CLOSED SUBMANIFOLDS WITH CONSTANT ν-TH MEAN CURVATURE RELATED WITH A VECTOR FIELD IN A RIEMANNIAN MANIFOLD

By

Yoshie KATSURADA

Introduction. H. Liebmann (1900) [1][1], proved the following theorem: The only ovaloids with constant mean curvature H in an Euclidean space E^3 are the spheres.

Extension of this theorem to a convex hypersurface in an n-dimensional Euclidean space E^n has been given by W. Süss (1929) [2], (cf. also [3], p. 118, and [4]). Then H. Hope (1951) [5], and A. D. Alexandrov (1958) [6], have shown the results that the convexity is not necessary for the validity of the Liebmann–Süss theorem.

Recently the analogous problem for closed hypersurfaces in an n-dimensional Riemannian manifold R^n has been discussed by the present author [8], [9], [10], A. D. Alexandrov [7], K. Yano [13], T. Ôtsuki [15], M. Tani [16], and K. Nomizu [17], [18]. And also for a submanifold of codimension 2 in an odd dimensional sphere, M. Okumura has treated the analogue [19].

In the previous papers [11], [12], which are common works by T. Nagai, H. Kôjyo and the present author, we have given a certain extension of this problem to an m-dimensional closed submanifold V^m $(1\leqq m\leqq n-1)$ in the n-dimensional Riemannian manifold R^n admitting a vector field ξ. However we have given there a restriction such that at each point on V^m, the vector ξ lies in the vector space spanned by the tangent space of V^m and the Euler–Schouten vector $\underset{E}{n}$.

The purpose of this paper is to give more general results except this restriction.

§ 1. Some integral formulas for a submanifold. We suppose an n-dimensional Riemannian manifold R^n $(n\geqq 3)$ of class C^r $(r\geqq 3)$ wich admits an one-parameter continuous group G of transformations generated by an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x)\delta\tau \tag{1.1}$$

1) Numbers in brackets refer to the references at the end of the paper.

(where x^i are local coordinates in R^n and ξ^i are the components of a contravariant vector ξ). If ξ is a Killing vector, a homothetic Killing vector, a conformal Killing vector, etc. ([14], p. 32), then the group G is called isometric, homothetic, conformal, etc.

In R^n, we consider a domain M. If the domain M is simply covered by the orbits of the transformations generated by ξ, and ξ is everywhere of class C^r and $\neq 0$ in M; then we call M a regular domain with respect to the vector field ξ.

Let us denote by V^m an m-dimensional closed orientable submanifold of class C^3 imbedded in a regular domain M with respect to the vector field ξ, locally given by

$$x^i = x^i(u^\alpha) \qquad \begin{array}{l} i = 1, \cdots, n \\ \alpha = 1, \cdots, m\,, \end{array} \tag{1.2}$$

where u^α are local coordinates of V^m. Throughout the present paper Latin indices run from 1 to n and Greek indices from 1 to m. We assume that at any point on V^m the vector ξ is not on its tangent space.

We shall indicate by $\underset{p}{n}{}^i$ $(p=m+1, \cdots, n)$ the contravariant unit vectors normal V^m and suppose that they are mutually orthogonal. Let $\underset{\xi}{n}$ be in the vector space spanned by $m+1$ independent vectors $\dfrac{\partial x^i}{\partial u^\alpha}$ $(\alpha=1, \cdots, m)$ and ξ and be the unit vector normal V^m. Then, we may consider $\underset{\xi}{n}$ as one of the unit normal vectors of V^m, that is, $\underset{m+1}{n}{}^i = \underset{\xi}{n}{}^i$.

Let us consider a differential form of $m-1$ degree at a point of V^m, defined by

$$\begin{aligned} ((\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \underbrace{dx, \cdots, dx}_{m-1})) &\overset{\text{def.}}{=} \sqrt{g}\, (\underset{\xi}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx) \\ &= \sqrt{g} \left((\underset{\xi}{n}, \cdots, \underset{n}{n}, \xi, \frac{\partial x}{\partial u^{\alpha_1}}, \cdots, \frac{\partial x}{\partial u^{\alpha_{m-1}}} \right) du^{\alpha_1} \wedge \cdots \wedge du^{\alpha_{m-1}}, \end{aligned} \tag{1.3}$$

