# Closed Formulas for the Sums of Squares of Generalized Fibonacci Numbers 

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#### Abstract

In this paper, closed forms of the sum formulas for the squares of generalized Fibonacci numbers are presented. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery. Our work generalize second order recurrence relations.


Keywords: Fibonacci numbers; Lucas numbers; Pell numbers; Jacobsthal numbers; sum formulas.
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## 1 INTRODUCTION

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci and

Lucas sequences are very well-known examples of second order recurrence sequences. The Fibonacci numbers are perhaps most famous for appearing in the rabbit breeding problem,

[^0]introduced by Leonardo de Pisa in 1202 in his book called Liber Abaci. The Fibonacci sequences are a source of many nice and interesting identities. A similar interpretation exists for Lucas sequence.

The sequence of Fibonacci numbers $\left\{F_{n}\right\}$ is defined by

$$
F_{n}=F_{n-1}+F_{n-2}, \quad n \geq 2, \quad F_{0}=0, \quad F_{1}=1 .
$$

and the sequence of Lucas numbers $\left\{L_{n}\right\}$ is defined by
$L_{n}=L_{n-1}+L_{n-2}, \quad n \geq 2, \quad L_{0}=2, L_{1}=1$.

The Fibonacci numbers, Lucas numbers and their generalizations have many interesting properties and applications to almost every field. In 1965, Horadam [1] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence
$\left\{W_{n}\left(W_{0}, W_{1} ; r, s\right)\right\}$, or simply $\left\{W_{n}\right\}$, as follows:
$W_{n}=r W_{n-1}+s W_{n-2} ; W_{0}=a, W_{1}=b, \quad(n \geq 2) \quad(1.1)$
where $W_{0}, W_{1}$ are arbitrary complex numbers and $r, s$ are real numbers, see also Horadam [2], [3] and [4]. Now these generalized Fibonacci numbers $\left\{W_{n}(a, b ; r, s)\right\}$ are also called Horadam numbers. The sequence $\left\{W_{n}\right\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$
W_{-n}=-\frac{r}{s} W_{-(n-1)}+\frac{1}{s} W_{-(n-2)}
$$

for $n=1,2,3, \ldots$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer $n$.

For some specific values of $a, b, r$ and $s$, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of $r, s$ and initial values.

Table 1. A few special case of generalized Fibonacci sequences.

| Name of sequence | Notation: $W_{n}(a, b ; r, s)$ | OEIS: [5] |
| :---: | :---: | :---: |
| Fibonacci | $F_{n}=W_{n}(0,1 ; 1,1)$ | A000045 |
| Lucas | $L_{n}=W_{n}(2,1 ; 1,1)$ | A000032 |
| Pell | $P_{n}=W_{n}(0,1 ; 2,1)$ | A000129 |
| Pell-Lucas | $Q_{n}=W_{n}(2,2 ; 2,1)$ | A002203 |
| Jacobsthal | $J_{n}=W_{n}(0,1 ; 1,2)$ | A001045 |
| Jacobsthal-Lucas | $j_{n}=W_{n}(2,1 ; 1,2)$ | A014551 |

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$
\sum_{i=1}^{n} F_{i}^{2}=F_{n} F_{n+1}
$$

and

$$
\sum_{i=1}^{n} Q_{i}^{2}=\frac{1}{2}\left(Q_{n} Q_{n+1}-4\right) .
$$

In this work, we derive expressions for sums of second powers of generalized Fibonacci numbers. Our results generalize second order linear recurrence relations.

We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers

| Name of sequence | sums of second powers | sums of third powers | sums of powers |
| :---: | :---: | :---: | :---: |
| Generalized Fibonacci | $[6,7,8,9,10]$ | $[11,12]$ | $[13,14,15]$ |
| Generalized Tribonacci | $[16]$ |  |  |
| Generalized Tetranacci | $[17,18]$ |  |  |

An application of the sum of the squares of the numbers is circulant matrix. Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized $m$ step Fibonacci sequences require the sum of the squares of the numbers of the sequences. For generalized $m$-step Fibonacci sequences see for example Soykan [19]. If $m=2, m=3$ and $m=4$, we get the generalized Fibonacci sequence, generalized Tribonacci sequence and generalized Tetranacci sequence, respectively. Next, we recall some information on circulant (r-circulant, geometric circulant) matrices and Frobenius norm, spectral norm, maximum
column length norm and maximum row length norm.

Circulant matrices have been around for a long time and have been extensively used in many scientific areas. In some scientific areas such as image processing, coding theory and signal processing we often encounter circulant matrices. These matrices also have many applications in numerical analysis, optimization, digital image processing, mathematical statistics and modern technology.

Let $n \geq 2$ be an integer and $r$ be any real or complex number. An $n \times n$ matrix $C_{r}$ is called a $r$-circulant matrix if it of the form

$$
C_{r}=\left(\begin{array}{cccccc}
c_{0} & c_{1} & c_{2} & \cdots & c_{n-2} & c_{n-1} \\
r c_{n-1} & c_{0} & c_{1} & \cdots & c_{n-3} & c_{n-2} \\
r c_{n-2} & r c_{n-1} & c_{0} & \cdots & c_{n-4} & c_{n-3} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
r c_{1} & r c_{2} & r c_{3} & \cdots & r c_{n-1} & c_{0}
\end{array}\right)_{n \times n} .
$$

and the $r$-circulant matrix $C_{r}$ is denoted by $C_{r}=\operatorname{Circ}_{r}\left(c_{0}, c_{1}, \ldots, c_{n-1}\right)$. If $r=1$ then 1-circulant matrix is called as circulant matrix and denoted by $C=\operatorname{Circ}\left(c_{0}, c_{1}, \ldots, c_{n-1}\right)$. Circulant matrix was first proposed by Davis in [20]. This matrix has many interesting properties, and it is one of the most important research subject in the field of the computational and pure mathematics (see for example references given in Table 3). For instance, Shen and Cen [21] studied on the norms of $r$-circulant matrices with Fibonacci and Lucas numbers. Then, later Kızılateş and Tuglu [22] defined a new geometric circulant matrix as follows:

