CLOSURE OPERATIONS FOR SCHUNCK CLASSES

Dedicated to the memory of Hanna Neumann

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In his Canberra lectures on finite soluble groups, [3], Gaschütz observed that a Schunck class (sometimes called a saturated homomorph) is $\{Q, E_\phi, D_0\}$-closed but not necessarily R_0-closed(*). In Problem 7.8 of the notes he then asks whether every $\{Q, E_\phi, D_0\}$-closed class is a Schunck class. We show below with an example† that this is not the case, and then we construct a closure operation $\bar{R}_0$ satisfying $D_0 < \bar{R}_0 < R_0$ such that $\mathfrak{X}$ is a Schunck class if and only if $\mathfrak{X} = \{Q, E_\phi, \bar{R}_0\}\mathfrak{X}$. In what follows the class of finite soluble groups is universal. Let $\mathfrak{P}$ denote the class of primitive groups. We recall that a Schunck class $\mathfrak{X}$ is one which satisfies:

(a) $\mathfrak{X} = Q\mathfrak{X}$, and

(b) $\mathfrak{X}$ contains all groups G such that $Q(G) \cap \mathfrak{P} \subseteq \mathfrak{X}$.

EXAMPLE. Let $\mathfrak{Y}$ denote the class comprising groups of order 1, groups of order 2 and non-Abelian groups of order 6. Set $\mathfrak{X} = E_\phi D_0 \mathfrak{Y}$. Since $E_\phi D_0$ is a closure operation (see [4]), $\mathfrak{X}$ is $\{E_\phi, D_0\}$-closed. It is not difficult to see that $D_0\mathfrak{Y}$ is Q-closed, and since $E_\phi Q$ is a closure operation (again see [4]), it follows that $Q\mathfrak{X} = QE_\phi D_0\mathfrak{Y} \subseteq E_\phi Q D_0 \mathfrak{Y} = E_\phi D_0 \mathfrak{Y} = \mathfrak{X}$, and hence that $\mathfrak{X}$ is Q-closed. Let G denote the extension of an elementary Abelian group of order 9 by an inverting involution. Then clearly $G \notin \mathfrak{X}$ and every primitive epimorphic image of G does belong to $\mathfrak{X}$, even to $\mathfrak{Y}$. Therefore $\mathfrak{X} = \{Q, E_\phi, D_0\}\mathfrak{X}$ but $\mathfrak{X}$ is not a Schunck class.

In order to formulate the closure operation $\bar{R}_0$ we need the concept of a crown, due to Gaschütz [2]. Let H/K be a complemented chief factor of a group G and M one of the maximal subgroups of G complementing it. Writing $C = C_G(H/K)$ it is well known that $\mathrm{Core}(M) = M \cap C$ and that $C/C \cap M$ is a chief factor

* The closure operation of taking finite direct products is denoted by D_0; the other closure operations mentioned are defined in [1]. A more detailed analysis of their properties appears in [4].

† I am pleased to acknowledge a similar example constructed by John Cossey of which I was unaware when I submitted this note.

G-isomorphic with H/K. Let R be the intersection of all normal subgroups T of G such that $C/T \cong_G H/K$. C/R is called the *crown* of H/K. A crown of G is a normal factor C/R associated in this way with some complemented chief factor. The following lemma shows that a normal subgroup of G either covers a crown C/R or is properly contained in C.

LEMMA. *Let C/R be a crown of G and $N \lhd G$. Then the following statements are equivalent*:

(1) *The image of C/R under the natural homomorphism $G \to G/N$ is a crown of G/N*;

(2) *N does not cover the factor C/R*;

(3) *$RN < C$.*

PROOF. Since a crown is by definition a non-trivial normal factor it is clear that (1) implies (2). Assume (2) holds, and set $L = C \cap NR\ [= (C \cap N)R]$. Then $L$is a normal subgroup of G properly contained in C. Now $[N, C] \leqq C \cap L \leqq L$, and since C is the centralizer in G of any non-trivial normal factor of G between C and R, we have $N \leqq C_G(C/L) = C$. Therefore $NR = NR \cap C = L < C$, and (3) is true. Finally assume condition (3) is satisfied. If C/K is a chief factor of G with $K \geqq R$, then C/K is complemented and C/R is the crown associated with it. By (3) we may choose such a K containing RN. Let $(C/N)/(T/N)$ be a chief factor of G/N isomorphic with $(C/N)/(K/N)$; then $C/T \cong_G C/K$, so $R \leqq T$. Since C/RN is a semi-simple (in fact, homogeneous) G-module, RN is the intersection of such T. Hence $(C/N)/(RN/N)$ is the crown of G/N associated with $(C/N)/(K/N)$.

DEFINITION. If $\mathfrak{X}$ is a class of groups, define $\bar{R}_0\mathfrak{X}$ as follows:

$G \in \bar{R}_0\mathfrak{X}$ if and only if G has a set $\{N_i\}_{i=1}^t$ of normal subgroups N_i satisfying

(α) $G/N_i \in \mathfrak{X}$ for $i = 1, \cdots, t$,

(β) $\bigcap_{i=1}^t N_i = 1$, and

(γ) for each crown C/R of G, there exists $i \in \{1, \cdots, t\}$ such that N_i does not cover C/R.

