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Closed Loop Two-Echelon Repairable Item Systems

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Abstract

In this paper we consider closed loop two-echelon repairable item systems with repair facilities both at a number of local service centers (called bases) and at a central location (the depot). The goal of the system is to maintain a number of production facilities (one at each base) in optimal operational condition. Each production facility consists of a number of identical machines which may fail incidentally. Each repair facility may be considered to be a multi-server station, while any transport from the depot to the bases is modeled as an ample server. At all bases as well as at the depot, ready-for-use spare parts (machines) are kept in stock. Once a machine in the production cell of a certain base fails, it is replaced by a ready-for-use machine from that base's stock, if available. The failed machine is either repaired at the base or repaired at the central repair facility. In the case of local repair, the machine is added to the local spare parts stock as a ready-for-use machine after repair. If a repair at the depot is needed, the base orders a machine from the central spare parts stock to replenish its local stock, while the failed machine is added to the central stock after repair. Orders are satisfied on a first-come-first-served basis while any requirement that cannot be satisfied immediately either at the bases or at the depot is backlogged. In case of a backlog at a certain base, that base's production cell performs worse.
To determine the steady state probabilities of the system, we develop a slightly aggregated system model and propose a special near-product-form solution that provides excellent approximations of relevant performance measures. The depot repair shop is modeled as a server with state-dependent service rates, of which the parameters follow from an application of Norton's theorem for Closed Queuing Networks. A special adaptation to a general Multi-Class MDA algorithm is proposed, on which the approximations are based. All relevant performance measures can be calculated with errors which are generally less than one percent, when compared to simulation results.

Keywords: Multi-echelon system, repairable item, limited repair capacity, open queuing network

Mathematics Subject Classification: 90B05, 90B15, 90B25

1 Introduction

Repairable inventory theory involves designing inventory systems for items which are repaired and returned to use rather than discarded. The items are less expensive to repair than to replace. Such items can for example be found in the military, aviation, copying machines, transportation equipment and electronics. The repairable inventory problem is typically concerned with the optimal stocking of parts at bases and a central depot facility which repairs failed units returned from bases while providing some predetermined level of service. Different performance measures may be used, such as

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cost, backorders and availability.
Over the past 30 years there has been considerable interest in multi-echelon inventory theory. Much of this work originates from a model called METRIC, which was first reported in the literature by Sherbrooke [7]. The model was developed for the US Air Force at the Rand Corporation for a multi-echelon repairable-item inventory system. In this model an item at failure is replaced by a spare if one is available. If none are available a spare is backordered. Of the failed items a certain proportion is repaired at the base and the rest at a repair depot, thereby creating a two-echelon repairable-item system. Items are returned from the depot using a one-for-one reordering policy. The METRIC model determines the optimal level of spares to be maintained at each of the bases and at the depot. A shortfall of the METRIC model is that it assumes that failures are Poisson from an infinite source and that the repair capacity is unlimited. Therefore, others have continued the research to gain results more useful for real life applications. Gross [5], Albright et al. [1] and Albright [2] focused their attention on closed queuing network models, thereby dropping the assumption of Poisson failures from an infinite source. The intensity by which machines enter the repair shops depends on the number of machines operating in the production cell. In case of a backlog at a base, this intensity is therefore smaller than in the optimal case where the maximum number of machines is operating in the production cell. Also the assumption of unlimited repair capacity is dropped by Gross and Albright.
This paper deals with similar models. It handles closed queuing network models with limited repair. However, the approximation method differs considerably. The approximation method builds on the method by Avsar and Zijm [3]. Avsar and Zijm considered an open queuing network model with limited repair. By a small aggregation step, the system is changed into a system with a special near-product-form solution that provides an approximation for the steady state distribution. From the steady state distribution all relevant performance measures can be computed. We will perform a similar aggregation step in this paper and again a special near-product-form solution will be obtained. However, as opposed to open systems, in a system with finite sources, the demand rates to the depot also become state dependent; moreover, these demand rates are clearly influenced by the efficiency of the base repair stations. Nevertheless, we are able to develop relatively simple approximation algorithms to obtain the relevant performance measures. These performance measures can ultimately be used within an optimization model to determine such quantities as the optimal repair capacities and the optimal inventory levels. However, these optimization procedures are beyond the scope of this paper.

The organization of this paper is as follows: In the next section we consider a very simple two-echelon system, consisting of one base, a base repair shop and a central repair shop. The repair shops are modeled as single servers. This model mainly serves to explain the essential elements of the aggregation step. We present the modified system with near-product-form solution and numerical results to show the accuracy of the approximation. Next, in Section 3, we turn to more general repairable item network structures, containing multiple bases and transport lines from the depot to the bases. The repair shops are modeled as multi-servers. The approximation method leading to an adapted Multi-Class MDA algorithm is presented and some numerical results are discussed. In the last section, we summarize our results and discuss a number of extensions that are currently being investigated.

2 Analysis of a simple two-echelon system with single server facilities

In this section a simplified repairable item system is discussed, to explain how a slight modification turns this system into a near-product form network that can be completely analyzed. In the next section we turn to more complex systems.

2.1 The single base model without transportation

Consider the system as depicted in Figure 1. The system consists of a single base and a depot. At the base a maximum of J_1 machines can be operational in the production cell. Operational machines

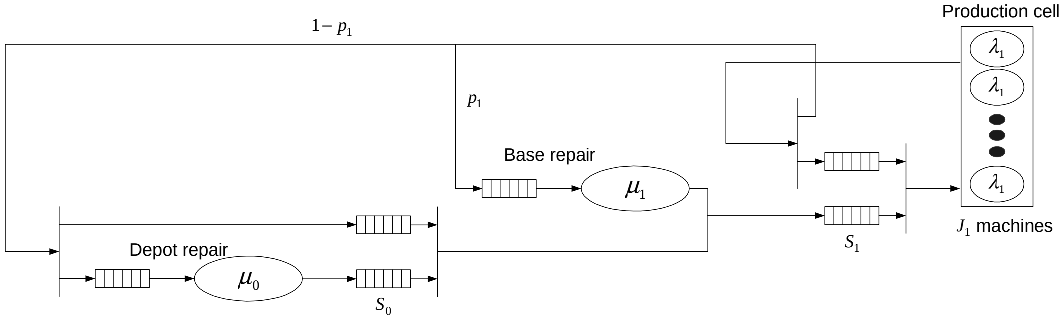

Figure 1: The single base repairable item system

fail at exponential rate λ_1 and are replaced by a machine from the base stock (if available). Both at the base and at the depot there is a repair shop. Failed machines are base-repairable with probability p_1 and consequently depot-repairable with probability $1-p_1$. The repair shops are modeled as single servers with exponential service rate μ_0 for the depot and exponential service rate μ_1 for the base. In addition to the J_1 machines another group of S_1 machines is dedicated to the base to act as spares. When a machine fails, the failed machine goes to a repair shop while at the same time a spare machine from the base stock is placed in the production cell. If there are no spare machines at the base, a backlog occurs. As soon as there is a repaired machine available, it becomes operational. A number of S_0 machines is dedicated to the depot to act as spares. When a failed machine cannot be repaired at the base and hence is sent to the depot, a spare machine is shipped from the depot to the base to replenish the base stock, or - in case of a backlog - to become operational immediately. When no spares are available at the depot, a backorder is created. In that case, as soon as a machine is repaired at the depot repair shop, it is sent to the base. In this simple model, transport times from the base to the depot and vice versa are not taken into account.
In Figure 1, the matching of a request and a ready-for-use machine is modeled as a synchronization queue, both at the base and at the depot. At the base however, some reflection reveals that the synchronization queue can be seen as a normal queue where machines are waiting to be moved into the production cell. This is only possible when the production cell does not contain the maximum number of machines, that is, if a machine in the production cell has failed. This leads to the model in Figure 2. In this figure the variables n_1, n_2, k, m_{11} and m_{12} indicate the lengths of the various

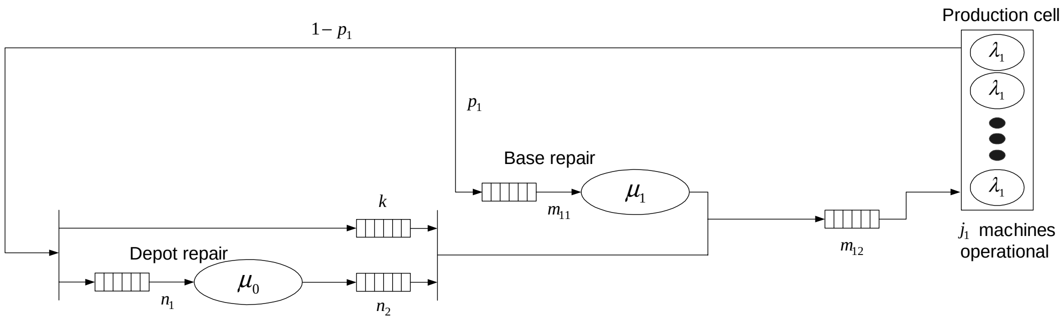

Figure 2: The modified single base system

queues in the system. The number of machines in (or awaiting) depot repair is denoted by the random variable $\underline{n}_1$, the number of spare machines at the depot is denoted by the random variable $\underline{n}_2$ and the backlog of machines at the depot is denoted by $\underline{k}$. At the base there are $\underline{m}_{11}$ machines waiting for repair or being repaired and $\underline{m}_{12}$ machines are acting as spares. In the production cell $\underline{j}_1$ machines are operational. As a result of the operating inventory control policies, for $\underline{n}_1 = n_1$, $\underline{n}_2 = n_2$, $\underline{k} = k$, $\underline{m}_{11} = m_{11}$, $\underline{m}_{12} = m_{12}$ and $\underline{j}_1 = j_1$ the following equations must hold:

$$n_1 + n_2 - k = S_0, \quad (1)$$
$$n_2 \cdot k = 0, \quad (2)$$

$$k + m_{11} + m_{12} + j_1 \quad = \quad S_1 + J_1, \tag{3}$$

$$m_{12} \cdot (J_1 - j_1) \quad = \quad 0, \tag{4}$$

where Equations (2) and (4) follow from the fact that it is impossible to have a backlog and spare machines available at the same time. If spare machines are available, a request is satisfied immediately. In case of a backlog, a request is not satisfied until a repair completion. The repaired machine is merged with the longest waiting request.
From these relations it follows immediately that n_1 and m_{11} completely determine the state of the system, including the values of n_2, k, m_{12} and j_1. Therefore, the system can be modeled as a continuous time Markov Chain with state description (n_1, m_{11}). The corresponding transition diagram is displayed in Figure 3.