$$
C_{r^{*}}=\left(\begin{array}{cccccc}
c_{0} & c_{1} & c_{2} & \cdots & c_{n-2} & c_{n-1} \\
r c_{n-1} & c_{0} & c_{1} & \cdots & c_{n-3} & c_{n-2} \\
r^{2} c_{n-2} & r c_{n-1} & c_{0} & \cdots & c_{n-4} & c_{n-3} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
r^{n-1} c_{1} & r^{n-2} c_{2} & r^{n-3} c_{3} & \cdots & r c_{n-1} & c_{0}
\end{array}\right)_{n \times n} .
$$

and then they obtained the bounds for the spectral norms of geometric circulant matrices with the generalized Fibonacci number and Lucas numbers. When the parameter satisfies $r=1$, we get the classical circulant matrix. See also Polatll [23] for the spectral norms of $r$-circulant matrices with a type of Catalan triangle numbers.
The Frobenius (or Euclidean) norm and spectral norm of a matrix $A=\left(a_{i j}\right)_{m \times n} \in M_{m \times n}(\mathbb{C})$ are defined respectively as follows:

$$
\|A\|_{F}=\left(\sum_{i=1}^{m} \sum_{j=1}^{n}\left|a_{i j}\right|^{2}\right)^{1 / 2} \text { and }\|A\|_{2}=\left(\max _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{1 / 2}
$$

where $\lambda_{i}$ 's are the eigenvalues of the matrix $A^{*} A$ and $A^{*}$ is the conjugate of transpose of the matrix $A$. The maximum column length norm $c_{1}($.$) and the maximum row length norm r_{1}($.$) of an matrix of$
order $n \times n$ are defined as follows:

$$
c_{1}(A)=\max _{1 \leq j \leq n}\left(\sum_{i=1}^{n}\left|a_{i j}\right|^{2}\right)^{1 / 2} \text { and } r_{1}(A)=\max _{1 \leq i \leq n}\left(\sum_{j=1}^{n}\left|a_{i j}\right|^{2}\right)^{1 / 2} .
$$

The following inequality holds for any matrix $A=M_{n \times n}(\mathbb{C})$ :

$$
\frac{1}{\sqrt{n}}\|A\|_{F} \leq\|A\|_{2} \leq\|A\|_{F} .
$$

Calculations of the above norms $\|A\|_{F},\|A\|_{2}, c_{1}(A)$ and $r_{1}(A)$ require the sum of the squares of the numbers $a_{i j}$. As in our case, the numbers $a_{i j}$ can be chosen as elements of second, third or higher order linear recurrence sequences.

In the following Table 3, we present a few special study on the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized $m$-step Fibonacci sequences which require sum formulas of second powers of numbers in $m$-step Fibonacci sequences ( $m=2,3,4$ ).

Table 3. Papers on the norms.

| Name of sequence | Papers |
| :---: | :---: |
| second order $\downarrow$ | second order $\downarrow$ |
| Fibonacci, Lucas | $[24,22,25,26,27,28,29,21,30,31,32,33]$ |
| Pell, Pell-Lucas | $[34,35]$ |
| Jacobsthal, Jacobsthal-Lucas | $[36,37,38,39]$ |
| third order $\downarrow$ | third order $\downarrow$ |
| Tribonacci, Tribonacci-Lucas | $[40,41]$ |
| Padovan, Perrin | $[42,43,44]$ |
| fourth order $\downarrow$ | fourth order $\downarrow$ |
| Tetranacci, Tetranacci-Lucas | $[45]$ |

Also linear summing formulas of the generalized $m$-step Fibonacci sequences are required for the computation of various norms of circulant matrices circulant matrices with the generalized $m$-step Fibonacci sequences. We present some works on summing formulas of the numbers in the following Table 4.

Table 4. A few special study of sum formulas.

| Name of sequence | Papers which deal with summing formulas |
| :---: | :---: |
| Pell and Pell-Lucas | $[46],[47,48]$ |
| Generalized Fibonacci | $[49,50,51,52,53]$ |
| Generalized Tribonacci | $[54,55,55,56]$ |
| Generalized Tetranacci | $[57,58,59]$ |
| Generalized Pentanacci | $[60,61]$ |
| Generalized Hexanacci | $[62]$ |

## 2 SUMMING FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 2.1. For $n \geq 1$ we have the following formulas: if $(s+1)(r+s-1)(r-s+1) \neq 0$ then
(a)

$$
\sum_{i=1}^{n} W_{i}^{2}=\frac{(1-s) W_{n+2}^{2}+\left(1-s-r^{2}-r^{2} s\right) W_{n+1}^{2}+2 r s W_{n+1} W_{n+2}+(s-1) W_{1}^{2}+s^{2}(s-1) W_{0}^{2}-2 r s W_{1} W_{0}}{(s+1)(r+s-1)(r-s+1)}
$$

(b)

$$
\sum_{i=1}^{n} W_{i+1} W_{i}=\frac{r W_{n+2}^{2}+r s^{2} W_{n+1}^{2}+\left(1-r^{2}-s^{2}\right) W_{n+1} W_{n+2}-r W_{1}^{2}-r s^{2} W_{0}^{2}+s\left(-r^{2}+s^{2}-1\right) W_{1} W_{0}}{(s+1)(r+s-1)(r-s+1)} .
$$

Proof. Using the recurrence relation

$$
W_{n+2}=r W_{n+1}+s W_{n}
$$

i.e.

$$
s W_{n}=W_{n+2}-r W_{n+1}
$$

we obtain

$$
\begin{aligned}
s^{2} W_{n}^{2}= & W_{n+2}^{2}+r^{2} W_{n+1}^{2}-2 r W_{n+2} W_{n+1} \\
s^{2} W_{n-1}^{2}= & W_{n+1}^{2}+r^{2} W_{n}^{2}-2 r W_{n+1} W_{n} \\
s^{2} W_{n-2}^{2}= & W_{n}^{2}+r^{2} W_{n-1}^{2}-2 r W_{n} W_{n-1} \\
s^{2} W_{n-3}^{2}= & W_{n-1}^{2}+r^{2} W_{n-2}^{2}-2 r W_{n-1} W_{n-2} \\
s^{2} W_{n-4}^{2}= & W_{n-2}^{2}+r^{2} W_{n-3}^{2}-2 r W_{n-2} W_{n-3} \\
& \vdots \\
s^{2} W_{2}^{2}= & W_{4}^{2}+r^{2} W_{3}^{2}-2 r W_{4} W_{3} \\
s^{2} W_{1}^{2}= & W_{3}^{2}+r^{2} W_{2}^{2}-2 r W_{3} W_{2} .
\end{aligned}
$$

