Closing gaps in problems related to Hamilton cycles in random graphs and hypergraphs

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Submitted: Feb 5, 2015; Accepted: Feb 17, 2015; Published: Mar 6, 2015
Mathematics Subject Classifications: 05C80,05C65

Abstract

We show how to adjust a very nice coupling argument due to McDiarmid in order to prove/reprove in a novel way results concerning Hamilton cycles in various models of random graphs and hypergraphs. In particular, we firstly show that for $k \geqslant 3$, if $pn^{k-1}/\log n$ tends to infinity, then a random k-uniform hypergraph on n vertices, with edge probability p, with high probability (w.h.p.) contains a loose Hamilton cycle, provided that $(k-1)|n$. This extends results of Frieze, Dudek and Frieze, and reproves a result of Dudek, Frieze, Loh and Speiss. Secondly, we show that there exists $K > 0$ such for every $p \geqslant (K\log n)/n$ the following holds: Let $G_{n,p}$ be a random graph on n vertices with edge probability p, and suppose that its edges are being colored with n colors uniformly at random. Then, w.h.p the resulting graph contains a Hamilton cycle for which all the colors appear on its edges (a rainbow Hamilton cycle). Bal and Frieze proved the latter statement for graphs on an even number of vertices, where for odd n their p was $\omega((\log n)/n)$. Lastly, we show that for $p = (1+o(1))(\log n)/n$, if we randomly color the edge set of a random directed graph $D_{n,p}$ with $(1+o(1))n$ colors, then w.h.p. one can find a rainbow Hamilton cycle where all the edges are directed in the same way.

1 Introduction

In this paper we show how to adjust a very nice coupling argument due to McDiarmid [7] in order to prove/reprove problems related to the existence of Hamilton cycles in various random graphs/hypergraphs models. The first problem we consider is related to the existence of a loose Hamilton cycle in a random k-uniform hypergraph.

A k-uniform hypergraph is a pair $\mathcal{H} = (V, \mathcal{E})$, where V is the set of *vertices* and $\mathcal{E} \subseteq \binom{[n]}{k}$ is the set of *edges*. In the special case where $k = 2$ we simply refer to it as a *graph* and denote it by $G = (V, E)$. The random k-uniform hypergraph $H_{n,p}^{(k)}$ is defined by adding each possible edge with probability p independently at random, where for the case $k = 2$ we denote it by $G_{n,p}$ (the usual binomial random graph). For $1 \leqslant \ell < k$ we define an ℓ-Hamilton cycle as a cyclic ordering of V for which the edges consist of k consecutive vertices, and for each two consecutive edges e_i and e_{i+1} we have $|e_i \cap e_{i+1}| = \ell$ (where we consider $n + 1 = 1$). In the special case $\ell = 1$ a 1-Hamilton cycle is referred to as a *loose* Hamilton cycle. It is easy to verify that if n is not divisible by $k - 1$ then such a cycle cannot exist.

Frieze [4] and Dudek and Frieze [2] showed that for $p = \omega(\log n/n)$, the random k-uniform hypergraph $H_{n,p}^{(k)}$ w.h.p. (with high probability) contains a loose Hamilton cycle in $H_{n,p}^{(k)}$ whenever $2(k-1)|n$. Formally, they showed:

Theorem 1. *The following hold:*

(*a*) *(Frieze) Suppose that $k = 3$. Then there exists a constant $c > 0$ such that for $p \geqslant (c \log n)/n$ the following holds*

$$\lim_{4|n\to\infty} \Pr\left[H_{n,p}^{(3)} \textit{ contains a loose Hamilton cycle}\right] = 1.$$

(*b*) *(Dudek and Frieze) Suppose that $k \geqslant 4$ and that $pn^{k-1}/\log n$ tends to infinity. Then*

$$\lim_{2(k-1)|n\to\infty} \Pr\left[H_{n,p}^{(k)} \textit{ contains a loose Hamilton cycle}\right] = 1.$$

The assumption $2(k-1)|n$ is clearly artificial, and indeed in [3] Dudek, Frieze, Loh and Speiss removed it and showed analog statement to 1 with the only restriction on n to be divisible by $k - 1$ (which is optimal).