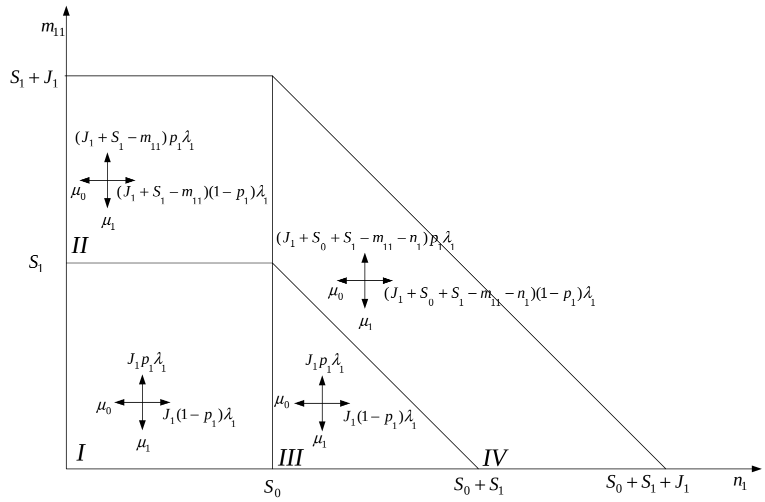

Figure 3: Transition diagram for state description (n_1, m_{11})

Let $P(n_1, m_{11}) = P(\underline{n}_1 = n_1, \underline{m}_{11} = m_{11})$ be the steady state probability of being in state (n_1, m_{11}). This steady state probability can be found by solving the global balance equations of the system. These can be deduced from the transition diagram. Nevertheless, it is not possible to find an algebraic expression for the steady state probabilities. Moreover, for larger systems with e.g. multiple bases, the computational effort becomes prohibitive. Therefore the system will be slightly adjusted in the next subsection, in order to arrive at a near-product form network.

2.2 Approximation

A first step towards an approximation for the steady state probabilities is to aggregate the state space. The most difficult parts of the transition diagram are regions I and II, that is, the parts with $n_1 \leq S_0$ or, equivalently, the parts with $k = 0$. The parts with $k > 0$ are equivalent to the states with $n_1 = k + S_0$. A natural aggregation of the system is a description through the states (k, m_{11}). The states (n_1, m_{11}) with $n_1 = 0, 1, \ldots, S_0$ are then aggregated into one state $(0, m_{11})$. Denote the steady state probabilities for the new model by $\tilde{P}$ then the following holds for any m_{11}:

$$\tilde{P}(\underline{k} = 0, \underline{m}_{11} = m_{11}) \quad = \quad \sum_{n_1=0}^{S_0} P(\underline{n}_1 = n_1, \underline{m}_{11} = m_{11}) \tag{5}$$

$$\tilde{P}(\underline{k} = k, \underline{m}_{11} = m_{11}) \quad = \quad P(\underline{n}_1 = S_0 + k, \underline{m}_{11} = m_{11}) \tag{6}$$

The transition diagram corresponding to the alternative state space description is displayed in Figure 4. The rates only differ from the transition diagram in Figure 3 for the case $k = 0$. Let $q(m_{11})$ be the steady state probability that an arriving request for a machine at the depot has to wait, given that it finds no other waiting requests in front of it ($k = 0$) and $\underline{m}_{11} = m_{11}$. Given the (aggregated)

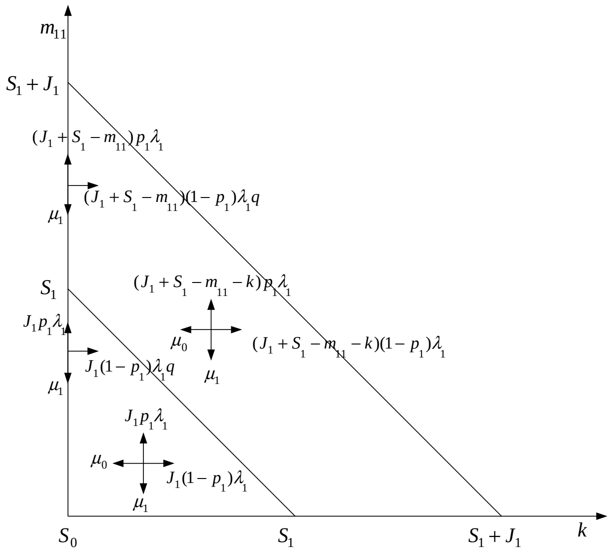

Figure 4: Transition diagram for state description (k, m_{11})

state $(0, m_{11})$, the state does not change in case of an arriving request with probability $1 - q(m_{11})$, because spares are available. With probability $q(m_{11})$ no spares are available and the state changes into $(1, m_{11})$. The transition rate from $(0, m_{11})$ to $(1, m_{11})$ equals $j_1(1 - p_1)\lambda_1 q(m_{11})$. To determine $q(m_{11})$ one needs

$$q(m_{11}) = P(\underline{n}_1 = S_0 | \underline{n}_1 \leq S_0, \underline{m}_{11} = m_{11}). \tag{7}$$

However, to compute this, one needs to know the steady state distribution of the original system, which is exactly what we attempt to approximate. Therefore, we approximate the $q(m_{11})$'s by their weighted average, i.e. we focus on the conditional probability q defined by

$$q = \sum_{m_{11}} q(m_{11}) P(\underline{m}_{11} = m_{11} | \underline{n}_1 \leq S_0) = P(\underline{n}_1 = S_0 | \underline{n}_1 \leq S_0) \tag{8}$$

and for every m_{11} we replace $q(m_{11})$ in the transition diagram by this q. In the next section will be explained how a reasonable approximation for this q can easily be found by means of an application of Norton's theorem.

Lemma 1 *The steady state probabilities for the model with state description (k, m_{11}) and transition rates as denoted in Figure 4 with $q(m_{11})$ replaced by arbitrary q have a product form.*

Proof. To find the steady state probabilities, consider both the original model in Figure 2 and the alternative model in Figure 5. In Figure 5 the depot repair shop with synchronization queue is replaced by a typical server. For jobs that find the server idle the server has infinite service rate with probability $(1 - q)$ (the case spares are available) and service rate μ_0 with probability q (the case no spares are available). Let $\underline{b}_1$ be the random variable equal to $\underline{m}_{12} + \underline{j}_1$, then by looking at the system with the typical server, and conditioning on the fact that the network contains exactly $J_1 + S_1$ jobs, it is easily verified that the following expression for $\tilde{P}(\underline{k} = k, \underline{m}_{11} = m_{11}, \underline{b}_1 = b_1)$ satisfies the balance equations of the TCQN:

$$\tilde{P}(k, m_{11}, b_1) = \begin{cases} \tilde{G} q (\frac{p_1}{\mu_1})^{m_{11}} (\frac{1-p_1}{\mu_0})^k (\frac{\frac{1}{\lambda_1}^{b_1}}{J_1! J_1^{b_1 - J_1}}), & b_1 > J_1, k > 0 \\ \tilde{G} q (\frac{p_1}{\mu_1})^{m_{11}} (\frac{1-p_1}{\mu_0})^k (\frac{\frac{1}{\lambda_1}^{b_1}}{b_1!}), & b_1 \leq J_1, k > 0 \\ \tilde{G} (\frac{p_1}{\mu_1})^{m_{11}} (\frac{\frac{1}{\lambda_1}^{b_1}}{J_1! J_1^{b_1 - J_1}}), & b_1 > J_1, k = 0 \\ \tilde{G} (\frac{p_1}{\mu_1})^{m_{11}} (\frac{\frac{1}{\lambda_1}^{b_1}}{b_1!}), & b_1 \leq J_1, k = 0 \end{cases} \tag{9}$$

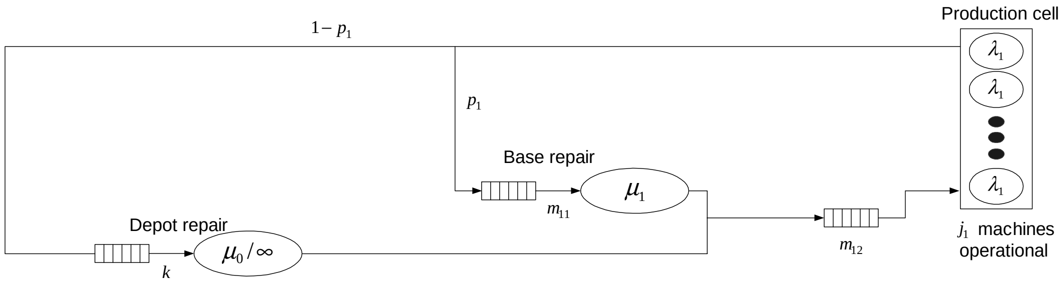

Figure 5: Typical-server Closed Queuing Network (TCQN)

with $k + m_{11} + b_1 = J_1 + S_1$ and $\tilde{G}$ the normalization constant. ■

Expressed in terms of the state variables (k, m_{11}), this result immediately leads to:

Lemma 2 *The steady state distribution for the aggregate model is given by*

$$\tilde{P}(k, m_{11}) \quad = \quad \begin{cases} \frac{Gq}{J_1! J_1^{S_1-k-m_{11}}} (\frac{p_1\lambda}{\mu_1})^{m_{11}} (\frac{(1-p_1)\lambda_1}{\mu_0})^k, & k + m_{11} \leq S_1, k > 0 \\ \frac{Gq}{(S_1+J_1-k-m_{11})!} (\frac{p_1\lambda}{\mu_1})^{m_{11}} (\frac{(1-p_1)\lambda_1}{\mu_0})^k, & k + m_{11} > S_1, k > 0 \\ \frac{G}{J! J^{S_1-m_{11}}} (\frac{p_1\lambda_1}{\mu_1})^{m_{11}}, & m_{11} \leq S_1, k = 0 \\ \frac{G}{(S_1+J_1-m_{11})!} (\frac{p_1\lambda_1}{\mu_1})^{m_{11}}, & m_{11} > S_1, k = 0 \end{cases} \tag{10}$$

with G the normalization constant.

The previous lemma gives an explicit expression for the steady state probabilities. For large systems it may be difficult to calculate the normalization constant G. However, since we are dealing with a product form network, Marginal Distribution Analysis (see e.g. Buzacott and Shanthikumar [4]) can be used to calculate the appropriate performance measures directly.

