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Cluster algebras via cluster categories with infinite-dimensional morphism spaces

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Abstract

We apply our previous work on cluster characters for Hom-infinite cluster categories to the theory of cluster algebras. We give a new proof of Conjectures 5.4, 6.13, 7.2, 7.10 and 7.12 of Fomin and Zelevinsky's *Cluster algebras IV* [Compositio Math. **143** (2007), 112–164] for skew-symmetric cluster algebras. We also construct an explicit bijection sending certain objects of the cluster category to the decorated representations of Derksen, Weyman and Zelevinsky, and show that it is compatible with mutations in both settings. Using this map, we give a categorical interpretation of the E-invariant and show that an arbitrary decorated representation with vanishing E-invariant is characterized by its **g**-vector. Finally, we obtain a substitution formula for cluster characters of not necessarily rigid objects.

1. Introduction

Since their introduction by Fomin and Zelevinsky in [FZ02], cluster algebras have been found to enjoy connections with several branches of mathematics; see for instance the survey papers [GLS08, Kel09, Zel07] or the talks of the ICM 2010 [Fom, Lec, Rei]. Cluster algebras are commutative algebras generated by *cluster variables* grouped into sets of fixed finite cardinality called *clusters*. Of particular importance are cluster algebras *with coefficients*, as most known examples of cluster algebras are of this kind. In this paper, we will work with cluster algebras of geometric type with coefficients.

In [FZ07], the authors developed a combinatorial framework allowing the study of coefficients in cluster algebras. Important tools that the authors introduced are the F-polynomials and **g**-vectors. In particular, they proved that the behavior of the coefficients in any cluster algebra is governed by that of the coefficients in a cluster algebra *with principal coefficients*, using the F-polynomials (see [FZ07, Theorem 3.7]).

The authors phrased a number of conjectures, mostly regarding F-polynomials and **g**-vectors. We list some of them here:

(5.4) every F-polynomial has constant term 1;

(6.13) the **g**-vectors of the cluster variables of any given seed are *sign-coherent* in a sense to be defined;

(7.2) cluster monomials are linearly independent;

(7.10) different cluster monomials have different **g**-vectors, and the **g**-vectors of the cluster variables of any cluster form a basis of $\mathbb{Z}^r$;

(7.12) the mutation rule for **g**-vectors can be expressed using a certain piecewise-linear transformation.

Work on these conjectures includes:

– a proof of (7.2) by Sherman and Zelevinsky [SZ04] for Dynkin and affine types of rank two;
– a proof of (7.2) by Caldero and Keller [CK08] for Dynkin type;
– a proof of (7.2) by Dupont [Dup11] for affine type A;
– a proof of (7.2) by Ding, Xiao and Xu [DXX09] for affine type;
– a proof of (7.2) by Cerulli Irelli [Cer11] in type $A_2^{(1)}$ by explicit computations;
– a proof of (5.4) by Schiffler [Sch10] for cluster algebras arising from unpunctured surfaces;
– a proof of (7.2) by Demonet [Dem10] for certain skew-symmetrizable cluster algebras;
– a proof of all five conjectures by Fu and Keller [FK10] for cluster algebras categorified by Hom-finite 2-Calabi–Yau Frobenius or triangulated categories, using work of Dehy and Keller [DK08];
– a proof of (7.2) by Geiss *et al.* [GLS07] for acyclic cluster algebras;
– a proof of (5.4), (6.13), (7.10) and (7.12) in full generality by Derksen *et al.* [DWZ10] using decorated representations of quivers with potentials;
– a proof of (5.4), (6.13), (7.10) and (7.12) by Nagao [Nag10], under a certain finiteness assumption, using Donaldson–Thomas theory (see for instance [Bri10, JS08, KS08]).

In this paper, we use (generalized) cluster categories to give a new proof of (5.4), (6.13), (7.10) and (7.12) in full generality, and to prove (7.2) for any skew-symmetric cluster algebra of geometric type whose associated matrix is of full rank.

More precisely, we use the cluster category introduced by Buan *et al.* [BMRRT06] (and independently by Caldero *et al.* [CCS06] in the A_n case) and generalized to any quiver with potential by Amiot in [Ami09]. Note that this category can be Hom-infinite. We obtain applications to cluster algebras via the cluster character of Palu [Pal08], which generalized the map introduced by Caldero and Chapoton in [CC06]. It was extended in [Pla11] to possibly Hom-infinite cluster categories. In particular, we have to restrict the cluster character to a suitable subcategory $\mathcal{D}$ of the cluster category.

Using this cluster character, we give categorical interpretations of F-polynomials and **g**-vectors which allow us to prove the conjectures mentioned above. We prove (7.2), (6.13), (7.10) and (7.12) in §3.2 and (5.4) in §3.3. Some of our results concerning rigid objects in §3.1 and indices in §3.2 are used in [IIKKN10a, IIKKN10b]. The methods we use are mainly generalizations of those used for the Hom-finite case in [DK08, FK10].

The key tool that we use in our proofs is the multiplication formula proved in [Pla11, Proposition 3.16], while the proofs of Derksen, Weyman and Zelevinsky rely on a substitution formula [DWZ10, Lemma 5.2].

We also show in §4 that the setup used in [DWZ10] is closely related to the cluster-categorical approach. We prove in §4.1 that (isomorphism classes of) decorated representations over a quiver with potential are in bijection with (isomorphism classes of) objects in the subcategory $\mathcal{D}$ of the cluster category. In §§4.2 and 4.3, we give an interpretation of the F-polynomial, **g**-vector, **h**-vector and E-invariant of a decorated representation in cluster-categorical terms. In particular,

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we prove a stronger version of [DWZ10, Lemma 9.2] in Corollary 4.16. Using the substitution formula for F-polynomials [DWZ10, Lemma 5.2], we also obtain a substitution formula for cluster characters of not necessarily rigid objects (Corollary 4.14).

2. Recollections

2.1 Background on cluster algebras

We give a brief summary of the definitions and results we will need concerning cluster algebras. Our main source for the material in this section is [FZ07].

2.1.1 *Cluster algebras with coefficients.* We will first recall the definition of (skew-symmetric) cluster algebras (of geometric type).

The *tropical semifield* $\mathrm{Trop}(u_1, u_2, \ldots, u_n)$ is the abelian group (written multiplicatively) freely generated by the u_i, with an *addition* $\oplus$ defined by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)}.$$

A *quiver* is an oriented graph. Thus, it is given by a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of *vertices*, Q_1 is the set of *arrows* and s (respectively t) is a map from Q_1 to Q_0 which sends each arrow to its *source* (respectively its *target*). A quiver is *finite* if it has finitely many vertices and arrows. An *ice quiver* (see [FK10]) is a pair (Q, F), where Q is a quiver and F is a subset of Q_0. The elements of F are the *frozen vertices* of Q. It is *finite* if Q is finite.

Let r and n be integers such that $1 \leqslant r \leqslant n$. Let us denote by $\mathbb{P}$ the tropical semifield $\mathrm{Trop}(x_{r+1}, \ldots, x_n)$. Let $\mathcal{F}$ be the field of fractions of the ring of polynomials in r indeterminates with coefficients in $\mathbb{QP}$.

Let (Q, F) be a finite ice quiver, where Q has no oriented cycles of length $\leqslant 2$, and F and Q_0 have r and n elements, respectively. We will denote the elements of $Q_0 \backslash F$ by the numbers $1, 2, \ldots, r$ and those of F by $(r+1), (r+2), \ldots, n$. Let i be in $Q_0 \backslash F$. One defines the *mutation of* (Q, F) *at* i as the ice quiver $\mu_i(Q, F) = (Q', F')$ constructed from (Q, F) as follows:

- the sets Q_0' and F' are equal to Q_0 and F, respectively;
- all arrows not incident with i in Q are kept;
- for each subquiver of Q of the form $j \to i \to \ell$, an arrow from j to ℓ is added;
- all arrows incident to i are reversed;
- arrows from a maximal set of pairwise disjoint oriented cycles of length two in the resulting quiver are removed.

A *seed* is a pair $((Q, F), \mathbf{x})$, where (Q, F) is an ice quiver as above, and $\mathbf{x} = \{x_1, \ldots, x_r\}$ is a free generating set of the field $\mathcal{F}$. Given a vertex i of $Q_0 \backslash F$, the *mutation* of the seed $((Q, F), \mathbf{x})$ at the vertex i is the pair $\mu_i((Q, F), \mathbf{x}) = ((Q', F'), \mathbf{x}')$, where:

- (Q', F') is the mutated ice quiver $\mu_i(Q, F)$;
- $\mathbf{x}' = \mathbf{x} \backslash \{x_i\} \cup \{x_i'\}$, where x_i' is obtained from the *exchange relation*

$$x_i x_i' = \prod_{\substack{\alpha \in Q_1 \\ s(\alpha) = i}} x_{t(\alpha)} + \prod_{\substack{\alpha \in Q_1 \\ t(\alpha) = i}} x_{s(\alpha)}.$$

The mutation of a seed is still a seed, and the mutation at a fixed vertex is an involution.