As a first result in this paper, we give a very short proof for the result of Dudek, Frieze, Loh and Speiss while weakening (a) a bit. Formally, we prove the following theorem:

Theorem 2. *The following hold:*

(*a*) *Suppose that $k = 3$. Then for every $\varepsilon > 0$ there exists a constant $c > 0$ such that for $p \geqslant (c \log n)/n$ the following holds*

$$\lim_{2|n\to\infty} \Pr\left[H_{n,p}^{(3)} \textit{ contains a loose Hamilton cycle}\right] \geqslant 1 - \varepsilon.$$

(*b*) *Suppose that $k \geqslant 4$ and that $pn^{k-1}/\log n$ tends to infinity. Then*

$$\lim_{(k-1)|n\to\infty} \Pr\left[H_{n,p}^{(k)} \textit{ contains a loose Hamilton cycle}\right] = 1.$$

Another problem we handle with is the problem of finding a *rainbow* Hamilton cycle in a randomly edge-colored random graph. For an integer c, let us denote by $G^c_{n,p}$ the random graph $G_{n,p}$, where each of its edges is being colored, uniformly at random with a color from $[c]$. A Hamilton cycle in $G^c_{n,p}$ is called *rainbow* if all its edges receive distinct colors. Clearly, a rainbow Hamilton cycle can not exist whenever $c < n$. Bal and Frieze [1] showed that for some constant $K > 0$, if $p \geqslant (K \log n)/n$, the $G^n(n,p)$ w.h.p. contains a rainbow Hamilton cycle, provided that n is even. For the odd case they proved similar statement but only for $p = \omega((\log n)/n)$. We overcome this small difficulty and show the following:

Theorem 3. *There exists a constant $K > 0$ such that $G^n_{n,p}$ w.h.p. contains a rainbow Hamilton cycle.*

It is well known (see e.g. [6]) that a Hamilton cycle appears (w.h.p.) in $G_{n,p}$ for $p \approx (\log n)/n$. Therefore, one would expect to prove an analog for Theorem 3 in this range of p. However, it is easy to see that in this range, while randomly color the edges of $G_{n,p}$ with n colors, w.h.p. not all the colors appear. Frieze and Loh [5] proved that for $p = (1+\varepsilon)(\log n)/n$ and for $c = n + \Theta(n/\log\log n)$, a graph $G^c_{n,p}$ w.h.p. contains a rainbow Hamilton cycle. It is thus natural to consider the same problem for a randomly edge-colored directed random graph, denoted by $D^c_{n,p}$ (we allow edges to go in both directions). Note that in directed graphs we require to have a directed Hamilton cycle, which is a Hamilton cycle with all arcs pointing to the same direction.

The following theorem will follow quite immediately from our auxiliary lemmas in the following section:

Theorem 4. *Let $p = (1+\varepsilon)(\log n)/n$ and let $c = n + \Theta(n/\log\log n)$. Then $D^c_{n,p}$ w.h.p. contains a rainbow Hamilton cycle.*

2 Auxiliary results

In this section we present some variants of a very nice argument by McDiarmid [7]. For the convenience of the reader we briefly sketch the proofs, and the omitted details will be left as easy exercises. Before stating our lemmas, let us define the *directed* random k-uniform hypergraph $D^{(k)}_{n,p}$ in the following way. Each ordered k-tuple $(x_1, \ldots, x_k)$ consisting of k distinct elements of $[n]$ appears as an *arc* with probability p, independently at random. In the special case where $k = 2$ we simply write $D_{n,p}$. For $1 \leqslant \ell < k$, a *directed* ℓ-Hamilton cycle is an ℓ-Hamilton cycle where consecutive vertices are now arcs of $D^{(k)}_{n,p}$ and the last ℓ vertices of every arc are the first of the consecutive one. In the following lemma we show that the probability for $D^{(k)}_{n,p}$ to have a directed ℓ-Hamilton cycle is lower bounded by the probability for $H^{(k)}_{n,p}$ to have one.

