(PROVISIONAL VERSION)

Closed Formulae for the Statistical Weights

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In the analysis of multiple production experiments from high energy collisions, the need arises to know the branching ratio between channels with the same multiplicity but different charges of the end particles. E.G. between the reactions

$$p+p \rightarrow \begin{cases} pp\,\pi^+\pi^- \\ pp\,\pi^0\pi^0 \\ pn\,\pi^+\pi^0 \\ nn\,\pi^+\pi^+ \end{cases}$$

If the following assumptions are made

a) conservation of isospin and charge,
b) equal weight or each possible isospin end state with fixed T and T_3 ("statistical hypothesis")

it is possible to compute those branching ratios.

Indeed, one knows the total isospin $T^{(i)}$ and $T_3^{(i)}$ from the initial state of the colliding particles. If we choose an orthonormal system of base vectors $|T,T_3,\alpha\rangle$ in which to describe our isospin end states, then the state after the collision is given - in isospace - by

$$\sum_{\alpha} c_{\alpha} \left| T^{(i)}, T_3^{(i)}, \alpha \right\rangle \qquad (1)$$

Here α stands for all additional quantum numbers required to describe the n-particle state completely in isospin space. But the system of the $|T, T_3, \alpha\rangle$ is not the one used by the experimentalist who measures the end products of a collision. He describes the nature of the final particles and their charges, his measurement states that the state vector of the system in isospace is of the form

$$|t_1\theta_1, t_2\theta_2, \ldots t_n\theta_n\rangle \qquad (2)$$

where

t_i : isospin of the ith particle

$\theta_i = (t_3)_i$ = 3rd component of $\vec{t}_i$.

Assumptions a) and b) mean that the transition matrix element squared

$$\left|\langle T^{(f)}, T_3^{(f)}, \alpha^{(f)}; \pi^{(f)} | \mathcal{T} | T^{(i)}\, T_3^{(i)} \quad ; \alpha^{(i)}\rangle\right|^2$$

is independent of $\alpha^{(f)}$ [$\pi^{(i)}$ and $\pi^{(f)}$ represent the quantum numbers for the state vector in ordinary space], and vanishes for $T^{(f)} \neq T^{(i)}$ and $T_3^{(f)} \neq T_3^{(i)}$. Because the initial state is a two-particle state $T^{(i)}$ and $T_3^{(i)}$ suffice to define it in isospace. The probability for state (2) is therefore in general

$$\left|\langle t_1\theta_1, \ldots, t_n\theta_n; \pi^{(f)} | \mathcal{T} | T, T_3 \quad \pi^{(i)}\rangle\right|^2$$

$$= \sum_{\alpha} \left|\langle t_1\theta_1, \ldots, t_n\theta_n | T, T_3, \alpha\rangle\right|^2 \left|\langle T, T_3\, \alpha; \pi^{(f)} | \mathcal{T} | T, T_3, \quad ; \pi^{(i)}\rangle\right|^2$$

but because of b) (3)

$$= F(\pi^{(f)}) \sum_{\alpha} \left|\langle t_1\theta_1, \ldots, t_n\theta_n | T, T_3, \alpha\rangle\right|^2 \qquad (4)$$

where $F(\pi^{(f)})$ is a function depending on the nature of the particles produced, on the configuration in p-space, and also of course on the initial condition, before the collision, but not on $\alpha^{(f)}$.

We see that the probability of certain charges $\theta_1,\ldots,\theta_n$ will be given by

$$P^{(T)}_{\theta_1,\ldots,\theta_n} = \left[\sum_\alpha \left|\langle t_1\theta_1,\ldots,t_n\theta_n \mid T,T_3,\alpha\rangle\right|^2\right]\left[\sum_{\theta_1,\ldots,\theta_n}\sum_\alpha \left|\langle t_1\theta_1\ldots \mid T,T_3\,\alpha\rangle\right|^2\right]^{-1} \tag{5}$$

and this is independent of the configuration in p-space.

The method used up to now *) to compute $P^{(T)}_{\theta_1,\ldots,\theta_n}$, was to choose explicitly a complete orthonormal system $|T,T_3,\alpha\rangle$, and to carry the summation $\sum_\alpha$ out, using of course the special properties of the system chosen to introduce simplifications whenever possible.

What one does in fact is to compute the projection of $|t_1\theta_1,\ldots,t_n\theta_n\rangle$ on all base vectors of $|T,T_3,\alpha\rangle$ and build the sum of the squares. This is nothing but the square of the projection of $|t_1\theta_1,\ldots,t_n\theta_n\rangle$ on the subspace spanned by the base $|T,T_3,\alpha\rangle$ (for T,T_3 given). Consequently, if one could compute directly the length of this projection, one could dispense with the computation of the components.

