Notre Dame Journal of Formal Logic
Volume 46, Number 3, 2005

Club Guessing and the Universal Models

Mirna Džamonja

Abstract We survey the use of club guessing and other PCF constructs in the context of showing that a given partially ordered class of objects does not have a largest, or a universal, element.

1 Introduction

A natural question in mathematics is, given some partially ordered or a quasi-ordered set or a class, is there the largest element in it? An aspect of this question appears in the theory of order where one concentrates on the properties of the set and the partial order ignoring the properties of the individual elements of the set. A very different view of this question is obtained when one takes the viewpoint that it is the structure of individual elements that is of interest. An instance of this is the question of universality. For this question we are given a class or a set of objects and a notion of embedding among them, and we ask whether there is an object in the class that embeds all the others. To simplify our exposition here we shall always assume that we are working with a set of objects and we shall discuss the smallest cardinality of the subset of that set that has the property of embedding all the other objects in the set. We shall refer to this question as *the universality problem*; the class and the embedding to which the problem refers will always be clear from the context. The number mentioned above will be then referred to as *the universality number*.

Instances of the universality problem have been of a continuous interest to mathematicians, especially those studying the mathematics of the infinite—even Cantor's work on the uniqueness of the rational numbers as the countable dense linear order with no endpoints is a result of this type. For some more recent examples see Argyros and Benyamini [1], Füredi and Komjáth [13], and Todorčević [45]. Apart from its intrinsic interest, the universality problem has an application in model theory, more specifically in classification theory. There it is used to distinguish among various kinds of unstable theories. For more on this program the reader may consult

Printed August 22, 2005
2000 Mathematics Subject Classification: Primary, 03C55, 03E04, 03C45
Keywords: universal models, club guessing

the introduction to Džamonja and Shelah [9] and some of the main results will be mentioned in §4 below. A good source for the classical results on universality is the introduction to Kojman and Shelah [17]. The study of the universality problem can naturally be divided into a "positive" part and a "negative" part. On the positive side one tries to show that the universality number has at most the given value 1, for example. Proofs here are often explicit constructions of universal objects or forcing constructions (see Chang and Keisler [4], Shelah [28], or Džamonja and Shelah [10]). On the negative side one does the opposite showing that the universality number is at least some given value. In this paper we shall concentrate on the negative side of the universality problem, in particular, on the instances of it that are obtained using a specific method that has appeared as a consequence of the discovery of the PCF theory, namely, the club guessing method. There are several other methods that appear in the study of the negative side of the universality problem, notably the σ-functor of Todorčević [45], but here we shall only concentrate on the club guessing method. This method is due to Kojman and Shelah [17]. We shall start by recalling the basic principles behind it and then give some applications including the original one from [17] to linear orders. Let us also note that in this subject it is often not difficult to construct a specific universe of set theory in which the desired negative universality result holds. For example, one can find in [17] a proof of the fact that when one forces over a model of *GCH* to add λ^{++} Cohen subsets to a regular cardinal λ, then in the resulting universe there is neither a universal graph of size λ nor a universal linear order of size λ nor a universal model of any first-order countable theory unstable in λ. The point of the negative universality results that are obtained by the club guessing method is that there are implications between a certain PCF statement and the desired negative universality result, so they hold in more than just one specifically constructed universe.

This paper is a survey of some of the existing techniques and results on this subject. Due to the extensive literature it will not be possible to mention all the relevant results, so our apologies go to the authors of the many deserving papers which we fail to mention.

2 Invariants and Linear Orders

Let us start with an easy example of a negative universality result: we remind the reader of why it is that for an infinite cardinal λ there is no well order of size λ to which there is an order-preserving embedding from any well order of size λ. The reason for this is that any well order of size λ is ordered in a type $\zeta < \lambda^+$ and hence cannot embed ordinals larger than ζ. This simple proof has three important elements: invariants, construction, and preservation. Specifically, to each well order we have associated an *invariant*, namely, its order type, then we observed that the invariant is *preserved* in the sense that it can only increase under embedding, and finally, we have *constructed* a family of well orders of size λ where many different values of the invariant are present (namely, the ordinals in $[\lambda, \lambda^+]$), so showing that no single well order of size λ can embed them all.

These same principles are present in many settings, for example, [1] and, in particular, in the club guessing proofs. The matters of course tend to be more complex. We shall now show the original Kojman-Shelah example of the use of club guessing for a universality result about linear orders (see [17]).

Let λ be a regular cardinal and let $\mathcal{K}$ be the class of all linear orders whose size is λ. By identifying the elements of $\mathcal{K}$ that are isomorphic to each other we obtain a set of size at most 2^λ which we shall call $\mathcal{K}$ again. We may without loss of generality assume that the universe of each element of $\mathcal{K}$ is the λ itself. We are interested in the universality number of $\mathcal{K}$ where for our notion of the embedding we take an injective order-preserving function. For future purposes let us also fix a ladder system $\bar{C} = \langle C_\delta : \delta \in S \subseteq \lambda\rangle$ such that each C_δ is a club of the corresponding δ and S is some stationary subset of the set of limit ordinals below λ. Let $C_\delta = \langle \alpha_i^\delta : i < i^*(\delta)\rangle$ be the increasing enumeration.

Every member L of $\mathcal{K}$ can be easily represented as a continuous increasing union $L = \bigcup_{j<\lambda} L_j$ of linear orders of size $< \lambda$, and such a representation is of course not unique. Any such sequence $\bar{L} = \langle L_j : j < \lambda\rangle$ is called a *filtration* of L. Next we shall define invariants for elements of L, but the definition of an invariant will depend both on the filtration $\bar{L}$ and on the specified ladder system $\bar{C}$.

Definition 2.1 Suppose that $L, \bar{L}, \bar{C}$ are as above and $\delta \in S$ is such that the universe of L_δ is δ. We define the *invariant* $\mathrm{inv}_{\bar{L},\bar{C}}(\delta)$ as the set

$$\{i < i^*(\delta) : (\exists \beta \in L_{\alpha^\delta_{i+1}} \setminus L_{\alpha^\delta_i})[\{x \in L_{\alpha^\delta_i} : x <_L \beta\} = \{x \in L_{\alpha^\delta_i} : x <_L \delta\}]\}.$$

So with this notation, the invariant of δ is a subset of i^*_δ that codes the "reflections" of δ along the places in the filtration $\bar{\lambda}$ that are determined by $\bar{C}$. It is easy to check that the set of δ, for which the universe of L_δ is δ, is a club of λ, so since S is stationary there are stationary many (a club in S) ordinals δ for which $\mathrm{inv}_{\bar{L},\bar{C}}(\delta)$ is well defined. It is also easy to see that for any two filtrations $\bar{L}$ and $\bar{L}' = \langle L'_j : j < \lambda\rangle$ of L there is a club of δ such that $L_\delta = L'_\delta$, hence the dependence of the invariant on the filtration is only up to a club. This is not the case with its dependence on the ladder system, and in fact only certain ladder systems are of interest to us.

Definition 2.2 A ladder system $\bar{C} = \langle C_\delta : \delta \in S\rangle$ is said to be a *club guessing sequence* if and only if for every club E of λ there is $\delta \in S$ such that $C_\delta \subseteq E$.

Club guessing sequences were introduced by Shelah in [32] as a tool toward the development of PCF theory and have since found many applications in various contexts. There is also a number of variants of this concept; for example, a number of interesting results about various kinds of club sequences appears in Ishiu's thesis [14]. However, for the moment we shall concentrate on the simple club guessing mentioned above. An interesting question is when such sequences exist, and one of the most important theorems in this vein is the following.