Lemma 5. *Let $k \geqslant 3$ and let $1 \leqslant \ell < k$. Then, for every $p := p(n) \in (0,1)$ we have*

$$\Pr\left[D^{(k)}_{n,p} \text{ contains a directed } \ell\text{-Hamilton cycle}\right] \geqslant \Pr\left[H^{(k)}_{n,p} \text{ contains an } \ell\text{-Hamilton cycle}\right].$$

Proof. (McDiarmid) Let us define the following sequence of random directed hypergraphs $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$, where $N = \binom{n}{k}$ in the following way: Let $e_1, \ldots, e_N$ be an arbitrary enumeration of all the (unordered) k-tuples contained in $[n]$. For each e_i one can define $k!$ different orientations. Now, in Γ_i, for every $j \leqslant i$ and for each of the $k!$ possible orderings of e_j, we add the corresponding arc with probability p, independently at random. For every $j > i$, we include all possible orderings of e_j or none with probability p, independently at random. Note that Γ_0 is $H^{(k)}_{n,p}$ while Γ_N is $D^{(k)}_{n,p}$. Therefore, in order to complete the proof it is enough to show that

$$\begin{aligned}&\Pr\left[\Gamma_i \text{ contains a directed } \ell\text{-Hamilton cycle}\right]\\ &\geqslant \Pr\left[\Gamma_{i-1} \text{ contains a directed } \ell\text{-Hamilton cycle}\right].\end{aligned}$$

To this end, expose all arcs but those obtained from e_i. There are three possible scenarios:

(a) Γ_{i-1} contains a directed loose Hamilton cycle without considering e_i, or

(b) Γ_{i-1} does not contain a directed loose Hamilton cycle even if we add all possible orderings of e_i, or

(c) Γ_{i-1} contains a directed loose Hamilton cycle using at least one of the orderings of e_i.

Note that in (a) and (b) there is nothing to prove. In case (c), the probability for Γ_{i-1} to have a directed loose Hamilton cycle is p, where the probability for Γ_i to have such a cycle is at least p. This completes the proof of the lemma. ☐

In the second lemma, we show that given an integer c, one can lower bound the probability of $D^c_{n,p}$ to have a rainbow directed Hamilton cycle by the probability of $G^c_{n,p}$ to have such a cycle.

Lemma 6. *Let c be a positive integer. Then, for every $p := p(n) \in (0,1)$ and $q \in (0,1)$ for which $q - q^2 = p$ we have*

$$\begin{aligned}&\Pr\left[D^c_{n,q} \textit{ contains a rainbow directed Hamilton cycle}\right]\\ &\geqslant \Pr\left[G^c_{n,p} \textit{ contains a rainbow Hamilton cycle}\right].\end{aligned}$$

Proof. (Sketch)

Define the following sequence of random edge-colored directed graphs $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$, where $N = \binom{n}{2}$ in the following way: Let $e_1, \ldots, e_N$ be an arbitrary enumeration of all the (unordered) pairs contained in $[n]$. Now, in Γ_i, for every $j \leqslant i$ and for each of the 2 possible orientations of e_j, we add the corresponding arc with probability q, independently at random. For every $j > i$, we include both orientations of e_j or none with probability p, independently at random. At the end, all the obtained arcs are being colored by colors from $[c]$, independently, uniformly at random. Note that Γ_0 is $G^c_{n,p}$ while Γ_N is $D^c_{n,q}$. Therefore, in order to complete the proof it is enough to show that

$$\Pr\left[\Gamma_i \text{ contains a rainbow Hamilton cycle }\right] \geqslant \Pr\left[\Gamma_{i-1} \text{ contains a rainbow Hamilton cycle}\right].$$

To this end, expose all arcs but those obtained from e_i. There are three possible scenarios:

(a) Γ_{i-1} contains a rainbow Hamilton cycle without considering e_i, or

(b) Γ_{i-1} does not contain a rainbow Hamilton cycle even if we add all possible orderings of e_i colored with any color, or

(c) Γ_{i-1} contains a rainbow Hamilton cycle using at least one of the orderings of e_i which is also getting colored "correctly".