The vectors $|t_1\theta_1,\ldots,t_n\theta_n\rangle$ span in fact a representation space of the rotation group. If one performs a rotation (defined by the Euler angles α,β,γ) these vectors transform according to

$$|t_1\theta_1,\ldots,t_n\theta_n\rangle = \sum_{(\theta')} \mathfrak{D}^{t_1}_{\theta_1\theta'_1}(\alpha\beta\gamma)\,\mathfrak{D}^{t_2}_{\theta_2\theta'_2}(\alpha\beta\gamma)\ldots\mathfrak{D}^{t_n}_{\theta_n\theta'_n}(\alpha\beta\gamma)\,|t_1\theta'_1,\ldots,t_n\theta'_n\rangle \tag{6}$$

and the matrices

$$D_{\theta_1,\theta_2\ldots;\theta'_1,\theta'_2\ldots}(\alpha\beta\gamma) \equiv \mathfrak{D}^{t_1}_{\theta_1,\theta'_1}(\alpha\beta\gamma)\ldots\mathfrak{D}^{t_n}_{\theta_n,\theta'_n}(\alpha\beta\gamma) \tag{7}$$

form a representation of the rotation $(\alpha\beta\gamma)$.

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*) F. Cerulus, Suppl.Nuovo Cimento 15, 402 (1960); here a coupling scheme is used where the (i+1)st particle is coupled to the i previous ones. This leads to recurrence formulae involving Clebsch-Gordan coefficients.

A. Pais, (preprint); the α are quantum numbers characterising the symmetry of the wave functions. For T=0 and T=1, this is sufficient to define a complete base for many-pion states.

This representation can be reduced into a direct sum of irreducible representations, by a suitable transformation. The reduction is equivalent to finding the invariant subspaces in the space of $|t_1\theta_1,\ldots,t_n\theta_n\rangle$. In particular to the irreducible representation of degree 2T+1 corresponds an invariant subspace of dimension 2T+1 whose base vectors are transformed into themselves by any rotation $(\alpha\beta\gamma)$. The vectors $|T,T_3,\alpha\rangle$ belong necessarily to such a subspace. There may be more than one subspace of dimension 2T+1, because the representation in the space of the $|t_1\theta_1,\ldots,t_n\theta_n\rangle$ may contain more than once the irreducible representation equivalent to $\mathfrak{D}^T$. Let us call the direct sum of these subspaces $\Gamma^{(T)}$.

We now recall the form of the projection operator of a vector from the $|t_1\theta_1,\ldots,t_n\theta_n\rangle$ space onto the subspace $\Gamma^{(T)}$.

$$P^{\Gamma^{(T)}} = \frac{2T+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} d\beta \sin\beta \; \chi^{(T)^*}(\alpha\beta\gamma)\; D_{\theta_1\theta_2\ldots;\theta'_1\theta'_2\ldots} \tag{8}$$

where $\frac{1}{8\pi^2}\int d\alpha \int d\gamma \int d\beta \sin\beta$ is the invariant integral in group space $\chi^{(T)}(\alpha\beta\gamma)$ is the character of the rotation $(\alpha\beta\gamma)$ in the representation of dimension 2T+1.

$D_{\theta_1\theta_2,\ldots;\theta'_1\theta'_2}$ is the rotation operator in the space of $|t_1\theta_1,\ldots,t_n\theta_n\rangle$, and is given by (7).

(8) is the application to the rotation group of a general theorem on such projection operators for finite groups or continuous groups which allow an invariant integration in group space *).

To get the square of the projection of $|t_1\theta_1,\ldots,t_n\theta_n\rangle$ on $\Gamma^{(T)}$ we build simply

$$\left|\,\left|P^{\Gamma^{(T)}}\right|t_1\theta_1,\ldots,t_n\theta_n\rangle\right|^2 = \langle t_1\theta_1,\ldots,t_n\theta_n\left|P^{\Gamma^{(T)}}\right|t_1\theta_1,\ldots,t_n\theta_n\rangle$$

because for any projection operator $P^2 = P$ and P is real.

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*) E. Wigner, Gruppentheorie und Quantenmechanik, Chapter XII.