Theorem 2.3 ([32]) *Suppose that κ and λ are regular cardinals such that $\kappa^+ < \lambda$. Then there is a club guessing sequence of the form $\bar{C} = \langle C_\delta : \delta \in S^\lambda_\kappa\rangle$.*

We have used the notation S^λ_κ to denote the set of $\alpha < \lambda$ whose cofinality is κ. A club guessing sequence of the form appearing in this theorem will be referred to as an S^λ_κ*-club guessing sequence*. Note that for $\kappa \geq \aleph_1$ we may without loss of generality, by intersecting with a club of order type κ if necessary, assume that each C_δ has order type κ. In fact, a stronger theorem than Theorem 2.3 has been proved by Shelah showing that in addition to this restriction on the order type the club guessing sequence above may be assumed to have a squarelike property, or in the terminology of [9], to be a *truly tight* (κ, λ) *club guessing sequence*.

Theorem 2.4 (Shelah [31]) *Suppose that κ and λ are regular cardinals such that $\kappa^+ < \lambda$. Then there is a stationary set $S \subseteq S^\lambda_\kappa$, a club guessing sequence of the form $\bar{C} = \langle C_\delta : \delta \in S\rangle$, and a sequence $\bar{P} = \langle P_\alpha : \alpha < \lambda\rangle$ such that each $P_\alpha \subseteq \mathcal{P}(\alpha)$ has size $< \lambda$ and the sequences $\bar{C}$ and $\bar{P}$ interact in the sense that*

$$\alpha \in C_\delta \setminus [\sup(C_\delta \cap \alpha) + 1] \implies C_\delta \cap \alpha \in \bigcup_{\beta<\alpha} P_\beta.$$

A nicely written proof of Theorem 2.3 appears in [17]. Theorem 2.4 can be read off from the conjunction of the claims in §1 of [31], and a direct proof is given in Shelah [25], 1.3.(a). Theorem 2.3 refers to cardinals κ, λ which have at least one successor gap, $\kappa^+ < \lambda$. It is natural to ask what happens at $\lambda = \aleph_1$. Clearly, in the presence of ♣ there is an $S^{\aleph_1}_{\aleph_0}$-guessing sequence. A theorem of Shelah (Theorem III, 7.1 in [36]) shows that it is consistent to have $2^{\aleph_0} = \aleph_2$ and that there is no $S^{\aleph_1}_{\aleph_0}$-club guessing sequence (in fact the theorem shows something stronger than what is being quoted here). This also follows from the conjunction of a result of Shelah in [28] which shows it is consistent to have the negation of *CH* and the existence of a universal linear order of size $\aleph_1$, and the Kojman-Shelah theorem (Theorem 2.5 below) which implies that in a model in which *CH* fails and there is an $S^{\aleph_1}_{\aleph_0}$-guessing sequence there cannot be a universal linear order of size $\aleph_1$. In the other direction, a consistency result showing how to add club guessing sequences with strong guessing properties was introduced by Komjáth and Foreman who, in [12], give a cardinal-preserving forcing which, for any regular λ and a given stationary set S in λ, keeps S stationary and adds a sequence $\langle C_\delta : \delta \in S\rangle$ such that for every club E of λ there is a club C such that for all $\alpha \in C \cap S$ a nonempty final segment of C_δ is included in E.

We shall now state and give a sketch of the proof of the Kojman-Shelah theorem on linear orders using the notions of an invariant and a club guessing sequence introduced above. For the case of $\kappa = \aleph_0$ we only need to use an S^λ_κ-club guessing sequence while the case $\kappa > \aleph_0$ is handled using Theorem 2.4.

Theorem 2.5 ([17]) *Suppose that κ and λ are regular such that $\lambda < 2^\kappa$. Further suppose that either*

(a) $\kappa = \aleph_0$, $\lambda = \aleph_1$, *and there is an $S^{\aleph_1}_{\aleph_0}$-club guessing sequence; or*
(b) $\kappa^+ < \lambda$.

Then there is no universal linear order of size λ; moreover, the universality number of the class of linear orders of size λ is at least 2^κ.

Sketch of Proof The proof uses the method of Construction and Preservation. First let us fix a club guessing sequence $\bar{C} = \langle C_\delta : \delta \in S \subseteq S^\lambda_\kappa\rangle$ and if $\kappa > \aleph_0$ assume also that this sequence is chosen in conjunction with a sequence $\bar{P}$ to form a truly tight (κ, λ)-guessing sequence. In particular, we assume that the order type $i^*(\delta)$ of C_δ is always κ. Recall the notation $\langle \alpha^\delta_i : i < \kappa\rangle$ for the increasing enumeration of C_δ.

Lemma 2.6 (Construction Lemma) *For every $A \subseteq \kappa$, there is a linear order L_A and its filtration $\bar{L}_A$ such that for a club C of λ, we have*

$$(\delta \in C \ \&\ C_\delta \subseteq \delta) \implies \mathrm{inv}_{\bar{L}_A, \bar{C}}(\delta) = A.$$

Sketch of Proof Construction Lemma 2.6 has a simpler proof for case (2.5) of Theorem 2.5 which is the case we shall prove. Then we shall comment on the changes needed to cover case (2.5). So assume that $\kappa = \aleph_0$, $\lambda = \aleph_1$, and $\bar{C}$ is an $S^{\aleph_1}_{\aleph_0}$-club guessing sequence. Let $A \subseteq \omega$ be given.

Recall that by a *cut* of a linear order we mean an initial segment of the order, and we say that the cut is *realized* if it has the least upper bound. If $L \subseteq L'$ are linear orders and D is a cut of L, then a cut D' of L' *extends* D if $D' \cap L = D$. When speaking of a linear order $(L, <_L)$ we may refer to cuts of L or cuts of $<_L$ as is more convenient for the context. By η we denote the order type of the rationals.

We shall define the order $<_{L_A}$ on ω_1 by inductively defining a strictly increasing sequence $\langle \gamma_i : i < \omega_1 \rangle$ of countable ordinals and defining $<_i \stackrel{\text{def}}{=} <_{L_A} \restriction \gamma_i$ at the step i of the induction. The requirements of this induction will be

(i) for every $i < j$ there is a cut of γ_i realized in γ_{j+1} and not realized in γ_j;
(ii) if $i < j < k$ and D is a cut of γ_i realized in γ_j but not in γ_i, then there is a cut of γ_k that extends D and that is realized in γ_{k+1} but not in γ_k;
(iii) if D is a cut of γ_i realized in γ_{i+1} and not realized in γ_i, then the $<_{i+1}$-order type of $\{x \in \gamma_{i+1} : x \text{ realizes } D\}$ is η.

The starting point of the induction is $\gamma_0 = \omega$ where we let $<_0$ be ω-ordered in the order type of the rationals. At limit i we define γ_i to be the $\sup_{j<i} \gamma_j$ and the order is defined as the union of the orders constructed so far.

At the stage γ_{i+1} we ask ourselves whether i is a *good point* of the construction, which would mean that i is a limit ordinal and $\gamma_i = i$. If so, we then ask whether C_δ consists of good points. If the answer to both of these questions is affirmative, we proceed to define a sequence $\langle D_n : n < \omega \rangle$ of cuts such that each D_n is a cut of α^i_n not realized in $<_{\alpha^i_n}$ and $D_n \subseteq D_{n+1}$. In addition, we require that D_n is realized in $<_{\alpha^i_{n+1}}$ if and only if $n \in A$. If $A = \varnothing$ we let D_0 be a cut of $<_0$ not realized in i, which exists as there are $2^{\aleph_0}$ cuts of $<_0$ and i is countable. We let D_n be the extension of D_0 to α^i_n, so $D_n \stackrel{\text{def}}{=} \{x < \alpha^i_n : (\exists y \in D)(x <_{\alpha^i_n} y)\}$. Otherwise, let n_0 be the first element of A and let D_{n_0} be a cut of $\alpha^i_{n_0}$ that is realized in $\alpha^i_{n_0+1}$ and not in $\alpha^i_{n_0}$. For $m < n_0$ let $D_m = D_{n_0} \cap \alpha^i_m$. If $A = \{n_0\}$ then since there is the order type η of elements of $\alpha^i_{n_0+1}$ that realize D_{n_0}—by requirement (ii)—there is a cut D of i that extends D_{n_0} and that is not realized in i. For $n > n_0$ we let D_n be $D \cap \alpha^i_n$. Otherwise, let $n_1 = \min A \setminus (n_0 + 1)$ and let D_{n_1} be a cut of $\alpha^i_{n_1}$ extending D_{n_0}, which is realized in $\alpha^i_{n_1+1}$ and not in α_{n_1} and which exists by (iii) above. Then we continue similarly to the previous case. In any case, we have constructed the increasing sequence of cuts as required, and letting D^* be their union we then let i realize D^* in $<_L \restriction (i + 1)$. We extend the order by transitivity.