Fix an *initial seed* $((Q, F), \mathbf{x})$.

- The sets $\mathbf{x}'$ obtained by repeated mutation of the initial seed are the *clusters*.
- The elements of the clusters are the *cluster variables*.
- The $\mathbb{Z}\mathbb{P}$-subalgebra of $\mathcal{F}$ generated by all cluster variables is the *cluster algebra* $\mathcal{A} = \mathcal{A}((Q, F), \mathbf{x})$.

Suppose that $n = 2r$. A cluster algebra has *principal coefficients at a seed* $((Q', F'), \mathbf{x}')$ if there is exactly one arrow from $(r + \ell)$ to ℓ (for $\ell = 1, 2, \ldots, r$), and if these are the only arrows whose source or target lies in F'.

2.1.2 *Cluster monomials and* $\mathbf{g}$*-vectors.* Given an ice quiver (Q, F), we associate to it an $(n \times r)$ matrix $\tilde{B} = (b_{ij})$, where each entry b_{ij} is the number of arrows from i to j minus the number of arrows from j to i.

Let $((Q, F), \mathbf{x})$ be a seed of a cluster algebra $\mathcal{A}$. A *cluster monomial* in $\mathcal{A}$ is a product of cluster variables lying in the same cluster.

To define $\mathbf{g}$*-vectors*, we shall need a bit of notation. For any integer j between 1 and r, let $\hat{y}_j$ be defined as

$$\hat{y}_j = \prod_{\ell \in Q_0} x_\ell^{b_{\ell j}}.$$

Let $\mathcal{M}$ be the set of non-zero elements of $\mathcal{A}$ which can be written in the form

$$z = R(\hat{y}_1, \ldots, \hat{y}_r) \prod_{j=1}^{n} x_j^{g_j},$$

where $R(u_1, \ldots, u_r)$ is an element of $\mathbb{Q}(u_1, \ldots, u_r)$. Note that all cluster monomials belong to $\mathcal{M}$. By [FZ07, Proposition 7.8], if the matrix $\tilde{B}$ is of full rank r, then any element of $\mathcal{M}$ can be written in a unique way in the form above, with R *primitive* (that is, R can be written as a ratio of two polynomials, neither of which is divisible by any of the u_j). In that case, if z is an element of $\mathcal{M}$ written as above with R primitive, the vector

$$\mathbf{g}(z) = (g_1, \ldots, g_r)$$

is the $\mathbf{g}$*-vector* of z.

Let us now state Conjectures 7.2, 7.10 and 7.12 of [FZ07].

(7.2) Cluster monomials are linearly independent over $\mathbb{Z}\mathbb{P}$.

(7.10) Different cluster monomials have different $\mathbf{g}$-vectors; the $\mathbf{g}$-vectors of the cluster variables of any cluster form a $\mathbb{Z}$-basis of $\mathbb{Z}^r$.

(7.12) Let $\mathbf{g} = (g_1, \ldots, g_r)$ and $\mathbf{g}' = (g_1', \ldots, g_r')$ be the $\mathbf{g}$-vectors of one cluster monomial with respect to two clusters t and t' related by one mutation at the vertex i. Then we have

$$g_j' = \begin{cases} -g_i & \text{if } j = i, \\ g_j + [b_{ji}]_+ g_i - b_{ji} \min(g_i, 0) & \text{if } j \neq i, \end{cases}$$

where $B = (b_{j\ell})$ is the matrix associated to the seed t, and we set $[x]_+ = \max(x, 0)$ for any real number x.

2.1.3 *F-polynomials.* Let $\mathcal{A}$ be a cluster algebra with principal coefficients at a given seed $((Q, F), \mathbf{x})$. Let t be a seed of $\mathcal{A}$ and ℓ be a vertex of Q that is not in F. Then the ℓth cluster

variable of t can be written as a subtraction-free rational function in variables $x_1, \ldots, x_{2r}$. Following [FZ07, Definition 3.3], we define the *F-polynomial* $F_{\ell,t}$ as the specialization of this rational function at $x_1 = \cdots = x_r = 1$. It was proved in [FZ07, Proposition 3.6] that $F_{\ell,t}$ is indeed a polynomial.

We now state Conjecture 5.4 of [FZ07]: *every F-polynomial has constant term* 1.

2.1.4 *Y-seeds and their mutations.* We now recall the notion of *Y*-seeds from [FZ07]. As above, let $1 \leqslant r \leqslant n$ be integers, and let $\mathbb{P}$ be the tropical semifield in the variables $x_{r+1}, \ldots, x_n$.

A *Y-seed* is a pair $(Q, \mathbf{y})$, where:

− Q is a finite quiver without oriented cycles of length $\leqslant 2$; and

− $\mathbf{y} = (y_1, \ldots, y_r)$ is an element of $\mathbb{P}^r$.

Let $(Q, \mathbf{y})$ be a *Y*-seed, and let i be a vertex of Q. The *mutation of $(Q, \mathbf{y})$ at the vertex i* is the *Y*-seed $\mu_i(Q, \mathbf{y}) = (Q', \mathbf{y}')$, where:

− Q is the mutated quiver $\mu_i(Q)$; and

− $\mathbf{y}' = (y_1', \ldots, y_r')$ is obtained from $\mathbf{y}$ using the mutation rule

$$y_j' = \begin{cases} y_i^{-1} & \text{if } i = j, \\ y_j y_i^m (y_i \oplus 1)^{-m} & \text{if there are } m \text{ arrows from } i \text{ to } j, \\ y_j (y_i \oplus 1)^m & \text{if there are } m \text{ arrows from } j \text{ to } i. \end{cases}$$

If, to any seed $((Q, F), \mathbf{x})$ of a cluster algebra, we associate a *Y*-seed $(Q, \mathbf{y})$ defined by

$$y_j = \prod_{i=r+1}^{n} x_i^{b_{ij}},$$

then for any such seed and its associated *Y*-seed, and for any vertex i of Q, we have that the *Y*-seed associated to $\mu_i((Q, F), \mathbf{x})$ is $\mu_i(Q, \mathbf{y})$. This was proved in [FZ07] after Definition 2.12.

2.2 Quivers with potentials and their mutations

We recall the notion of a quiver with potential from [DWZ08]. Let Q be a finite quiver. Denote by $\widehat{kQ}$ its *completed path algebra*, that is, the k-algebra whose underlying k-vector space is

$$\prod_{w \text{ path}} kw$$

and whose multiplication is deduced from the composition of paths by distributivity (by convention, we compose paths from right to left). It is a topological algebra for the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the ideal of $\widehat{kQ}$ generated by the arrows of Q. A *potential* on Q is an element W of the space

$$\mathrm{Pot}(Q) = \widehat{kQ}/C,$$

where C is the closure of the commutator subspace $[\widehat{kQ}, \widehat{kQ}]$ in $\widehat{kQ}$. In other words, it is a (possibly infinite) linear combination of cyclically inequivalent oriented cycles of Q. The pair (Q, W) is a *quiver with potential*.

Given any arrow a of Q, the *cyclic derivative with respect to a* is the linear map ∂_a from $\mathrm{Pot}(Q)$ to $\widehat{kQ}$ whose action on (equivalence classes of) oriented cycles is given by

$$\partial_a(b_r \cdots b_2 b_1) = \sum_{b_i = a} b_{i-1} b_{i-2} \cdots b_1 b_r b_{r-1} \cdots b_{i+1}.$$

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The *Jacobian algebra* $J(Q,W)$ of a quiver with potential (Q,W) is the quotient of the algebra $\widehat{kQ}$ by the closure of the ideal generated by the cyclic derivatives $\partial_a W$, as a ranges over all arrows of Q.

The above map is generalized as follows. For any path p of Q, define ∂_p as the linear map from $\mathrm{Pot}(Q)$ to $\widehat{kQ}$ whose action on any (equivalence class of) oriented cycle c is given by

$$\partial_p(c) = \sum_{c=upv} vu + \sum_{\substack{c=p_1 w p_2 \\ p=p_2 p_1}} w,$$

where the sums are taken over all decompositions of c into paths of smaller length, with u, v and w possibly trivial paths, and p_1 and p_2 non-trivial paths.