where the symbol ( ) means a determinant of order n whose colums are the components of respective vectors, dx is a displacement along V^m, g is the determinant of the metric tensor g_{ij} of R^n. Then, the exterior differential of the differential form (1.3) divided by $m\,!$ becomes as follows

$$\begin{aligned} \frac{1}{m\,!} d\, ((\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) = \frac{1}{m\,!} \Big\{ &((\delta \underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) \\ &+ ((\underset{\xi}{n}, \delta \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) + \cdots \\ &+ ((\underset{\xi}{n}, \cdots, \underset{n-1}{n}, \delta \underset{n}{n}, \xi, dx, \cdots, dx)) + ((\underset{\xi}{n}, \cdots, \underset{n}{n}, \delta\xi, dx, \cdots, dx)) \Big\} \end{aligned} \tag{1.4}$$

where δv means $v_{;\alpha}\, du^{\alpha}$ and the symbol " ; " the operation of D-symbol due to van der Waerden–Bortolotti ([20] p. 254).

Let C^i_j be $\sum_{p=m+1}^{n} \underset{p}{n}^i \underset{p}{n}_j$ ($\underset{m+1}{n} = \underset{\xi}{n}$) and $\underset{\lambda}{i}$ ($\lambda = 1, \cdots, m$) mutually orthogonal unit tangent vectors of V^m. Then we have

$$\underset{p}{n}^i_{;\alpha} = C^i_{j;k} \underset{p}{n}^j \frac{\partial x^k}{\partial u^\alpha} = -\sum_{\lambda=1}^{m} \left(\underset{\lambda}{i}_{j;k} \underset{p}{n}^j \frac{\partial x^k}{\partial u^\alpha} \right) \underset{\lambda}{i}^i .$$

Therefore we may put

$$\underset{p}{n}^i_{;\alpha} = \underset{p}{\gamma}^{\gamma}_{\alpha} \frac{\partial x^i}{\partial u^\gamma} .$$

Since we have

$$g_{ij} \left(\frac{\partial x^i}{\partial u^\beta} \right)_{;\alpha} \underset{p}{n}^j = -g_{ij} \frac{\partial x^i}{\partial u^\beta} \underset{p}{n}^j_{\alpha} ,$$

we obtain

$$\underset{p}{n}^i_{;\alpha} = -\underset{p}{b}^{\gamma}_{\alpha} \frac{\partial x^i}{\partial u^\gamma} \qquad (p = \xi,\ m+2, \cdots, n) \tag{1.5}$$

where $\underset{p}{b}^{\gamma}_{\alpha}$ means $g^{\gamma\beta} \underset{p}{b}_{\alpha\beta}$ and $\underset{p}{b}_{\alpha\beta} \equiv \left(\frac{\partial x^i}{\partial u^\alpha} \right)_{;\beta} \underset{p}{n}_i$, and $g^{\gamma\beta}$ is the contravariant metric tensor of V^m.

From (1. 5) the first term of the right-hand member of (1. 4) becomes

$$\frac{1}{m!} ((\delta \underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) = (-1)^{(n-m)(n-1)} \underset{\xi}{H}_1 \underset{\xi}{n}_i \xi^i dA , \tag{1.6}$$

where dA is the area element of V^m and $\underset{\xi}{H}_1$ means the first mean curvature of V^m with respect to the normal direction $\underset{\xi}{n}^i$. Similarly, for every integer p satisfying $m+2 \leqq p \leqq n$ we have

$$\begin{aligned} \frac{1}{m!} ((\underset{\xi}{n}, \cdots, \delta \underset{p}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) &= (-1)^{(n-m)(n-1)} \underset{p}{H}_1 \underset{p}{n}_i \xi^i dA . \\ &= 0 \end{aligned} \tag{1.7}$$

because ξ lies in the vector space spanned by $m+1$ independent vectors $\frac{\partial x^i}{\partial u^\alpha}$ ($\alpha = 1, \cdots, m$) and $\underset{\xi}{n}$.

On the other hand the last term of the right-hand member of 1. 4 becomes

$$\begin{aligned} &\frac{1}{m!} ((\underset{\xi}{n}, \cdots, \underset{n}{n}, \delta\xi, dx, \cdots, dx)) \\ &= (-1)^{(n-m)(n-1)} \frac{1}{2m} (\underset{\xi}{L}\, g_{ij}) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta} dA , \end{aligned} \tag{1.8}$$

where $\underset{\xi}{L}\, g_{ij}$ is the Lie derivative of g_{ij} with respect to ξ ([14], p. 5).