Consequently

$$P^{(T)}_{\theta_1 \theta_2 \ldots \theta_n} = \frac{2T+1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} d\beta \sin\beta \; \chi^{(T)*}(\alpha\beta\gamma)\, \mathcal{D}^{(t_1)}_{\theta_1\theta_1}(\alpha\beta\gamma) \ldots \mathcal{D}^{(t_n)}_{\theta_n\theta_n}(\alpha\beta\gamma) \tag{9}$$

It is now a metter of straightforward computation to find $P^{(T)}_{\theta_1\theta_2\ldots}$ in each case. Indeed, let us consider the expressions for $\chi^{(T)}$ and $\mathcal{D}^{(t)}$:

$$\mathcal{D}^{(j)}_{m,m}(\alpha\beta\gamma) = \left(\frac{1+\cos\beta}{2}\right)^m \mathbb{P}^{0,2m}_{T-m}(\cos\beta)\, e^{-im(\alpha+\gamma)} \tag{10}$$ *)

$$\chi^{(j)}(\alpha\beta\gamma) = \sum_m \mathcal{D}^{(j)}_{m,m}(\alpha\beta\gamma) \tag{10a}$$

with

$$\mathbb{P}^{0,2m}_{T-m}(x) = \text{Jacobi polynomial}$$

$$= 2^{-T+m} \sum_{\nu=0}^{T-m} \binom{T-m}{\nu} \binom{T+m}{T-m-\nu} (x-1)^{T-m-\nu} (x+1)^{\nu} \tag{10b}$$

With these expressions (9) can be transformed into

$$P^{(T)}_{\theta_1,\theta_2,\ldots,\theta_n} = \frac{2T+1}{8\pi^2} \int d\alpha \int d\gamma \int d\beta \sin\beta \left(\frac{1+\cos\beta}{2}\right)^{\theta_1+\theta_2+\ldots+\theta_n} \left[\prod_{i=1}^{n} \mathbb{P}^{0,2\theta_i}_{t_i-\theta_i}(\cos\beta)\right] \times$$

$$\times\, e^{-i(\theta_1+\theta_2+\ldots+\theta_n)(\alpha+\gamma)} \left[\sum_{m=-T}^{+T} \left(\frac{1+\cos\beta}{2}\right)^m \mathbb{P}^{0,2m}_{T-m}(\cos\beta)\, e^{+im(\alpha+\gamma)}\right] \tag{11}$$

$$= (2T+1)2^{-2m-1} \int_{-1}^{+1} dx\, (1+x)^{2m} \mathbb{P}^{0,2m}_{T-m}(x) \left[\prod_{i=1}^{n} \mathbb{P}^{0,2\theta_i}_{t_i-\theta_i}(x)\right] \tag{12}$$

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*) cf. e.g. : A.R. Edmonds, Angular momentum in quantum mechanics, CERN 55-26

where we have used that $\frac{1}{2\pi}\int_0^{2\pi} e^{im\alpha}\, d\alpha = \delta_{0,m}$ (Kronecke δ) and where $m \equiv \theta_1+\theta_2+\ldots+\theta_n = T_3$ and $x = \cos\theta$.

The integrand is a polynomial of degree $N = T + \sum_{i=1}^{n} t_i - 2m$. If we represent it by

$$\sum_{j=0}^{N} a_j x^j$$

then

$$P^{(T)}_{\theta_1,\ldots,\theta_n} = (2T+1)\; 2^{-2m} \sum_{j=0}^{[N/2]} \frac{a_{2j}}{2j+1} \tag{13}$$

Formula (10) assumes that $m \geqq 0$. This is no restriction because one can prove easily *) that

$$P^{(T)}_{\theta_1,\theta_2,\ldots,\theta_n} = P^{(T)}_{-\theta_1,-\theta_2,\ldots,-\theta_n}$$

It is obvious from the foregoing that $P^{(T)}_{\theta_1,\ldots,\theta_n}$ does not depend on the order of the θ_i.

If one does not label explicitly the particles of the same kind (which are in general distinguishable because they have different momenta), one shall want to sum over the different permutations of the $\theta_1,\ldots,\theta_n$ which give rise to distinguishable states. Permutations which shuffle only particles of the same kind and change among themselves do not lead to distinguishable states. In order to find then the probability of a certain charge distribution one has to multiply the coefficient $P^{(T)}_{\theta_1,\ldots,\theta_n}$ with the number of permutations of the $\theta_1,\ldots,\theta_n$ which give rise to a reordering of the numbers θ_1 to θ_n. One permutes of course only the θ's belonging to the same kind of particles. This coefficient we shall denote by $\overset{*}{P}{}^{T}_{\theta_1,\ldots,\theta_n}$.

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*) F. Cerulus, loc.cit.