The results presented so far hold true for any value of $q \in [0, 1]$. In the derivation of the lemmas above the interpretation of q as the conditional probability that a request at the depot has to wait given that it finds no other requests in front of it (see (8)), has not been used. Therefore any $q \in [0, 1]$ will do, but it is expected that a good approximation will be obtained by using a q that does correspond to this interpretation. In the next subsection Norton's theorem will be used to find a q with a meaningful interpretation that gives good results.

2.3 Applying Norton's theorem to approximate q

Although we have stated in the previous section that the product form does not depend on q, it is still needed to find a q that gives a good approximation for the performance measures. In this section, the basic idea of Norton's theorem (see Harrison and Patel [6] for an overview) is used to find an approximation for q that gives good results. This basic idea is that a product form network can be analyzed by replacing subnetworks by state dependent servers. Norton's theorem states that the joint distributions for the numbers of customers in the subnetworks and the queue lengths at the replacing state dependent servers are the same.

To use this idea, first recall the original model as shown in Figure 2. We want to find q, the conditional probability that a request corresponding with a machine failure finds no spare parts in stock at the depot, although there was no backlog so far. The base, consisting of the production cell and the base repair shop, is taken apart and replaced by a state dependent server. The new network with the state dependent server is displayed in Figure 6 (left graph). In order to find the service rates for this state dependent server, the original network is short circuited by setting the service rate at the depot repair facility to infinity. This short circuited network is also depicted in Figure 6 (right graph). The

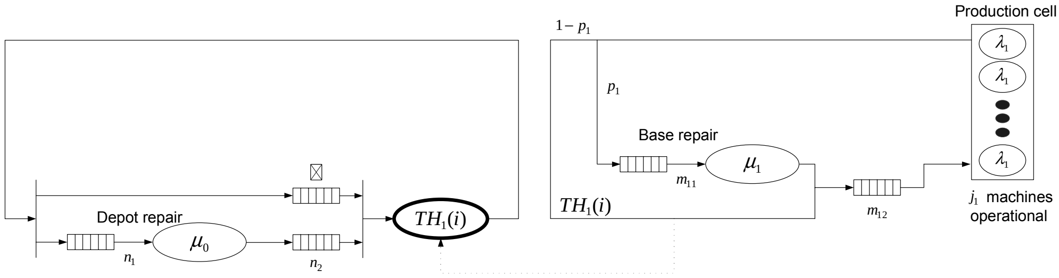

Figure 6: The new network with state dependent server (left graph) and the short circuited network (right graph)

service rate for the new state dependent server with i jobs present is equal to the throughput of the short circuited network with i jobs present, denoted by $TH_1(i)$.

The evolution of $\underline{n}_1 = n_1$, the number of machines in or awaiting depot repair, can be described as a birth-death process. The transition diagram is shown in Figure 7. Note that this is just an

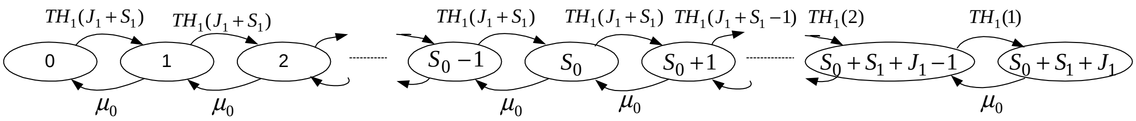

Figure 7: Transition diagram for n_1

approximation due to the fact that Norton's theorem is only valid for product form networks. In case $S_0 = 0$, we would have a product form network and the results would be exact. From the diagram one can observe that

$$P(\underline{n}_1 = n_1)\, TH_1(J_1 + S_1 - (n_1 - S_0)^+) = P(\underline{n}_1 = n_1 + 1)\, \mu_0 \tag{11}$$

for $n_1 = 1, \ldots, J_1 + S_1 + S_0$. In principle one can derive an approximation of the distribution of $\underline{n}_1$ from this. However, by the definition of q (see (8)), we only need to study the behavior for $\underline{n}_1 \leq S_0$. For these states, the service rate of the state dependent server is equal to $TH_1(J_1 + S_1)$. Let $\delta = TH_1(J_1 + S_1)/\mu_0$. From (11) we observe that $P(\underline{n}_1 = n_1) = \delta^{n_1} P(\underline{n}_1 = 0)$ for $n_1 = 0, \ldots, S_0$ so

$$\begin{aligned} q &= \frac{P(\underline{n}_1 = S_0)}{P(\underline{n}_1 \leq S_0)} = \frac{\delta^{S_0} P(\underline{n}_1 = 0)}{\sum_{n_1=0}^{S_0} P(\underline{n}_1 = n_1)} = \frac{\delta^{S_0} P(\underline{n}_1 = 0)}{\sum_{n_1=0}^{S_0} \delta^{n_1} P(\underline{n}_1 = 0)} = \frac{\delta^{S_0}}{\frac{1-\delta^{S_0+1}}{1-\delta}} \\ &= \delta^{S_0} \frac{1-\delta}{1-\delta^{S_0+1}}. \end{aligned} \tag{12}$$

It remains to find the throughput of the short circuited network in Figure 6 (right graph) with $J_1 + S_1$ jobs present. A simple observation reveals that $P(\underline{b}_1 = b_1)\ min(b_1, J_1)\ \lambda_1\ p_1 = P(\underline{b}_1 = b_1 - 1)\mu_1$ for $b_1 = 1, \ldots, J_1 + S_1$ from which the steady state probabilities of $\underline{b}_1$ are immediately deduced. Moreover, the throughput satisfies

$$\begin{aligned} TH_1(J_1 + S_1) &= (1 - p_1) \sum_{b_1=1}^{J_1+S_1} P(\underline{b}_1 = b_1)\, min(b_1, J_1)\, \lambda_1 \\ &= \frac{1-p_1}{p_1}\, \mu_1\, (1 - P(\underline{b}_1 = J_1 + S_1)). \end{aligned} \tag{13}$$

We can determine q with (12) and (13). This q can be used to approximate the steady state distribution using (10) or using Marginal Distribution Analysis. Results of this approximation are presented in the next section.

2.4 Results

In this section numerical results obtained by the approximation described above will be presented. To be able to judge the approximation the results are compared to exact results. The exact results are obtained by solving the balance equations for the original model.

The performance measures we are interested in are the availability, i.e. the probability that the maximum number of machines is working in the production cell, denoted by A, and the expected number of machines operating in the production cell ($E\underline{j}_1$). These are defined as follows:

$$A = P(\underline{j}_1 = J_1) = P(\underline{b}_1 \geq J_1) = P(\underline{k} + \underline{m}_{11} \leq S_1) \tag{14}$$

$$E\underline{j}_1 = E(J_1 - [\underline{k} + \underline{m}_{11} - S_1]^+) = \sum_{k,m_{11}} (J_1 - [k + m_{11} - S_1]^+) P(k, m_{11}) \tag{15}$$

The performance measures are computed for several values of $J_1, S_0, S_1, p_1, \lambda_1, \mu_0$ and μ_1. The results are given in Table 1 and in Tables 4 and 5 in the Appendix. Also, the percentage deviation is given. The numbers reveal that in these systems, the approximation gives an error of at most 1 %. In all other cases that we tested, we got similar results. The largest errors are attained in the cases with only a small number of spares ($S_0 > 0$) in the system. For the case $S_0 = 0$ the results are exact.

3 General two-echelon repairable item systems

In this section the simple system from Section 2 will be extended to a more realistic one. The system will contain multiple bases and transport lines. Furthermore, the single servers that are used in the repair shops are replaced by multi-servers. These adjustments will make the analysis of the system more complicated. Nevertheless, the basic idea of the aggregation step will be the same.

3.1 The multi-base model with transportation

The system in this section consists of multiple bases, where the number of bases is denoted by L. A graphical representation of the system is given in Figure 8 for the case $L = 2$. As in the simple system

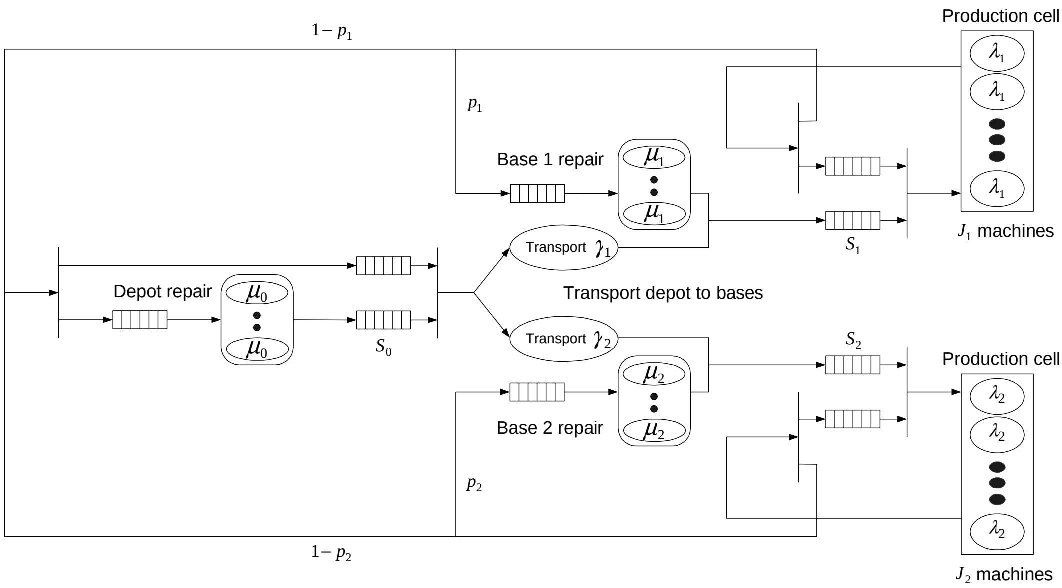

Figure 8: The multi-base repairable item system for $L = 2$

Table 1: Results for the simple single base model, $p_1 = 0.5, \lambda_1 = 1, \mu_0 = 2J_1, \mu_1 = J_1$