From (1. 6), (1. 7) and (1. 8), (1. 4) is rewritten as follows

$$\text{(1.9)}\quad \begin{aligned} \frac{1}{m!} d((\underset{\xi}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) &= (-1)^{(n-m)(n-1)} \Big\{ H_1 \underset{\xi}{n}_i \underset{\xi}{\xi}^i dA \\ &\quad + \frac{1}{2m} (\underset{\xi}{L}\, g_{ij}) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} g^{\alpha\beta} dA \Big\} . \end{aligned}$$

Integrating both members of (1. 9) over the whole submanifold and applying Stokes' theorem, we obtain

$$\begin{aligned} &\frac{1}{m!} \int_{\partial V^m} ((\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, dx, \cdots, dx)) \\ &\qquad = (-1)^{(n-m)(n-1)} \left\{ \int_{V^m} H_1 \underset{\xi}{n}_i \underset{\xi}{\xi}^i dA + \frac{1}{2m} \int_{V^m} g^{*ij} \underset{\xi}{L}\, g_{ij} dA \right\}, \end{aligned}$$

where ∂V^m means the boundary of V^m and g^{*ij} is $\dfrac{\partial x^i}{\partial u^\alpha} \dfrac{\partial x^j}{\partial u^\beta} g^{\alpha\beta}$. Making use of the fact that V^m is closed, we have

$$\text{(I}''\text{)}\quad \int_{V^m} H_1 \underset{\xi}{n}_i \underset{\xi}{\xi}^i dA + \frac{1}{2m} \int_{V^m} g^{*ij} \underset{\xi}{L}\, g_{ij} dA = 0 .$$

If the manifold R^n assumes of constant Riemann curvature which includes an Euclidean space, then we consider the following differential form of $m-1$ degree

$$\text{(1.10)}\quad \begin{aligned} &((\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \underbrace{\delta \underset{\xi}{n}, \cdots, \delta \underset{\xi}{n}}_{\nu}, \underbrace{dx, \cdots, dx}_{m-\nu-1})) \\ &\overset{\text{def.}}{=} \sqrt{g}\, (\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \delta \underset{\xi}{n}, \cdots, \delta \underset{\xi}{n}, dx, \cdots, dx) \end{aligned}$$

for a fixed integer ν satisfying $m-1 \geqq \nu \geqq 1$.

As well-known, a submanifold V^m in R^n has the following property:

$$\underset{\xi}{b}_{\alpha\beta;\gamma} - \underset{\xi}{b}_{\alpha\gamma;\beta} = -R_{ijkl} \underset{\xi}{n}^i \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} \frac{\partial x^l}{\partial u^\gamma} \qquad \text{([20], p. 226)},$$

where R_{ijkl} is the curvature tensor of R^n. Since R^n is of constant Riemann curvature, we have

$$\text{(1.11)}\quad \underset{\xi}{n}^i{}_{,\alpha;\beta} - \underset{\xi}{n}^i{}_{;\beta;\alpha} = 0 .$$

Consequently differentiating exteriorly the differential form (1. 10), we have

$$
(1.12)\quad
\begin{aligned}
d((\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \underset{\xi}{\delta n}, \cdots, \underset{\xi}{\delta n}, dx, \cdots, dx))
&= ((\underset{\xi}{\delta n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \underset{\xi}{\delta n}, \cdots, \underset{\xi}{\delta n}, dx, \cdots, dx)) \\
&\quad + ((\underset{\xi}{n}, \delta \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \underset{\xi}{\delta n}, \cdots, \underset{\xi}{\delta n}, dx, \cdots, dx)) \\
&\quad + \cdots + ((\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{\delta n}, \xi, \underset{\xi}{\delta n}, \cdots, \underset{\xi}{\delta n}, dx, \cdots, dx)) \\
&\quad + ((\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{u}{n}, \delta\xi, \underset{\xi}{\delta n}, \cdots, \underset{\xi}{\delta n}, dx, \cdots, dx)),
\end{aligned}
$$

because $((\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \delta\underset{\xi}{\delta n}, \underset{\xi}{\delta n}, \cdots, \underset{\xi}{\delta n}, dx, \cdots, dx))=0$ from (1.11).