Let (Q,W) be a quiver with potential. In order to define the *mutation of (Q,W) at a vertex ℓ*, we must recall the process of reduction of a quiver with potential. Let Λ be the k-algebra given by $\bigoplus_{i \in Q_0} ke_i$, where e_i is the idempotent associated to the vertex i. Two quivers with potentials (Q,W) and (Q',W') are *right-equivalent* if $Q_0 = Q'_0$ and there exists a Λ-algebra isomorphism $\varphi : \widehat{kQ} \longrightarrow \widehat{kQ'}$ sending the class of W to the class of W' in $\mathrm{Pot}(Q')$.

A quiver with potential (Q,W) is *trivial* if W is a (possibly infinite) linear combination of paths of length at least 2, and $J(Q,W)$ is isomorphic to Λ. It is *reduced* if W has no terms which are cycles of length at most 2.

THEOREM 2.1 [DWZ08, Theorem 4.6, Proposition 4.5]. *Any quiver with potential (Q,W) is right-equivalent to a direct sum of a reduced one $(Q_\mathrm{red}, W_\mathrm{red})$ and a trivial one $(Q_\mathrm{triv}, W_\mathrm{triv})$, both unique up to right-equivalence. Moreover, $J(Q,W)$ and $J(Q_\mathrm{red}, W_\mathrm{red})$ are isomorphic.*

We can now define the mutation of quivers with potentials. Let (Q,W) be a quiver with potential, and let ℓ be a vertex of Q not involved in any cycle of length $\leqslant 2$. Assume that W is written as a series of oriented cycles which do not begin or end in ℓ (W is always cyclically equivalent to such a potential). The *mutation of (Q,W) at vertex ℓ* is the new quiver with potential $\mu_\ell(Q,W)$ obtained from (Q,W) as follows.

(i) For any subquiver $i \xrightarrow{a} \ell \xrightarrow{b} j$ of Q, add an arrow $i \xrightarrow{[ba]} j$.

(ii) Delete any arrow a incident with ℓ and replace it by an arrow $a^\star$ going in the opposite direction; the first two steps yield a new quiver $\widetilde{Q}$.

(iii) Let $\widetilde{W}$ be the potential on $\widetilde{Q}$ defined by $\widetilde{W} = [W] + \sum a^\star b^\star [ba]$, where the sum is taken over all subquivers of Q of the form $i \xrightarrow{a} \ell \xrightarrow{b} j$, and where $[W]$ is obtained from W by replacing each occurrence of ba in its terms by $[ba]$. The first three steps yield a new quiver with potential $\widetilde{\mu}_\ell(Q,W) = (\widetilde{Q}, \widetilde{W})$.

The mutation $\mu_\ell(Q,W)$ is then defined as the reduced part of $\widetilde{\mu}_\ell(Q,W)$. Note that $\mu_\ell(Q,W)$ might contain oriented cycles of length 2, even if (Q,W) did not. This prevents us from performing iterated mutations following an arbitrary sequence of vertices.

A vertex i of (Q,W) which is not involved in any oriented cycle of length $\leqslant 2$ (and thus at which mutation can be performed) is an *admissible vertex*. An *admissible sequence of vertices* is a sequence $\underline{i} = (i_1, \ldots, i_s)$ of vertices of Q such that i_1 is an admissible vertex of (Q,W), and i_m is an admissible vertex of $\mu_{m-1}\mu_{m-2}\cdots\mu_1(Q,W)$, for $1 < m \leqslant s$. In that case, we denote by $\mu_{\underline{i}}(Q,W)$ the mutated quiver with potential $\mu_s \mu_{s-1} \cdots \mu_1(Q,W)$.

2.3 Decorated representations and their mutations

We now recall from [DWZ08, §10] the notion of a decorated representation of a quiver with potential.

Let (Q, W) be a quiver with potential, and let $J(Q, W)$ be its Jacobian algebra. A *decorated representation* of (Q, W) is a pair $\mathcal{M} = (M, V)$, where M is a finite-dimensional nilpotent $J(Q, W)$-right module and V is a finite-dimensional Λ-module (recall that Λ is given by $\bigoplus_{i \in Q_0} k e_i$).

We now turn to the *mutation* of decorated representations. Given a decorated representation $\mathcal{M} = (M, V)$ of (Q, W), and given any admissible vertex ℓ of (Q, W), we construct a decorated representation $\widetilde{\mu}_\ell(\mathcal{M}) = (\overline{M}, \overline{V})$ of $\widetilde{\mu}_\ell(Q, W)$ as follows.

We view M as a representation of the opposite quiver Q^{op} (we must work over the opposite quiver, since we use *right* modules). In particular, to each vertex i is associated a vector space M_i (which is equal to Me_i), and to each arrow $a : i \to j$ is associated a linear map $M_a : M_j \to M_i$. For any path $p = a_r \cdots a_2 a_1$, we denote by M_p the linear map $M_{a_1} M_{a_2} \cdots M_{a_r}$ and, for any (possibly infinite) linear combination $\sigma = \sum_{i \in I} \lambda_i p_i$ of paths, we denote by M_σ the linear map $\sum_{i \in I} \lambda_i M_{p_i}$ (this sum is finite since M is nilpotent). If σ is zero in $J(Q, W)$, then M_σ is the zero map. Define the vector spaces M_{in} and M_{out} by

$$M_{\mathrm{in}} = \bigoplus_{\substack{a \in Q_1 \\ s(a) = \ell}} M_{t(a)} \quad \text{and} \quad M_{\mathrm{out}} = \bigoplus_{\substack{b \in Q_1 \\ t(b) = \ell}} M_{s(b)}.$$

Define the linear map $\alpha : M_{\mathrm{in}} \longrightarrow M_\ell$ as the map given in matrix form by the line vector $(M_a : M_{t(a)} \to M_\ell \mid a \in Q_1, s(a) = \ell)$. Similarly, define $\beta : M_\ell \longrightarrow M_{\mathrm{out}}$ as the map given in matrix form by the column vector $(M_b : M_\ell \to M_{s(b)} \mid b \in Q_1, t(b) = \ell)$. Define a third map $\gamma : M_{\mathrm{out}} \longrightarrow M_{\mathrm{in}}$ as the map given in matrix form by

$$(M_{\partial_{ab} W} : M_{s(b)} \to M_{t(a)} \mid a, b \in Q_1, s(a) = t(b) = \ell).$$

Now construct $\widetilde{\mu}_\ell(\mathcal{M}) = (\overline{M}, \overline{V})$ as follows.

 – For any vertex $i \neq \ell$, set $\overline{M}_i = M_i$ and $\overline{V}_i = V_i$.
 – Define $\overline{M}_\ell$ and $\overline{V}_\ell$ by

$$\overline{M}_\ell = \frac{\mathrm{Ker}\, \gamma}{\mathrm{Im}\, \beta} \oplus \mathrm{Im}\, \gamma \oplus \frac{\mathrm{Ker}\, \alpha}{\mathrm{Im}\, \gamma} \oplus V_\ell \quad \text{and} \quad \overline{V}_\ell = \frac{\mathrm{Ker}\, \beta}{\mathrm{Ker}\, \beta \cap \mathrm{Im}\, \alpha}.$$

 – For any arrow a of Q not incident with ℓ, set $\overline{M}_a = M_a$.
 – For any subquiver of the form $i \xrightarrow{a} \ell \xrightarrow{b} j$, set $\overline{M}_{[ba]} = M_{ba}$.
 – The actions of the remaining arrows are encoded in the maps

$$\overline{\alpha} = \begin{pmatrix} -\pi\rho \\ -\gamma \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \overline{\beta} = (0 \quad \iota \quad \iota\sigma \quad 0),$$

where:

 ∗ the map $\rho : M_{\mathrm{out}} \to \mathrm{Ker}\, \gamma$ is such that its composition with the inclusion map of $\mathrm{Ker}\, \gamma$ gives the identity map of $\mathrm{Ker}\, \gamma$;
 ∗ the map $\pi : \mathrm{Ker}\, \gamma \to \mathrm{Ker}\, \gamma / \mathrm{Im}\, \beta$ is the natural projection map;

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* the map $\sigma : \operatorname{Ker} \alpha / \operatorname{Im} \gamma \to \operatorname{Ker} \alpha$ is such that its composition with the projection map $\operatorname{Ker} \alpha \to \operatorname{Ker} \alpha / \operatorname{Im} \gamma$ gives the identity map of $\operatorname{Ker} \alpha / \operatorname{Im} \gamma$;
* the map $\iota : \operatorname{Im} \gamma \to M_{\mathrm{in}}$ is the natural inclusion map.