Now we still have to assure that the requirements of the induction are preserved, which can be done by amalgamating countably many ordinals to $i + 1$ in the way requested by the requirements. The sup of all these ordinals is then defined to be γ_{i+1}. If i is not a good point we do not have to take special care of i but instead proceed just as in this paragraph.

At the end of the induction we let $<_{L_A}$ be the union of the orders constructed, $\bar{L}_A = \langle <_i : i < \omega_1 \rangle$, and E a club of good points of the construction. The construction was made so that at any $\delta \in E$ the required invariant is guaranteed to be achieved only if $C_\delta \subseteq E$.

This finishes the proof of Construction Lemma 2.6 for the case $\kappa = \aleph_0$. For larger κ things become more complex as one also has to handle the limit points of cofinality $< \kappa$. This is a difficulty familiar from classical constructions such as that of a Suslin tree from a $\diamondsuit$, where one in addition uses a $\square$-sequence at cardinals larger than $\aleph_1$. In this case the construction can be carried through thanks to the squarelike properties of a truly tight guessing sequence. See [17] for details. $\square$

We also need the Preservation Lemma.

Lemma 2.7 (Preservation Lemma) *Suppose that L and L' are linear orders with universe λ and with filtrations $\bar{L}$ and $\bar{L}'$, respectively, while $f : L \to L'$ is an order-preserving injection. Then there is a club E of λ such that for every $\delta \in S^\lambda_\kappa$ satisfying $C_\delta \subseteq E$, we have*

$$\mathrm{inv}_{\bar{L},\bar{C}}(\delta) = \mathrm{inv}_{\bar{L}',\bar{C}}(f(\delta)).$$

Sketch of Proof We start by defining a model M with universe λ, order relations $<_L$, $<_{L'}$, and $<$ (the ordinary order on the ordinals), and the function f. Let E be a club of $\delta < \lambda$ such that $\delta \in E$ implies that $M \restriction \delta \prec M$ and the universe of both L_δ and L'_δ is δ. Suppose that $\delta \in E$ is such that $C_\delta \subseteq E$, and we shall prove that $\mathrm{inv}_{\bar{L},\bar{C}}(\delta) = \mathrm{inv}_{\bar{L}',\bar{C}}(f(\delta))$. The more difficult direction of the proof is the inclusion $\subseteq$. So suppose that $i \in \mathrm{inv}_{\bar{L},\bar{C}}(\delta)$, hence there is $\beta \in L_{\alpha^\delta_{i+1}} \setminus L_{\alpha^\delta_i}$ satisfying that $\{x <_L \beta : x \in L_{\alpha^\delta_i}\} = \{x <_L \delta : x \in L_{\alpha^\delta_i}\}$. We would like to claim that $f(x)$ witnesses that $i \in \mathrm{inv}_{\bar{L}',\bar{C}}(f(\delta))$, and it does follow from the choice of E and the fact that $C_\delta \subseteq E$ that

$$\{y <_{L'} f(\beta) : y \in f``(L_{\alpha^\delta_i})\} = \{y <_{L'} f(\delta) : y \in f``(L_{\alpha^\delta_i})\}.$$

However, the problem is that f is not necessarily onto. As L' is a linear order we have $f(\beta) <_{L'} f(\delta)$ or $f(\delta) <_{L'} f(\beta)$ (equality cannot occur by the choice of E). Let us suppose that the former is true; the latter case is symmetric. Suppose that $f(\beta)$ does not witness that $i \in \mathrm{inv}_{\bar{L}',\bar{C}}(f(\delta))$; this then means that there is $\gamma \in L'_{\alpha^\delta_i}$ (so $\gamma < \alpha^\delta_i$) such that $f(\beta) <_{L'} \gamma <_{L'} f(\delta)$. Observe that there is no $\varepsilon \in L_{\alpha^\delta_i}$ such that $\gamma \leq_{L'} f(\varepsilon) \leq_{L'} f(\delta)$, by the choice of β.

Consider $T \stackrel{\mathrm{def}}{=} \{x : (\neg\exists q \in L_{\alpha^\delta_i})\, \gamma <_{L'} f(q) <_{L'} x\}$. We claim that $T \cap \alpha^\delta_i$ is exactly the set $\{\zeta < \alpha^\delta_i : \zeta <_{L'} f(\delta)\}$. Namely, if $\zeta < \alpha^\delta_i$ and $\zeta <_{L'} f(\delta)$ and $\zeta \notin T$ then there is $\varepsilon \in L_{\alpha^\delta_i}$ such that $\gamma <_{L'} f(\varepsilon) <_{L'} \zeta <_{L'} f(\delta)$—a contradiction. On the other hand, if for some $\zeta < \alpha^\delta_i$ we have $f(\delta) <_{L'} \zeta$ then in M it is true that there is q such that $\gamma <_{L'} f(q) <_{L'} \zeta$, as δ is such a q. By elementarity it is true that there is such $q \in L_{\alpha^\delta_i}$, so $\zeta \notin T$. Hence we have shown the existence of a ξ such that $\{\zeta < \alpha^\delta_i : \zeta <_{L'} \xi\}$ is exactly

$$\{x < \alpha^\delta_i : (\neg\exists q \in L_{\alpha^\delta_i})\, \gamma <_{L'} f(q) <_{L'} x\}.$$

By elementarity there must be such $\xi \in L'_{\alpha^\delta_{i+1}} \setminus L'_{\alpha^\delta_i}$, and then this ξ shows that $i \in \mathrm{inv}_{\bar{L}',\bar{C}}(f(\delta))$. $\square$

To finish the proof of Theorem 2.5, suppose that there were a family $\{L_i : i < i^*\}$ of linear orders of size λ for some $i^* < 2^\kappa$ such that every linear order of size λ embeds into some L_i. We may assume that the universe of each L_i is λ. Let $\bar{L}_i$

be any filtration of L_i and let $\mathcal{B}$ be the family of all $B \subseteq \kappa$ such that for some x and i we have $\mathrm{inv}_{\bar{L}_i,\bar{C}}(x) = B$. Then the size of $\mathcal{B}$ is at most $\lambda \cdot |i^*|$, which is $< 2^\kappa$. Hence there is $A \subseteq \kappa$ with $A \notin B$ and, by Construction Lemma 2.6, a linear order L_A with universe λ and its filtration $\bar{L}_A$ such that for a club C of λ we have $\delta \in C \implies \mathrm{inv}_{\bar{L}_A,\bar{C}}(\delta) = A$. Suppose that $f : L_A \to L_i$ is an embedding and let E be a club guaranteed to exist by Preservation Lemma 2.7. Let $\delta \in C$ be such that $C_\delta \subseteq E$. Then $\mathrm{inv}_{\bar{L}_i,\bar{C}}(f(\delta)) = A$—a contradiction with the choice of A. □

3 A Few More Words on Orders and Orderable Structures

The Kojman-Shelah method can be ramified to give results on cardinals λ that are not necessarily regular. Using another PCF staple, the covering number $\mathrm{cov}(\lambda, \mu, \theta, \sigma)$, they were able to strap together the negative universality results on the regular cardinals below a given singular cardinal to obtain a negative universality result for the singular. Note that by classical results for special models (see [4]) there is a universal linear order (or any other first-order theory of a sufficiently small size) in any strong limit uncountable cardinality. The question then becomes what happens if, for example, $\aleph_\omega$ is not a strong limit. The answer is that then there is no universal linear order. Specifically, we have the following theorem.