On substituting $\underset{\xi}{n}^{i}{}_{;\alpha} = -\underset{\xi}{b}{}^{\beta}_{\alpha} \dfrac{\partial x^{i}}{\partial u^{\beta}}$ into the first term of the right-hand member of (1.12), we get

$$
(1.13)\quad
\begin{aligned}
&((\underset{\xi}{\delta n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \underset{\xi}{\delta n}, \cdots, \underset{\xi}{\delta n}, dx, \cdots, dx)) \\
&\qquad = m!(-1)^{(n-m)(n-1)-\nu} \underset{\xi}{H}_{\nu+1} \underset{\xi}{n}_{i} \xi^{i} dA ,
\end{aligned}
$$

where $\underset{\xi}{H}_{\nu+1}$ denotes the $\nu+1$-th mean curvature of V^m with respect to the normal direction $\underset{\xi}{n}^{i}$ and if we indicate by $\underset{\xi}{k}_1, \underset{\xi}{k}_2, \cdots, \underset{\xi}{k}_m$ the principal curvatures of V^m for the normal vector $\underset{\xi}{n}$, $\underset{\xi}{H}_{\nu+1}$ is defined to be the $\nu+1$-th elementary symmetric function of $\underset{\xi}{k}_\alpha$ $(\alpha=1, \cdots, m)$ divided by the number of terms, that is,

$$
\binom{m}{\nu+1} \underset{\xi}{H}_{\nu+1} = \sum_{\alpha_1<\alpha_2<\cdots<\alpha_{\nu+1}} \underset{\xi}{k}_{\alpha_1} \underset{\xi}{k}_{\alpha_2} \cdots \underset{\xi}{k}_{\alpha_{\nu+1}} .
$$

Also, by virtue of (1.5) we can see that the vectors

$$
\underset{\xi}{n} \times \delta \underset{m+2}{n} \times \underset{m+3}{n} \times \cdots \times \underset{n}{n} \times \underbrace{\underset{\xi}{\delta n} \times \cdots \times \underset{\xi}{\delta n}}_{\nu} \times \underbrace{dx \times \cdots \times dx}_{m-\nu-1} ,
$$

$$
\underset{\xi}{n} \times \underset{m+2}{n} \times \delta \underset{m+3}{n} \times \cdots \times \underset{n}{n} \times \underset{\xi}{\delta n} \times \cdots \times \underset{\xi}{\delta n} \times dx \times \cdots \times dx ,
$$

$$
\cdots
$$

and

$$
\underset{\xi}{n} \times \underset{m+2}{n} \times \cdots \times \underset{n-1}{n} \times \underset{n}{\delta n} \times \underset{\xi}{\delta n} \times \cdots \times \underset{\xi}{\delta n} \times dx \times \cdots \times dx
$$

have the same direction to the covariant vectors $\underset{m+2}{n}, \underset{m+3}{n}, \cdots$ and $\underset{n}{n}$ respectively. Thus we obtain

$$
((\underset{\xi}{n}, \delta \underset{m+2}{n}, \underset{m+3}{n}, \cdots, \underset{n}{n}, \xi, \underset{\xi}{\delta n}, \cdots, \underset{\xi}{\delta n}, dx, \cdots, dx)) = 0 ,
$$

$$(1.14)\qquad \begin{aligned} &((\underset{\xi}{n}, \underset{m+2}{n}, \delta \underset{m+3}{n}, \cdots, \underset{n}{n}, \xi, \delta \underset{\xi}{n}, \cdots, \delta \underset{\xi}{n}, dx, \cdots, dx)) = 0\,, \\ &\qquad\qquad\cdots, \\ &((\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n-1}{n}, \delta \underset{n}{n}, \xi, \delta \underset{\xi}{n}, \cdots, \delta \underset{\xi}{n}, dx, \cdots, dx)) = 0\,, \end{aligned}$$

because ξ lies in the vector space spanned by $m+1$ independent vectors $\frac{\partial x^i}{\partial u^\alpha}$ $(\alpha=1, \cdots, m)$ and $\underset{\xi}{n}$.