J	S_0	S_1	A_{exact}	A_{appr}	% dev	$E\underline{j}_1 exact$	$E\underline{j}_1 appr$	% dev
3	1	0	0.5651	0.5674	0.4185	2.4225	2.4246	0.0853
3	3	0	0.5889	0.5892	0.0543	2.4572	2.4576	0.0145
3	5	0	0.5901	0.5901	0.0041	2.4589	2.4590	0.0012
3	1	1	0.7945	0.7952	0.0934	2.7283	2.7286	0.0098
3	3	1	0.8110	0.8111	0.0154	2.7506	2.7507	0.0036
3	5	1	0.8120	0.8120	0.0014	2.7518	2.7518	0.0004
3	1	3	0.9506	0.9506	0.0012	2.9349	2.9348	0.0057
3	3	3	0.9554	0.9554	0.0000	2.9412	2.9412	0.0006
3	5	3	0.9557	0.9557	0.0000	2.9416	2.9416	0.0000
3	1	4	0.9755	0.9754	0.0012	2.9677	2.9676	0.0036
3	3	4	0.9779	0.9779	0.0004	2.9709	2.9709	0.0005
3	5	4	0.9781	0.9781	0.0000	2.9711	2.9711	0.0000
5	1	0	0.5369	0.5387	0.3314	4.3147	4.3160	0.0318
5	3	0	0.5625	0.5628	0.0461	4.3581	4.3584	0.0064
5	5	0	0.5639	0.5639	0.0037	4.3604	4.3604	0.0006
5	1	1	0.7759	0.7765	0.0761	4.6703	4.6704	0.0006
5	3	1	0.7940	0.7941	0.0127	4.6978	4.6979	0.0012
5	5	1	0.7950	0.7950	0.0012	4.6994	4.6994	0.0002
5	1	3	0.9453	0.9453	0.0012	4.9198	4.9196	0.0041
5	3	3	0.9506	0.9506	0.0000	4.9276	4.9276	0.0005
5	5	3	0.9510	0.0000	0.0000	4.9281	4.9281	0.0000
5	1	4	0.9727	0.9727	0.0009	4.9601	4.9600	0.0025
5	3	4	0.9755	0.9755	0.0003	4.9641	4.9640	0.0004
5	5	4	0.9757	0.9757	0.0000	4.9643	4.9643	0.0000
10	1	0	0.5091	0.5102	0.2178	9.1830	9.1837	0.0073
10	3	0	0.5363	0.5365	0.0328	9.2375	9.2377	0.0017
10	5	0	0.5379	0.5379	0.0028	9.2406	9.2406	0.0002
10	1	1	0.7565	0.7569	0.0507	9.5979	9.5977	0.0016
10	3	1	0.7762	0.7762	0.0087	9.6321	9.6321	0.0000
10	5	1	0.7774	0.7774	0.0008	9.6341	9.6341	0.0000
10	1	3	0.9395	0.9395	0.0006	9.9006	9.9004	0.0020
10	3	3	0.9455	0.9455	0.0001	9.9104	9.9104	0.0003
10	5	3	0.9458	0.9458	0.0000	9.9110	9.9110	0.0000
10	1	4	0.9698	0.9698	0.0007	9.9504	9.9503	0.0012
10	3	4	0.9728	0.9728	0.0002	9.9554	9.9554	0.0002
10	5	4	0.9730	0.9730	0.0000	9.9557	9.9557	0.0000

described before, at base $l = 1, \ldots, L$ at most J_l machines are operating in the production cell. The machines fail at exponential rate λ_l and are always replaced by a machine from the corresponding base stock (if available). Failed machines from base l are base-repairable with probability p_l and depot-repairable with probability $1 - p_l$. In contrast to the simple model described before, the repair shops are modeled as multi-servers. That is, at the repair shop of base $l = 1, \ldots, L$ R_l repairmen are working, each at exponential rate μ_l. At the depot repair shop R_0 repairmen are working at exponential rate μ_0. Consistent with the simple model S_l machines are dedicated to base l to act as spares and S_0 spare machines are dedicated to the depot. Broken machines at a certain base l that are base-repairable are sent to the base l repair shop. After repair they fill up the spares buffer at base l or, in case of a backlog at that base, become operational immediately. Broken machines from base l that are considered depot-repairable are sent to the depot repair shop. When depot spares are available, a spare is immediately sent to the stock of base l. In case there are no spares available a backlog occurs. Machines that have completed repair are sent to the base that has been waiting the longest. That is, a FCFS return policy is used. In this model the transportation from the depot to the bases is taken into account explicitly. The transport lines are modeled as ample servers with exponential service rate γ_l for the transport to base $l = 1, \ldots, L$. The number of machines in transport to base l is denoted by the random variable $\underline{t}_l$. The transport from the bases to the depot is not taken into account.
As in the simple model, the synchronization queues at the bases can be replaced by ordinary queues as is depicted in Figure 9.

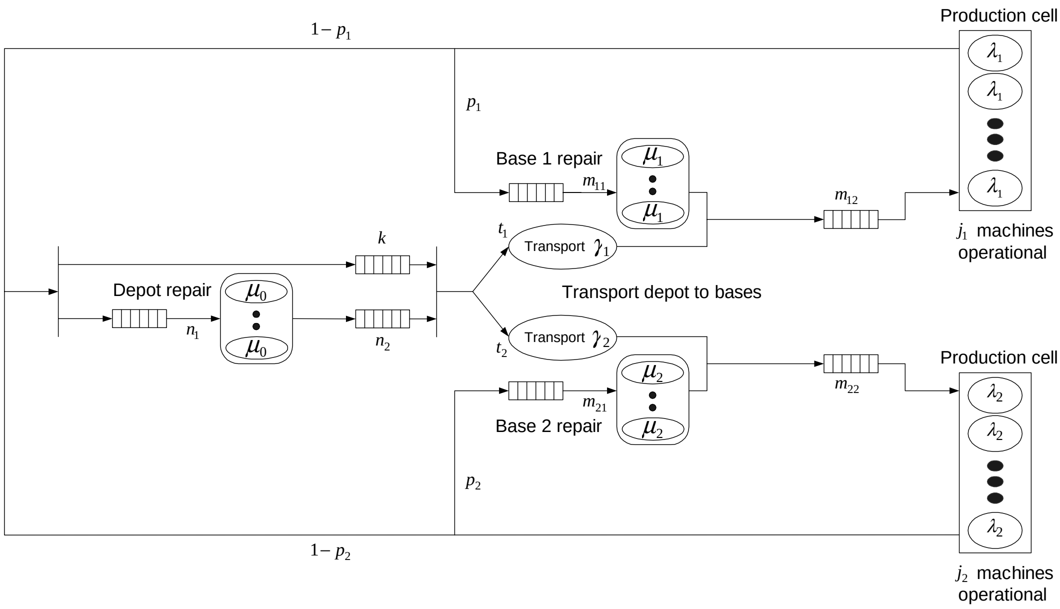

Figure 9: The modified multi-base system for $L = 2$

The vector $\mathbf{m}_1 = (m_{11}, m_{21}, \ldots, m_{L1})$ denotes the number of machines in base repair $(l = 1, \ldots, L)$ and the vector $\mathbf{m}_2 = (m_{12}, m_{22}, \ldots, m_{L2})$ denotes the number of spares at the bases $(l = 1, \ldots, L)$. The variable n_1 stands for the number of machines in depot repair and n_2 is the number of spare machines at the depot. The vector $\mathbf{k}_0 = (k_{01}, k_{02}, \ldots, k_{0L})$ denotes the backorders at the depot, originating from base l $(l = 1, \ldots, L)$. The total number of backorders at the depot equals $k = \sum_{l=1}^{L} k_{0l}$. The machines in transit to the bases are given by the vector $\mathbf{t} = (t_1, t_2, \ldots, t_L)$ and the numbers of machines operating in the production cells are expressed in vector $\mathbf{j} = (j_1, j_2, \ldots, j_L)$. The sum of the number of machines in base stock and the number of machines operating in the production cell is denoted in the vector $\mathbf{b} = (b_1, b_2, \ldots, b_L)$, where $b_l = m_{l2} + j_l$.

As a result of the operating inventory control policies, for $\underline{n}_1 = n_1$, $\underline{n}_2 = n_2$, $\underline{\mathbf{k}}_0 = \mathbf{k}_0$, $\underline{\mathbf{t}} = \mathbf{t}$, $\underline{\mathbf{m}}_1 = \mathbf{m}_1$, $\underline{\mathbf{m}}_2 = \mathbf{m}_2$ and $\underline{\mathbf{j}} = \mathbf{j}$ the following equations must hold:

$$n_1 + n_2 - k = S_0 \qquad (16)$$
$$n_2 \cdot k = 0 \qquad (17)$$

and for $l = 1, 2, \ldots, L$:

$$k_{0l} + t_l + m_{l1} + m_{l2} + j_l = S_l + J_l \tag{18}$$

$$m_{l2} \cdot (J_l - j_l) = 0 \tag{19}$$

From these relations it follows immediately that $\mathbf{k}_0$, n_1, $\mathbf{t}$ and $\mathbf{m}_1$ completely determine the state of the system. Therefore, the system can be modeled as a continuous time Markov Chain with state description $(\mathbf{k}_0, n_1, \mathbf{t}, \mathbf{m}_1)$.

Remark 3 *In the vector that denotes the number of backorders originating from the bases,* $\boldsymbol{k}_0 = (k_{01}, k_{02}, \ldots, k_{0L})$*, it is not taken into account that the order of the backorders matters. Since a FCFS return policy is assumed, this order should be known. Nevertheless, in this model all states with similar numbers of backorders per base, are aggregated into one state. This aggregation step will not have a big influence on the results, but it will considerably simplify the analysis.*

3.2 Approximation

In correspondence with the simple model as described in Section 2 a similar aggregation step is performed to tackle this extended model. Once more, all states with $0 \leq n_1 \leq S_0$ are aggregated into one state. The aggregation step is performed as follows

$$P(\underline{\mathbf{k}}_0 = \mathbf{0}, \underline{k} = 0, \underline{\mathbf{t}} = \mathbf{t}, \underline{\mathbf{m}}_1 = \mathbf{m}_1) = \sum_{n_1=0}^{S_0} P(\underline{\mathbf{k}}_0 = \mathbf{0}, \underline{n}_1 = n_1, \underline{\mathbf{t}} = \mathbf{t}, \underline{\mathbf{m}}_1 = \mathbf{m}_1) \tag{20}$$

$$P(\underline{\mathbf{k}}_0 = \mathbf{k}_0, \underline{k} = k, \underline{\mathbf{t}} = \mathbf{t}, \underline{\mathbf{m}}_1 = \mathbf{m}_1) = P(\underline{\mathbf{k}}_0 = \mathbf{k}_0, \underline{n}_1 = S_0 + k, \underline{\mathbf{t}} = \mathbf{t}, \underline{\mathbf{m}}_1 = \mathbf{m}_1) \tag{21}$$

The aggregated system can be described by $(\mathbf{k}_0, k, \mathbf{t}, \mathbf{m}_1)$. Furthermore, because $k = \sum_{l=1}^{L} k_{0l}$ the state space can also be described by $(\mathbf{k}_0, \mathbf{t}, \mathbf{m}_1)$.