If we add the above equations by side by, we get

$$
\begin{equation*}
s^{2} \sum_{i=1}^{n} W_{i}^{2}=\sum_{i=3}^{n+2} W_{i}^{2}+r^{2} \sum_{i=2}^{n+1} W_{i}^{2}-2 r \sum_{i=2}^{n+1} W_{i+1} W_{i} . \tag{2.1}
\end{equation*}
$$

Note that

$$
\begin{aligned}
\sum_{i=3}^{n+2} W_{i}^{2} & =-W_{1}^{2}-W_{2}^{2}+W_{n+1}^{2}+W_{n+2}^{2}+\sum_{i=1}^{n} W_{i}^{2}, \\
\sum_{i=2}^{n+1} W_{i}^{2} & =-W_{1}^{2}+W_{n+1}^{2}+\sum_{i=1}^{n} W_{i}^{2}, \\
\sum_{i=2}^{n+1} W_{i+1} W_{i} & =-W_{2} W_{1}+W_{n+2} W_{n+1}+\sum_{i=1}^{n} W_{i+1} W_{i} .
\end{aligned}
$$

We put them into the (2.1) we obtain

$$
\begin{align*}
s^{2} \sum_{i=1}^{n} W_{i}^{2}= & \left(-W_{1}^{2}-W_{2}^{2}+W_{n+1}^{2}+W_{n+2}^{2}+\sum_{i=1}^{n} W_{i}^{2}\right)+r^{2}\left(-W_{1}^{2}+W_{n+1}^{2}+\sum_{i=1}^{n} W_{i}^{2}\right)  \tag{2.2}\\
& -2 r\left(-W_{2} W_{1}+W_{n+2} W_{n+1}+\sum_{i=1}^{n} W_{i+1} W_{i}\right)
\end{align*}
$$

Next we calculate $\sum_{i=1}^{n} W_{i+1} W_{i}$. Again, using the recurrence relation

$$
W_{n+2}=r W_{n+1}+s W_{n}
$$

i.e.

$$
s W_{n}=W_{n+2}-r W_{n+1}
$$

we obtain

$$
\begin{aligned}
s W_{n+1} W_{n}= & W_{n+2} W_{n+1}-r W_{n+1}^{2} \\
s W_{n} W_{n-1}= & W_{n+1} W_{n}-r W_{n}^{2} \\
s W_{n-1} W_{n-2}= & W_{n} W_{n-1}-r W_{n-1}^{2} \\
& \vdots \\
s W_{3} W_{2}= & W_{4} W_{3}-r W_{3}^{2} \\
s W_{2} W_{1}= & W_{3} W_{2}-r W_{2}^{2} .
\end{aligned}
$$

If we add the above equations by side by, we get

$$
\begin{equation*}
s \sum_{i=1}^{n} W_{i+1} W_{i}=\sum_{i=2}^{n+1} W_{i+1} W_{i}-r \sum_{i=2}^{n+1} W_{i}^{2} . \tag{2.3}
\end{equation*}
$$

Note that

$$
\begin{aligned}
\sum_{i=2}^{n+1} W_{i+1} W_{i} & =-W_{2} W_{1}+W_{n+2} W_{n+1}+\sum_{i=1}^{n} W_{i+1} W_{i}, \\
\sum_{i=2}^{n+1} W_{i}^{2} & =-W_{1}^{2}+W_{n+1}^{2}+\sum_{i=1}^{n} W_{i}^{2} .
\end{aligned}
$$

If we put them into the (2.3) then it follows that

$$
\begin{equation*}
s \sum_{i=1}^{n} W_{i+1} W_{i}=\left(-W_{2} W_{1}+W_{n+2} W_{n+1}+\sum_{i=1}^{n} W_{i+1} W_{i}\right)-r\left(-W_{1}^{2}+W_{n+1}^{2}+\sum_{i=1}^{n} W_{i}^{2}\right) . \tag{2.4}
\end{equation*}
$$

Then, using

$$
W_{2}=\left(r W_{1}+s W_{0}\right)
$$

and solving the system (2.2)-(2.4), the required results of (a) and (b) follow.
Taking $r=s=1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.
Proposition 2.1. If $r=s=1$ then for $n \geq 1$ we have the following formulas:
(a) $\sum_{i=1}^{n} W_{i}^{2}=\frac{1}{2}\left(-2 W_{n+1}^{2}+2 W_{n+2} W_{n+1}-2 W_{1} W_{0}\right)$.
(b) $\sum_{i=1}^{n} W_{i+1} W_{i}=\frac{1}{2}\left(W_{n+2}^{2}+W_{n+1}^{2}-W_{n+1} W_{n+2}-W_{1}^{2}-W_{0}^{2}-W_{1} W_{0}\right)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_{n}=F_{n}$ with $F_{0}=0, F_{1}=1$ ).

Corollary 2.2. For $n \geq 1$, Fibonacci numbers have the following properties:
(a) $\sum_{i=1}^{n} F_{i}^{2}=\frac{1}{2}\left(-2 F_{n+1}^{2}+2 F_{n+2} F_{n+1}\right)$.
(b) $\sum_{i=1}^{n} F_{i+1} F_{i}=\frac{1}{2}\left(F_{n+2}^{2}+F_{n+1}^{2}-F_{n+1} F_{n+2}-1\right)$.

Taking $W_{n}=L_{n}$ with $L_{0}=2, L_{1}=1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.3. For $n \geq 1$, Lucas numbers have the following properties:
(a) $\sum_{i=1}^{n} L_{i}^{2}=\frac{1}{2}\left(-2 L_{n+1}^{2}+2 L_{n+2} L_{n+1}-4\right)$.
(b) $\sum_{i=1}^{n} L_{i+1} L_{i}=\frac{1}{2}\left(L_{n+2}^{2}+L_{n+1}^{2}-L_{n+1} L_{n+2}-7\right)$.