From that the vector $\underset{\xi}{n} \times \underset{m+2}{n} \times \cdots \times \underset{n}{n} \times \delta \underset{\xi}{n} \times \cdots \times \delta \underset{\xi}{n} \times dx \times \cdots \times dx$ is orthogonal to the normal vectors $\underset{\xi}{n}, \underset{m+2}{n}, \cdots$ and $\underset{n}{n}$, and $\delta \underset{\xi}{n}^i = -\underset{\xi}{b}{}_\alpha^\beta \frac{\partial x^i}{\partial u^\beta} du^\alpha$, the last term of the right-hand member of (1.12) becomes as follows

$$(1.15)\qquad \begin{aligned} &((\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \delta\xi, \delta \underset{\xi}{n}, \cdots, \delta \underset{\xi}{n}, dx, \cdots, dx)) \\ &\qquad = m!\,(-1)^{(n-m)(n-1)-\nu} \frac{1}{2m} \underset{\xi}{H}{}_\nu^{\alpha\beta} \underset{\xi}{L}\, g_{\alpha\beta} dA\,, \end{aligned}$$

where $\underset{\xi}{L}\, g_{\alpha\beta} = (\underset{\xi}{L}\, g_{ij}) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}$ and

$$\underset{\xi}{H}{}_\nu^{\alpha\beta} = \frac{1}{(m-1)!} \varepsilon^{\alpha\alpha_1\cdots\alpha_{m-1}} \varepsilon^{\beta\beta_1\cdots\beta_{m-1}} \underset{\xi}{b}_{\alpha_1\beta_1} \cdots \underset{\xi}{b}_{\alpha_\nu\beta_\nu} g_{\alpha_{\nu+1}\beta_{\nu+1}} \cdots g_{\alpha_{m-1}\beta_{m-1}}\,,$$

and $\varepsilon^{\alpha\alpha_1\cdots\alpha_{m-1}}$ denotes the ε-symbol of the submanifold V^m. Accordingly we have

$$(1.16)\qquad \begin{aligned} &\frac{1}{m!} d\,((\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \delta \underset{\xi}{n}, \cdots, \delta \underset{\xi}{n}, dx, \cdots, dx)) \\ &\qquad = (-1)^{(n-m)(n-1)-\nu} \left\{ \underset{\xi}{H}_{\nu+1} \underset{\xi}{n}_i \xi^i dA + \frac{1}{2m} \underset{\xi}{H}{}_\nu^{\alpha\beta} \underset{\xi}{L}\, g_{\alpha\beta} dA \right\}. \end{aligned}$$

Integrating both members of (1.16) over the whole submanifold V^m and applying Stokes' theorem, we have

$$\begin{aligned} &\frac{1}{m!} \int_{\partial V^m} ((\underset{\xi}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi, \delta \underset{\xi}{n}, \cdots, \delta \underset{\xi}{n}, dx, \cdots, dx)) \\ &\qquad = (-1)^{(n-m)(n-1)-\nu} \left\{ \int_{V^m} \underset{\xi}{H}_{\nu+1} \underset{\xi}{n}_i \xi^i dA + \frac{1}{2m} \int_{V^m} \underset{\xi}{H}{}_\nu^{\alpha\beta} \underset{\xi}{L}\, g_{\alpha\beta} dA \right\}. \end{aligned}$$

Thus, for a closed orientable submanifold V^m we obtain

$$(\mathrm{II}'')\qquad \int_{V^m} \underset{\xi}{H}_{\nu+1} \underset{\xi}{n}_i \xi^i dA + \frac{1}{2m} \int_{V^m} \underset{\xi}{H}{}_\nu^{\alpha\beta} \underset{\xi}{L}\, g_{\alpha\beta} dA = 0\,.$$

If $m=n-1$, that is, V^m is the hypersurface in R^n, the formulas (I$''$) and

(II″) are coincide with the formulas (I) and (II) given in the previous paper [8]. Especially if the vector $\underset{\xi}{n}$ is coincide with the Euler-Schouten unit vector $\underset{E}{n}$ at each point on V^m, then the formulas (I″) and (II″) become the formulas (I′) and (II′) given in the previous paper [12].