Theorem 3.1 ([17]) *Suppose that λ is a singular cardinal which is not a strong limit and it satisfies that either*

(a) $\aleph_\lambda > \lambda$, *or*
(b) $\aleph_\lambda = \lambda$ *but* $|\{\mu < \lambda : \aleph_\mu = \mu\}| < \lambda$ *and either* $\mathrm{cf}(\lambda) = \aleph_0$ *or* $2^{<\mathrm{cf}(\lambda)} < \lambda$,

then there is no universal linear order of size λ.

Linear orders are representatives of theories T that have the *strict order property* which means that there is a formula $\varphi(\bar{x}; \bar{y})$ such that in the monster model $\mathfrak{C}$ of T there are $\bar{a}_n$, for $n < \omega$, such that for any $m, n < \omega$

$$\mathfrak{C} \models \text{“}(\exists \bar{x})[\neg\varphi(\bar{x}; \bar{a}_m) \wedge \varphi(\bar{x}; \bar{a}_n)]\text{” iff } m < n.$$

Other examples of first-order theories that have the strict order property are Boolean algebras, partial orders, lattices, ordered fields, ordered groups, and any unstable complete theory that does not have the independence property (see Shelah [23]). Using the fact that the strict order property of T allows for coding of orders into models of T and that there is a quantifier-free definable order in the above (noncomplete) theories, §5 of [17] shows that the existence of a universal element in any of these theories at a cardinal λ implies the existence of a universal linear order of size λ. Therefore the negative universality results stated above also apply to these theories.

A different approach to drawing conclusions about the universality problem in one class knowing the behavior of another class is taken by Thompson ([43] and [44]) who uses functors that preserve the embedding structure. She reproves the Kojman-Shelah conclusion about universality of partial orders versus that of linear orders and connects certain classes of graphs with certain classes of strict orders. This approach is also useful when one moves from the first-order context, for example, to the class of orders that omit chains of a certain type. The simplest case is the one of orders that omit infinite descending sequence. Universality is resolved trivially in the class of well orders, as follows from the example of the ordinals, but by changing the context to that of well-founded partial orders with some extra requirements one obtains a different situation. This type of problem is the subject of [43].

4 Model Theory

There is a model-theoretic motivation behind an attempt to deliver a general method of approach to the universality problem stemming from Shelah's program of classification theory. This is very well described in §5 of Shelah's paper on open questions in model theory [37]. Namely, it may be hoped that the behavior of a theory with respect to the universality would classify the theory as "good" if it can admit a small number of universal models even when the relevant instances of *GCH* fail, while a bad theory would rule out small universal families as soon as *GCH* would be sufficiently violated. (Recall that the situation in the presence of *GCH* is information-free here, since all first-order countable theories, for example, have a universal element in every uncountable cardinal; see [4]). Such a division would be used to classify unstable theories with the hope for a result similar to the classification of stable versus unstable theories where a model-theoretic property of stability of a countable theory was closely connected with the number of pairwise nonisomorphic models a theory may have at an uncountable cardinal through the celebrated Shelah's Main Gap Theorem (see [23]). The idea of using universality in a similar manner has proved to be quite successful, and although no precise model-theoretic equivalent has been found as of yet, there is much information available about the existing model-theoretic properties. One can find a rather detailed description of the present state of knowledge in [9] where there is also a precise definition of the proposed division from the set-theoretic point of view, that is, what is meant by being "good" (referred to as *amenable*) and "very-bad" (*highly nonamenable*) from the universality point of view. In this paper we mostly concentrate on the highly nonamenable theories which can be defined as follows.

Definition 4.1 A theory T is said to be *highly nonamenable* if and only if for every large enough regular cardinal λ and $\kappa < \lambda$ such that there is a truly tight (κ, λ) club guessing sequence, the smallest number of models of T of size λ needed to embed all models of T of that size is at least 2^κ. T is *highly nonamenable up to* κ^* if the above characterization is not necessarily true, but true whenever $\kappa < \kappa^*$.

In model theory one usually works with complete theories while our examples above were not necessarily so (for example, we worked with the theory of a linear order). We adopt the convention that when speaking of a complete theory by an embedding we mean an elementary embedding, and otherwise we just mean an ordinary embedding. With this clause, Definition 4.1 makes sense in both contexts, and the work of Kojman and Shelah presented in §2 showed that linear orders and theories with the strict order property are highly nonamenable. Shortly after this work, the same authors in [18] proceeded to show—in Theorems 4.1 and 5.1 of that work—the following.

Theorem 4.2 ([18]) *Countable stable unsuperstable theories are highly nonamenable up to* $\aleph_1$. *In general, stable unsuperstable theories* T *are highly nonamenable up to their stability cardinal* $\kappa(T)$.

Here we use the notion of the stability cardinal $\kappa(T)$ defined as the minimal cardinal κ such that for every set $A \subseteq \mathfrak{C}$ and a type p over A there is $B \subseteq A$ such that $|B| < \kappa$ and p does not fork over B. It is proved in Shelah's Stability Spectrum Theorem [23] that for any stable T we have $\kappa(T) \leq |T|^+$ and for every λ we have T is stable in λ if and only if $\lambda = \lambda^{<\kappa(T)}$ and either $\lambda \geq 2^{\aleph_0}$ or $\lambda \geq$ the number

$D(T)$ of parameter-free types of T in $\mathfrak{C}$. A countable complete first-order theory T is stable unsuperstable if and only if $\kappa(T) = \aleph_1$. In Theorem 4.2, as well as in many other applications of the method, a major issue is how to define an invariant. Suppose that T is, for simplicity, a complete countable stable unsuperstable theory.

Definition 4.3 Let λ be regular, $\bar{C} = \langle C_\delta : \delta \in S\rangle$ a truly tight (κ, λ) club guessing sequence, and N a model of T of size λ given with a continuous increasing filtration $\bar{N} = \langle N_i : i < \lambda\rangle$. We define for a $\delta \in S$ and a tuple $\bar{a}$ of N,

$$\mathrm{inv}_{\bar{N},\bar{C}}(\bar{a}) \stackrel{\mathrm{def}}{=} \{i < \kappa : \text{ the type of } \bar{a} \text{ over } N_{\alpha_i^\delta} \text{ forks over } N_{\alpha_{i+1}^\delta}\}.$$

As before, we have used the notation $\langle \alpha_i^\delta : i < \kappa\rangle$ for the increasing enumeration of C_δ. In the case of $\kappa = \aleph_0$ it suffices to deal with ordinary club guessing sequences. We do not have the space to introduce the notion of forking here, but the intuition behind Definition 4.3 is similar to the idea behind the invariant for linear orders: i is in the invariant if and only if "something new happens" at the stage α_{i+1}^δ, something that "reflects" the behavior of $\bar{a}$ with respect to $\bigcup_{i<\kappa} N_{\alpha_i^\delta}$. Recalling that $\kappa < \kappa(T)$ is assumed may give a hint of how the (rather complex) proof in [18] proceeds.

One can also try to capture what is meant by the "good" universality behavior, and in [9] we have tried to capture this using the notion of *amenability*.

Definition 4.4 A theory T is *amenable* if and only if whenever λ is an uncountable cardinal larger than the size of T and satisfies $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$, while θ satisfies $\mathrm{cf}(\theta) > \lambda^+$, there is a λ^+-cc $(< \lambda)$-closed forcing notion that forces 2^λ to be θ and assures that in the extension there is a family $\mathcal{F}$ of $< \theta$ models of T of size λ^+ such that every model of T of size λ^+ embeds into one of the models in $\mathcal{F}$. Localizing at a specific λ we obtain the definition of amenability at λ.