Taking $r=2, s=1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.
Proposition 2.2. If $r=2, s=1$ then for $n \geq 0$ we have the following formulas:
(a) $\sum_{i=1}^{n} W_{i}^{2}=\frac{1}{2}\left(-2 W_{n+1}^{2}+W_{n+2} W_{n+1}-W_{1} W_{0}\right)$.
(b) $\sum_{i=1}^{n} W_{i+1} W_{i}=\frac{1}{4}\left(W_{n+2}^{2}+W_{n+1}^{2}-2 W_{n+2} W_{n+1}-W_{1}^{2}-W_{0}^{2}-2 W_{1} W_{0}\right)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_{n}=P_{n}$ with $P_{0}=0, P_{1}=1$ ).

Corollary 2.4. For $n \geq 1$, Pell numbers have the following properties.
(a) $\sum_{i=1}^{n} P_{i}^{2}=\frac{1}{2}\left(-2 P_{n+1}^{2}+P_{n+2} P_{n+1}\right)=\frac{1}{2} P_{n} P_{n+1}$.
(b) $\sum_{i=1}^{n} P_{i+1} P_{i}=\frac{1}{4}\left(P_{n+2}^{2}+P_{n+1}^{2}-2 P_{n+2} P_{n+1}-1\right)$.

Taking $W_{n}=Q_{n}$ with $Q_{0}=2, Q_{1}=2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.5. For $n \geq 1$, Pell-Lucas numbers have the following properties:
(a) $\sum_{i=1}^{n} Q_{i}^{2}=\frac{1}{2}\left(-2 Q_{n+1}^{2}+Q_{n+2} Q_{n+1}-4\right)=\frac{1}{2}\left(Q_{n} Q_{n+1}-4\right)$.
(b) $\sum_{i=1}^{n} Q_{i+1} Q_{i}=\frac{1}{4}\left(Q_{n+2}^{2}+Q_{n+1}^{2}-2 Q_{n+2} Q_{n+1}-16\right)$.

If $r=1, s=2$ then $(s+1)(r+s-1)(r-s+1)=0$ so we can't use Theorem 2.1 directly. Therefore we need another method to find $\sum_{i=1}^{n} W_{i}^{2}$ and $\sum_{i=1}^{n} W_{i+1} W_{i}$ which is given in the following theorem.

Theorem 2.6. If $r=1, s=2$ then for $n \geq 1$ we have the following formulas:
(a) $\sum_{i=1}^{n} W_{i}^{2}=\frac{1}{9}\left(W_{n+2}^{2}-W_{n+1}^{2}-4\left(W_{0}+W_{1}\right) W_{0}+\left(2 W_{0}-W_{1}\right)^{2} n\right)$.
(b) $\sum_{i=1}^{n} W_{i+1} W_{i}=\frac{1}{36}\left(5 W_{n+2}^{2}+4 W_{n+1}^{2}+\left(-9 W_{1}^{2}-20 W_{0}^{2}-20 W_{1} W_{0}\right)-4\left(W_{1}-2 W_{0}\right)^{2} n\right)$.

Proof.
(a) The proof will be by induction on $n$. Before the proof, we recall some information on generalized Jacobsthal numbers. A generalized Jacobsthal sequence $\left\{W_{n}\right\}_{n \geq 0}=\left\{W_{n}\left(W_{0}, W_{1}\right)\right\}_{n \geq 0}$ is defined by the second-order recurrence relations

$$
\begin{equation*}
W_{n}=W_{n-1}+2 W_{n-2} ; \quad W_{0}=a, W_{1}=b, \quad(n \geq 2) \tag{2.5}
\end{equation*}
$$

with the initial values $W_{0}, W_{1}$ not all being zero. The sequence $\left\{W_{n}\right\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$
W_{-n}=-\frac{1}{2} W_{-(n-1)}+\frac{1}{2} W_{-(n-2)}
$$

for $n=1,2,3, \ldots$. Therefore, recurrence (2.5) holds for all integer $n$. The first few generalized Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 5.

Table 5. A few generalized Jacobsthal numbers

| $n$ | $W_{n}$ | $W_{-n}$ |
| :---: | :---: | :---: |
| 0 | $W_{0}$ |  |
| 1 | $W_{1}$ | $-\frac{1}{2} W_{0}+\frac{1}{2} W_{1}$ |
| 2 | $2 W_{0}+W_{1}$ | $\frac{3}{4} W_{0}-\frac{1}{4} W_{1}$ |
| 3 | $2 W_{0}+3 W_{1}$ | $-\frac{5}{8} W_{0}+\frac{3}{8} W_{1}$ |
| 4 | $6 W_{0}+5 W_{1}$ | $\frac{11}{16} W_{0}-\frac{5}{16} W_{1}$ |
| 5 | $10 W_{0}+11 W_{1}$ | $-\frac{21}{32} W_{0}+\frac{11}{32} W_{1}$ |
| 6 | $22 W_{0}+21 W_{1}$ | $\frac{43}{64} W_{0}-\frac{21}{64} W_{1}$ |

Binet formula of generalized Jacobsthal sequence can be calculated using its characteristic equation which is given as

$$
t^{2}-t-2=0
$$

The roots of characteristic equation are

$$
\alpha=2, \quad \beta=-1
$$

and the roots satisfy the following

$$
\alpha+\beta=1, \alpha \beta=-2, \alpha-\beta=3
$$

Using these roots and the recurrence relation, Binet formula can be given as

$$
\begin{equation*}
W_{n}=\frac{A \alpha^{n}-B \beta^{n}}{\alpha-\beta}=\frac{A \times 2^{n}-B(-1)^{n}}{3} \tag{2.6}
\end{equation*}
$$

where $A=W_{1}-W_{0} \beta=W_{1}+W_{0}$ and $B=W_{1}-W_{0} \alpha=W_{1}-2 W_{0}$.
We now prove (a) by induction on $n$. If $n=1$ we see that the sum formula reduces to the relation

$$
\begin{equation*}
W_{1}^{2}=\frac{1}{9}\left(W_{3}^{2}-W_{2}^{2}+W_{1}^{2}-8 W_{1} W_{0}\right) \tag{2.7}
\end{equation*}
$$

Since

$$
\begin{aligned}
& W_{2}=2 W_{0}+W_{1} \\
& W_{3}=2 W_{0}+3 W_{1}
\end{aligned}
$$