It is shown in [DWZ08, Proposition 10.7] that $\widetilde{\mu}_\ell(\mathcal{M})$ is indeed a decorated representation of $\widetilde{\mu}_\ell(Q, W)$.

2.4 Some invariants of decorated representations

In this section, we recall from [DWZ10, DWZ08] the definitions of F-polynomial, **g**-vector, **h**-vector and E-invariant of a decorated representation.

We fix a quiver with potential (Q, W) and a decorated representation $\mathcal{M} = (M, V)$ of (Q, W). We number the vertices of Q from 1 to n.

The F-*polynomial* of $\mathcal{M}$ is the polynomial of $\mathbb{Z}[u_1, \ldots, u_n]$ defined by

$$F_{\mathcal{M}}(u_1, \ldots, u_n) = \sum_e \chi(\operatorname{Gr}_e(M)) \prod_{i=1}^n u_i^{e_i}.$$

The **g**-*vector* of $\mathcal{M}$ is the vector $\mathbf{g}_{\mathcal{M}} = (g_1, \ldots, g_n)$ of $\mathbb{Z}^n$, where

$$g_i = \dim \operatorname{Ker} \gamma_i - \dim M_i + \dim V_i,$$

where γ_i is the map $\gamma : M_{\mathrm{out}} \longrightarrow M_{\mathrm{in}}$ defined in § 2.3.

The **h**-*vector* of $\mathcal{M}$ is the vector $\mathbf{h}_{\mathcal{M}} = (h_1, \ldots, h_n)$ of $\mathbb{Z}^n$, where

$$h_i = -\dim \operatorname{Ker} \beta_i,$$

where β_i is the map $\beta : M_i \longrightarrow M_{\mathrm{out}}$ defined in § 2.3.

The E-*invariant* of $\mathcal{M}$ is the integer

$$E(\mathcal{M}) = \dim \operatorname{Hom}_{J(Q,W)}(M, M) + \sum_{i=1}^n g_i \dim M_i,$$

where $(g_1, \ldots, g_n)$ is the **g**-vector of $\mathcal{M}$.

Let $\mathcal{N} = (N, U)$ be another decorated representation of (Q, W). The E-invariant can also be defined using the two integer-valued invariants

$$E^{\mathrm{inj}}(\mathcal{M}, \mathcal{N}) = \dim \operatorname{Hom}_{J(Q,W)}(M, N) + \sum_{i=1}^n (\dim M_i) g_i(\mathcal{N})$$

and

$$E^{\mathrm{sym}}(\mathcal{M}, \mathcal{N}) = E^{\mathrm{inj}}(\mathcal{M}, \mathcal{N}) + E^{\mathrm{inj}}(\mathcal{N}, \mathcal{M}).$$

We have that $E(\mathcal{M}) = E^{\mathrm{inj}}(\mathcal{M}, \mathcal{M}) = (1/2) E^{\mathrm{sym}}(\mathcal{M}, \mathcal{M})$.

2.5 Cluster categories and cluster characters

In this section, we recall the definition of the (generalized) cluster category of a quiver with potential from [Ami09] and some results on cluster characters from [Pla11].

2.5.1 *Cluster categories.* Let (Q, W) be any quiver with potential. We first recall a construction of [Gin06]. Define a graded quiver $\overline{Q}$ from Q in the following way:

– the quiver $\overline{Q}$ has the same vertices as the quiver Q;
– the set of arrows of Q is contained in that of $\overline{Q}$, and these arrows have degree zero;

– for each arrow $a : i \to j$ of Q, add an arrow $a^* : j \to i$ of degree -1 in $\overline{Q}$;

– for each vertex i of Q, add a loop $t_i : i \to i$ of degree -2 in $\overline{Q}$.

From the graded quiver $\overline{Q}$, we construct a differential graded algebra (dg algebra for short) $\Gamma = \Gamma_{Q,W}$ as follows. As a graded algebra, Γ is the completed path algebra of $\overline{Q}$. In particular, its degree ℓ component is

$$\Gamma^\ell = \prod_{\substack{w \text{ path in } \overline{Q} \\ \deg(w)=\ell}} kw,$$

for any integer ℓ.

The differential d of Γ is the unique continuous and k-linear differential acting as follows on the arrows:

– for any arrow a of Q, $d(a) = 0$;

– for any arrow a of Q, $d(a^*) = \partial_a W$;

– for any vertex i of Q, $d(t_i) = e_i(\sum_{a \in Q_1}(aa^* - a^*a))e_i$.

The dg algebra thus defined is the *complete Ginzburg dg algebra*. Notice that $\mathrm{H}^0\Gamma$ is canonically isomorphic to $J(Q, W)$.

Consider now the derived category $\mathcal{D}\Gamma$ of Γ (for background material on the derived category of a dg algebra, see, for example, [Kel94] or [KY11]). Let $\mathrm{per}\,\Gamma$ be the *perfect derived category* of Γ, that is, the smallest triangulated full subcategory of $\mathcal{D}\Gamma$ containing Γ and closed under taking direct summands. Denote by $\mathcal{D}_{\mathrm{fd}}\Gamma$ the full subcategory of $\mathcal{D}\Gamma$ whose objects are those of $\mathcal{D}\Gamma$ with finite-dimensional total homology. It was proved in [KY11, Theorem 2.17] that $\mathcal{D}_{\mathrm{fd}}\Gamma$ is a triangulated subcategory of $\mathrm{per}\,\Gamma$.

Following [Ami09, Definition 3.5] (and [KY11, §4] in the case where $J(Q, W)$ is infinite dimensional), we define the *cluster category of* (Q, W) as the idempotent completion of the triangulated quotient $\mathrm{per}\,\Gamma/\mathcal{D}_{\mathrm{fd}}\Gamma$. We denote it by $\mathcal{C} = \mathcal{C}_{Q,W}$.

2.5.2 *Cluster characters.* Let $\mathcal{T}$ be a triangulated category, with suspension functor Σ. Let $T = \bigoplus_{i=1}^n T_i$ be a rigid object of $\mathcal{T}$, where the T_i are indecomposable and pairwise non-isomorphic (a *rigid* object is an object with no self-extensions). Assume that $\mathrm{pr}_\mathcal{T}T$ is a Krull–Schmidt subcategory, where $\mathrm{pr}_\mathcal{T}T$ is the full subcategory of $\mathcal{T}$ whose objects are those X for which there exists a triangle

$$T_1^X \longrightarrow T_0^X \longrightarrow X \longrightarrow \Sigma T_1^X$$

with T_0^X and T_1^X in $\mathrm{add}\,T$.

In this situation, following [DK08, Pal08], we define the *index of X with respect to T* as the element of the Grothendieck group $\mathrm{K}_0(\mathrm{add}\,T)$ given by

$$\mathrm{ind}_T X = [T_0^X] - [T_1^X].$$

This is well defined since we assumed $\mathrm{pr}_\mathcal{T}T$ to be Krull–Schmidt. We then write $\mathrm{ind}_T X = \sum_{i=1}^n [\mathrm{ind}_T X : T_i][T_i]$, where $[\mathrm{ind}_T X : T_i]$ is an integer for $i = 1, 2, \ldots, n$.

Our preferred example is the case when $\mathcal{T}$ is the cluster category $\mathcal{C}$ of a quiver with potential (Q, W), and T is the object $\Sigma^{-1}\Gamma$ (or any mutation of $\Sigma^{-1}\Gamma$ in the sense of [KY11], see §2.6). We assume now that we work in that case.

Consider the full subcategory $\mathcal{D}$ of $\mathcal{C}$ whose objects are those M of $\mathrm{pr}_{\mathcal{C}} T \cap \mathrm{pr}_{\mathcal{C}} \Sigma T$ such that $\mathrm{Hom}_{\mathcal{C}}(T, M)$ is finite dimensional. Following [Pla11, Definition 3.10], we define the *cluster character with respect to T* as the map sending each (isomorphism class of) object M of $\mathcal{D}$ to the element (notation is explained below)

$$X'_M = x^{\mathrm{ind}_T \Sigma^{-1} M} \sum_{e \in \mathbb{N}^n} \chi(\mathrm{Gr}_e(\mathrm{Hom}_{\mathcal{C}}(T, M))) x^{-\iota(e)}$$

of $\mathbb{Q}(x_1, \ldots, x_n)$.