The point is that no theory can be both amenable and highly nonamenable. Namely, suppose that a theory T is both amenable and highly nonamenable, and let λ be a large enough regular cardinal while $V = L$ or simply $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(S_\lambda^{\lambda^+})$ holds. Let P be the forcing exemplifying that T is amenable. Clearly there is a truly tight (λ, λ^+) club guessing sequence $\bar{C}$ in V, and since the forcing P is λ^+-cc, every club of λ^+ in V^P contains a club of λ^+ in V, hence $\bar{C}$ continues to be a truly tight (λ, λ^+) club guessing sequence in V^P. Then, on the one hand, we have that in V^P, the universality number of models of T of size λ, $\mathrm{univ}(T, \lambda^+)$, is at least 2^λ by the high nonamenability, while $\mathrm{univ}(T, \lambda^+) < 2^\lambda$ by the choice of P—a contradiction. In [10], building on the earlier work of Shelah in [24], we gave an axiomatization of elementary classes that guarantees that the underlying theory is amenable. Shelah proved in [33] that all countable simple theories are amenable at all successors of regular κ satisfying $\kappa^{<\kappa} = \kappa$. (Note that even though all simple theories are stable, this is not in contradiction with Theorem 4.2 as there it is only proved that countable stable unsuperstable theories are highly nonamenable up to $\aleph_1$). In that same paper Shelah introduced a hierarchy of complexity for first-order theories and showed that high nonamenability appears as soon as a certain level on that hierarchy is passed. Details of this hierarchy are given in the following definition.

Definition 4.5 Let $n \geq 3$ be a natural number. A formula $\varphi(\bar{x}, \bar{y})$ is said to exemplify the *n-strong order property* of T, SOP_n, if $lg(\bar{x}) = lg(\bar{y})$, and there are $\bar{a}_k$ for $k < \omega$, each of length $lg(\bar{x})$ such that

(a) $\models \varphi[\bar{a}_k, \bar{a}_m]$ for $k < m < \omega$,

(b) $\models \neg(\exists \bar{x}_0, \ldots, \bar{x}_{n-1})[\bigwedge\{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n \text{ and } k = \ell + 1 \mod n\}]$.

The following were proved in [33]: the hierarchy above describes a sequence SOP_n ($3 \leq n < \omega$) of properties of strictly increasing strength such that the theory of a dense linear order possesses all the properties, while on the other hand, no simple theory can have the weakest among them, SOP_3. The property SOP_4 of a theory T implies that T is highly nonamenable. In light of these results it might then be asked whether SOP_4 is a characterization of high nonamenability. A partial answer appears in [9]. There we considered a property of theories that we called *oak property*, as its prototypical example is a tree of the form ${}^{\kappa\geq}\lambda$ equipped with restriction where we can express that $\eta \restriction \alpha = \nu$ for $\eta \in {}^{\kappa}\lambda$, $\alpha < \kappa$ and $\nu \in {}^{\kappa>}\lambda$. This property is also a generalization of the theory of infinitely-many independent equivalence relations T^*_{feq} (see [9]). Following is the formal definition.

Definition 4.6 A theory T is said to *satisfy* the *oak property as exhibited by a formula* $\varphi(\bar{x}, \bar{y}, \bar{z})$ if and only if for any infinite λ, κ there are $\bar{b}_\eta (\eta \in {}^{\kappa>}\lambda)$, $\bar{c}_\nu (\nu \in {}^{\kappa}\lambda)$, and $\bar{a}_i (i < \kappa)$ such that

(a) $[\eta \lhd \nu \;\&\; \nu \in {}^{\kappa}\lambda] \implies \varphi[\bar{a}_{lg(\eta)}, \bar{b}_\eta, \bar{c}_\nu]$,

(b) if $\eta \in {}^{\kappa>}\lambda$ and $\eta\hat{\ }\langle\alpha\rangle \lhd \nu_1 \in {}^{\kappa}\lambda$ and $\eta\hat{\ }\langle\beta\rangle \lhd \nu_2 \in {}^{\kappa}\lambda$, while $\alpha \neq \beta$ and $i > lg(\eta)$, <u>then</u> $\neg\exists\bar{y}\,[\varphi(\bar{a}_i, \bar{y}, \bar{c}_{\nu_1}) \wedge \varphi(\bar{a}_i, \bar{y}, \bar{c}_{\nu_2})]$,[1]

and, in addition, φ satisfies

(c) $\varphi(\bar{x}, \bar{y}_1, \bar{z}) \wedge \varphi(\bar{x}, \bar{y}_2, \bar{z}) \implies \bar{y}_1 = \bar{y}_2$.

Shelah proved in [24] that T^*_{feq} exhibits a nonamenability behavior provided that some cardinal arithmetic assumptions close to the failure of the singular cardinal hypothesis are satisfied. This does not necessarily imply high nonamenability as was proved also in [24] that this theory is in fact amenable at any cardinal which is the successor of a cardinal κ satisfying $\kappa^{<\kappa} = \kappa$. In [9] we generalized the first of these two results by showing that any theory with oak property satisfies the same nonamenability results as those of T^*_{feq}, and we gave some more circumstances, given in terms of PCF theory for when such nonamenability results hold. The oak property cannot be made a part of the SOP_n hierarchy, as [9] gave a theory which has oak, and is not SOP_3, while the model completion of the theory of triangle-free graphs is an example of a SOP_3 theory which does not satisfy the oak property. On the other hand, it is also proved in [9] that no oak theory is simple. Further considerations of the oak property appear in Shelah [26] where it is proved that (under an interpretation of what it means for a class to have oak) that the class of groups has this property. That paper also gives further universality results in the context of Abelian groups.

5 Some Applications in Analysis and Topology

There is a rich literature concerning the universality problem in the various classes of compact spaces coming from analysis, such as Corson and Eberlein compacta, where by an embedding we usually mean the existence of a continuous surjection (see, e.g., [1]). Many of these questions were nicely resolved by the σ-functor of Todorčević [45] which gives for every such space K another space $\sigma(K)$ in the same class such that $\sigma(K)$ is not a continuous image of K. The class of uniform Eberlein compacta (UEC), which are those compact spaces that are homeomorphic to a weakly compact subspace of a Hilbert space, seems to be an odd one out in these problems since neither the method of generalized Szlenk invariants as employed in

[1] nor the σ-functor gives any results in this class. The reason is, as the authors of [1] observed, that their invariant defined for all Eberlein spaces trivializes in the case where the Eberlein is uniform, while $\sigma(K)$ for a uniform Eberlein compact space K is Eberlein compact but not necessarily uniform. Bell in [2] made a major advance in the universality problem of the uniform Eberlein compacta which had been completely open since it was posed in the 1977 paper of Benyamini, Rudin, and Wage [3]. Namely, Bell defined a certain algebraic structure, the so-called c-algebra, and he proved that there is a universal UEC of weight λ if and only if there is a universal c-algebra of size λ. In the same paper Bell showed that if $2^{<\lambda} = \lambda$, there is a c-algebra of size λ which is universal not merely under ordinary embeddings but also under a stronger notion of a c-embedding. We shall call such algebras c-*universal* (see Definition 5.1 below). He also provided negative consistency results in models obtained by adding Cohen subsets to a regular cardinal.

Definition 5.1

(1) A subset C of a Boolean algebra B has *the nice property* if for no finite $F \subseteq C$ do we have $\bigvee F = 1$. A Boolean algebra B is a c-*algebra* if and only if there is a family $\langle A_n : n < \omega\rangle$ of pairwise disjoint subsets of B, each consisting of pairwise disjoint elements whose union has the nice property and generates B.
(2) If B_l for $l \in \{0, 1\}$ are c-algebras with fixed sequences $\langle A^l_n : n < \omega\rangle$ of subsets exemplifying that B_l is a c-algebra, then a 1-1 Boolean homomorphism $f : B_0 \to B_1$ is a c-*embedding* iff $f``A^0_n \subseteq A^1_n$ for all $n < \omega$.

Note that the notion of a c-algebra is not first-order so the Kojman-Shelah results from [17] do not directly apply. We showed in Džamonja [8] that for no regular cardinal $\lambda > \aleph_1$ with $2^{\aleph_0} > \lambda$ can there exist $< 2^{\aleph_0}$ c-algebras of size λ such that every c-algebra of size λ embeds into one of them. These results were continued in Džamonja [7] which contains both negative and positive results about the existence of universal c-algebras and UEC. On the other hand, we proved a positive consistency result showing that under certain non-*GCH* assumptions there can be a family of UEC of a relatively small size ($\lambda^{++} < 2^{\lambda^+}$) each of which has weight λ^+ and which are jointly universal for UEC of weight λ^+. The negative results are the ones relevant to this paper and they were obtained using the method of Kojman-Shelah invariants. The appropriate definition in this context turned out to be the following.