(2.7) is true. Assume that the relation in (a) is true for $n=m$, i.e.,

$$
\sum_{i=1}^{m} W_{i}^{2}=\frac{1}{9}\left(W_{m+2}^{2}-W_{m+1}^{2}-4\left(W_{0}+W_{1}\right) W_{0}+\left(2 W_{0}-W_{1}\right)^{2} m\right)
$$

Then we get

$$
\begin{aligned}
\sum_{i=1}^{m+1} W_{i}^{2} & =W_{m+1}^{2}+\sum_{i=1}^{m} W_{i}^{2} \\
& =W_{m+1}^{2}+\frac{1}{9}\left(W_{m+2}^{2}-W_{m+1}^{2}-4\left(W_{0}+W_{1}\right) W_{0}+\left(2 W_{0}-W_{1}\right)^{2} m\right) \\
& =\frac{1}{9}\left(W_{m+2}^{2}+8 W_{m+1}^{2}-4\left(W_{0}+W_{1}\right) W_{0}+\left(2 W_{0}-W_{1}\right)^{2} m\right) \\
& =\frac{1}{9}\left(W_{m+2}^{2}+8 W_{m+1}^{2}-\left(2 W_{0}-W_{1}\right)^{2}-4\left(W_{0}+W_{1}\right) W_{0}+\left(2 W_{0}-W_{1}\right)^{2}(m+1)\right) \\
& =\frac{1}{9}\left(W_{m+3}^{2}-W_{m+2}^{2}-4\left(W_{0}+W_{1}\right) W_{0}+\left(2 W_{0}-W_{1}\right)^{2}(m+1)\right) \\
& =\frac{1}{9}\left(W_{(m+1)+2}^{2}-W_{(m+1)+1}^{2}-4\left(W_{0}+W_{1}\right) W_{0}+\left(2 W_{0}-W_{1}\right)^{2}(m+1)\right)
\end{aligned}
$$

where

$$
\begin{equation*}
W_{m+2}^{2}+8 W_{m+1}^{2}-\left(2 W_{0}-W_{1}\right)^{2}=W_{m+3}^{2}-W_{m+2}^{2} . \tag{2.8}
\end{equation*}
$$

(2.8) can be proved by using Binet formula of $W_{n}$. Hence, the relation in (a) holds also for $n=m+1$.
(b) We now prove (b) by induction on $n$. If $n=1$ we see that the sum formula reduces to the relation

$$
\begin{equation*}
W_{2} W_{1}=\frac{1}{36}\left(5 W_{3}^{2}+4 W_{2}^{2}-13 W_{1}^{2}-36 W_{0}^{2}-4 W_{1} W_{0}\right) . \tag{2.9}
\end{equation*}
$$

Since

$$
\begin{aligned}
& W_{2}=2 W_{0}+W_{1}, \\
& W_{3}=2 W_{0}+3 W_{1},
\end{aligned}
$$

(2.9) is true. Assume that the relation in (b) is true for $n=m$, i.e.,

$$
\sum_{i=1}^{m} W_{i+1} W_{i}=\frac{1}{36}\left(5 W_{m+2}^{2}+4 W_{m+1}^{2}+\left(-9 W_{1}^{2}-20 W_{0}^{2}-20 W_{1} W_{0}\right)-4\left(W_{1}-2 W_{0}\right)^{2} m\right) .
$$

Then we get

$$
\begin{aligned}
\sum_{i=1}^{m+1} W_{i+1} W_{i}= & W_{m+2} W_{m+1}+\sum_{i=1}^{m} W_{i+1} W_{i} \\
= & \frac{1}{36}\left(5 W_{m+2}^{2}+4 W_{m+1}^{2}+36 W_{m+2} W_{m+1}+\left(-9 W_{1}^{2}-20 W_{0}^{2}-20 W_{1} W_{0}\right)-4\left(W_{1}-2 W_{0}\right)^{2} m\right) \\
= & \frac{1}{36}\left(5 W_{m+2}^{2}+4 W_{m+1}^{2}+36 W_{m+2} W_{m+1}+4\left(W_{1}-2 W_{0}\right)^{2}+\left(-9 W_{1}^{2}-20 W_{0}^{2}-20 W_{1} W_{0}\right)\right. \\
& \left.-4\left(W_{1}-2 W_{0}\right)^{2}(m+1)\right) \\
= & \frac{1}{36}\left(5 W_{m+3}^{2}+4 W_{m+2}^{2}+\left(-9 W_{1}^{2}-20 W_{0}^{2}-20 W_{1} W_{0}\right)-4\left(W_{1}-2 W_{0}\right)^{2}(m+1)\right) \\
= & \frac{1}{36}\left(5 W_{(m+1)+2}^{2}+4 W_{(m+1)+1}^{2}+\left(-9 W_{1}^{2}-20 W_{0}^{2}-20 W_{1} W_{0}\right)-4\left(W_{1}-2 W_{0}\right)^{2}(m+1)\right)
\end{aligned}
$$

where

$$
\begin{equation*}
5 W_{m+2}^{2}+4 W_{m+1}^{2}+36 W_{m+2} W_{m+1}+4\left(W_{1}-2 W_{0}\right)^{2}=5 W_{m+3}^{2}+4 W_{m+2}^{2} \tag{2.10}
\end{equation*}
$$

(2.10) can be proved by using Binet formula of $W_{n}$. Hence, the relation in (b) holds also for $n=m+1$.

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_{n}=J_{n}$ with $J_{0}=0, J_{1}=1$ ).

Corollary 2.7. For $n \geq 1$, Jacobsthal numbers have the following property:
(a) $\sum_{i=1}^{n} J_{i}^{2}=\frac{1}{9}\left(J_{n+2}^{2}-J_{n+1}^{2}+n\right)$.
(b) $\sum_{i=1}^{n} J_{i+1} J_{i}=\frac{1}{36}\left(5 J_{n+2}^{2}+4 J_{n+1}^{2}-9-4 n\right)$.

Taking $W_{n}=j_{n}$ with $j_{0}=2, j_{1}=1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.8. For $n \geq 1$, Jacobsthal-Lucas numbers have the following property:
(a) $\sum_{i=1}^{n} j_{i}^{2}=\frac{1}{9}\left(j_{n+2}^{2}-j_{n+1}^{2}-24+9 n\right)$.
(b) $\sum_{i=1}^{n} j_{i+1} j_{i}=\frac{1}{36}\left(5 j_{n+2}^{2}+4 j_{n+1}^{2}-129-36 n\right)$.