Define q as before, that is q is the conditional probability that an arriving request at the depot cannot be fulfilled immediately, given that there are no other waiting requests. In a formula it says $q = P(\underline{n}_1 = S_0 | \underline{n}_1 \leq S_0)$. So, given there is no backlog at the depot, an arriving request has to wait with probability q. The waiting time depends on the number of spares already in the queue. The first

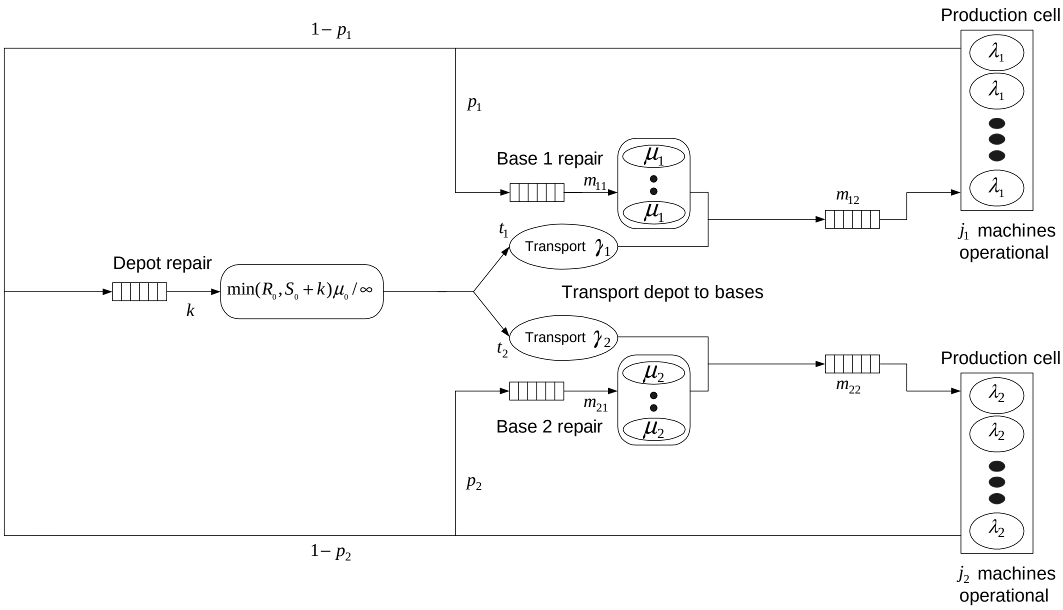

Figure 10: The Typical-server Closed Queuing Network

spare that finishes repair will fulfill the just arrived request. With probability $1-q$ spares are available and the arriving request does not have to wait. This aggregated network is depicted as a Typical-server Closed Queuing Network in Figure 10. The depot repair shop is modeled as a state dependent server. In case of no backlog ($k=0$) the service rate equals infinity with probability $1-q$ and equals $min(S_0, R_0)\mu_0$ with probability q. In all other cases ($k>0$) the service rate equals $min(k+S_0, R_0)\mu_0$.

To determine q Norton's theorem is used once more. As in Subsection 2.3 each base (the transport line, the base repair and the production cell) is replaced by a state dependent server. To determine the transition rate of this state dependent server, each base-part of the network is short circuited and its throughput is calculated. This throughput operates as the transition rate of the state dependent server. The new network with the state dependent servers and the short circuited networks are depicted in Figure 11.

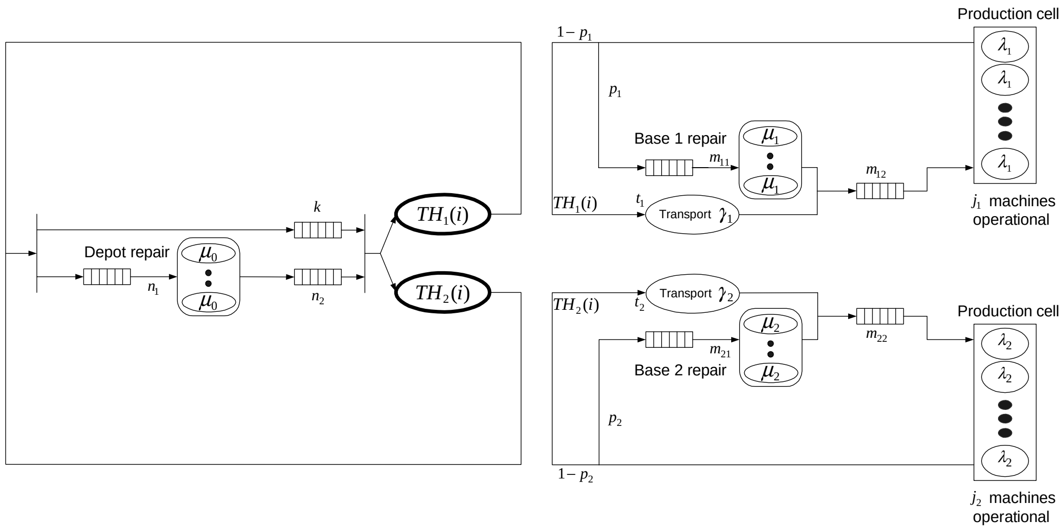

Figure 11: The new network with state dependent servers (left graph) and the short circuited networks (right graphs)

Once again the evolution of $\underline{n}_1$ can be described as a birth-death process. The (approximated) transition diagram for $n_1 = 0, \ldots, S_0$ is given in Figure 12. Let $TH_l(i)$ be the throughput of the

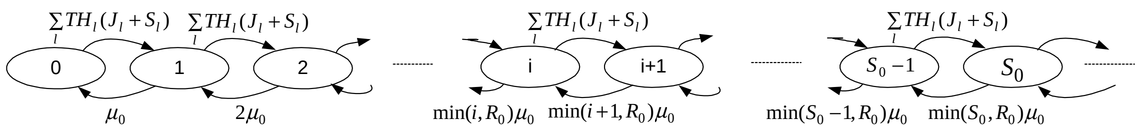

Figure 12: Transition diagram for n_1

subnetwork replacing base l ($l = 1, \ldots, L$) with i jobs present. As in the simple model only the behavior for $\underline{n}_1 \leq S_0$ needs to be studied to determine q. Take $\delta = \sum_l TH_l(J_l + S_l)/\mu_0$, then

$$P(\underline{n}_1 = n_1) = \frac{1}{\prod_{k=1}^{n_1} min(k, R_0)} \delta^{n_1} P(\underline{n}_1 = 0) \quad \text{for} \quad n_1 = 0, \ldots, S_0 \tag{22}$$

and

$$\begin{aligned} q &= \frac{P(\underline{n}_1 = S_0)}{P(\underline{n}_1 \leq S_0)} = \frac{P(\underline{n}_1 = S_0)}{\sum_{n_1=0}^{S_0} P(\underline{n}_1 = n_1)} = \frac{\delta^{S_0} \frac{1}{\prod_{k=1}^{S_0} min(k,R_0)} P(\underline{n}_1 = 0)}{\sum_{n_1=0}^{S_0} \delta^{n_1} \frac{1}{\prod_{k=1}^{n_1} min(k,R_0)} P(\underline{n}_1 = 0)} \\ &= \frac{\delta^{S_0} \frac{1}{\prod_{k=1}^{S_0} min(k,R_0)}}{\sum_{n_1=0}^{S_0} \delta^{n_1} \frac{1}{\prod_{k=1}^{n_1} min(k,R_0)}}. \end{aligned} \tag{23}$$

The throughputs can be obtained by applying a standard MDA algorithm (see [4]) on the short circuited product form networks as shown in Figure 11.

The steady state marginal probabilities as well as the main performance measures for the aggregated system can be found by using an adapted Multi-Class Marginal Distribution Analysis algorithm (see Buzacott and Shanthikumar [4] for ordinary Multi-Class MDA). To see this, introduce tokens of class l with $l = 1, \ldots, L$ that either represent machines present at base l (in the production cell, in the base repair shop, in the base stock or in transit to this base) or represent requests to the depot stock emerging from a failure of a machine at base l that cannot be repaired locally. Recall that machines that have to be repaired in the depot repair shop, in fact lose their identity, i.e. after completion they are placed in the depot stock, from which they can in principle be shipped to any arbitrary base. However, the request arriving jointly with that broken machine at the depot, maintains its identity, meaning that it is matched with the first spare machine available, after which the combination is shipped to the base the request originated from. Therefore, a token can be seen as connected to a machine as long as that machine is at the base (in any status) and connected with the corresponding request as soon as the machine is sent to the depot. This request matches with an available machine from stock (which generally is different from the one sent to the depot, unless $S_0 = 0$) and the combination returns to the base that generated the request. Hence, in this way, a multi-class network arises in a natural way.

The adapted algorithm is given below. An important aspect of an MDA algorithm is the computation of the expected sojourn time in the stations. Since the depot repair shop is modeled as a state dependent server, the standard sojourn time as described in [4] will not do for this station. As denoted before, in case of no backlog ($k = 0$) the service rate equals infinity with probability $1 - q$ and equals $min(S_0, R_0)\mu_0$ with probability q. In all other cases ($k > 0$) the service rate equals $min(k + S_0, R_0)\mu_0$. The expected sojourn time of an arriving request is the time it takes until all requests in front of it (k) are fulfilled and the request itself is fulfilled. That is, the time until $k + 1$ machines come out of repair. In case $k = 0$ with probability $1 - q$ the sojourn time equals 0 because a spare fulfills the request. This adaptation of the sojourn time reveals itself in the algorithm in step 4. Another adaptation of the ordinary algorithm is found in step 6. The transition rates from the states with 0 machines in depot repair to the states with 1 machine in depot repair now equal q times the throughput, instead of just the throughput.

