJ, Pavier MJ. Measurement of residual stress in thick composite laminates using the deep-hole method, *International Journal of Mechanical Sciences*, 47, 1718-1739, 2005.
4. Lekhnitskii SG, *Anisotropic Plates*, Gordon and Breach, 1968
5. Mahmoudi AH, Hossain S, Truman CE, Smith DJ, Pavier MJ. A new procedure to measure near yield residual stresses using the deep hole drilling technique, *Experimental Mechanics*, 49, 595–604, 2009
6. Schajer GS and Yang L. Residual stress measurement in orthotropic materials using the hole drilling method, *Experimental Mechanics*, 34, 324-333, 1994.
7. Shokrieh MM, Ghasemi K AR. Determination of calibration factors of the hole drilling method for orthotropic composites using an exact solution, *J. Composite Materials*, 41, 2293-2311, 2007
8. Pagliaro P, Zuccarello B. Residual stress analysis of orthotropic materials by the through-hole drilling method, *Experimental Mechanics*, 47, 217-236, 2007
9. Baldi, A. Full field methods and residual stress analysis in orthotropic material. I: Linear approach, *Int. J. Solids and Structures*, 44, 8229-8243, 2007
10. Baldi, A. Full field methods and residual stress analysis in orthotropic material. II: Nonlinear approach, *Int. J. Solids and Structures*, 44, 8244-8258, 2007
11. Baldi A. Residual stress analysis of orthotropic materials using integrated digital image correlation, *Experimental Mechanics*, 54, 1279-1292, 2014.

12. Pisarev VS, Eleonsky SI, Chernov AV. Residual stress determination in orthotropic composites by displacement measurements near through hole, *Experimental Mechanics*, 55, 1225-1238, 2015
13. Smith CB. Effect of elliptic or circular holes on the stress distribution in plates of wood or plywood considered as orthotropic materials. USDA Forest Products Laboratory, Mimeo 1510, 1944.
14. Dassault Systèmes. Abaqus 6.12 Online Documentation. *Notes* 92, 815–821 (2012).