Note that in (a) and (b) there is nothing to prove. We therefore only consider Case (c). Suppose that $e_i = \{x, y\}$, and let $\mathcal{C}_1$ be the set of all colors for which if we add the arc xy to Γ_{i-1} colored with any color from $\mathcal{C}_1$, then it contains a rainbow Hamilton cycle. Similarly, define $\mathcal{C}_2$ with respect to the arc yx. By assumption we have $\mathcal{C}_1 \cup \mathcal{C}_2 \neq \emptyset$ (note that these sets are not necessarily disjoint). Let c_i denote the size of $\mathcal{C}_i$ ($i \in \{1,2\}$), and let $0 < c_0 = |\mathcal{C}_1 \cup \mathcal{C}_2| \leqslant c_1 + c_2$. Note that the probability for Γ_{i-1} to have a rainbow Hamilton cycle is $\frac{pc_0}{n}$. For Γ_i, the probability to have a rainbow Hamilton cycle is $\frac{qc_1}{n} + \frac{qc_2}{n} - \frac{q^2c_1c_2}{n^2} \geqslant \frac{qc_0}{n} - \frac{q^2c_0}{n} \geqslant \frac{pc_0}{n}$. This completes the proof of the lemma.

□

Note that by combining the result of Bal and Frieze [1] with Lemma 6 we immediately obtain the following corollary:

Corollary 7. *There exists a constant $K > 0$ such that for every $p \geqslant (K \log n)/n$ we have*

$$\Pr\left[D^n_{n,p} \text{ contains a rainbow Hamilton cycle}\right] = 1,$$

provided that n is even.

3 Proofs of our main results

In this section we prove Theorems 2, 3 and 4. We start with proving Theorem 2.

Proof of Theorem 2: Suppose that $(k-1)|n$ and that $2(k-1)$ does not divide n. Let $f^2(n)$ be a function that tends arbitrarily slowly to infinity and suppose that $p = \frac{f^2(n)\log n}{n^{k-1}}$. Note that by deleting the orderings of the edges in a $D^{(k)}_{n,q}$, using a similar argument as a multi-round exposure (we refer the reader to [6] for more details), we obtain a $H^{(k)}_{n,s}$ where $(1-q)^{k!} = 1 - s$ (one can just think about $D^{(k)}_{n,q}$ as an undirected hypergraph such that for every $e \in \binom{[n]}{k}$ there are $k!$ independent trials to decide whether to add it).

Now, let us choose q in such a way that $(1-p/2)(1-q)^{k!f(n)} = 1-p$, and observe that $q \geqslant \frac{p}{2k!f(n)} = \omega\left(\log n/n^{k-1}\right)$. We generate $H^{(k)}_{n,p}$ in a multi-round exposure and present it as a union $\bigcup_{i=0}^{f(n)} H_i$, where H_0 is $H^{(k)}_{n,p/2}$ and H_i is $D^{(k)}_{n,q}$ (which, as stated above, is like $H^{(k)}_{n,s}$ with $(1-q)^{k!} = 1-s$) for each $1 \leqslant i \leqslant f(n)$ (of course, ignoring the orientations). In addition, all the H_i's are considered to be independent.

