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CLOSED SYSTEMS OF FUNCTIONS AND PREDICATES

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In this paper we show that there is a one to one correspondence between systems of functions defined on a finite set A and systems of predicates defined on A. This result implies that a complete set of invariants for a universal algebra on A is given by predicates defined on A. Conversely functions on A provide a complete system of invariants for sets of predicates closed under conjunction, change of variable and application of the existential quantifier.

We begin in § 2 by giving a definition of closure for systems of functions and predicates. This is followed by a definition of commutivity of a function and a predicate which gives a correspondence between the two types of systems. In Theorems 1 and 2 of § 3 we show that the correspondence is a Galois connection. In Theorem 3 we consider sets of predicates closed under the existential quantifier and show that the corresponding systems are determined by functions defined for all values of the arguments. In Theorems 4 and 5 we include disjunction and then negation in the definition of closure of a set of predicates. We also require that equality be among the predicates. The corresponding systems consist of essentially first order functions and essentially first order permutations respectively. We conclude in § 4 with some comments on the infinite case and some general comments on these results.

2. Basic definitions.

Associated with any subset of A^{n+1}, the set of all sequences of length $n+1$ with elements in A, is the n-th order function $f(x_1, \cdots, x_n)$ which may be many valued and may not be defined on all of A^n. A system of functions $\mathscr{L}$ is defined to be closed if the following conditions are satisfied:

(i) $\mathscr{L}$ is closed under composition.

(ii) If $f(x_1, \cdots, x_n) \in \mathscr{L}$ is associated with the subset $P \subset A^{n+1}$ then any $g(x_1, \cdots, x_n)$ associated with $Q \subset P$ is in $\mathscr{L}$.

(iii) For any n, $\mathscr{L}$ contains all functions f defined on A^n such that $f(x_1, \cdots, x_n) = x_i$.

In defining closed systems of predicates the author has the following model in mind. We are given a sequence $A_1, A_2, A_3, \cdots$ of sets of predicates, each A_i containing all subsets of A^i. For each A_i a set of operators isomorphic to $\mathscr{S}_i$ the symmetric group is given which maps A_i onto A_i. These correspond to permutations of the variables of predicates in A_i. There is an operator R: $A_{i+1} \to A_i$ which takes $P((x_1, \cdots, x_{i+1})$ to $P(x_1, x_1, x_2, \cdots, x_i)$ and an operator E: $A_{i+1} \to A_i$ which takes $P(x_1, \cdots, x_{i+1})$ to $(\exists y)P(y, x_1, \cdots, x_i)$. Also there is an operator A: $A_i \to A_{i+1}$ which corresponds to the cartesian product with A or to the introduction of a dummy variable. Thus $(x_1, \cdots, x_{i+1}) \in AP$ if and only if $(x_2, \cdots, x_{i+1}) \in P$. A predicate in A_i will be said to have order i. A system $\mathscr{P}$ of predicates is defined to be closed if it satisfies the following conditions:

(i) If $P \in \mathscr{P}$ and $Q \in \mathscr{P}$ and P and Q have the same order then $P \cap Q \in \mathscr{P}$.

(ii) If $P \in \mathscr{P}$ then any predicate obtained from P by permuting the variables is in $\mathscr{P}$.

(iii) If $P \in \mathscr{P}$ then AP and RP are contained in $\mathscr{P}$.

(iv) $\mathscr{P}$ contains the first order predicate A.

Now we define commutivity of a function and a predicate. Let M be an $n \times m$ matrix with elements in A, then we write $M \subset P$ where P is an m-th order predicate if each row of M is a sequence contained in P. If N is an $m \times n$ matrix and f is an n-th order function then $f(N)$ is the $m \times 1$ column matrix obtained by letting f operate on each row of N. If f is not defined for some row of N we say that $f(N)$ is not defined. The predicate P commutes with the function f if for every $M \subset P$ the row matrix $f(M^T)^T$ when defined is a sequence contained in P. Here M^T is the transpose matrix of M. If $\mathscr{L}$ and $\mathscr{P}$ are systems of functions and predicates we write $\mathscr{L}^*$ and $\mathscr{P}^*$ for the systems of predicates and functions respectively which commute with $\mathscr{L}$ and $\mathscr{P}$.

3. Main results. It can be verified that $\mathscr{L}^*$ and $\mathscr{P}^*$ are closed systems. We will show that if $\mathscr{L}$ and $\mathscr{P}$ are closed systems then $\mathscr{L} = \mathscr{L}^{**}$ and $\mathscr{P} = \mathscr{P}^{**}$.