§ 2. The integral formulas concerning some special transformations. In this section, we shall discuss the formulas (I″) and (II″) for a special infinitesimal transformation. Let the group G of transformations be conformal, that is, ξ^i satisfies an equation: $\underset{\xi}{L}g_{ij}\equiv\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ ([14], p. 32). Then we obtain

$$g^{*ij}\underset{\xi}{L}g_{ij}=2m\phi\,,\qquad \underset{\xi}{H}{}_\nu^{\alpha\beta}\underset{\xi}{L}g_{\alpha\beta}=2m\phi\underset{\xi}{H}{}_\nu\,.$$

Therefore (I″) and (II″) are rewritten in the following forms:

$$\text{(I″)}_c\qquad \int_{V^m}\underset{\xi}{H}{}_1\underset{\xi}{n}{}_i\xi^i dA+\int_{V^m}\phi dA=0\,,$$

$$\text{(II″)}_c\qquad \int_{V^m}\underset{\xi}{H}{}_{\nu+1}\underset{\xi}{n}{}_i\xi^i dA+\int_{V^m}\phi\underset{\xi}{H}{}_\nu dA=0\qquad(1\leqq\nu\leqq m-1)$$

and we can see

$$\text{(I″)}_h\qquad \int_{V^m}\underset{\xi}{H}{}_1\underset{\xi}{n}{}_i\xi^i dA+c\int_{V^m}dA=0\,,$$

$$\text{(II″)}_h\qquad \int_{V^m}\underset{\xi}{H}{}_{\nu+1}\underset{\xi}{n}{}_i\xi^i dA+c\int_{V^m}\underset{\xi}{H}{}_\nu dA=0\qquad(1\leqq\nu\leqq m-1)^{2)}$$

in case of ϕ = constant ($\equiv c$) (G being homothetic), and

$$\text{(I″)}_i\qquad \int_{V^m}\underset{\xi}{H}{}_1\underset{\xi}{n}{}_i\xi^i dA=0\,,$$

$$\text{(II″)}_i\qquad \int_{V^m}\underset{\xi}{H}{}_{\nu+1}\underset{\xi}{n}{}_i\xi^i dA=0\qquad(1\leqq\nu\leqq m-1)$$

in case of $\phi=0$ (G being isometric).

Especially if our manifold R^n is an Euclidean space E^n and if ξ is the homothetic Killing vector field on E^n with components $\xi^i=x^i$, x^i being rectangular coordinates with a point in the interior of V^m as origin in the space E^n, then the orbits of the transformations generated by ξ are the lines through

2) In this case, R^n becomes an Euclidean space, because if R^n with constant Riemann curvature admits an one-parameter group G of homothetic transformations, then either R^n is E^n or the group G is isometric.

the origin and we have

$$\underset{\xi}{L} g_{ij} = 2g_{ij} .$$

Consequently, from (I$''$)$_h$ and (II$''$)$_h$ we obtain

$$\text{(I*)} \qquad \int_{V^n} \underset{\xi}{H_1} p dA + \int_{V^m} dA = 0 ,$$

$$\text{(II*)} \qquad \int_{V^m} \underset{\xi}{H_{\nu+1}} p dA + \int_{V^m} \underset{\xi}{H_\nu} dA = 0 ,$$

where $p = \underset{\xi}{n_i} x^i$. This means that the formulas (I*) and (II*) are generalization of those formulas given by C. C. Hsiung [4] for a closed orientable hypersurface in an n-dimensional Euclidean space E^n.

§ 3. Some properties of a closed orientable submanifold related with a vector field. In this section we suppose again that the group G is conformal. Then we shall prove the following four theorems for a closed orientable submanifold V^m in a Riemannian manifold R^n with constant Riemann curvature.

Theorem 3.1. *If in R^n, there exists such a group G of conformal transformations as ρ is positive (or negative) at each point of V^m and if $\underset{\xi}{H_1}$ is constant, then every point of V^m is umbilic with respect to the normal vector $\underset{\xi}{n}$, where ρ denotes $\underset{\xi}{n_i} \xi^i$.*

Proof. Multiplying the formula (I$''$)$_c$ by $\underset{\xi}{H_1} = \text{const.}$, we have

$$\int_{V^m} \underset{\xi}{H_1^2} \rho dA + \int_{V^m} \phi \underset{\xi}{H_1} dA = 0 .$$

On the other hand, from (II$''$)$_c$ we have

$$\int_{V^m} \underset{\xi}{H_2} \rho dA + \int_{V^m} \phi \underset{\xi}{H_1} dA = 0 .$$

Consequently it follows that

$$\int_{V^m} (\underset{\xi}{H_1^2} - \underset{\xi}{H_2}) \rho dA = 0 .$$

From our assumption about ρ, this holds if and only if $\underset{\xi}{H_1^2} - \underset{\xi}{H_2} = 0$, since

$$\underset{\xi}{H_1^2} - \underset{\xi}{H_2} = \frac{1}{m^2(m-1)} \sum_{\alpha<\beta} (\underset{\xi}{k_\alpha} - \underset{\xi}{k_\beta})^2 \geqq 0 .$$

Therefore at each point of V^m we obtain

$$\underset{\xi}{k_1} = \underset{\xi}{k_2} = \cdots = \underset{\xi}{k_m} .$$

Accordingly every point of V^m is umbilic with respect to $\underset{\xi}{n}$.