Our strategy goes as follows: First, take $H_0 = H^{(k)}_{n,p/2}$ and pick an arbitrary edge $e* = \{x_1 \dots, x_k\}$. Trivially, H_0 contains an edge w.h.p. (and this was the only use of H_0). Now, fix an arbitrary ordering $(x_1, \dots, x_k)$ of e^* and let $V^* = ([n] \setminus \{x_1, \dots, x_k\}) \cup \{e^*\}$ (that is, V^* is obtained by deleting all the elements of e^* and adding an auxiliary vertex e^*). For each $i \geqslant 1$, whenever we expose H_i we define an auxiliary k-uniform directed random hypergraph D_i on a vertex set V^* in the following way. Every arc e of H_i is being added to D_i if it satisfies one of the following:

- $e \cap e^* = \emptyset$, or
- $e \cap e^* = \{x_1\}$, and x_1 is not the first vertex of the arc e, or
- $e \cap e^* = \{x_k\}$ and x_k is the first vertex of the arc e.

Note that indeed, by definition, every k-tuple of V^* now appear with probability p, independently at random and that $|V^*| = n-(k-1)$. Therefore, we clearly have that each of the D_i's is an independent $D^{(k)}_{n-(k-1),q}$. Moreover, note that $2(k-1)|n-(k-1)$ and that each directed loose Hamilton cycle of D_i with the special vertex $e*$ as a starting/ending vertex of the edges touching it corresponds to a (undirected) loose Hamilton cycle of $H^{(k)}_{n,p}$. To see the latter, suppose that $e^* v_2 \dots v_t e^*$ is such a cycle in D_i. Now, by definition we have that both $x_k v_2, \dots v_k$ and $v_{t-k+2} \dots v_t x_1$ are arcs of H_i, and therefore, by replacing e^* with its entries $x_1 \dots x_k$, one obtains a loose Hamilton cycle in H_i.

Next, by combining Theorem 1 with Lemma 5, we observe that w.h.p. D_i contains a directed loose Hamilton cycle. Note that by symmetry we have that the probability for e^* to be an endpoint of an edge on the Hamilton cycle is $2/k$. Therefore, after exposing all the D_i's, the probability to fail in finding such a cycle is $(1-2/k)^{f(n)} = o(1)$ as desired. This completes the proof. □

Next we prove Theorem 3.

Proof of Theorem 3: Let us assume that n is odd (since otherwise there is nothing to prove) and that $K >$ is a sufficiently large constant for our needs. Now, let q be such that $(1-p/2)(1-q)^2 = 1-p$, and present $G^n_{n,p}$ as a union $G_1 \cup G_2$, where G_1 is $G^n_{n,p/2}$ and G_2 is $D^n_{n,q}$ (as in the proof of Theorem 2, by ignoring orientations one can see $D^n_{n,q}$ as $G^n_{n,s}$ with s satisfying $(1-q)^2 = 1-s$). Next, let $e^* = (x, y)$ be an arbitrary edge of G_1 (trivially, w.h.p. there exists an edge), let c_1 denote its color, and define an auxiliary edge-colored random directed graph D as follows. The vertex set of D is $V^* = ([n] \setminus x, y) \cup \{e^*\}$ (that is, we delete x and y and add an auxiliary vertex e^*). The arc set of D consist of all arcs uv of G_2 with colors distinct than c_1 for which one of the following holds:

- $\{u, v\} \cap \{x, y\} = \emptyset$, or
- $v = x$, or
- $u = y$.

A moment's thought now reveals that D is $D^{n-1}_{n-1,s}$, where $s = (1-1/n)q$, that $n-1$ is even, and that a rainbow Hamilton cycle of D corresponds to a rainbow Hamilton cycle of $G^n_{n,p}$. Now, since $s \geqslant (K' \log n)/n$ for some K' (we can take it to be arbitrary large), it follows from Corollary 7 that w.h.p. D contains a rainbow Hamilton cycle, and this completes the proof. □

Lastly, we prove Theorem 4.

Proof of Theorem 4: The proof is an immediate corollary of the result of Frieze and Loh [5] and Lemma 6. □

Acknowledgment

The author would like to thank Alan Frieze for helpful comments and for pointing out that there is also a small gap in Bal and Frieze [1].

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