THEOREM 1. *If $\mathscr{L}$ is a closed system of functions then $\mathscr{L} = \mathscr{L}^{**}$.*

Since $\mathscr{L} \subset \mathscr{L}^{**}$ we need only show that for any function $g(x_1, \cdots, x_m)$ not in $\mathscr{L}$ there exists a predicate in $\mathscr{L}^*$ which does not commute with g. Assume that g is defined only on the sequences $s_1, s_2, \cdots, s_k$. We form the $k \times m$ matrix T with i-th row equal to s_i. For any function $f(x_1, \cdots, x_r)$ in $\mathscr{L}$ and any $k \times r$ matrix F with columns taken from T we form the column matrix $f(F)$. If $f(F)$ is not a column of T we adjoin it to T and get a $k \times (m + 1)$ matrix T_1. In this way we can adjoin columns to T until we finally reach a matrix T_0 with k rows such that for any function f in $\mathscr{L}$ and any matrix F with columns from T_0 the column matrix $f(F)$

will be in T_0 if it is defined. If $g(T)$ is a column of T_0 then g can be derived from functions in $\mathscr{L}$ so we can assume that $g(T)$ is not in T_0. From T_0 we form the k-th order predicate P_0 which contains all the rows of T_0^T. It is evident that P_0 is in $\mathscr{L}^*$ but does not commute with g. Thus $\mathscr{L} = \mathscr{L}^{**}$.

THEOREM 2. *If $\mathscr{P}$ is a closed system of predicates then $\mathscr{P} = \mathscr{P}^{**}$.*

Since $\mathscr{P} \subset \mathscr{P}^{**}$ we need only show that for any n-th order predicate Q not in $\mathscr{P}$ there exists a function in $\mathscr{P}^*$ which does not commute with Q. Let P be the intersection of all n-th order predicates of $\mathscr{P}$ which contain Q. Let $s_1, s_2, \cdots, s_k$ be all the $1 \times n$ matrices contained in Q and let N be the $k \times n$ matrix with i-th row s_i. Let t be any row matrix in P but not in Q. Then there exists a k-th order function f defined only on the rows of N^T such that $f(N^T) = t^T$. We wish to show that any predicate in $\mathscr{P}$ commutes with f. By way of contradiction suppose that the m-th order predicate $P_1 \in \mathscr{P}$ does not commute with f and that every predicate obtained from P_1 by identification of variables does commute with f. Then there exists a $j \times m$ matrix $N_1 \subset P_1$ such that $f(N_1^T) = t_1^T$ and t_1 is not contained in P_1.

Since every identification of variables in P leads to a predicate which commutes with f we must have that each pair r_i, $f(r_i) i = 1, \cdots, m$ where r_i is the i-th row of N_1^T and $f(r_i)$ is the corresponding element of t_1^T, is distinct from any other pair r_j, $f(r_j)$. Thus each pair is the same as a row element pair taken from N^T and t^T. We can find a $k \times n$ matrix $N_2 \subset A^{n-m}P_1$ and row matrix t_2 such that the last m rows of N_2^T and elements of t_2^T are equal to r_i, $f(r_i)$. Also the first n-m pairs can be chosen so that there is a one to one correspondence between pairs taken from N^T, t^T and pairs taken from N_2^T, t_2^T. By permuting the variables of $A^{n-m}P_1$ we can arrive at a predicate P_3 which contains N and does not contain t. Since P_3 is in $\mathscr{P}$ we get that P is not the least n-th order predicate which contains Q. Thus we have a contradiction and f must commute with every predicate of $\mathscr{P}$. Thus $\mathscr{P} = \mathscr{P}^{**}$.

Now we consider systems of predicates which are closed under the existential quantifier. Let $\mathscr{L}$ be a closed system of functions and assume that for any $f(x_1, \cdots, x_n) \in \mathscr{L}$ with restricted domain of definition, there exists a $g(x_1, \cdots, x_n) \in \mathscr{L}$ which is defined on all of A^n and equals f where f is defined. Then it can be verified that $\mathscr{L}^*$ is closed under the existential quantifier.

THEOREM 3. *If $\mathscr{P}$ is a closed system of predicates which is* *closed under the existential quantifier then every function in* $\mathscr{P}^*$ *can be extended to a function in* $\mathscr{P}^*$ *which is defined for all values of the arguments.*

We assume that the elements of A are the integers from 1 to n. Let $f(x_1, \cdots, x_m) \in \mathscr{P}^*$ be defined on the sequences $s_1, s_2, \cdots, s_k$ and let s be any other sequence in A^m. We define the n functions f_i such that $f_i(s_j) = f(s_j)$ and $f_i(s) = i$ for $i = 1, \cdots, n$ and show that for some i, f_i is in $\mathscr{P}^*$. By way of contradiction suppose that for each f_i there exists a $P_i \supset N_i$ where $P_i \in \mathscr{P}$ and N_i is a matrix such that $f_i(N_i^T)^T$ is not in P_i. We can assume that each N_i has s^T in the first column and every other column is an s_i^T, if N_i has more than one occurence of s^T then by identifying variables in P_i we can arrive at a new P_i which has only one occurence of s^T in the corresponding N_i. Also after permuting the variables of P_i we can assume that s^T occurs as the first column of N_i. Let

$$P_1(x, x_1, \cdots, x_p),\ P_2(x, y_1, \cdots, y_q), \cdots, P_n(x, z_1, \cdots, z_r)$$