## 3 SUMMING FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1. For $n \geq 1$ we have the following formulas: If $(s+1)(r+s-1)(r-s+1) \neq 0$ then
(a)

$$
\sum_{i=1}^{n} W_{-i}^{2}=\frac{\begin{array}{c}
(s-1) W_{-n+1}^{2}+\left(r^{2}+r^{2} s+s-1\right) W_{-n}^{2}-2 r s W_{-n+1} W_{-n}+2 r s W_{1} W_{0}+(1-s) W_{1}^{2} \\
+\left(1-s-r^{2}-r^{2} s\right) W_{0}^{2}
\end{array}}{(s+1)(r+s-1)(r-s+1)} .
$$

(b)

$$
\sum_{i=1}^{n} W_{-i+1} W_{-i}=\frac{-r W_{-n+1}^{2}-r s^{2} W_{-n}^{2}+\left(r^{2}+s^{2}-1\right) W_{-n+1} W_{-n}+\left(1-r^{2}-s^{2}\right) W_{1} W_{0}+r W_{1}^{2}+r s^{2} W_{0}^{2}}{(s+1)(r+s-1)(r-s+1)} .
$$

Proof. Using the recurrence relation

$$
W_{-n+2}=r W_{-n+1}+s W_{-n} \Rightarrow W_{-n}=-\frac{r}{s} W_{-n+1}+\frac{1}{s} W_{-n+2}
$$

i.e.

$$
s W_{-n}=W_{-n+2}-r W_{-n+1}
$$

we obtain

$$
\begin{aligned}
s^{2} W_{-n}^{2}= & W_{-n+2}^{2}+r^{2} W_{-n+1}^{2}-2 r W_{-n+2} W_{-n+1} \\
s^{2} W_{-n+1}^{2}= & W_{-n+3}^{2}+r^{2} W_{-n+2}^{2}-2 r W_{-n+3} W_{-n+2} \\
s^{2} W_{-n+2}^{2}= & W_{-n+4}^{2}+r^{2} W_{-n+3}^{2}-2 r W_{-n+4} W_{-n+3} \\
s^{2} W_{-n+3}^{2}= & W_{-n+5}^{2}+r^{2} W_{-n+4}^{2}-2 r W_{-n+5} W_{-n+4} \\
& \vdots \\
s^{2} W_{-3}^{2}= & W_{-1}^{2}+r^{2} W_{-2}^{2}-2 r W_{-1} W_{-2} \\
s^{2} W_{-2}^{2}= & W_{0}^{2}+r^{2} W_{-1}^{2}-2 r W_{0} W_{-1} \\
s^{2} W_{-1}^{2}= & W_{1}^{2}+r^{2} W_{0}^{2}-2 r W_{1} W_{0} .
\end{aligned}
$$

If we add the above equations by side by, we get

$$
\begin{align*}
s^{2} \sum_{i=1}^{n} W_{-i}^{2}= & \left(W_{1}^{2}+W_{0}^{2}-W_{-n+1}^{2}-W_{-n}^{2}+\sum_{i=1}^{n} W_{-i}^{2}\right)+r^{2}\left(W_{0}^{2}-W_{-n}^{2}+\sum_{i=1}^{n} W_{-i}^{2}\right)  \tag{3.1}\\
& -2 r\left(W_{1} W_{0}-W_{-n+1} W_{-n}+\sum_{i=1}^{n} W_{-i+1} W_{-i}\right)
\end{align*}
$$

Next we calculate $\sum_{i=1}^{n} W_{-i+1} W_{-i}$. Again using the recurrence relation

$$
W_{-n+2}=r W_{-n+1}+s W_{-n} \Rightarrow W_{-n}=-\frac{r}{s} W_{-n+1}+\frac{1}{s} W_{-n+2}
$$

i.e.

$$
s W_{-n}=W_{-n+2}-r W_{-n+1}
$$

we obtain

$$
\begin{aligned}
s W_{-n+1} W_{-n}= & W_{-n+2} W_{-n+1}-r W_{-n+1}^{2} \\
s W_{-n+2} W_{-n+1}= & W_{-n+3} W_{-n+2}-r W_{-n+2}^{2} \\
s W_{-n+3} W_{-n+2}= & W_{-n+4} W_{-n+3}-r W_{-n+3}^{2} \\
s W_{-n+4} W_{-n+3}= & W_{-n+5} W_{-n+4}-r W_{-n+4}^{2} \\
& \vdots \\
s W_{-2} W_{-3}= & W_{-1} W_{-2}-r W_{-2}^{2} \\
s W_{-1} W_{-2}= & W_{0} W_{-1}-r W_{-1}^{2} \\
s W_{0} W_{-1}= & W_{1} W_{0}-r W_{0}^{2} .
\end{aligned}
$$

If we add the above equations by side by, we get

$$
\begin{equation*}
s \sum_{i=1}^{n} W_{-i+1} W_{-i}=\left(W_{1} W_{0}-W_{-n+1} W_{-n}+\sum_{i=1}^{n} W_{-i+1} W_{-i}\right)-r\left(W_{0}^{2}-W_{-n}^{2}+\sum_{i=1}^{n} W_{-i}^{2}\right) . \tag{3.2}
\end{equation*}
$$

Then, solving the system (3.1)-(3.2), the required results of (a) and (b) follow.

Taking $r=s=1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.
Proposition 3.1. If $r=s=1$ then for $n \geq 1$ we have the following formulas:
(a) $\sum_{i=1}^{n} W_{-i}^{2}=\frac{1}{2}\left(2 W_{-n}^{2}-2 W_{-n+1} W_{-n}+2 W_{1} W_{0}-2 W_{0}^{2}\right)$.
(b) $\sum_{i=1}^{n} W_{-i+1} W_{-i}=\frac{1}{2}\left(-W_{-n+1}^{2}-W_{-n}^{2}+W_{-n+1} W_{-n}-W_{1} W_{0}+W_{1}^{2}+W_{0}^{2}\right)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_{n}=F_{n}$ with $F_{0}=0, F_{1}=1$ ).