Here, χ is the Euler–Poincaré characteristic and, for any $\mathrm{End}_{\mathcal{C}}(T)$-module Z, $\mathrm{Gr}_e(Z)$ is the quiver Grassmannian of dimension vector e of Z, that is, the projective variety whose points are the submodules of Z with dimension vector e. For any dimension vector e, $\iota(e)$ is the expression $\mathrm{ind}_T Y + \mathrm{ind}_T \Sigma^{-1} Y$ for any object Y of $\mathcal{D}$ such that e is the dimension vector of $\mathrm{Hom}_{\mathcal{C}}(T, Y)$ (it was shown in [Pla11, Lemma 3.6] that $\iota(e)$ does not depend on the choice of such an object Y). Finally, for any element $\underline{t} = \sum_{i=1}^n \lambda_i [T_i]$ of $\mathrm{K}_0(\mathrm{add}\, T)$, we denote by $x^{\underline{t}}$ the product $\prod_{i=1}^n x_i^{\lambda_i}$.

As proved in [Pla11, Proposition 3.6], this cluster character satisfies the following *multiplication formula*: for any objects X and Y of $\mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y)$ is one dimensional, if

$$X \longrightarrow E \longrightarrow Y \longrightarrow \Sigma X \quad \text{and} \quad Y \longrightarrow E' \longrightarrow X \longrightarrow \Sigma Y$$

are non-split triangles, then E and E' lie in $\mathcal{D}$, and we have the equality

$$X'_X X'_Y = X'_E + X'_{E'}.$$

2.6 Mutations as derived equivalences

Let (Q, W) be a quiver with potential. Assume that Q has no loops, and that i is a vertex of Q not contained in a cycle of length 2. Let (Q', W') be the mutated quiver with potential $\widetilde{\mu}_i(Q, W)$. Let Γ and Γ' be the complete Ginzburg dg algebras associated to (Q, W) and (Q', W'), respectively.

We recall here some results of [KY11] on the mutation of Γ in $\mathcal{D}\Gamma$. Let Γ_i^* be the cone in $\mathcal{D}\Gamma$ of the morphism

$$\Gamma_i \longrightarrow \bigoplus_{\alpha} \Gamma_{t(\alpha)}$$

whose components are given by left multiplication by α. Similarly, let $\Sigma \overline{\Gamma}_i^*$ be the cone of the morphism

$$\bigoplus_{\beta} \Gamma_{s(\beta)} \longrightarrow \Gamma_i$$

whose components are given by left multiplication by β.

Then it was proved in the discussion after [KY11, Lemma 4.4] that the morphism $\varphi_i : \Sigma \Gamma_i^* \longrightarrow \Sigma \overline{\Gamma}_i^*$ given in matrix form by

$$\begin{pmatrix} -\beta^* & -\partial_{\alpha\beta} W \\ t_i & a^* \end{pmatrix}$$

becomes an isomorphism in $\mathcal{C}$.

Remark 2.2. In particular, the composition of the morphisms

$$\bigoplus_{\alpha} \Gamma_{t(\alpha)} \longrightarrow \Gamma_i^* \xrightarrow{\Sigma^{-1}\varphi_i} \overline{\Gamma}_i^* \longrightarrow \bigoplus_{\beta} \Gamma_{s(\beta)}$$

is given in matrix form by $(-\partial_{ab} W)$.

The theorem below describes how the mutation of an indecomposable summand of Γ can be interpreted as a derived equivalence.

THEOREM 2.3 [KY11, Theorem 3.2]. (i) *There exists a triangle equivalence $\widetilde{\mu}_i^+$ from $\mathcal{D}(\Gamma')$ to $\mathcal{D}(\Gamma)$ sending Γ_j' to Γ_j if $i \neq j$ and to the cone Γ_i^* of the morphism*

$$\Gamma_i \longrightarrow \bigoplus_\alpha \Gamma_{t(\alpha)}$$

whose components are given by left multiplication by α if $i = j$. The functor $\widetilde{\mu}_i^+$ restricts to triangle equivalences from per Γ' to per Γ and from $\mathcal{D}_{\mathrm{fd}}\Gamma'$ to $\mathcal{D}_{\mathrm{fd}}\Gamma$.

(ii) *Let Γ_{red} and Γ'_{red} be the complete Ginzburg dg algebras of the reduced part of (Q, W) and $\tilde{\mu}_i(Q, W)$, respectively. The functor $\widetilde{\mu}_i^+$ induces a triangle equivalence $(\mu_i^+)_{\mathrm{red}} : \mathcal{D}(\Gamma'_{\mathrm{red}}) \longrightarrow \mathcal{D}(\Gamma_{\mathrm{red}})$ which restricts to triangle equivalences from per Γ'_{red} to per Γ_{red} and from $\mathcal{D}_{\mathrm{fd}}\Gamma'_{\mathrm{red}}$ to $\mathcal{D}_{\mathrm{fd}}\Gamma_{\mathrm{red}}$.*

We will denote a quasi-inverse of $\widetilde{\mu}_i^+$ by the symbol $\widetilde{\mu}_i^-$. Note that these equivalences induce equivalences on the level of cluster categories, which we will also denote by $\widetilde{\mu}_i^+$ and $\widetilde{\mu}_i^-$.

In § 4.1, we will need a concrete description of $\widetilde{\mu}_i^+$ and $\widetilde{\mu}_i^-$. The functor $\widetilde{\mu}_i^+$ is the derived functor $? \otimes_{\Gamma'}^L T$, where T is the Γ'–Γ-bimodule described below. The functor $\widetilde{\mu}_i^-$ is then $\mathcal{H}om_\Gamma(T, ?)$.

As a right Γ-module, T is a direct sum $\bigoplus_{j=1}^n T_j$, where T_j is isomorphic to $e_j\Gamma$ if $i \neq j$ and T_i is the cone of the morphism

$$e_i\Gamma \longrightarrow \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} e_{t(\alpha)}\Gamma$$

whose components are given by left multiplication by α. Thus, as a graded module, T_i is isomorphic to

$$P_{\Sigma i} \oplus \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} P_\alpha,$$

where $P_{\Sigma i}$ is a copy of $\Sigma(e_i\Gamma)$, and each P_α is a copy of $e_t(\alpha)\Gamma$. We will denote by $e_{\Sigma i}$ the idempotent of $P_{\Sigma i}$ and by e_α the idempotent of P_α.

The left Γ'-module structure of T is described in terms of a homomorphism of dg algebras $\Gamma' \longrightarrow \mathcal{H}om_\Gamma(T, T)$, using the left $\mathcal{H}om_\Gamma(T, T)$-module structure of T. We will need the description of the image of some elements of Γ' under this homomorphism. This description is given below.

For any vertex j of Q, the element e_j is sent to the identity of T_j.

Any arrow δ not incident with i is sent to the map which is left multiplication by δ.

For any arrow α of Q such that $s(\alpha) = i$, the element $\alpha^\star$ is sent to the map $f_{\alpha^\star} : T_{t(\alpha)} \longrightarrow T_i$ defined by $f_{\alpha^\star}(a) = e_\alpha a$.

For any arrow β of Q such that $t(\beta) = i$, the element $\beta^\star$ is sent to the map $f_{\beta^\star} : T_i \longrightarrow T_{s(\beta)}$ defined by $f_{\beta^\star}(e_{\Sigma i}a_i + \sum_{s(\rho)=i} e_\rho a_\rho) = -\beta^* a_i - \sum_{s(\rho)=i}(\partial_{\rho\beta}W)a_\rho$.

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3. Application to skew-symmetric cluster algebras

3.1 Rigid objects are determined by their index

This section is the Hom-infinite equivalent of [DK08, § 2].

Let $\mathcal{C}$ be a triangulated category, and let $T = \bigoplus_{i=1}^{n} T_i$ be a rigid object of $\mathcal{C}$, where the T_i are indecomposable and pairwise non-isomorphic. Assume that $\mathrm{pr}_{\mathcal{C}} T$ is a Krull–Schmidt category, and that $B = \mathrm{End}_{\mathcal{C}} T$ is the completed Jacobian algebra $J(Q, W)$ of a quiver with potential (Q, W). An example of such a situation is the cluster category $\mathcal{C}_{Q,W}$, with $T = \Sigma^{-1}\Gamma$.

The main result of this section is the following.

PROPOSITION 3.1. *With the above assumptions, if X and Y are rigid objects in $\mathrm{pr}_{\mathcal{C}} T$ such that $\mathrm{ind}_T X = \mathrm{ind}_T Y$, then X and Y are isomorphic.*

The rest of the section is devoted to the proof of the proposition.