be the predicates which satisfy these conditions, since $\mathscr{P}$ is closed the predicate $P(x, x_1, \cdots, x_p, y_1, \cdots, y_q, \cdots, z_1, \cdots, z_r)$ equivalent to the conjunction of the P_i is in $\mathscr{P}$. Also P contains a matrix N derived from the N_i with first column s^T and each remaining column equal to an s_i^T. Now EP contains the matrix N_0 which is N with its first column deleted. Since EP is in $\mathscr{P}$ we have that $f(N^T)^T$ is in EP. Thus P contains a sequence $i, f(N^T)^T$ for some i. But this contradicts the assumption that $f_i(N_i^T)^T$ is not in P_i. Thus f can be extended to a function defined for all values of the variables.

Now we consider single valued functions which are defined for all values of their arguments. If $\mathscr{P}$ is a system of predicates we redefine $\mathscr{P}^*$ as the set of single valued functions defined for all values of the arguments which commute with $\mathscr{P}$. Also we assume that $\mathscr{P}$ is closed, contains $e(x_1, x_2) \Leftrightarrow (x_1 = x_2)$ and is closed under the existential quantifier. We will give necessary and sufficient conditions on $\mathscr{P}^*$ in order that $\mathscr{P}$ be closed under disjunction and negation.

First we define the predicates $D(x_1, x_2, x_3, x_4) \Leftrightarrow (x_1 = x_2) \vee (x_3 = x_4)$ and $Q_n(x_1, \cdots, x_n)$ which holds in case $x_i \neq x_j$ for all $1 \leqq i < j \leqq n$. We have the following equivalences for a closed system $\mathscr{P}$.

(1) $\mathscr{P}^*$ consists of essentially first order functions if and only if $D \in \mathscr{P}$.

(2) When $\mathscr{P}$ is defined on a set A with n elements then $\mathscr{P}^*$ consists of essentially first order permutations if and only if $D, Q_n \in \mathscr{P}$.

We only prove that if $D \in \mathscr{P}$ then $\mathscr{P}^*$ consists of essentially first order functions. Let $g(x_1, \cdots, x_n)$ be a function in $\mathscr{P}^*$ which depends essentially on the variables x_1 and x_2. Then there exist sequences $(a_1, a_2, \cdots, a_n) = s_1$, $(a_0, a_2, \cdots, a_n) = s_2$, $(b_1, b_2, \cdots, b_n) = s_3$ and $(b_1, b_0, b_3, \cdots, b_n) = s_4$ such that $g(s_1) \neq g(s_2)$ and $g(s_3) \neq g(s_4)$. We construct the $4 \times n$ matrix M with i-th row s_i. Then $M^T \subset D$ but $g(M)^T$ is not in D so g cannot be in $\mathscr{P}^*$. The other implications also follow easily. From these equivalences we get:

THEOREM 4. *$\mathscr{P}$ is closed under disjunction if and only if $\mathscr{P}^*$ consists of essentially first order functions.*

THEOREM 5. *$\mathscr{P}$ is closed under negation if and only if $\mathscr{P}^*$ consists of first order permutations.*

4. Comments and applications. First we consider the case where A is an infinite set. Craig R. Platt has found in this case that we need to add the following condition to the definition of closure of a set of functions or predicates. A set of functions $\mathscr{L}$ is locally closed if, for any n-th order function g and for every finite $H \subset A^{n+1}$ there exists an $f \in \mathscr{L}$ such that $g \cap H = f \cap H$, then $g \in \mathscr{L}$. A similar definition is given for sets of predicates. Then it follows, if $\mathscr{L}$ and $\mathscr{P}$ are any sets of functions and predicates, that $\mathscr{L}^*$ and $\mathscr{P}^*$ are locally closed sets and Theorems 1 and 2 hold when $\mathscr{L}$ and $\mathscr{P}$ are locally closed. Also a theorem has been found in the infinite case which specializes to Theorem 3.

Theorems 1 and 2 can be summarized in the following way. Let $\mathscr{L}$ and $\mathscr{P}$ be the sets of all functions and predicates on a set and let C be a binary relation which holds between elements in $\mathscr{L}$ and $\mathscr{P}$ if and only if they commute. Then C is a difunctional relation [1, p. 193] that is $CC^*C = C$. Here C^* is the converse relation to C. Then CC^* and C^*C are congruence relations on $\mathscr{L}$ and $\mathscr{P}$ and C establishes a one to one correspondence between the congruence classes. Alternately we may say that there exists a set S and mappings ϕ: $\mathscr{L} \rightarrow S$ and π: $\mathscr{P} \rightarrow S$ such that two elements $f \in \mathscr{L}$ and $P \in \mathscr{P}$ commute if and only if $\phi(f) = \pi(P)$.

In [2] Post has given a classification of two valued systems of functions. This gives a classification of two valued systems of predicates containing equality and closed under the existential quantifier. Finding these systems can be simplified using theorems of this paper.

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