Algorithm 4 *The depot repair shop is defined as station* 0 *and all other stations are defined as station* li*, where* l *denotes the number of the base* ($l = 1, \ldots, L$) *and* i *denotes the specific station associated with that base. The production cell is denoted by* $i = b$*, the base repair shop by* $i = m$ *and the transport line from the depot to the base by* $i = t$*.*
Let $V_j^{(r)}$ *be the visit ratio of station* j *for class* r *type machines. Let* z *denote the number of machines in the system and* $\boldsymbol{z} = (z_1, \ldots, z_r, \ldots, z_L)$ *the vector denoting the state that indicates the number of machines per class. The steady state probability that* y *machines are in station* j*, given vector* $\boldsymbol{z}$ *is denoted by* $p_j(y|\boldsymbol{z})$*. The expected sojourn time for type* r *machines arriving at station* j *given that* $\boldsymbol{z}$ *machines are wandering through the system is given by* $EW_j^{(r)}(\boldsymbol{z})$ *and* $TH_j^{(r)}(\boldsymbol{z})$ *denotes the throughput of type* r *machines given state* $\boldsymbol{z}$*. The algorithm is executed as follows:*

1. *(Initialization) For* $l = 1, \ldots, L$ *set* $V_0^{(l)} = 1$, $V_{lb}^{(l)} = \frac{1}{1-p_l}$, $V_{lm}^{(l)} = \frac{p_l}{1-p_l}$ *and* $V_{lt}^{(l)} = 1$*. For* $l = 1, \ldots, L$, $r = 1, \ldots, L$, $r \neq l$, $i \in \{b, m, t\}$ *set* $V_{li}^{(r)} = 0$*. Set* $z = 0$ *and* $p_j(0|\boldsymbol{0}) = 1$ *for* $j \in \bigcup_l \{lb, lm, lt\} \cup \{0\}$*.*
2. *z:=z+1.*
3. *For all states* $\boldsymbol{z} \in \{\boldsymbol{z} | \sum_{l=1}^{L} z^{(l)} = z \text{ and } z^{(l)} \leq J_l + S_l\}$ *execute steps 4 through 6.*
4. *Compute the sojourn times for* $l = 1, \ldots, L$ *for which* $z^{(l)} > 0$ *from:*

$$\begin{aligned} EW_0^{(l)}(\boldsymbol{z}) &= \sum_{k=1}^{z-1} \frac{k+1}{min(R_0, S_0 + k + 1)\mu_0} p_0(k|\boldsymbol{z} - \boldsymbol{e}_l) \\ &+ \frac{q}{min(R_0, S_0 + 1)\mu_0} p_0(0|\boldsymbol{z} - \boldsymbol{e}_l), \end{aligned}$$

$$
\begin{aligned}
EW_{lb}^{(l)}(\boldsymbol{z}) &= \sum_{b_l=J_l}^{z-1} \frac{b_l - J_l + 1}{J_l \lambda_l} p_{lb}(b_l|\boldsymbol{z} - \boldsymbol{e}_l) + \frac{1}{\lambda_l}, \\
EW_{lm}^{(l)}(\boldsymbol{z}) &= \sum_{m_{l1}=R_l}^{z-1} \frac{m_{l1} - R_l + 1}{R_l \mu_l} p_{lm}(m_{l1}|\boldsymbol{z} - \boldsymbol{e}_l) + \frac{1}{\mu_l}, \\
EW_{lt}^{(l)}(\boldsymbol{z}) &= \frac{1}{\gamma_l}.
\end{aligned}
$$

5. *Compute* $TH_0^{(l)}(\boldsymbol{z})$ *for* $l = 1, \ldots, L$ *if* $z^{(l)} > 0$ *from:*

$$
TH_0^{(l)}(\boldsymbol{z}) = \frac{z^{(l)}}{V_0^{(l)} EW_0^{(l)} + \sum_{i \in \{b,m,t\}} V_{li}^{(l)} EW_{li}^{(l)}},
$$

and if $z^{(l)} = 0$ *then* $TH_0^{(l)}(\boldsymbol{z}) = 0$*. Compute* $TH_{li}^{(l)}(\boldsymbol{z})$ *for* $l = 1, \ldots, L$ *and* $i \in \{b, m, t\}$ *from:*

$$
TH_{li}^{(l)}(\boldsymbol{z}) = V_{li}^{(l)} TH_0^{(l)}(\boldsymbol{z}).
$$

6. *Compute the marginal probabilities for all stations from:*

$$
\begin{aligned}
\mu_0 \, min(R_0, S_0 + 1) \, p_0(1|\boldsymbol{z}) &= \sum_{l=1}^{L} TH_0^{(l)}(\boldsymbol{z}) \, q \, p_0(0|\boldsymbol{z} - \boldsymbol{e}_l), \\
\mu_0 \, min(R_0, S_0 + k) \, p_0(k|\boldsymbol{z}) &= \sum_{l=1}^{L} TH_0^{(l)}(\boldsymbol{z}) \, p_0(k-1|\boldsymbol{z} - \boldsymbol{e}_l) \quad \textit{for } k = 1, \ldots, z,
\end{aligned}
$$

and for $l = 1, \ldots, L$ *from:*

$$
\begin{aligned}
\lambda_l \, min(J_l, b_l) \, p_{lb}(b_l|\boldsymbol{z}) &= TH_{lb}^{(l)}(\boldsymbol{z}) \, p_{lb}(b_l - 1|\boldsymbol{z} - \boldsymbol{e}_l) \quad \textit{for } b_l = 1, \ldots, z, \\
\mu_l \, min(R_l, m_{l1}) \, p_{lm}(m_{l1}|\boldsymbol{z}) &= TH_{lm}^{(l)}(\boldsymbol{z}) \, p_{lm}(m_{l1} - 1|\boldsymbol{z} - \boldsymbol{e}_l) \quad \textit{for } m_{l1} = 1, \ldots, z, \\
\gamma_l \, t_l \, p_{lt}(t_l|\boldsymbol{z}) &= TH_{lt}^{(l)}(\boldsymbol{z}) \, p_{lt}(t_l - 1|\boldsymbol{z} - \boldsymbol{e}_l) \quad \textit{for } t_l = 1, \ldots, z.
\end{aligned}
$$

Compute $p_j(0|\boldsymbol{z})$ *for* $j \in \bigcup_l \{lb, lm, lt\} \cup \{0\}$ *from:*

$$
p_j(0|\boldsymbol{z}) = 1 - \sum_{y=1}^{z} p_j(y|\boldsymbol{z}).
$$

7. *If* $z = \sum_{l=1}^{L} J_l + S_l$ *then stop; else go to step 2.*

With the adapted Multi-Class MDA algorithm presented above, the marginal probabilities of the system as well as the throughputs and the sojourn times can be approximated. From these, various performance measures can be computed. In the next section some results obtained by the algorithm will be compared with results from simulation.

3.3 Results

In this section results obtained by the adapted Multi-Class MDA algorithm from the previous section will be presented. They will be compared to results obtained by simulation. For each base we are interested in the availability, that is the probability that the maximum number of machines is operating in the production cell. For base l this is denoted by A_l for $l = 1, \ldots, L$. Furthermore we are interested in the expected number of machines operating in the production cell, denoted by $E\underline{j}_l$ for base $l = 1, \ldots, L$. For $l = 1, \ldots, L$ the performance measures can be computed by

$$A_l = P(\underline{j}_l = J_l) = P(\underline{b}_l \geq J_l) = P(\underline{k}_{0l} + \underline{m}_{l1} \leq S_l) \quad (24)$$

$$E\underline{j}_l = E(J_l - [\underline{k}_{0l} + \underline{m}_{l1} - S_l]^+) = \sum_{k_{0l}, m_{l1}} (J_l - [k_{0l} + m_{l1} - S_l]^+) P(k_{0l}, m_{l1}) \quad (25)$$

In Table 2 and Tables 6 and 7 in the Appendix, the parameters for some representative test problems are given. It is obvious that a large number of input parameters is required to specify a given problem. This makes it difficult to vary these parameters in a totally systematic manner. In Albright [2] it is shown that traffic intensities are good indicators of whether a system will work well (minimal backorders) and better indicators than the stock levels. Therefore we selected most of the test problem parameter settings by selecting values of the traffic intensities, usually well less than 1, and then selecting parameters to achieve these traffic intensities. For the base l repair facility, the traffic intensity ρ_l is defined as

$$\rho_l = J_l \lambda_l p_l / R_l \mu_l, \quad (26)$$

Table 2: Parameter settings for test problems multi-base model with transportation (1)

Problem	L	J_l	S_l S_0	λ_l	μ_l μ_0	R_l R_0	p_l	γ_l	ρ_l ρ_0
1	2	10 10	2 2 1	1 1	10 10 20	1 1 1	0.5 0.5	∞ ∞	0.5 0.5 0.5
2	2	5 5	2 2 1	1 1	5 5 10	1 1 1	0.5 0.5	∞ ∞	0.5 0.5 0.5
3	2	5 5	2 2 1	1 1	5 5 10	2 2 2	0.5 0.5	∞ ∞	0.25 0.25 0.25
4	2	5 5	2 2 1	1 1	5 5 10	1 1 1	0.5 0.5	10 10	0.5 0.5 0.5
5	2	5 5	2 2 1	1 1	5 5 10	1 1 1	0.5 0.5	2 2	0.5 0.5 0.5
6	2	5 5	2 2 1	1 1	5 5 10	2 2 2	0.5 0.5	2 2	0.25 0.25 0.25
7	2	5 5	2 2 1	1 1	1 1 2	5 5 5	0.5 0.5	2 2	0.5 0.5 0.5
8	2	5 5	2 2 7	1 1	1 1 2	5 5 5	0.5 0.5	2 2	0.5 0.5 0.5
9	2	5 5	5 5 5	1 1	3 3 6	1 1 1	0.5 0.5	∞ ∞	0.83 0.83 0.83
10	2	5 5	5 5 5	1 1	5 5 10	1 1 1	0.5 0.5	∞ ∞	0.5 0.5 0.5

the maximum failure rate divided by the maximum repair rate. Similarly, the depot traffic intensity ρ_0 is defined as

$$\rho_0 = \sum_{l=1}^{L} J_l \lambda_l (1 - p_l)/R_0 \mu_0. \tag{27}$$

The results are given in Table 3 and Table 8 in the Appendix. The simulation leads to 95 % confidence intervals. To compare the approximations with the simulation results, the deviation from the approximation to the midpoint of the confidence interval is calculated. These percentage deviations are given as well.