Theorem 3.2. *If in R^n, there exists such a group G of conformal transformations as ρ is positive (or negative) at each point of V^m, and if the principal curvatures $\underset{\xi}{k_1}, \underset{\xi}{k_2}, \cdots, \underset{\xi}{k_m}$ at each point of V^m are positive and $\underset{\xi}{H_\nu}$ is constant for any ν $(1<\nu\leqq m-1)$, then every point of V^m is umbilic with respect to the normal vector $\underset{\xi}{n}$.*

Proof. Multiplying the formula $(\mathrm{I}'')_c$ by $\underset{\xi}{H_\nu}=$const., we obtain

$$(3.1) \qquad \int_{V^m} \underset{\xi}{H_1}\underset{\xi}{H_\nu}\rho dA + \int_{V^m} \phi \underset{\xi}{H_\nu} dA = 0 .$$

By vatue of $(\mathrm{II}'')_c$ and (3.1), we have

$$\int_{V^m} (\underset{\xi}{H_1}\underset{\xi}{H_\nu} - \underset{\xi}{H_{\nu+1}})\rho dA = 0 .$$

From our assumption, this holds if and only if $\underset{\xi}{H_1}\underset{\xi}{H_\nu} - \underset{\xi}{H_{\nu+1}}=0$, since

$$\underset{\xi}{H_1}\underset{\xi}{H_\nu} - \underset{\xi}{H_{\nu+1}} = \frac{\nu!(m-\nu-1)!}{mm!} \sum \underset{\xi}{k_{\alpha_1}} \cdots \underset{\xi}{k_{\alpha_{\nu-1}}} (\underset{\xi}{k_{\alpha_\nu}} - \underset{\xi}{k_{\alpha_{\nu+1}}})^2 \geqq 0 .$$

Then at each point of V^m, we obtain

$$\underset{\xi}{k_1} = \underset{\xi}{k_2} = \cdots = \underset{\xi}{k_m} .$$

Accordingly every point of V^m is umbilic with respect to $\underset{\xi}{n}$.

Theorem 3.3. *If in R^n, there exists such a group G of conformal transformations as ρ is positive (or negative) at each point of V^m, for which $\underset{\xi}{H_1}\rho+\phi\geqq 0$ (or $\leqq 0$) at all points of V^m, then every point of V^m is umbilic with respect to $\underset{\xi}{n}$.*

Proof. If we express the formula $(\mathrm{I}'')_c$ as follows

$$\int_{V^m} (\underset{\xi}{H_1}\rho + \phi) dA = 0 ,$$

then from our assumption we have the relation:

$$(3.2) \qquad \phi = -\underset{\xi}{H_1}\rho .$$

Substituting (3.2) into $(\mathrm{II}'')_c$ for $\nu=1$, we have

$$\int_{V^m} (\underset{\xi}{H_1^2} - \underset{\xi}{H_2})\rho dA = 0 .$$

Thus, we can see the conclusion.

Theorem 3.4. *If $\underset{\xi}{H_1}$ is positive (or negative) at all points of V^m and if R^n admits such a group G of conformal transformations as ϕ is positive (or negative), for which either $\rho \geqq \frac{-\phi}{\underset{\xi}{H_1}}$ or $\rho \leqq \frac{-\phi}{\underset{\xi}{H_1}}$ at all points of V^m, then every point of V^m is umbilic with respect to $\underset{\xi}{n}$.*

Proof. The formula $(\mathrm{I}'')_c$ is rewritten as follows

$$\int_{V^m} \underset{\xi}{H_1}\left(\rho + \frac{\phi}{\underset{\xi}{H_1}}\right) dA = 0 .$$

By virtue of our assumptions $\underset{\xi}{H_1} > 0$ (or < 0) and $\rho + \frac{\phi}{\underset{\xi}{H_1}} \geqq 0$ (or $\leqq 0$) at all points of V^m, we have the following relation

$$\rho = -\frac{\phi}{\underset{\xi}{H_1}} . \tag{3.3}$$

Substituting (3.3) into $(\mathrm{II}'')_c$ for $\nu = 1$, we obtain

$$\int_{V^m} \frac{\phi}{\underset{\xi}{H_1}} (\underset{\xi}{H_1^2} - \underset{\xi}{H_2}) dA = 0 ,$$

which holds if and only if $\underset{\xi}{H_1^2} - \underset{\xi}{H_2} = 0$. Thus we obtain the conclusion.