Evidently $\mathfrak{X} \subseteq \bar{R}_0\mathfrak{X}$ and, if $X \subseteq \mathfrak{Y}$, then $\bar{R}_0\mathfrak{X} \subseteq \bar{R}_0\mathfrak{Y}$. To prove that $\bar{R}_0$ is a closure operation, it remains to show it is idempotent. Let $G \in \bar{R}_0{}^2\mathfrak{X} = \bar{R}_0(\bar{R}_0\mathfrak{X})$. Then G has normal subgroups $\{N_i\}_{i=1}^t$ with $G/N_i \in \bar{R}_0\mathfrak{X}$ satisfying conditions (β) and (γ) above. Thus each G/N_i has normal subgroups $\{N_{ij}/N_i\}_{j=1}^{t_i}$ such that

(a) $G/N_{ij} \in \mathfrak{X}$ for $j = 1, \cdots, t_i$,

(b) $\bigcap_{j=1}^{t} N_{ij} = N_i$, and

(c) each crown of G/N_i is nor covered by at least one N_{ij}/N_i.

The full set $\{N_{ij} \mid j = 1, \cdots, t_i,\ i = 1, \cdots, t\}$ of normal subgroups of G clearly satisfies conditions (α) and (β) of the Definition. Let C/R be a crown of G. There exists an $i \in \{1, \cdots, t\}$ such that N_i does not cover C/R. By the Lemma $(C/N_i)/(RN_i/N_i)$ is a crown of G/N_i and by condition (c) above there exists a

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$j \in \{1, \cdots, t_i\}$ such that N_{ij} does not cover it. Again by the Lemma we have $RN_{ij} < C$ and therefore the set $\{N_{ij}\}$ also satisfies (γ). Thus $G \in \bar{R}_0\mathfrak{X}$. It follows that $\bar{R}_0{}^2 = \bar{R}_0$ and that $\bar{R}_0$ is a closure operation.

It is obvious that $\bar{R}_0 \leqq R_0$. To see that $D_0 \leqq \bar{R}_0$, let $G = G_1 \times \cdots \times G_t$ with $1 \neq G_i \in \mathfrak{X}$ for $i = 1, \cdots, t$. Set $N_i = \prod_{j \neq i} G_j$. Then the normal subgroups $\{N_i\}_{i=1}^t$ clearly satisfy conditions (α) and (β). It follows easily from the properties of a direct product that each chief factor is centralized by at least one N_i and that a factor of the form N_i/R is never a crown. Hence (γ) is also satisfied and we have $G \in \bar{R}_0\mathfrak{X}$. It remains to prove the following

THEOREM. *The condition $\mathfrak{X} = \{Q, E_\phi, \bar{R}_0\}\mathfrak{X}$ is both necessary and sufficient for $\mathfrak{X}$ to be a Schunck class.*

PROOF. Let $\mathfrak{X}$ be a Schunck class and let $G \in \bar{R}_0\mathfrak{X}$. G has a family $\{N_i\}_{i=1}^t$ of normal subgroups satisfying conditions (α), (β) and (γ). Let G/K be a primitive epimorphic image of G. Let C/K denote the monolith of G/K and C/R the crown of G associated with C/K. It follows from the hypothesis and the Lemma that there is an $i \in \{1, \cdots, t\}$ such that $N_iR < C$. Let C/T be a chief factor of G with $T \geqq N_iR$. Then $G/K \cong G/T \in Q(G/N_iK) \leqq Q\mathfrak{X} = \mathfrak{X}$. Thus $Q(G) \cap \mathfrak{P} \subseteq \mathfrak{X}$, and so $G \in \mathfrak{X}$. This shows that $\mathfrak{X} = \bar{R}_0\mathfrak{X}$. Since Schunk classes are Q-closed and E_ϕ-closed, the necessity of the condition is established.

We prove the sufficiency arguing by contradiction. Suppose there exists a $\{Q, E_\phi, \bar{R}_0\}$-closed class $\mathfrak{X}$ which is not a Schunck class. Let G be a group of minimal order subject to satisfying $Q(G) \cap \mathfrak{P} \subseteq \mathfrak{X}$ and $G \notin \mathfrak{X}$. If $1 \neq N \lhd G$, then $Q(G/N) \cap \mathfrak{P} \subseteq \mathfrak{X}$, and by minimality $G/N \in \mathfrak{X}$. Hence by the E_ϕ-closure of $\mathfrak{X}$ we have $\Phi(G) = 1$. Let S denote the set of all minimal normal subgroups of G. If $|S| = 1$, $G \in \mathfrak{P}$ and so $G \in \mathfrak{X}$, a contradiction. If $|S| > 1$, $\bigcap \{N \mid N \in S\} = 1$. If C/R is a crown of G, either $R > 1$, or $R = 1$ and $C \notin S$. In either case, there is an $N \in S$ such that $RN < C$. Thus the set S of all minimal normal subgroups of G satisfies conditions (α), (β) and (γ) of the Definition. Hence $G \in \bar{R}_0\mathfrak{X} = \mathfrak{X}$. This final contradiction completes the proof.

References

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