Table 3: Results for test problems from Table (2)

Problem	$A_l sim$	$A_l appr$	% dev	$E\underline{j}_l sim$	$E\underline{J}_l appr$	% dev
1	(0.8529,0.8563)	0.8542	0.05	(9.7533,9.7615)	9.7562	0.01
	(0.8505,0.8559)	0.8542	0.11	(9.7489,9.7611)	9.7562	0.01
2	(0.8638,0.8750)	0.8683	0.13	(4.7957,4.8161)	4.8043	0.03
	(0.8636,0.87210	0.8683	0.05	(4.7983,4.8116)	4.8043	0.01
3	(0.9695,0.9714)	0.9701	0.04	(4.9626,4.9655)	4.9633	0.02
	(0.9689,0.9709)	0.9701	0.02	(4.9617,4.9649)	4.9633	0.00
4	(0.8311,0.8403)	0.8353	0.04	(4.7461,4.7640)	4.7543	0.02
	(0.8274,0.8346)	0.8353	0.51	(4.7399,4.7549)	4.7543	0.15
5	(0.6548,0.6639)	0.6605	0.18	(4.4542,4.4737)	4.4672	0.07
	(0.6583,0.6652)	0.6605	0.18	(4.4610,4.4753)	4.4672	0.02
6	(0.7490,0.7539)	0.7514	0.00	(4.6463,4.6545)	4.6521	0.04
	(0.7458,0.7529)	0.7514	0.27	(4.6416,4.6538)	4.6521	0.09
7	(0.2938,0.3008)	0.2978	0.17	(3.6284,3.6497)	3.6445	0.15
	(0.2949,0.3001)	0.2978	0.09	(3.6352,3.6529)	3.6445	0.01
8	(0.3781,0.3883)	0.3800	0.83	(3.8866,3.9096)	3.8907	0.19
	(0.3779,0.3836)	0.3800	0.19	(3.8855,3.9000)	3.8907	0.05
9	(0.8165,0.8361)	0.8234	0.34	(4.6622,4.7032)	4.6770	0.12
	(0.8142,0.8304)	0.8234	0.13	(4.6582,4.6941)	4.6770	0.02
10	(0.9854,0.9894)	0.9875	0.01	(4.9785,4.9851)	4.9817	0.00
	(0.9874,0.9894)	0.9875	0.09	(4.9815,4.9851)	4.9817	0.03

From the results it can be concluded that the approximations are extremely accurate. The maximum deviation is well less than 1 % and all approximating values lie within the confidence intervals. Furthermore, all types of problems exhibited similar levels of accuracy.

4 Summary and possible extensions

In this paper we have analyzed a closed two-echelon repairable item system with a fixed number of items circulating in the network. The system consists of several bases and a central repair facility (depot). Each base consists of a production cell and a base repair shop. There are transport lines leading from the depot to the bases. Transport from bases to the depot is not taken into account. The repair shops are modeled as multi-servers and the transport lines as ample servers. Repair shops at the depot as well as at the bases are able to keep a number of ready-for-use items in stock. Machines that have failed in the production cell of a certain base are immediately replaced by a ready-for-use machine from that base's stock, if available. The failed machine is sent to either the base repair facility or to the depot repair facility, in the latter case a spare machine is sent from the depot to the base, to deplete the base's stock of ready-for-use items. Once the machine at the depot is repaired, it is added to the central stock. Orders are satisfied on a first-come-first-served basis while any requirement that cannot be satisfied immediately either at a base or at the depot is backlogged. In case of a backlog at a certain base, that base's production cell performs worse. This also means that the expected total rate at which machines fail at the production cell is smaller than in the case of no backlog.

The exact analysis of a Markov chain model for this system with multiple bases and many machines or with large inventories, is difficult to handle. Therefore, we aggregated a number of states and adjusted some rates to obtain a special near-product-form solution. The new system can be observed as a Typical-server Closed Queuing Network (TCQN). The notion *typical* comes from modeling the central repair facility together with the synchronization queue, as a typical server with state dependent service rates. These state dependent service rates follow from an application of Norton's theorem for Closed Queuing Networks. An adapted Multi-Class Marginal Distribution Analysis algorithm is developed to compute the steady state probabilities. From these steady state probabilities several performance measures can be obtained, such as the availability and the expected number of machines operating in the production cells. Numerical results show that the approximations are extremely accurate, when compared to simulation results.

A disadvantage of the adapted Multi-Class Marginal Distribution Analysis algorithm is the computational slowness. Especially for large systems with multiple bases, many machines and large inventories, the algorithm is not very fast. Here, further aggregation steps may speed up the system evaluation considerably, unfortunately at the cost of some accuracy.
Furthermore, the model considered is quite a realistic model. However, it could be more realistic by including transport from the bases to the depot and to allow for more complicated networks in the repair facilities. In the model described in this paper, each repair shop is modeled as a multi-server. An interesting extension to this, is to consider the repair facility to be a job shop and model it as a limited capacity open queuing network, as has been done in [3] for the case of an open multi-echelon repairable item system. Then, it is easy to include transport to the depot repair facility as just an additional node in the job shop.
Last but not least, it is interesting to find an optimization algorithm to determine optimal inventory levels at both the central and local facilities in combination with optimal repair capacities. This will be the subject of future research.

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Appendix

Table 4: Results for the simple single base model, $p_1 = 0.5, \lambda_1 = 1, \mu_0 = J_1, \mu_1 = J_1$

J	S_0	S_1	A_{exact}	A_{appr}	% dev	$E\underline{j}_1 exact$	$E\underline{j}_1 appr$	% dev
3	1	0	0.5056	0.5100	0.8575	2.3178	2.3225	0.2037
3	3	0	0.5749	0.5771	0.3784	2.4338	2.4368	0.1227
3	5	0	0.5874	0.5880	0.1066	2.4544	2.4553	0.0379
3	1	1	0.7322	0.7340	0.2516	2.6331	2.6345	0.0545
3	3	1	0.7948	0.7961	0.1590	2.7264	2.7279	0.0531
3	5	1	0.8082	0.8087	0.0578	2.7463	2.7469	0.0217
3	1	3	0.9171	0.9172	0.0114	2.8875	2.8873	0.0061
3	3	3	0.9465	0.9466	0.0106	2.9287	2.9287	0.0005
3	5	3	0.9535	0.9536	0.0055	2.9385	2.9385	0.0014
3	1	4	0.9538	0.9538	0.0008	2.9376	2.9374	0.0058
3	3	4	0.9722	0.9722	0.0001	2.9630	2.9629	0.0022
3	5	4	0.9766	0.9766	0.0006	2.9691	2.9691	0.0003
5	1	0	0.4690	0.4722	0.6947	4.1654	4.1688	0.0817
5	3	0	0.5452	0.5470	0.3263	4.3224	4.3250	0.0595
5	5	0	0.5602	0.5607	0.0987	4.3529	4.3538	0.0209
5	1	1	0.7045	0.7059	0.2070	4.5407	4.5416	0.0187
5	3	1	0.7748	0.7758	0.1318	4.6643	4.6654	0.0237
5	5	1	0.7905	0.7909	0.0486	4.6915	4.6920	0.0108
5	1	3	0.9068	0.9069	0.0094	4.8573	4.8570	0.0059
5	3	3	0.9403	0.9404	0.0078	4.9111	4.9110	0.0016
5	5	3	0.9484	0.9484	0.0040	4.9240	4.9240	0.0001
5	1	4	0.9480	0.9480	0.0007	4.9207	4.9205	0.0045
5	3	4	0.9689	0.9689	0.0006	4.9537	4.9536	0.0022
5	5	4	0.9740	0.9740	0.0002	4.9617	4.9617	0.0005
10	1	0	0.4318	0.4339	0.4703	8.9658	8.9676	0.0206
10	3	0	0.5150	0.5162	0.2363	9.1819	9.1836	0.0182
10	5	0	0.5329	0.5333	0.0756	9.2279	9.2286	0.0073
10	1	1	0.6746	0.6756	0.1407	9.4175	9.4177	0.0023
10	3	1	0.7535	0.7542	0.0907	9.5842	9.5848	0.0057
10	5	1	0.7718	0.7721	0.0336	9.6225	9.6228	0.0031
10	1	3	0.8953	0.8953	0.0058	9.8165	9.8161	0.0036
10	3	3	0.9335	0.9335	0.0043	9.8880	9.8879	0.0017
10	5	3	0.9428	0.9428	0.0021	9.9054	9.9053	0.0004
10	1	4	0.9414	0.9414	0.0009	9.8980	9.8978	0.0024
10	3	4	0.9652	0.9652	0.0011	9.9415	9.9414	0.0015
10	5	4	0.9711	0.9711	0.0002	9.9522	9.9522	0.0004

Table 5: Results for the simple single base model, $p_1 = 0.25, \lambda_1 = 1, \mu_0 = 2J_1, \mu_1 = J_1$

J	S_0	S_1	A_{exact}	A_{appr}	% dev	$E\underline{j}_1 exact$	$E\underline{j}_1 appr$	% dev
3	1	0	0.5348	0.5383	0.6612	2.3402	2.3436	0.1475
3	3	0	0.6743	0.6783	0.5878	2.5726	2.5777	0.1978
3	5	0	0.7282	0.7310	0.3796	2.6619	2.6658	0.1468
3	1	1	0.7201	0.7208	0.0913	2.5951	2.5956	0.0176
3	3	1	0.8384	0.8394	0.1194	2.7746	2.7757	0.0405
3	5	1	0.8906	0.8914	0.0958	2.8537	2.8548	0.0381
3	1	3	0.8705	0.8705	0.0007	2.8110	2.8109	0.0007
3	3	3	0.9311	0.9311	0.0019	2.8999	2.8999	0.0001
3	5	3	0.9613	0.9613	0.0023	2.9442	2.9443	0.0007
3	1	4	0.9075	0.9075	0.0001	2.8649	2.8649	0.0003
3	3	4	0.9505	0.9505	0.0000	2.9278	2.9278	0.0002
3	5	4	0.9726	0.9726	0.0002	2.9602	2.9602	0.0000
5	1	0	0.4900	0.4923	0.4515	4.1493	4.1514	0.0483
5	3	0	0.6429	0.6455	0.4015	4.4641	4.4675	0.0762
5	5	0	0.7066	0.7085	0.2621	4.5946	4.5975	0.0620
5	1	1	0.6814	0.6818	0.0605	4.4558	4.4560	0.0042
5	3	1	0.8147	0.8154	0.0774	4.6983	4.6990	0.0142
5	5	1	0.8761	0.8767	0.0617	4.8098	4.8105	0.0149
5	1	3	0.8477	0.8477	0.0002	4.7371	4.7370	0.0005
5	3	3	0.9182	0.9182	0.0007	4.8597	4.8596	0.0003
5	5	3	0.9540	0.9540	0.0011	4.9219	4.9219	0.0001
5	1	4	0.8904	0.8904	0.0002	4.8106	4.8106	0.0002
5	3	4	0.9409	0.9409	0.0002	4.8980	4.8980	0.0002
5	5	4	0.9672	0.9672	0.0000	4.9436	4.9436	0.0001
10	1	0	0.4390	0.4401	0.2503	8.8481	8.8489	0.0094
10	3	0	0.6051	0.6064	0.2198	9.2890	9.2906	0.0176
10	5	0	0.6807	0.6817	0.1415	9.4891	9.4906	0.0158
10	1	1	0.6338	0.6340	0.0316	9.2282	9.2282	0.0000
10	3	1	0.7843	0.7846	0.0387	9.5703	9.5706	0.0024
10	5	1	0.8574	0.8576	0.0300	9.7364	9.7367	0.0032
10	1	3	0.8177	0.8177	0.0001	9.6112	9.6112	0.0003
10	3	3	0.9007	0.9007	0.0001	9.7898	9.7898	0.0002
10	5	3	0.9440	0.9440	0.0002	9.8828	9.8828	0.0001
10	1	4	0.8675	0.8675	0.0002	9.7170	9.7170	0.0001
10	3	4	0.9277	0.9277	0.0002	9.8460	9.8460	0.0001
10	5	4	0.9597	0.9597	0.0001	9.9146	9.9146	0.0001