Let X be an object of $\mathrm{pr}_{\mathcal{C}} T$, and let the triangle $T_1^X \xrightarrow{f^X} T_0^X \longrightarrow X \longrightarrow \Sigma T_1^X$ be an (add T)-presentation of X. The group $\mathrm{Aut}_{\mathcal{C}}(T_1^X) \times \mathrm{Aut}_{\mathcal{C}}(T_0^X)$ acts on the space $\mathrm{Hom}_{\mathcal{C}}(T_1^X, T_0^X)$, with action defined by $(g_1, g_0) f' = g_0 f'(g_1)^{-1}$. The orbit of f^X under this action is the image of the map

$$\Phi : \mathrm{Aut}_{\mathcal{C}}(T_1^X) \times \mathrm{Aut}_{\mathcal{C}}(T_0^X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(T_1^X, T_0^X),$$
$$(g_1, g_0) \longmapsto g_0 f^X (g_1)^{-1}.$$

Our strategy is to show that if Y is another rigid object of $\mathrm{pr}_{\mathcal{C}} T$, then the orbits of f^X and f^Y must intersect (and thus coincide), proving that X and Y are isomorphic.

It was proved in [Pla11, Lemma 3.2] that the functor $F = \mathrm{Hom}_{\mathcal{C}}(T, ?)$ induces an equivalence of categories

$$\mathrm{pr}_{\mathcal{C}} T / (\Sigma T) \longrightarrow \mathrm{mod}\, B,$$

where mod B is the category of finitely presented right B-modules. Since T is rigid, this implies that F induces a fully faithful functor

$$\mathrm{add}\, T \longrightarrow \mathrm{mod}\, B.$$

Thus, we can often consider automorphisms and morphisms in the category mod B instead of in $\mathcal{C}$.

Now, let $\mathfrak{m}$ be the ideal of $J(Q, W)$ generated by the arrows of Q.

The group $A = \mathrm{Aut}_B(FT_1^X) \times \mathrm{Aut}_B(FT_0^X)$ is the limit of the finite-dimensional affine algebraic groups

$$A_n = \mathrm{Aut}_B(FT_1^X / (FT_1^X \mathfrak{m}^n)) \times \mathrm{Aut}_B(FT_0^X / (FT_0^X \mathfrak{m}^n))$$

with respect to the natural projection maps from A_{n+1} to A_n, for $n \in \mathbb{N}$.

Similarly, the vector space $H = \mathrm{Hom}_B(FT_1^X, FT_0^X)$ is the limit of the spaces

$$H_n = \mathrm{Hom}_B(FT_1^X / (FT_1^X \mathfrak{m}^n), FT_0^X / (FT_0^X \mathfrak{m}^n))$$

with respect to the natural projections. All the H_n are finite-dimensional spaces, and they are endowed with the Zariski topology. The projection maps are then continuous, and H is endowed with the limit topology.

Finally, for any integer n, we define a morphism $\Phi_n : A_n \to H_n$ which sends any element (g_1, g_0) of A_n to $g_0 f_n^X (g_1)^{-1}$, where f_n^X is the image of f^X in H_n under the canonical projection. Then the morphism Φ is the limit of the Φ_n.

The situation is summarized in the following commuting diagram.

$$
\begin{array}{ccccccccc}
A = \lim A_n & \cdots \longrightarrow & \cdots \longrightarrow & A_3 & \longrightarrow & A_2 & \longrightarrow & A_1 \\
\Big\downarrow \Phi & & & \Big\downarrow \Phi_3 & & \Big\downarrow \Phi_2 & & \Big\downarrow \Phi_1 \\
H = \lim H_n & \cdots \longrightarrow & \cdots \longrightarrow & H_3 & \longrightarrow & H_2 & \longrightarrow & H_1
\end{array}
$$

The next step is the following: we will prove that the image of Φ is the limit of the images of the Φ_n. This will follow from the lemma below.

LEMMA 3.2. *Let $(X_i)_{i \in \mathbb{N}}$ be a family of topological spaces. Let $(f_i : X_i \to X_{i-1})_{i \geqslant 1}$ be a family of continuous maps, and let $X = \lim X_i$. Let $(X_i')_{i \in \mathbb{N}}$ be another family of topological spaces, with continuous maps $(f_i' : X_i' \to X_{i-1}')_{i \geqslant 1}$, and let $X' = \lim X_i'$. Let $(u_i : X_i \to X_i')$ be a family of continuous maps such that $f_i' u_i = u_{i-1} f_i$ for all $i \geqslant 1$, and let $u = \lim u_i$. Denote by $p_i : X \to X_i$ and $p_i' : X' \to X_i'$ the canonical projections. For integers $i < j$, denote by f_{ij} (respectively f_{ij}') the composition $f_j f_{j-1} \cdots f_{i+1}$ (respectively $f_j' f_{j-1}' \cdots f_{i+1}'$). Let x' be an element of X' with the property that for all $i \in \mathbb{N}$, there exists $j \geqslant i$ such that for all $\ell \geqslant j$, $f_{i\ell}(u_\ell^{-1}(p_\ell'(x'))) = f_{ij}(u_j^{-1}(p_j'(x')))$.*

Then x' admits a preimage in X, that is, there exists $x \in X$ such that $u(x) = x'$.

Proof. This is a consequence of the Mittag–Leffler theorem; see for instance [Bou71, Corollary II.5.2]. □

The above lemma implies that the image of Φ is the limit of the images of the Φ_n. Indeed, the universal property of the limit gives an inclusion from the image of Φ to the limit of the images of the Φ_n. Let now x' be in the image of Φ, and let x_n' be its projection in the image of Φ_n. The set $\Phi_n^{-1}(x_n)$ is a closed subset of A_n and, for any $m \geqslant n$, the image of $\Phi_m^{-1}(x_m)$ in $\Phi_n^{-1}(x_n)$ is closed. Since A_n has finite dimension as a variety, the sequences of images of the $\Phi_m^{-1}(x_m)$ in $\Phi_n^{-1}(x_n)$ eventually becomes constant. Applying the above lemma, we get that x' has a preimage in A by Φ. This proves that the image of Φ is the limit of the images of the Φ_n.

We will now prove that the image of each Φ_n is open (and thus dense, since H_n is irreducible). To prove this, we pass to the level of Lie algebras. To lighten notation, we let $E_n = \operatorname{End}_B(FT_1^X / FT_1^X \mathfrak{m}) \times \operatorname{End}_B(FT_0^X / FT_0^X \mathfrak{m})$ be the Lie algebra of A_n for all positive integers n. To prove that the image of Φ_n is open, it is sufficient to show that the map

$$
\begin{aligned}
\Psi_n : E_n &\longrightarrow H_n, \\
(g_1, g_0) &\longmapsto g_0 f_n^X - f_n^X g_1
\end{aligned}
$$

is surjective.

The limit of the E_n is $E = \operatorname{End}_B(FT_1^X) \times \operatorname{End}_B(FT_0^X)$, and the limit of the Ψ_n is the map

$$
\begin{aligned}
\Psi : E &\longrightarrow H, \\
(g_1, g_0) &\longmapsto g_0 f^X - f^X g_1.
\end{aligned}
$$

The diagram below summarizes the situation.

$$
\begin{array}{ccccccccc}
E = \lim E_n & \cdots \longrightarrow & \cdots \longrightarrow & E_3 & \longrightarrow & E_2 & \longrightarrow & E_1 \\
\Big\downarrow \Psi & & & \Big\downarrow \Psi_3 & & \Big\downarrow \Psi_2 & & \Big\downarrow \Psi_1 \\
H = \lim H_n & \cdots \longrightarrow & \cdots \longrightarrow & H_3 & \longrightarrow & H_2 & \longrightarrow & H_1
\end{array}
$$

All the canonical projections are surjective.

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Lemma 3.3. *The map Ψ defined above is surjective.*

Proof. This proof is contained in the proof of [DK08, Lemma 2.1]. □

As a consequence, all the Ψ_n are surjective. Hence, the images of the Φ_n are open.

From this, we deduce that if Y is another rigid object of $\mathrm{pr}_{\mathcal{C}} T$ with $(\mathrm{add}\, T)$-presentation $T_0^X \xrightarrow{f^Y} T_1^X \longrightarrow Y \longrightarrow \Sigma T_1^X$, then X and Y are isomorphic. Indeed, by the above reasoning, the orbit of f^Y is the limit of the orbits of its projections in the H_n. But these orbits are open, and so they intersect (and coincide) with the images of the Φ_n defined above. Hence, the orbit of f^Y in H is the limit of the images of the Φ_n, and this is exactly the orbit of f^X. Therefore, X and Y are isomorphic.

The last step in proving Proposition 3.1 is to show that given $\mathrm{ind}_T X$, we can 'deduce' T_1^X and T_0^X.

An $(\mathrm{add}\, T)$-approximation $T_1^X \to T_0^X \to X \to \Sigma T_1^X$ is *minimal* if one of the following conditions hold.

