$$a_4 = \left(1/_{40} - \frac{31\gamma^2}{30} + \frac{16\gamma^4}{15}\right)(1+\gamma^2)^{1/2} + \left(\frac{\gamma^2}{2} - \frac{3\gamma^4}{2}\right)\{\ln[1+(1+\gamma^2)^{1/2}] - \ln\gamma\} - \frac{\gamma}{8} + 2\gamma^3 - \frac{16\gamma^5}{15} \quad (29)$$

$$\frac{a_5}{2} = \left(1/_{448} - \frac{726\gamma^2}{8400} + \frac{4468\gamma^4}{8400} - \frac{296\gamma^6}{525}\right)(1+\gamma^2)^{1/2} + \left(\frac{7\gamma^2}{160} - \frac{5\gamma^4}{16} + \frac{3\gamma^6}{4}\right)\{\ln[1+(1+\gamma^2)^{1/2}] - \ln\gamma\} - \frac{\gamma}{64} + \frac{3\gamma^3}{16} - \gamma^5 + \frac{296\gamma^7}{525} + \frac{\gamma^2 a_4}{2} \quad (30)$$

$$a_6 = \frac{\gamma^2}{2}\int_{-1/2}^{1/2} \xi^4 g(\xi) d\xi \quad (31)$$

$$g(\xi) = [(1/_2 - \xi)^2 + \gamma^2]^{1/2}[1/_2(1/_2 - \xi) + 4\xi] + [(1/_2 - \xi)^2 + \gamma^2]^{-1/2}\left[(1/_2 - \xi)\left(\gamma^2 - 6\xi^2 + \frac{\xi^4}{\gamma^2}\right) + 4\xi(\gamma^2 - \xi^2)\right] + [(1/_2 + \xi)^2 + \gamma^2]^{1/2}[1/_2(1/_2 + \xi) - 4\xi] + [(1/_2 + \xi)^2 + \gamma^2]^{-1/2}\left[(1/_2 + \xi)\left(\gamma^2 - 6\xi^2 + \frac{\xi^4}{\gamma^2}\right) - 4\xi(\gamma^2 - \xi^2)\right] + \left(6\xi^2 - \frac{3\gamma^2}{2}\right) \ln\left\{\frac{(1/_2 - \xi) + [(1/_2 - \xi)^2 + \gamma^2]^{1/2}}{-(1/_2 + \xi) + [(1/_2 + \xi)^2 + \gamma^2]^{1/2}}\right\}$$

With respect to the a_6 computation, it was found expedient to evaluate the integral numerically (by Simpson's rule) rather than analytically.

DISCUSSION

R. Viskanta[9]

The application of the variational method to the solution of integral equations encountered in the radiant heat-transfer problems is a welcome addition to the variational techniques already developed for the solution of Milne's integral equation [1].[10] Integration of equation (15) as well as the evaluation of a's and D's is quite involved even for a simple function $G(X)$ and kernel $K(X,Y)$. The extension of variational method to the solution of more complicated integral equations [2, 3] becomes extremely complex.

A solution of the Fredholm integral equation of the second kind (12) can also be obtained by the method of successive substitutions, due to Neumann and Liouville, which gives the unknown function $\beta(X)$ as an integral series in λ [4],

$$\beta(X) = G(X) + \lambda\int_a^b K(X,Y)\,G(Y)dY + \lambda^2\int_a^b K(X,Y)dY\int_a^b K(Y,Z)G(Z)dZ + \ldots + \quad (32)$$

where Z is a dummy integration variable.

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[10] Numbers in brackets designate References at end of discussion.

The convergence of the Neumann series as well as the existence and uniqueness of the solution can readily be demonstrated [5]. Now turning to the specific example in the analysis, the solution of equation (6) can simply be written as

$$\beta(X) = 1 + \lambda\int_{-1/2}^{1/2}\frac{dY}{[(X-Y)^2+\gamma^2]^{3/2}} + \lambda^2\int_{-1/2}^{1/2}\frac{dY}{[(X-Y)^2+\gamma^2]^{3/2}}\int_{-1/2}^{1/2}\frac{dz}{[(Y-Z)^2+\gamma^2]^{3/2}} + \ldots + \quad (33)$$

From the form of the solution it is evident that the convergence of the series is rapid when $\lambda \ll 1$. For case (a) numerical calculations yield the following values for the combined dimensionless flux $\beta(X) = 1.570$, 1.618 and 1.546 for $X = 0$, 0.2, and 0.4, respectively. Thus at $X = 0$ the prediction, as given by the three terms of the series (33), is in error by 4 per cent.

Both the variational method and the method of successive substitutions can give only approximate solutions, though the order of approximation can be extremely high. However, before attempting to solve a specific problem it is more advantageous to investigate the integral equation and the parameters to ascertain which of the several available methods is more suitable from the viewpoint of engineering accuracy and the effort expended in the calculations.

References

1 V. Kourganoff, "Basic Methods in Transfer Problems," Oxford, at the Clarendon Press, Oxford, England, 1952, pp. 126–52.

2 N. C. Francis, J. C. Stewart, L. S. Bohl, and T. J. Krieger, "Variational Solutions of the Transport Equation," Progress in Nuclear Energy, series I, Physics and Mathematics, vol. I, Pergamon Press, London, England, 1959, pp. 360–388.

3 R. Viskanta, "Heat Transfer in Thermal Radiation Absorbing and Scattering Medium," dissertation, Purdue University (in preparation).

4 E. T. Whittaker and G. N. Watson, "Modern Analysis," Cambridge University Press, Cambridge, England, fourth edition, 1952, p. 221.

5 E. Hopf, "Mathematical Problems of Radiative Equilibrium," Cambridge University Press, Cambridge, England, 1934, p. 37.

Author's Closure

Sincere thanks are extended to Mr. Viskanta for pointing out the procedure of successive approximations. As he has indicated, this is a fairly standard method and may be found in such well-known books as Hildebrand's "Methods of Applied Mathematics."[11]

In appraising the numerical results which Mr. Viskanta has presented, the author is surprised by their modest level of accuracy. The particular case considered was (a) in Table 2. As has already been stated in the paper, this is the least testing of all the cases shown in the table. Yet, the results from the successive approximate procedure are much less accurate than are those of the simple variation quadratic. But, even more surprising is that the β valves from the successive approximate method do not decrease monotonically from $X = 0$ to $X = 0.5$, as they should. Instead, a maximum is achieved somewhere between. In view of this, it would appear that further investigation is needed to clarify the utility of the successive approximation procedure.

[11] Prentice-Hall, Inc., Englewood-Cliffs, N. J., 1952, pp. 421–427.