Table 6: Parameter settings for test problems multi-base model with transportation (2)

Problem	L	J_l	S_l S_0	λ_l	μ_l μ_0	R_l R_0	p_l	γ_l	ρ_l ρ_0
11	2	5 5	5 5 5	1 1	5 5 10	1 1 1	0.5 0.5	10 10	0.5 0.5 0.5
12	2	5 5	2 2 3	1 1	2 2 10	1 1 1	0.2 0.2	∞ ∞	0.5 0.5 0.8
13	2	5 5	2 2 3	1 1	2 2 10	2 2 2	0.2 0.2	∞ ∞	0.25 0.25 0.4
14	2	5 5	2 2 3	1 1	2 2 10	2 2 2	0.2 0.2	5 5	0.25 0.25 0.4
15	2	5 5	1 1 2	1 1	5 5 5	1 1 1	0.5 0.5	∞ ∞	0.5 0.5 1
16	2	5 5	3 3 2	1 1	2 2 3	3 3 3	0.5 0.5	5 5	0.42 0.42 0.56
17	2	5 5	2 2 4	1 1	5 5 2	1 1 10	0.5 0.5	10 10	0.5 0.5 0.25
18	2	5 5	2 2 8	1 1	5 5 5	1 1 3	0.5 0.5	10 10	0.5 0.5 0.33
19	2	5 5	2 2 8	1 1	5 5 1	1 1 8	0.5 0.5	10 10	0.5 0.5 0.63
20	2	7 7	3 3 3	1 1	5 5 10	1 1 1	0.25 0.25	∞ ∞	0.35 0.35 1.05

Table 7: Parameter settings for test problems multi-base model with transportation (3)

Problem	L	J_l	S_l S_0	λ_l	μ_l μ_0	R_l R_0	p_l	γ_l	ρ_l ρ_0
21	2	5	1	1	5	1	0.5	∞	0.5
		10	3	1	10	1	0.5	∞	0.5
			3		10	1			0.75
22	2	2	1	1	2	1	0.5	∞	0.5
		8	3	1	8	1	0.7	∞	0.7
			3		5	1			0.68
23	2	5	2	1	5	1	0.5	∞	0.5
		7	2	1	5	1	0.5	∞	0.7
			1		10	1			0.6
24	3	5	2	1	5	1	0.5	10	0.5
		5	2	1	5	1	0.5	10	0.5
		5	2	1	5	1	0.5	10	0.5
			8		5	3			0.5
25	3	5	1	1	2	3	0.5	10	0.42
		5	1	1	2	3	0.5	10	0.42
		5	1	1	2	3	0.5	10	0.42
			1		4	8			0.23
26	3	2	1	2	3	1	0.5	5	0.67
		5	1	1	2	3	0.5	10	0.42
		7	1	1	5	3	0.5	10	0.23
			3		4	8			0.25
27	3	7	5	1	3	3	0.5	10	0.39
		7	5	2	3	3	0.2	10	0.31
		7	5	3	3	7	0.8	10	0.8
			5		3	7			0.9
28	3	7	0	1	5	2	0.5	10	0.35
		7	5	1	5	2	0.5	10	0.35
		7	10	1	5	2	0.5	10	0.35
			5		5	2			1.05
29	3	3	2	1	5	1	0.5	5	0.3
		3	2	1	5	2	0.5	5	0.15
		3	2	1	5	3	0.5	5	0.1
			2		5	2			0.45
30	4	5	2	1	5	2	0.5	10	0.25
		5	2	1	5	2	0.5	10	0.25
		5	2	1	5	2	0.5	10	0.25
		5	2	1	5	2	0.5	10	0.25
			2		5	4			0.5

Table 8: Results for test problems from Tables (6) and (7)

Problem	$A_l sim$	$A_l appr$	% dev	$E\underline{j}_l sim$	$E\underline{J}_l appr$	% dev
11	(0.9824,0.9853)	0.9840	0.02	(4.9735,4.9786)	4.9765	0.01
	(0.9826,0.9854)	0.9840	0.00	(4.9737,4.9792)	4.9765	0.00
12	(0.8148,0.8294)	0.8192	0.35	(4.7034,4.7322)	4.7129	0.10
	(0.8151,0.8266)	0.8192	0.20	(4.7062,4.7304)	4.7129	0.11
13	(0.9731,0.9756)	0.9731	0.12	(4.9666,4.9705)	4.9669	0.03
	(0.9720,0.9742)	0.9731	0.01	(4.9650,4.9690)	4.9669	0.00
14	(0.8536,0.8585)	0.8563	0.03	(4.8066,4.8149)	4.8118	0.02
	(0.8559,0.8607)	0.8563	0.23	(4.8108,4.8181)	4.8118	0.05
15	(0.5481,0.5550)	0.5526	0.19	(4.1956,4.2123)	4.2061	0.05
	(0.5462,0.5522)	0.5526	0.61	(4.1962,4.2112)	4.2061	0.06
16	(0.8470,0.8496)	0.8493	0.12	(4.7762,4.7823)	4.7804	0.02
	(0.8487,0.8510)	0.8493	0.06	(4.7788,4.7833)	4.7804	0.01
17	(0.8561,0.8626)	0.8594	0.01	(4.7876,4.7981)	4.7931	0.01
	(0.8567,0.8614)	0.8594	0.04	(4.7882,4.7982)	4.7931	0.00
18	(0.8684,0.8727)	0.8714	0.09	(4.8062,4.8150)	4.8113	0.01
	(0.8704,0.8752)	0.8714	0.16	(4.8103,4.8195)	4.8113	0.08
19	(0.8551,0.8594)	0.8555	0.20	(4.7859,4.7923)	4.7851	0.08
	(0.8526,0.8606)	0.8555	0.13	(4.7798,4.7945)	4.7851	0.04
20	(0.6532,0.6811)	0.6608	0.95	(6.2607,6.3417)	6.2806	0.33
	(0.6480,0.6813)	0.6608	0.58	(6.2491,6.3370)	6.2806	0.20
21	(0.7250,0.7325)	0.7305	0.25	(4.5786,4.5933)	4.5884	0.05
	(0.8776,0.8871)	0.8813	0.12	(9.7689,9.7944)	9.7783	0.03
22	(0.7985,0.8068)	0.8019	0.09	(1.7560,1.7676)	1.7607	0.06
	(0.7977,0.8020)	0.7994	0.05	(7.6077,7.6190)	7.6096	0.05
23	(0.8511,0.8587)	0.8561	0.14	(4.7765,4.7885)	4.7849	0.05
	(0.6898,0.6971)	0.6933	0.02	(6.4198,6.4366)	6.4237	0.07
24	(0.8703,0.8741)	0.8711	0.03	(4.8091,4.8166)	4.8109	0.00
	(0.8709,0.8756)	0.8711	0.08	(4.8098,4.8198)	4.8109	0.01
	(0.8676,0.8718)	0.8711	0.25	(4.8051,4.8128)	4.8109	0.08
25	(0.5066,0.5131)	0.5109	0.21	(4.2460,4.2587)	4.2558	0.06
	(0.5068,0.5144)	0.5109	0.06	(4.2466,4.2613)	4.2558	0.04
	(0.5041,0.5130)	0.5109	0.46	(4.2436,4.2618)	4.2558	0.07
26	(0.6456,0.6538)	0.6525	0.43	(1.5538,1.5658)	1.5638	0.26
	(0.5754,0.5821)	0.5790	0.04	(4.3828,4.3952)	4.3883	0.02
	(0.7056,0.7089)	0.7070	0.04	(6.5989,6.6031)	6.6016	0.01
27	(0.9577,0.9617)	0.9599	0.02	(6.9352,6.9433)	6.9400	0.01
	(0.7820,0.7903)	0.7859	0.03	(6.5683,6.5911)	6.5778	0.03
	(0.4492,0.4542)	0.4510	0.15	(5.8151,5.8303)	5.8196	0.05
28	(0.1745,0.1807)	0.1766	0.59	(5.0709,5.1143)	5.0859	0.13
	(0.8530,0.8624)	0.8575	0.03	(6.7166,6.7389)	6.7280	0.00
	(0.9845,0.9862)	0.9848	0.05	(6.9731,6.9760)	6.9738	0.01
29	(0.9430,0.9450	0.9443	0.04	(2.9308,2.9337)	2.9326	0.01
	(0.9649,0.9671	0.9670	0.10	(2.9595,2.9624)	2.9622	0.04
	(0.9674,0.9686)	0.9686	0.07	(2.9627,2.9644)	2.9644	0.03
30	(0.9251,0.9281)	0.9268	0.02	(4.9046,4.9094)	4.9074	0.01
	(0.9247,0.9274)	0.9268	0.08	(4.9039,4.9082)	4.9074	0.03
	(0.9249,0.9276)	0.9268	0.06	(4.9049,4.9087)	4.9074	0.01
	(0.9250,0.9273)	0.9268	0.07	(4.9047,4.9083)	4.9074	0.02