If e.g. we want the weight of the charge distribution of n pions, in isospin T-states, without consideration of their momenta, we get for this

$$\text{Probability for } n_+\pi^+,\ n_o\pi^o,\ n_-\pi^- \equiv {}^*P^{(T)}_{n_+,n_o,n_-} = \frac{n!}{n_+!n_o!n_-!}\ P^{(T)}_{\underbrace{++\dots}_{n_+}\underbrace{oo\dots}_{n_o}\underbrace{--\dots}_{n_-}}$$

The properties of the $P^{(T)}_{\theta_1,\dots,\theta_n}$ due to the unitarity of the transformation between the vectors $|t_1\theta_1,\dots,t_n\theta_n\rangle$ and $|T,T_3,\alpha\rangle$ have been discussed already *). Let us recall only that

$$\sum_{\substack{(\theta)\\ m_{\text{fixed}}}}{}^* P^{(T)}_{\theta_1,\dots,\theta_n} = \rho^{(T)}$$

where the sum is over all combinations of $\theta_1,\dots,\theta_n$ such that $\theta_1+\theta_2+\dots\theta_n = m$ remains constant. $\rho^{(T)}$ is the number of independent isospin functions that can be formed to a fixed T and T_3 with the given number of particles having isospin $t_1,t_2,\dots,t_n$. $\rho^{(T)}$ has already been computed **).

We shall now consider a few important cases and calculate the formula for each of them.

A. Case of n pions

Here we have only to consider $\mathfrak{D}^1_{m,m}$ in (9)

$$\mathfrak{D}^1_{+1,+1}(\alpha\beta\gamma) = e^{-i(\alpha+\gamma)}\,\tfrac{1}{2}(1+\cos\beta)$$

$$\mathfrak{D}^1_{0,0}\ (\alpha\beta\gamma) = \cos\beta \qquad\qquad \chi^{(1)} = \sum_{m=-1}^{+1}\mathfrak{D}^1_{m,m}$$

$$\mathfrak{D}^1_{-1,-1}(\alpha\beta\gamma) = e^{i(\alpha+\gamma)}\,\tfrac{1}{2}(1+\cos\beta)$$

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*) F. Cerulus, loc.cit.

**) Y. Yeivin and A. de-Shalit, Nuovo Cimento 1, 1147 (1955).
B.M. Barbašev and V.S. Barašenkov, Nuovo Cimento Suppl. 7, 19 (1957).

and (12) reduces to (n_+, n_o, n_- are the numbers of π^+, π^o, π^- respectively; $m = n_+ - n_-$)

$$P^T_{n_+,n_o,n_-} = (2T+1)\, 2^{-2n_+-1} \int_{-1}^{+1} (1+x)^{2n_+} x^{n_o} \mathbb{P}^{0,2m}_{T-m}(x)\, dx \tag{14}$$

which is easily expressed as a double sum

$$P^T_{n_+,n_o,n_-} = (2T+1)\, 2^{-2n_+-T+m} (-1)^{T-m} \sum_{\nu=0}^{T-m} \binom{T-m}{\nu} \binom{T+m}{T-m-\nu} (-1)^{\nu} \times$$

$$\times \left[A(2n_+ + \nu, n_o, T-m-\nu) + (-1)^{n_o} A(T-m-\nu, n_o, 2n_+ + \nu) \right]$$

with

$$A(a,b,c) \equiv \sum_{p=0}^{a} \binom{a}{p} \frac{(p+b)!\,c!}{(p+b+c+1)!} \tag{15}$$

Especially :

For T=0

$$P^{(0)}_{n_+,n_o,n_-} = \begin{cases} 2^{-2n_+} \displaystyle\sum_{q=0}^{n_+} \binom{2n_+}{2q} \frac{1}{n_o+2q+1} & \text{for } n_o \text{ even} \\ 2^{-n_+} \displaystyle\sum_{q=0}^{n_+-1} \binom{2n_+}{2q+1} \frac{1}{n_o+2q+2} & \text{for } n_o \text{ odd} \end{cases} \tag{16}$$