Remark I. If V^m is the hypersurface in R^n, these four theorems are coincide with the theorems given in the previous paper [8]. Especially if the vector $\underset{\xi}{n}$ is coincide with the Euler-Schouten unit vector $\underset{E}{n}$ at each point of V^m, then these four theorems become those theorems given in the previous paper [12].

Remark II. In all these sections we have treated the normal unit vector $\underset{\xi}{n}$ with respect to the vector field ξ and the mean curvature $\underset{\xi}{H_\nu}$. These are the notions due to R. E. Stong [21].

References

[1] H. LIEBMANN: Ueber die Verbiegung der geschlossenen Flächen positive Krümmung, Math. Ann. 53 (1900), 91–112.

[2] W. SÜSS: Zur relativen Differentialgeometrie V., Tôhoku Math. J., 30 (1929), 202–209.

[3] T. BONNESEN und W. FENCHEL: Theorie der Konvexen Körper, (Springer, Berlin 1934).

[4] C. C. HSIUNG: Some integral formulas for closed hypersurfaces, Math. Scand. 2 (1954), 286–294.

[5] H. HOPF: Ueber Flächen mit einer Relation zwischen den Hauptkrümmungen, Math. Nachr. 4 (1951), 232–249.

[6] A. D. ALEXANDROV: Uniqueness theorems for surfaces in the large, V. Vestnik Leningrad University 13 (1958), 5–8 (Russian, with English summary).

[7] A. D. ALEXANDROV: A characteristic property of spheres, Ann. di Mat. p. appl., 58 (1962), 303–315.

[8] Y. KATSURADA: Generalized Minkowski formulas for closed hypersurfaces in Riemann space, Ann. di Mat. p. appl., 57 (1962), 283–293).

[9] Y. KATSURADA: On a certain property of closed hypersurfaces in an Einstein space, Comment. Math. Helv., 38 (1964), 165–171.

[10] Y. KATSURADA: On the isoperimetric problem in a Riemann space, Comment. Math. Helv., 41 (1966), 18–29.

[11] Y. KATSURADA and T. NAGAI: On some properties of a submanifold with constant mean curvature in a Riemann space, Jour. Fac. Sci. Hokkaido Univ., 20 (1968), 79–89.

[12] Y. KATSURADA and H. KÔJYO: Some integral formulas for closed submanifolds in a Riemann space, Jour. Fac. Sci. Hokkaido Univ., 20 (1968), 90–100.

[13] K. YANO: Closed hypersurfaces with constant mean curvature in a Riemannian manifold, J. Math. Soc. Japan, 17 (1965), 333–340.

[14] K. YANO: The theory of Lie derivatives and its applications, North-Holland, Amsterdam, 1957.

[15] T. ÔTSUKI: Integral formulas for hypersurfaces in a Riemannian manifold and their applications, Tohoku Math J. 17 (1965), 335–348.

[16] M. TANI: On hypersurfaces with constant k-th mean curvature, Kodai Math. Sem. Rep., 20 (1968), 94–102.

[17] K. NOMIZU: I. Hypersurfaces with constant mean curvature in S^n, to appear.

[18] K. NOMIZU: II. Surfaces with constant mean curvature in S^3, to appear.

[19] M. OKUMURA: Compact orientable submanifold of codimension 2 in an odd dimensional sphere, Tôhoku Math. J., 20 (1968), 8–20.

[20] J. A. SCHOUTEN: Ricci-Calculus, (second edition) (Berlin 1954).

[21] R. E. STONG: Some Differential Geometric Properties of Submanifolds of Euclidean Spaces, Proc. Amer. Math. Soc., 12 (1961), 343–349).

Department of Mathematics,
Hokkaido University

(Received Nov. 26, 1968)