- The above triangle does not admit a direct summand of the form

$$R \xrightarrow{\mathrm{id}_R} R \longrightarrow 0 \longrightarrow \Sigma R.$$

- The morphism $f : T_0^X \to X$ in the presentation has the property that for any $g : T_0^X \to T_0^X$, the equality $fg = f$ implies that g is an isomorphism.

In fact, either of these two conditions implies the other.

Lemma 3.4. *The above two conditions are equivalent if $\mathrm{pr}_{\mathcal{C}} T$ is Krull–Schmidt.*

Proof. First suppose that the presentation has the form

$$T_1' \oplus R \xrightarrow{u \oplus 1_R} T_0' \oplus R \longrightarrow X \longrightarrow \Sigma T_1^X,$$

where $f = (f', 0)$ in matrix form. Then the endomorphism g of $T_0' \oplus R$ given by $g = 1_{T_0'} \oplus 0$ is not an isomorphism, and $fg = f$.

Now suppose that the presentation admits no direct summand of the form

$$R \xrightarrow{\mathrm{id}_R} R \longrightarrow 0 \longrightarrow \Sigma R.$$

Using the Krull–Schmidt property of $\mathrm{pr}_{\mathcal{C}} T$, we can decompose both T_0^X and T_1^X as a finite direct sum of objects with local endomorphism rings. In that case, the morphism f written in matrix form (in any basis) has no non-zero entries.

Let g be an endomorphism of T_0^X such that $fg = f$. Then $f(1_{T_0^X} - g) = 0$. Consider the morphism $(1_{T_0^X} - g)$ written in matrix form. If one of its entries is an isomorphism, then by a change of basis we can write $(1_{T_0^X} - g)$ as the matrix

$$\left(\begin{array}{c|c} * & 0 \\ \hline 0 & \phi \end{array} \right),$$

where ϕ is an isomorphism. In that case, it is impossible that $f(1_{T_0^X} - g) = 0$, since f has no non-zero entries. This implies that none of the entries of the matrix of $(1_{T_0^X} - g)$ is invertible. Therefore, the diagonal entries of g are invertible (since for any element x of a local ring, if $(1 - x)$ is not invertible, then x is), while the other entries are not, and g is an isomorphism. □

LEMMA 3.5. *If X is rigid and $T_1^X \xrightarrow{\alpha} T_0^X \longrightarrow X \xrightarrow{\gamma} \Sigma T_1^X$ is a minimal (add T)-presentation, then T_1^X and T_0^X have no direct summand in common.*

Proof. The first proof of [DK08, Proposition 2.2] works in this setting. We include here a similar argument for the convenience of the reader.

Suppose that T_i is a direct factor of T_0^X. Let us prove that it is not a direct factor of T_1^X.

Applying $F = \mathrm{Hom}_\mathcal{C}(T, ?)$ to the triangle above, we get a minimal projective presentation of FX. This yields an exact sequence

$$(FX, S_i) \longrightarrow (FT_0^X, S_i) \xrightarrow{F\alpha^*} (FT_1^X, S_i),$$

where S_i is the simple $\mathrm{End}_\mathcal{C}(T)$-module at the vertex i. Since the presentation is minimal, $F\alpha^*$ vanishes, and there exists a non-zero morphism $f : FX \to S_i$. In particular, f is surjective.

Let $g : FT_1^X \longrightarrow S_i$ be a morphism. Since FT_1^X is projective, there exists a morphism $h : FT_1^X \longrightarrow FX$ such that $fh = g$.

Lift S_i to an object ΣT_i^* of $\mathcal{C}$, and lift f, g and h to morphisms $\overline{f} : X \to \Sigma T_i^*$, $\overline{g} : T_1^X \to \Sigma T_i^*$ and $\overline{h} : T_1^X \to X$ of $\mathcal{C}$. Using [Pla11, Lemma 3.2(1)], we get that $\overline{f}\overline{h} = \overline{g}$.

$$
\begin{array}{ccccc}
\Sigma^{-1}X & \xrightarrow{\Sigma^{-1}\gamma} & T_1^X & \xrightarrow{\alpha} & T_0^X \\
& \overline{h} \searrow & \ \ \downarrow \overline{g} & \sigma \nearrow & \\
X & \xrightarrow{\overline{f}} & \Sigma T_i^* & &
\end{array}
$$

Since X is rigid, $\overline{h}\Sigma^{-1}\gamma$ vanishes, and thus so does $\overline{g}\Sigma^{-1}\gamma$. Then there exists a morphism $\sigma : T_0^X \to \Sigma T_i^*$ such that $\sigma\alpha = \overline{g}$. But, since $F\alpha^* = 0$, we get that $g = (F\sigma)(F\alpha)$ vanishes.

We have thus shown that there are no non-zero morphisms from FT_1^X to S_i. Therefore, T_i is not a direct factor of T_1^X. $\qquad\square$

By the above lemma, the knowledge of $\mathrm{ind}_T X$ is sufficient to deduce the isomorphism classes of T_1^X and T_0^X in any minimal (add)-presentation of X. Therefore, if Y is another rigid object of $\mathrm{pr}_\mathcal{C} T$ with $\mathrm{ind}_T X = \mathrm{ind}_T Y$, all of the above reasoning implies that X and Y are isomorphic. This finishes the proof of Proposition 3.1.

3.2 Index and g-vectors

It was proved in [FK10, Proposition 6.2] that, inside a certain Hom-finite cluster category $\mathcal{C}$, the index of an object M with respect to a cluster-tilting object T gives the **g**-vector of X'_M with respect to the associated cluster. The authors then used this result to prove conjectures of [FZ07] in this case. In this section, we will prove a similar result, dropping the assumption of Hom-finiteness.

Let (Q, F) be a finite ice quiver, where Q has no oriented cycles of length $\leqslant 2$. Suppose that the associated matrix B has full rank r. Denote by $\mathcal{A}$ the associated cluster algebra. Let W be a potential on Q, and let $\mathcal{C} = \mathcal{C}_{Q,W}$ be the associated cluster category. Denote by $\mathcal{D}$ the full subcategory of $\mathrm{pr}_\mathcal{C} \Gamma \cap \mathrm{pr}_\mathcal{C} \Sigma^{-1}\Gamma$ whose objects are those X such that $\mathrm{Hom}_\mathcal{C}(\Sigma^{-1}\Gamma, X)$ is finite dimensional.

Following [FK10], let $\mathcal{U}$ be the full subcategory of $\mathcal{D}$ defined by

$$\mathcal{U} = \{X \in \mathcal{D} \mid \mathrm{Hom}_\mathcal{C}(\Sigma^{-1}\Gamma_j, X) = 0 \text{ for } r+1 \leqslant j \leqslant n\}.$$

Note that $\mathcal{U}$ is invariant under iterated mutation of Γ at vertices $1, 2, \dots, r$.

Let $T = \bigoplus_{j=1}^{n} T_j = \bigoplus_{j=1}^{r} T_j \oplus \bigoplus_{j=r+1}^{n} \Gamma_j$ be a rigid object of $\mathcal{D}$ reachable from Γ by mutation at an admissible sequence of vertices of Q not in F, and let G be the functor $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, ?)$ from $\mathcal{C}$ to the category of $\mathrm{End}_{\mathcal{C}}(T)$-modules. Let $X'_?$ be the associated cluster character, defined by

$$X'_M = x^{\mathrm{ind}_T M} \sum_e (\chi(\mathrm{Gr}_e(GM)))x^{-\iota(e)},$$

where $\iota(e)$ is the vector $\mathrm{ind}_T Y + \mathrm{ind}_T \Sigma Y$ for any Y such that the dimension vector of FY is e (it was proved in [Pla11, Lemma 3.6] that this vector is independent of the choice of such a Y; see also [Pal08]).

Since we only allow mutations at vertices not in F, the Gabriel quiver of T can be thought of as an ice quiver (Q^T, F) with the same set of frozen vertices as (Q, F). Let $B^T = (b_{j\ell}^T)$ be the matrix associated to (Q^T, F). According to [GSV03, Lemma 1.2] and [BFZ05, Lemma 3.2], B^T is of full rank r if B is.

Suppose now that M is an object of $\mathcal{U}$. Let us prove that X'_M then admits a **g**-vector, that is, X'_M is in the set $\mathcal{M}$ defined in §2.1.2. In order to do this, let us compute $-\iota(\delta_j)$, where δ_j is the vector whose jth coordinate is 1 and all others are 0, for $j = 1, 2, \ldots, r$.