Definition 5.2 Let λ be a regular cardinal and $\langle C_\delta : \delta \in S\rangle$ a club guessing sequence on λ with $C_\delta = \langle \alpha^\delta_i : i < i^*\rangle$ an increasing enumeration (so all C_δ have the same order type). Let B be a c-algebra of size λ with a filtration $\bar{B}$, and we assume that $\langle A_n : n < \omega\rangle$ is a fixed sequence demonstrating that B is a c-algebra. Suppose that $\delta \in S$ and define for $\delta \in S$ and $b \in B \setminus B_\delta$

$$\mathrm{inv}_{\bar{B},\bar{C}}(b) \stackrel{\text{def}}{=} \{i < i^* : (\exists m \geq 1)(\exists y \in A_m \cap B_{\alpha^\delta_{i+1}} \setminus B_{\alpha^\delta_i})\,[y \geq b]\}.$$

A recent application of the method of invariants comes from Kojman-Shelah's work [15] on almost isometric embeddings between metric spaces. A map $f : X \to Y$ between metric spaces is said to be *Lipshitz with constant* $r > 0$ if for every $x, y \in X$ we have $d_Y(f(x), f(y)) < r \cdot d_X(x, y)$. X is *almost-isometrically embeddable into* Y if and only if for every $r > 1$ there is a continuous injection $f : X \to Y$ such that both f and f^{-1} are Lipshitz with constant r, which is called *bi-Lipshitz with constant* r. Among many interesting results about such embeddings, [15] also gives the following.

Theorem 5.3 ([15]) *If* $\aleph_1 < \lambda < 2^{\aleph_0}$ *is regular then for every* $\kappa < 2^{\aleph_0}$ *and metric spaces* $\{(X, d_i) : i < \kappa\}$ *of size* λ*, there exists a metric space of size* λ *that is not almost-isometrically embeddable into any* (X, d_i).

The proof of the theorem again uses the method of invariants but with a twist. Namely, one defines two kind of invariants, $\mathrm{inv}^{\mathrm{dom}}$ and $\mathrm{inv}^{\mathrm{rng}}$, as follows, where we are using the same notation for club guessing sequences as above.

Definition 5.4 Suppose that (X, d) is a metric space with universe λ, $\bar{C}$ is an $S^{\lambda}_{\aleph_0}$ club guessing sequence, $\delta \in S^{\lambda}_{\aleph_0}$, $\beta > \delta$, and $K \geq 1$ is an integer. We consider X as being given in the filtration $\bar{X} = \{\alpha : \alpha < \lambda\}$. Then

$$\mathrm{inv}^{\mathrm{dom}}_{\bar{C},X,\delta}(\beta) = \{n < \omega : d(\beta, \alpha^{\delta}_n)/d(\beta, \alpha^{\delta}_{n+1}) > 2K^2\},$$

and

$$\mathrm{inv}^{\mathrm{rng}}_{\bar{C},X,\delta}(\beta) = \{n < \omega : d(\beta, \alpha^{\delta}_n)/d(\beta, \alpha^{\delta}_{n+1}) > 4K^4\}.$$

The Preservation Lemma 2.7 then says, in particular, that if $f : X \to Y$, where both X and Y are metric spaces with universe λ, is bi-Lipshitz with constant K, then there is a club E of λ such that for every $\delta \in E \cap S^{\lambda}_{\aleph_0}$ and $\beta > \delta$, we have $f(\beta) > \delta$ and $\mathrm{inv}^{\mathrm{dom}}_{\bar{C},X,\delta}(\beta) = \mathrm{inv}^{\mathrm{rng}}_{\bar{C},Y,\delta}(f(\beta))$. Theorem 5.3 is to be contrasted with another theorem from [15] which says that for any regular cardinal λ it is consistent that $2^{\aleph_0} > \lambda^+$ and there are λ^+ separable metric spaces of size λ such that every separable metric space of size λ almost-isometrically embeds into one of them. Earlier results about universality of metric spaces under different kinds of embeddings and involving the method of invariants were obtained by Shelah in [35].

Model theory of metric spaces is also one of the subjects of Ustvyasov's Ph.D. thesis [46] and his joint work with Shelah in [40] where they concentrate on complete metric spaces. A model-theoretic approach to Banach spaces pays off in the Shelah-Ustvyasov paper [41] where they prove that the appropriately axiomatized theory of Banach spaces has SOP_n for all $n \geq 3$ and hence draw the negative universality results provided by SOP_4 (see §4 above) where the notion of embedding is isometry. Note that if $\lambda = 2^{<\lambda} > \aleph_0$ then there is an isometrically universal Banach space of size λ. Universality results in Banach spaces are quite well studied classically; maybe the most well-known result in this vein is that of Szlenk in [42] where it was proved that there is no universal reflexive separable Banach space.

The use of club guessing in topology is discussed as part of Dow's recent survey article [6] where he also gives a nice application to the nonexistence of a certain embeddability behavior between maximal almost disjoint families of countable subsets of ω_2 (Proposition 2.2 of that work).

6 Some Applications in Algebra

A very fruitful application of the Kojman-Shelah method of invariants has been in the theory of infinite Abelian groups which we shall take in their additive notation. In [19], Kojman and Shelah study the problem of universality in several kinds of groups under various kinds of embeddings. Many classes of groups simply have a universal element under ordinary embeddings in every infinite cardinality, namely, there is always a universal group, a universal p-group (for any prime p), a universal torsion group, and a universal torsion-free group (see [19]). On the other hand, there is no universal reduced p-group. The situation becomes different when one restricts the kind of embeddings and the kind of groups one considers. Of particular interest are *pure* embeddings where a group monomorphism $f : H \to G$ is a pure embedding if $f``H$ satisfies that for all $n < \omega$, $nf``H = nG \cap f``H$. In other words, $f``H$ is a *pure* subgroup of G.

The following is the appropriate notion of the invariant.

Definition 6.1 Suppose that $\lambda > \aleph_0$ is regular cardinal, G is an Abelian group of size λ given with its filtration $\bar{G}$, and $\langle C_\delta : \delta \in S \subseteq \lambda\rangle$ is a club guessing sequence on λ where for each δ the increasing enumeration of C_δ is $\langle \alpha_i^\delta : i < i_\delta^*\rangle$. For $g \in G$ and $\delta \in S$ we define

$$\mathrm{inv}_{\bar{G},\bar{C},\delta}(g) \stackrel{\mathrm{def}}{=} \{i < i_\delta^* : g \in \bigcup_{n<\omega}((G_{\alpha_{i+1}^\delta} + nG) \setminus (G_{\alpha_i^\delta} + nG))\}.$$

A preservation lemma can be proved for this type of invariant and pure embedding. The paper [19] gives a number of constructions of various types of groups with the prescribed invariant which allows for the proof of several theorems one of which is the following.

Theorem 6.2 ([19]) *Suppose that λ is regular and that for some μ, $\mu^+ < \lambda < \mu^{\aleph_0}$, while p is any prime. Then there is no*

(a) *purely universal separable p-group of size λ,*
(b) *universal reduced torsion-free group of size λ.*

Research on the universality in various classes of groups was continued by Shelah in [34], [35], and [38] where he considered various classes of groups under ordinary embeddings (so they are not assumed to be pure). In [34] the class considered is that of $(< \lambda)$-*stable* Abelian groups which means that for every subset A of G of size $< \lambda$ the closure in G of the subgroup $\langle A\rangle_G$ generated by A, defined as

$$\mathrm{cl}_G(\langle A\rangle_G) = \{x : \inf_{y\in\langle A\rangle_G}(\min_{i>1}\{2^{-i} : x - y \text{ divisible by } \Pi_{1<j<i} n_j\}) = 0\}$$

for some conveniently chosen and fixed increasing sequence $\langle n_i : i < \omega\rangle$ of natural numbers > 1. This notion in particular includes strongly λ-free groups and can be handled using the same definition of invariant as that of Definition 6.1. In [35] there is a deep analysis of how necessary this is, and it proceeds through a series of results about classes of trees with $\omega + 1$ levels with the thesis that these are a prototype for various classes of groups (deriving also some surprising results as to what kind of trees one needs to look at here). Of particular interest in [35] are reduced torsion-free groups and reduced separable Abelian p-groups, but the paper indeed gives a very rich selection of results on various classes of both groups and trees. This research was continued in [38] and Shelah [27], and as a combined result, one has almost a complete calculation of the universality spectrum of the reduced torsion-free Abelian groups and reduced separable p-groups.