Corollary 3.2. For $n \geq 1$, Fibonacci numbers have the following properties.
(a) $\sum_{i=1}^{n} F_{-i}^{2}=\frac{1}{2}\left(2 F_{-n}^{2}-2 F_{-n+1} F_{-n}\right)$.
(b) $\sum_{i=1}^{n} F_{-i+1} F_{-i}=\frac{1}{2}\left(-F_{-n+1}^{2}-F_{-n}^{2}+F_{-n+1} F_{-n}+1\right)$.

Taking $W_{n}=L_{n}$ with $L_{0}=2, L_{1}=1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.3. For $n \geq 1$, Lucas numbers have the following properties.
(a) $\sum_{i=1}^{n} L_{-i}^{2}=\frac{1}{2}\left(2 L_{-n}^{2}-2 L_{-n+1} L_{-n}-4\right)$.
(b) $\sum_{i=1}^{n} L_{-i+1} L_{-i}=\frac{1}{2}\left(-L_{-n+1}^{2}-L_{-n}^{2}+L_{-n+1} L_{-n}+3\right)$.

Taking $r=2, s=1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.
Proposition 3.2. If $r=2, s=1$ then for $n \geq 1$ we have the following formulas:
(a) $\sum_{i=1}^{n} W_{-i}^{2}=\frac{1}{2}\left(2 W_{-n}^{2}-W_{-n+1} W_{-n}-2 W_{0}^{2}+W_{1} W_{0}\right)$.
(b) $\sum_{i=1}^{n} W_{-i+1} W_{-i}=\frac{1}{4}\left(-W_{-n+1}^{2}-W_{-n}^{2}+2 W_{-n+1} W_{-n}+W_{1}^{2}+W_{0}^{2}-2 W_{1} W_{0}\right)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_{n}=P_{n}$ with $P_{0}=0, P_{1}=1$ ).

Corollary 3.4. For $n \geq 1$, Pell numbers have the following properties.
(a) $\sum_{i=1}^{n} P_{-i}^{2}=\frac{1}{2}\left(2 P_{-n}^{2}-P_{-n+1} P_{-n}\right)$.
(b) $\sum_{i=1}^{n} P_{-i+1} P_{-i}=\frac{1}{4}\left(-P_{-n+1}^{2}-P_{-n}^{2}+2 P_{-n+1} P_{-n}+1\right)$.

Taking $W_{n}=Q_{n}$ with $Q_{0}=2, Q_{1}=2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 3.5. For $n \geq 1$, Pell-Lucas numbers have the following properties.
(a) $\sum_{i=1}^{n} Q_{-i}^{2}=\frac{1}{2}\left(2 Q_{-n}^{2}-Q_{-n+1} Q_{-n}-4\right)$.
(b) $\sum_{i=1}^{n} Q_{-i+1} Q_{-i}=\frac{1}{4}\left(-Q_{-n+1}^{2}-Q_{-n}^{2}+2 Q_{-n+1} Q_{-n}\right)$.

If $r=1, s=2$ then $(s+1)(r+s-1)(r-s+1)=0$ so we can't use Theorem 3.1 directly. Therefore we need another method to find $\sum_{i=1}^{n} W_{-i}^{2}$ and $\sum_{i=1}^{n} W_{-i+1} W_{-i}$ which is given in the following theorem.

Theorem 3.6. If $r=1, s=2$ then for $n \geq 1$ we have the following formulas:
(a) $\sum_{i=1}^{n} W_{-i}^{2}=\frac{1}{9}\left(-W_{-n+1}^{2}+W_{-n}^{2}+\left(W_{1}^{2}-W_{0}^{2}\right)+\left(W_{1}-2 W_{0}\right)^{2} n\right)$.
(b) $\sum_{i=1}^{n} W_{-i+1} W_{-i}=\frac{1}{27}\left(-2 W_{-n+1}^{2}+4 W_{-n}^{2}-7 W_{-n+1} W_{-n}+\left(W_{1}+4 W_{0}\right)\left(2 W_{1}-W_{0}\right)-3\left(W_{1}-2 W_{0}\right)^{2} n\right)$.

Proof. (a) and (b) can be proved by mathematical induction.
(a) The proof will be by induction on $n$. If $n=1$ we see that the sum formula reduces to the relation

$$
\begin{equation*}
W_{-1}^{2}=\frac{1}{9}\left(2 W_{0}^{2}-4 W_{0} W_{1}+2 W_{1}^{2}+W_{-1}^{2}\right) . \tag{3.3}
\end{equation*}
$$

Since

$$
W_{-1}=\left(-\frac{1}{2} W_{0}+\frac{1}{2} W_{1}\right)
$$

(3.3) is true. Assume that the relation in (a) is true for $n=m$, i.e.

$$
\sum_{i=1}^{m} W_{-i}^{2}=\frac{1}{9}\left(-W_{-m+1}^{2}+W_{-m}^{2}+\left(W_{1}^{2}-W_{0}^{2}\right)+\left(W_{1}-2 W_{0}\right)^{2} m\right) .
$$

Then we get

$$
\begin{aligned}
\sum_{i=1}^{m+1} W_{-i}^{2} & =W_{-(m+1)}^{2}+\sum_{i=1}^{m} W_{-i}^{2} \\
& =W_{-m-1}^{2}+\frac{1}{9}\left(-W_{-m+1}^{2}+W_{-m}^{2}+\left(W_{1}^{2}-W_{0}^{2}\right)+\left(W_{1}-2 W_{0}\right)^{2} m\right) \\
& =\frac{1}{9}\left(-W_{-m+1}^{2}+W_{-m}^{2}+9 W_{-m-1}^{2}+\left(W_{1}^{2}-W_{0}^{2}\right)+\left(W_{1}-2 W_{0}\right)^{2} m\right) \\
& =\frac{1}{9}\left(-W_{-m+1}^{2}+W_{-m}^{2}+9 W_{-m-1}^{2}-\left(W_{1}-2 W_{0}\right)^{2}+\left(W_{1}^{2}-W_{0}^{2}\right)+\left(W_{1}-2 W_{0}\right)^{2}(m+1)\right) \\
& =\frac{1}{9}\left(-W_{-m}^{2}+W_{-m-1}^{2}+\left(W_{1}^{2}-W_{0}^{2}\right)+\left(W_{1}-2 W_{0}\right)^{2}(m+1)\right) \\
& =\frac{1}{9}\left(-W_{-(m+1)+1}^{2}+W_{-(m+1)}^{2}+\left(W_{1}^{2}-W_{0}^{2}\right)+\left(W_{1}-2 W_{0}\right)^{2}(m+1)\right)
\end{aligned}
$$

where

$$
\begin{equation*}
-W_{-m+1}^{2}+W_{-m}^{2}+9 W_{-m-1}^{2}-\left(W_{1}-2 W_{0}\right)^{2}=-W_{-m}^{2}+W_{-m-1}^{2} . \tag{3.4}
\end{equation*}
$$