For T=1

$$P^{(1)}_{n_+,n_o,n_-} = 3\, P^{(0)}_{n_+,n_o+1,n_-} \qquad m = 0$$

$$P^{(1)}_{n_+,n_o,n_-} = 3\, P^{(0)}_{n_+,n_o,n_-+1} \qquad m = 1 \tag{17}$$

B. Case of 2 nucleons and n pions (or 2 K-mesons and n pions)

Because $\mathcal{D}^{\frac{1}{2}}_{m,m} = e^{im(\alpha+\gamma)} \cos\frac{\beta}{2}$ we have, from (9) or (12)

$$P^{(T)}_{m_1,m_2;n_+,n_o,n_-} = (2T+1)\, 2^{-(n_+ + n_- + m)-2} \int_{-1}^{+1} (1+x)^{n_+ + n_- + m+1}\, x^{n_o}\, \mathbb{P}^{0,2m}_{T-m}(x)\, dx \tag{18}$$

as

$$n_+ + n_- + m = 2n_+ + m_1 + m_2 = \begin{cases} 2n_+ + 1 & \text{for } pp \\ 2n_+ & \text{"} \quad pn \\ 2n_+ - 1 & \text{"} \quad nn \end{cases}$$

we find, by comparing (18) with (14), the following relations with the coefficients for n pions only :

$$\begin{aligned} P^{(T)}_{pp;n_+,n_o,n_-} &= P^{(T)}_{n_+ +1,n_o,n_-} \\ P^{(T)}_{nn;n_+,n_o,n_-} &= P^{(T)}_{n_+,n_o,n_- +1} \\ P^{(T)}_{pn;n_+,n_o,n_-} &= \tfrac{1}{2}\left[P^{(T)}_{n_+,n_o,n_-} + P^{(T)}_{n_+,n_o+1,n_-}\right] \end{aligned} \tag{19}$$

C. Case of one nucleon and n pions

Formula (12) in this case becomes :

$$P^{(T)}_{m_1;n_+,n_o,n_-} = (2T+1)\, 2^{-(n_+ + n_- + m + \frac{3}{2})} \int_{-1}^{+1} (1+x)^{n_+ + n_- + m + \frac{1}{2}}\, x^{n_o}\, \mathbb{P}^{0,2m}_{T-m}(x)\, dx$$

Let us put

$$T = J + \tfrac{1}{2} \qquad\qquad m = M + \tfrac{1}{2}$$

Then

$$P^{(T)}_{m_1;n_+,n_o,n_-} = 2(J+1)\, 2^{-(2n_+ + m_1 + \frac{3}{2})} \int_{-1}^{+1} (1+x)^{2n_+ + m_1 + \frac{1}{2}}\, x^{n_o}\, \mathbb{P}^{0,2M+1}_{J-M}(x)\, dx$$

$$= (J+1)\, 2^{-(n_+ + n_- + J+1)} (-1)^{J-M} \sum_{\nu=0}^{J-M} (-1)^{\nu} \binom{J-M}{\nu} \binom{J+M+2}{J-M-\nu}$$

$$\left[A(2n_+ +1+\nu, n_o, J-M-\nu) + (-1)^{n_o} A(J-M-\nu, n_o, 2n_+ +1+\nu) \right] \qquad (20)$$

with the same definition of A as in (15). Formulae (20) and (15) are very similar, and suggest also in this case simple relations with the coefficients for n pions only.

Especially :

For $\begin{cases} T = 1/2 \\ m = 1/2 \end{cases}$ i.e. $J = 0$ and $M = 0$

$$\begin{aligned} P^{(1/2)}_{\frac{1}{2};\, n_+,n_o,n_-} &= P^{(0)}_{n_+,n_o,n_-} + P^{(0)}_{n_+,n_o+1,n_-} \\ P^{(1/2)}_{-\frac{1}{2};n_+,n_o,n_-} &= 2P^{(0)}_{n_+,n_o,n_-+1} \end{aligned} \qquad (21)$$

For $\begin{cases} T = 3/2 \\ m = 1/2 \end{cases}$ i.e. $J = 1$ and $M = 1$

$$\begin{aligned} P^{(3/2)}_{\frac{1}{2};\, n_+,n_o,n_-} &= 3P^{(0)}_{n_+,n_o+2,n_-} + 2\, P^{(0)}_{n_+,n_o+1,n_-} - P^{(0)}_{n_+,n_o,n_-} \\ P^{(3/2)}_{-\frac{1}{2};n_+,n_o,n_-} &= 6P^{(0)}_{n_+,n_o+1,n_-+1} - 2\, P^{(0)}_{n_+,n_o,n_-+1} \end{aligned} \qquad (22)$$

For $\begin{cases} T = 3/2 \\ m = 3/2 \end{cases}$ i.e. $J = 1$, $M = 1$ $n_+ - n_- = 1$ or 2

$$P^{3/2}_{\frac{1}{2};\, n_+,n_o,n_-} = 2\, P^{0}_{n_+,n_o,n_-+1} + 2\, P^{0}_{n_+,n_o+1,n_-+1}$$

$$P^{3/2}_{-\frac{1}{2};n_+,n_o,n_-} = 4\, P^{(0)}_{n_+,n_o,n_-+2}$$

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