Theorem 6.3 (Shelah) *Let λ be an infinite cardinal and $\mathcal{K}_\lambda$ the class of reduced torsion-free Abelian groups of size λ considered under ordinary embeddings.*

(a) *If $\lambda = \lambda^{\aleph_0}$ or λ is singular of countable cofinality and $(\forall\theta < \lambda)\theta^{\aleph_0} < \theta$, then there is a universal member of $\mathcal{K}_\lambda$.*

(b) *If* $\lambda < 2^{\aleph_0}$*, or for some* μ *we have* $2^{\aleph_0} < \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$*, then there is no universal member of* $\mathcal{K}_\lambda$.

Some of the remaining cases of possible cardinal arithmetic assumptions were reduced to some weak PCF assumptions, the consistency of whose failure is not known. In [35] there are also results about modules. Model theoretic properties of groups related to universality are studied in [41] where it is proved that if $\mathcal{G}$ is the "universal domain" (a monster model for groups) then it has SOP_3 and, surprisingly, it does not have SOP_4.

7 Some Applications in Graph Theory and a Representation Theorem

The universality problem in the class of graphs has a particularly long tradition; see, for example, Rado's well-known paper [22] with the construction of the Rado graph. If one considers graphs with the ordinary notion of embedding (so the edges are kept, but not necessarily non-edges—this is also called weak embedding), then under *GCH* there is a universal object in every infinite cardinality, as follows for uncountable cardinalities from the classical first-order model theory and which was also proved independently by Rado. Similar results hold for the class of graphs which omit the complete graph K_n of size n where $n \geq 3$. It is a very interesting result of Shelah's work that even when *CH* fails there can be a universal graph of size $\aleph_1$ (see Shelah [29] and Shelah [30]). Results in [33] and [10] imply that the theory of graphs is amenable. The situation becomes different when one restricts the graphs to those that omit a certain structure. For example, the model completion of the theory of triangle-free graphs is amenable but the model completion of the theory of directed graphs which omit directed cycles of length ≤ 4 has SOP_4 and is hence highly nonamenable (see [33] where a number of similar results are given). Passing to graphs that omit an infinite structure, so exiting the realm of the first-order theories, the situation immediately becomes very different. For example, it is mathematical folklore (see Komjáth and Shelah [20]) that there is no universal $K_{\aleph_0}$-free graph in any cardinality. In [20] the authors investigate the class of K_κ-free graphs and show that under *GCH* such a universal graph exists in λ if and only if κ is finite or $\mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$. They also give consistency results showing how much the universality number of this class can be when it is known that there is no one universal element. There is a very rich literature available on the problem of the existence of a universal member in various classes of graphs. For example, there is a complete classification of countable homogeneous directed graphs and countable homogeneous n-tournaments obtained by Cherlin in the memoir [5]. We cannot even begin to do justice to this rich literature in this survey, so we shall simply concentrate on the impact the club guessing method has had. This will also give us an opportune way of closing this paper by a theorem which very elegantly shifts the method of invariants from an arbitrary class of models of size $< 2^{\aleph_0}$ to a consideration of the structure of the subsets of the reals, namely, a representation theorem by Kojman.

A *ray* in a graph is a 1-way infinite path. A *tail* of such a ray is any infinite connected subgraph, and two rays are *tail equivalent* if they have a common ray. Consider the class $\mathcal{K}$ of graphs G that satisfy that for every vertex v of G the induced subgraph of G spanned by v has at most one ray, up to tail equivalence. This class can also be described in terms of forbidding certain structures. Among other theorems that Kojman proves about this class in [16] is that for a regular uncountable λ the smallest size of a family of graphs in $\mathcal{K}$ of size λ (denoted by $\mathcal{K}_\lambda$) needed to embed such graphs is at least $2^{\aleph_0}$. This theorem actually follows from a representation obtained in the following.

Theorem 7.1 (Kojman's Representation Theorem [16]) *If $\lambda > \aleph_1$ is regular then there is a surjective homomorphism from the structure $\mathcal{K}_\lambda$ partially ordered by the embedding relation to the structure $[\mathbb{R}]^{\leq\lambda}$ partially ordered by the subset relation.*

The method of invariants is still used here where its construction lemma corresponds to proving that the proposed map is surjective and its preservation lemma corresponds to showing that the map is a homomorphism. The Representation Theorem has some advantages over the construction and preservation approach because it allows for a smooth way to handle singular cardinals. In an upcoming paper, Džamonja and Thompson [11], we have used this method to consider well-founded partial orders under rank-preserving embeddings and some other classes and to prove negative universality results analogous to those in [16].

Let us finish by mentioning that guessing sequences stronger than club guessing are used in a recent Shelah paper [39] to obtain negative results about the universality of the class of graphs that omit complete bipartite graphs. The paper also gives a complete characterization of the universality problem in this class under *GCH*.

8 Some Questions

There are many open questions in this subject, and considering instances of universality in a specific class is an interesting pursuit per se. We have selected two more general questions that are in our view very important. The first is in model theory.

Question 8.1 *Does SOP_4 characterize high nonamenability? In other words, does every highly nonamenable theory have the SOP_4 property?*

The interest of this question is described in §4.

The second question is in set theory and calls for a finer understanding of our forcing iteration techniques. Namely, the reader may have noticed that Definition 4.4 does not refer to the existence of universal family $\mathcal{F}$ of size 1, namely, the universal model. The reason is that all we know how to do, in the generality of the axioms of [10] or in specific forcing proofs of universality such as Mekler and Väänänen [21] (where the authors produced consistently with *CH* a family of $\aleph_2$ trees of size $\aleph_1$ with no uncountable branches and universal under reductions) or the Kojman-Shelah theorem about separable metric spaces mentioned above, is to produce λ^+ models of size λ jointly universal for models of size λ. Hence we have this question.

Question 8.2 *Suppose that a theory T is amenable, λ is an uncountable cardinal larger than the size of T and satisfies $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$ while θ satisfies $\mathrm{cf}(\theta) > \lambda^+$. Can one find a cardinality preserving forcing extension in which $2^\lambda = \theta$ and T has a universal model of size λ^+?*

Note

1. Here '$\lhd$' stands for being a strict initial segment.

References

[1] Argyros, S. A., and Y. Benyamini, "Universal WCG Banach spaces and universal Eberlein compacts," *Israel Journal of Mathematics*, vol. 58 (1987), pp. 305–20. Zbl 0637.46010. MR 89d:46016. 283, 284, 292, 293

[2] Bell, M., "Universal uniform Eberlein compact spaces," *Proceedings of the American Mathematical Society*, vol. 128 (2000), pp. 2191–97. Zbl 0949.54023. MR 2000m:54019. 293

[3] Benyamini, Y., M. E. Rudin, and M. Wage, "Continuous images of weakly compact subsets of Banach spaces," *Pacific Journal of Mathematics*, vol. 70 (1977), pp. 309–24. Zbl 0374.46011. 293

[4] Chang, C. C., and H. J. Keisler, *Model Theory*, 2d edition, vol. 73 of *Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Co., Amsterdam, 1977. MR 58:27177. 284, 289, 290

[5] Cherlin, G. L., "The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments," *Memoirs of the American Mathematical Society*, vol. 131, no. 621 (1998). Zbl 0978.03029. MR 98k:03080. 296

[6] Dow, A., "Remarks on recent results in set-theoretic topology," preprint, 2005. 294