(3.4) can be proved by using Binet formula of $W_{n}$. Hence, the relation in (a) holds also for $n=m+1$.
(b) We now prove (b) by induction on $n$. If $n=1$ we see that the sum formula reduces to the relation

$$
\begin{equation*}
W_{0} W_{-1}=\frac{1}{27}\left(-W_{1}^{2}-18 W_{0}^{2}+4 W_{-1}^{2}+19 W_{0} W_{1}-7 W_{0} W_{-1}\right) . \tag{3.5}
\end{equation*}
$$

Since

$$
W_{-1}=\left(-\frac{1}{2} W_{0}+\frac{1}{2} W_{1}\right),
$$

(3.5) is true. Assume that the relation in (b) is true for $n=m$ i.e.,

$$
\sum_{i=1}^{m} W_{-i+1} W_{-i}=\frac{1}{27}\left(-2 W_{-m+1}^{2}+4 W_{-m}^{2}-7 W_{-m+1} W_{-m}+\left(W_{1}+4 W_{0}\right)\left(2 W_{1}-W_{0}\right)-3\left(W_{1}-2 W_{0}\right)^{2} m\right) .
$$

Then we get

$$
\begin{aligned}
\sum_{i=1}^{m+1} W_{-i+1} W_{-i}= & W_{-(m+1)+1} W_{-(m+1)}+\sum_{i=1}^{m} W_{-i+1} W_{-i} \\
= & W_{-m} W_{-m-1}+\frac{1}{27}\left(-2 W_{-m+1}^{2}+4 W_{-m}^{2}-7 W_{-m+1} W_{-m}\right. \\
& \left.+\left(W_{1}+4 W_{0}\right)\left(2 W_{1}-W_{0}\right)-3\left(W_{1}-2 W_{0}\right)^{2} m\right) \\
= & \frac{1}{27}\left(-2 W_{-m+1}^{2}+4 W_{-m}^{2}-7 W_{-m+1} W_{-m}+27 W_{-m} W_{-m-1}\right. \\
& \left.+\left(W_{1}+4 W_{0}\right)\left(2 W_{1}-W_{0}\right)-3\left(W_{1}-2 W_{0}\right)^{2} m\right) \\
= & \frac{1}{27}\left(-2 W_{-m+1}^{2}+4 W_{-m}^{2}-7 W_{-m+1} W_{-m}+27 W_{-m} W_{-m-1}+3\left(W_{1}-2 W_{0}\right)^{2}\right. \\
& \left.+\left(W_{1}+4 W_{0}\right)\left(2 W_{1}-W_{0}\right)-3\left(W_{1}-2 W_{0}\right)^{2}(m+1)\right) \\
= & \frac{1}{27}\left(-2 W_{-m}^{2}+4 W_{-m-1}^{2}-7 W_{-m} W_{-m-1}+\left(W_{1}+4 W_{0}\right)\left(2 W_{1}-W_{0}\right)\right. \\
& \left.-3\left(W_{1}-2 W_{0}\right)^{2}(m+1)\right) \\
= & \frac{1}{27}\left(-2 W_{-(m+1)+1}^{2}+4 W_{-(m+1)}^{2}-7 W_{-(m+1)+1} W_{-(m+1)}+\left(W_{1}+4 W_{0}\right)\left(2 W_{1}-W_{0}\right)\right. \\
& \left.-3\left(W_{1}-2 W_{0}\right)^{2}(m+1)\right)
\end{aligned}
$$

## where

$-2 W_{-m+1}^{2}+4 W_{-m}^{2}-7 W_{-m+1} W_{-m}+27 W_{-m} W_{-m-1}+3\left(W_{1}-2 W_{0}\right)^{2}=-2 W_{-m}^{2}+4 W_{-m-1}^{2}-7 W_{-m} W_{-m-1}$.
(3.6) can be proved by using Binet formula of $W_{n}$. Hence, the relation in (b) holds also for $n=m+1$.

From the last theorem, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_{n}=J_{n}$ with $J_{0}=0, J_{1}=1$ ).

Corollary 3.7. For $n \geq 1$, Jacobsthal numbers have the following property:
(a) $\sum_{i=1}^{n} J_{-i}^{2}=\frac{1}{9}\left(-J_{-n+1}^{2}+J_{-n}^{2}+1+n\right)$.
(b) $\sum_{i=1}^{n} J_{-i+1} J_{-i}=\frac{1}{27}\left(-2 J_{-n+1}^{2}+4 J_{-n}^{2}-7 J_{-n+1} J_{-n}+2-3 n\right)$.

Taking $W_{n}=j_{n}$ with $j_{0}=2, j_{1}=1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.8. For $n \geq 1$, Jacobsthal-Lucas numbers have the following property:
(a) $\sum_{i=1}^{n} j_{-i}^{2}=\frac{1}{9}\left(-j_{-n+1}^{2}+j_{-n}^{2}-3+9 n\right)$.
(b) $\sum_{i=1}^{n} j_{-i+1} j_{-i}=\frac{1}{27}\left(-2 j_{-n+1}^{2}+4 j_{-n}^{2}-7 j_{-n+1} j_{-n}-27 n\right)$.

## 4 CONCLUSION

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant ( $r$ circulant, geometric circulant, semicirculant) matrices with the generalized $m$-step Fibonacci sequences require the sum of the squares of the numbers of the sequences. Our future work will be investigation of the closed forms of the sum formulas for the squares of generalized Tribonacci numbers.

## COMPETING INTERESTS

Authors has declared that no competing interests exist.

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