[7] Džamonja, M., "Universality of uniform Eberlein compacta," forthcoming in the *Proceedings of the American Mathematical Society*. 293

[8] Džamonja, M., "On uniform Eberlein compacta and c-algebras," *Topology Proceedings*, vol. 23 (1998), pp. 143–50. Zbl 0949.54024. 293

[9] Džamonja, M., and S. Shelah, "On properties of theories which preclude the existence of universal models," forthcoming in the *Annals of Pure and Applied Logic*. 284, 285, 290, 291, 292

[10] Džamonja, M., and S. Shelah, "On the existence of universal models," *Archive for Mathematical Logic*, vol. 43 (2004), pp. 901–36. Zbl 02133514. 284, 291, 296, 297

[11] Džamonja, M., and K. Thompson, "Universality results for well-founded posets," submitted to *Sarajevo Journal of Mathematics*. 297

[12] Foreman, M., and P. Komjáth, "The club guessing ideal (commentary on a theorem of Gitik and Shelah)," preprint, 2003. 286

[13] Füredi, Z., and P. Komjáth, "Nonexistence of universal graphs without some trees," *Combinatorica*, vol. 17 (1997), pp. 163–71. MR 98m:05052. 283

[14] Ishiu, T., *Club Guessing Sequences and Filters*, Ph.D. thesis, University of California, Irvine, 2003. 285

[15] Kojman, M., and S. Shelah, "Almost isometric embeddings between metric spaces," preprint, 2003. 293, 294

[16] Kojman, M., "Representing embeddability as set inclusion," *Journal of the London Mathematical Society. Second Series*, vol. 58 (1998), pp. 257–70. MR 2000a:05067. 296, 297

[17] Kojman, M., and S. Shelah, "Nonexistence of universal orders in many cardinals," *The Journal of Symbolic Logic*, vol. 57 (1992), pp. 875–91. MR 94b:03064. 284, 286, 288, 289, 293

[18] Kojman, M., and S. Shelah, "The universality spectrum of stable unsuperstable theories," *Annals of Pure and Applied Logic*, vol. 58 (1992), pp. 57–72. Zbl 0769.03019. MR 93h:03043. 290, 291

[19] Kojman, M., and S. Shelah, "Universal Abelian groups," *Israel Journal of Mathematics*, vol. 92 (1995), pp. 113–24. Zbl 0840.20057. MR 96i:20071. 294, 295

[20] Komjáth, P., and S. Shelah, "Universal graphs without large cliques," *Journal of Combinatorial Theory. Series B*, vol. 63 (1995), pp. 125–35. Zbl 0830.03021. MR 95k:05056. 296

[21] Mekler, A., and J. Väänänen, "Trees and Π^1_1-subsets of ${}^{\omega_1}\omega_1$," *The Journal of Symbolic Logic*, vol. 58 (1993), pp. 1052–70. Zbl 0795.03068. MR 94k:03063. 297

[22] Rado, R., "Universal graphs and universal functions," *Acta Arithmetica*, vol. 9 (1964), pp. 331–40. Zbl 0139.17303. MR 30:2488. 296

[23] Shelah, S., *Classification Theory and the Number of Nonisomorphic Models*, 2d edition, vol. 92 of *Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Co., Amsterdam, 1990. MR 91k:03085. 289, 290

[24] Shelah, S., "The universality spectrum: Consistency for more classes," pp. 403–20 in *Combinatorics, Paul Erdös is Eighty, Vol. 1*, Bolyai Society Mathematical Studies, János Bolyai Mathematical Society, Budapest, 1993. Proceedings of the Meeting in honor of P. Erdös, Keszthely, Hungary, 1993. Improved version available at http://www.math.rutgers.edu/~shelarch. Zbl 0805.03020. MR 95a:03043. 291, 292

[25] Shelah, S., "Analytical guide and corrections to *Cardinal Arithmetic*," preprint at http://www.math.rutgers.edu/~shelarch, http://front.math.ucdavis.edu/math.LO/9906022. 286

[26] Shelah, S., "Existence of universal structures," in preparation. 292

[27] Shelah, S., "Notes on Abelian groups," in preparation. 295

[28] Shelah, S., "Independence results," *The Journal of Symbolic Logic*, vol. 45 (1980), pp. 563–73. Zbl 0451.03017. MR 82b:03099. 284, 286

[29] Shelah, S., "On universal graphs without instances of CH," *Annals of Pure and Applied Logic*, vol. 26 (1984), pp. 75–87. Zbl 0551.03032. MR 85h:03054. 296

[30] Shelah, S., "Universal graphs without instances of CH: Revisited," *Israel Journal of Mathematics*, vol. 70 (1990), pp. 69–81. Zbl 0709.03039. MR 94e:03048. 296

[31] Shelah, S., "Advances in cardinal arithmetic," pp. 355–83 in *Finite and Infinite Combinatorics in Sets and Logic (Banff AB, 1991)*, vol. 411 of *NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences*, Kluwer Academic Publishers, Dordrecht, 1993. Zbl 0844.03028. MR 95h:03112. 286

[32] Shelah, S., *Cardinal Arithmetic*, vol. 29 of *Oxford Logic Guides*, The Clarendon Press, New York, 1994. Zbl 0848.03025. MR 96e:03001. 285

[33] Shelah, S., "Toward classifying unstable theories," *Annals of Pure and Applied Logic*, vol. 80 (1996), pp. 229–55. Zbl 0874.03043. MR 97e:03052. 291, 292, 296

[34] Shelah, S., "Universal in $(< \lambda)$-stable Abelian group," *Mathematica Japonica*, vol. 44 (1996), pp. 1–9. Zbl 0889.03022. MR 98g:03092. 295

[35] Shelah, S., "Non-existence of universals for classes like reduced torsion free Abelian groups under embeddings which are not necessarily pure," pp. 229–86 in *Advances in Algebra and Model Theory (Essen, 1994; Dresden, 1995)*, vol. 9 of *Algebra, Logic and Applications*, Gordon and Breach, Amsterdam, 1997. Zbl 0936.20044. MR 2000f:20091. 294, 295, 296

[36] Shelah, S., *Proper and Improper Forcing*, 2d edition, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998. Zbl 0889.03041. MR 98m:03002. 286

[37] Shelah, S., "On what I do not understand (and have something to say), Model Theory," *Mathematica Japonica*, vol. 51 (2000), pp. 329–77. Zbl 0965.03042. MR 2001a:03073. 290

[38] Shelah, S., "Non-existence of universal members in classes of Abelian groups," *Journal of Group Theory*, vol. 4 (2001), pp. 169–91. Zbl 1035.20044. MR 2002f:20088. 295

[39] Shelah, S., "Is there universality among the graph omitting a complete bipartite graph?" preprint, 2004. 297

[40] Shelah, S., and A. Usvyatsov, "Model theoretic stability for complete metric spaces," in preparation. 294

[41] Shelah, S., and A. Usvyatsov, "Banach spaces and groups—order properties and universal models," preprint, 2002. 294, 296

[42] Szlenk, W., "The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces," *Studia Mathematica*, vol. 30 (1968), pp. 53–61. Zbl 0169.15303. MR 37:3327. 294

[43] Thompson, K., *Universality results for ordered and directed structures*, Ph.D. thesis, University of East Anglia, 2003. 289

[44] Thompson, K., "Universality for orders and graphs which omit large substructures," submitted to *Notre Dame Journal of Formal Logic*. 289

[45] Todorčević, S., "The functor $\sigma^2 x$," *Studia Mathematica*, vol. 116 (1995), pp. 49–57. Zbl 0839.46009. 283, 284, 292

[46] Usvyatsov, A., Ph.D. thesis, Hebrew University of Jerusalem, in preparation. 294

Acknowledgments

The author thanks EPSRC for their support on an Advanced Fellowship.

School of Mathematics
University of East Anglia
Norwich, NR4 7TJ
UNITED KINGDOM
M.Dzamonja@uea.ac.uk
http://www.mth.uea.ac.uk/people/md.html