Let T_j^* be an indecomposable object of $\mathcal{D}$ such that GT_j^* is the simple $\mathrm{End}_{\mathcal{C}}(T)$-module at the vertex j. It follows from the derived equivalence in [KY11, Theorem 3.2] that we have triangles

$$T_j \to \bigoplus_{\substack{\alpha \in Q_1^T \\ s(\alpha)=j}} T_{t(\alpha)} \to T_j^* \to \Sigma T_j \quad \text{and} \quad T_j^* \to \bigoplus_{\substack{\alpha \in Q_1^T \\ t(\alpha)=j}} T_{s(\alpha)} \to T_j \to \Sigma T_j^*.$$

We deduce from those triangles that for any $0 \leqslant \ell \leqslant n$, the ℓth entry of $-\iota(\delta_j)$ is the number of arrows in Q^T from ℓ to j minus the number of arrows from j to ℓ. This number is $b_{\ell j}^T$. Thus, with the notation of §2.1.2, we have that $x^{-\iota(\delta_j)} = \prod_{\ell=1}^{n} x_{\ell}^{b_{\ell j}} = \hat{y}_j$.

Therefore, since ι is additive, for M in $\mathcal{U}$, we have the equality

$$X'_M = x^{\mathrm{ind}_T M} \sum_e (\chi(\mathrm{Gr}_e(GM))) \prod_{j=1}^{r} \hat{y}_j^{e_j}$$

(notice that if M is in $\mathcal{U}$, then $\mathrm{Gr}_e(GM)$ is empty for all vectors e such that one of $e_{r+1}, \ldots, e_n$ is non-zero). Moreover, the rational function $R(u_1, \ldots, u_r) = \sum_e (\chi(\mathrm{Gr}_e(GM)))\prod_{j=1}^{r} u_j^{e_j}$ is in fact a polynomial with constant coefficient 1, and is thus primitive.

We have proved the following result.

PROPOSITION 3.6. *Any object M of $\mathcal{U}$ is such that X'_M admits a **g**-vector. This **g**-vector $(g_1, \ldots, g_r)$ is given by $g_j = [\mathrm{ind}_T M : T_j]$, for $1 \leqslant j \leqslant r$.*

These considerations allow us to prove the following theorem, whose parts (i), (iii) and (iv) were first shown in the same generality in [DWZ10] using decorated representations, and then in [Nag10] using Donaldson–Thomas theory.

We say that a collection of vectors of $\mathbb{Z}^r$ are *sign-coherent* if the ith coordinates of all the vectors of the collection are either all non-positive or all non-negative.

THEOREM 3.7. *Let (Q, F) be any ice quiver without oriented cycles of length $\leqslant 2$, and let $\mathcal{A}$ be the associated cluster algebra. Suppose that the matrix B associated to (Q, F) is of full rank r.*

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(i) *Conjecture 6.13 of [FZ07] holds for $\mathcal{A}$, that is, the $\mathbf{g}$-vectors of the cluster variables of any given cluster are sign-coherent.*

(ii) *Conjecture 7.2 of [FZ07] holds for $\mathcal{A}$, that is, the cluster monomials are linearly independent over $\mathbb{Z}\mathbb{P}$, where $\mathbb{P}$ is the tropical semifield in the variables $x_{r+1}, \ldots, x_n$.*

(iii) *Conjecture 7.10 of [FZ07] holds for $\mathcal{A}$, that is, different cluster monomials have different $\mathbf{g}$-vectors, and the $\mathbf{g}$-vectors of the cluster variables of any cluster form a $\mathbb{Z}$-basis of $\mathbb{Z}^r$.*

(iv) *Conjecture 7.12 of [FZ07] holds for $\mathcal{A}$, that is, if $\mathbf{g} = (g_1, \ldots, g_r)$ and $\mathbf{g}' = (g_1', \ldots, g_r')$ are the $\mathbf{g}$-vectors of one cluster monomial with respect to two clusters t and t' related by one mutation at the vertex i, then we have*

$$g_j' = \begin{cases} -g_i & \text{if } j = i, \\ g_j + [b_{ji}]_+ g_i - b_{ji} \min(g_i, 0) & \text{if } j \neq i, \end{cases}$$

where $B = (b_{j\ell})$ is the matrix associated to the seed t, and we set $[x]_+ = \max(x, 0)$ for any real number x.

Proof. Choose a non-degenerate potential W on Q, and let $\mathcal{C} = \mathcal{C}_{Q,W}$ be the associated cluster category. Let $X'_?$ be the cluster character associated to Γ.

We first prove Conjecture 6.13. We reproduce the arguments of [DK08, §2.4]. To any cluster t of $\mathcal{A}$, we associate (using [Pla11, Theorem 4.1]) a reachable rigid object T of $\mathcal{U}$, obtained by mutating at vertices not in F. Write T as the direct sum of the indecomposable objects $T_1, \ldots, T_n$. Then, for $1 \leqslant j \leqslant r$, we have that X'_{T_j} is a cluster variable lying in the cluster t. By Proposition 3.6, its $\mathbf{g}$-vector $(g_1^j, \ldots, g_r^j)$ is given by $g_\ell^j = [\text{ind}_\Gamma T_j : \Gamma_\ell]$. Now, by Lemma 3.5, any minimal add Γ-presentation of T

$$R_1 \longrightarrow R_0 \longrightarrow T \longrightarrow \Sigma R_1$$

is such that R_0 and R_1 have no direct factor in common. But this triangle is a direct sum of minimal presentations of $T_1, \ldots, T_n$. Therefore, the indices of these objects must be sign-coherent. This proves Conjecture 6.13.

Next, we prove Conjecture 7.2. We prove it in the same way as in [FK10, Corollary 4.4(b), Theorem 6.3(c)]. Using [Pla11, Theorem 4.1], we associate to any finite collection of clusters $(t_j)_{j \in J}$ of $\mathcal{A}$ a family of reachable rigid objects $(T^j)_{j \in J}$ of $\mathcal{U}$, obtained by mutating at vertices not in F (for the moment, we do not know if this assignment is unique). Let $(M_j)_{j \in J}$ be a family of pairwise non-isomorphic objects, where each M_j lies in add T^j (in particular, these objects are rigid). Any $\mathbb{Z}\mathbb{P}$-linear combination of cluster monomials can be written as a $\mathbb{Z}$-linear combination of some X'_{M_j}, where the M_j are as above. Thus, it is sufficient to show that the X'_{M_j} are linearly independent over $\mathbb{Z}$.

The key idea is to assign a degree to each x_j in such a way that each $\hat{y}_j$ is of degree one. Such an assignment is obtained by putting $\deg(x_j) = k_j$, where the k_j are rational numbers such that

$$(k_1, \ldots, k_n) B = (1, \ldots, 1).$$

This equation admits a solution, since the rank of B is r. Thus, the term of minimal degree in X'_M is $x^{\text{ind}_\Gamma M}$, for any M in $\mathcal{U}$.

Now let $(c_j)_{j \in J}$ be a family of real numbers such that $\sum_{j \in J} c_j X'_{M_j} = 0$. The term of minimal degree of this polynomial has the form $\sum_{\ell \in L} c_\ell x^{\text{ind}_\Gamma M_\ell}$ for some non-empty subset L of J, and this term must vanish. But, according to Proposition 3.1, the indices of the M_ℓ are pairwise distinct.

Thus, c_ℓ is zero for any $\ell \in L$. Repeating this argument, we get that c_j is zero for any $j \in J$. This proves the linear independence of cluster monomials.

The proof of Conjecture 7.10 goes as follows. Let $\{w_1, \dots, w_r\}$ be a cluster of $\mathcal{A}$, and let $w_1^{a_1} \cdots w_r^{a_r}$ be a cluster monomial. Let $T = \bigoplus_{j=1}^r T_j \oplus \bigoplus_{j=r+1}^n \Gamma_n$ be the rigid object of $\mathcal{C}$ associated to that cluster. Then the cluster character

$$X_M' = x^{\operatorname{ind}_\Gamma M} \sum_e (\chi(\operatorname{Gr}_e(\operatorname{Hom}_\mathcal{C}(\Sigma^{-1}\Gamma, M)))) x^{-\iota(e)}$$

sends the object $\bigoplus_{j=1}^r T_j^{a_j}$ to the cluster monomial $w_1^{a_1} \cdots w_r^{a_r}$. The **g**-vector of this cluster monomial is the index of $\bigoplus_{j=1}^r T_j^{a_j}$ by Proposition 3.6 and, by Proposition 3.1, this object is completely determined by its index. Therefore, two different cluster monomials, being associated to different rigid objects of $\mathcal{C}$, have different **g**-vectors.

Let us now prove that the **g**-vectors of $w_1, \dots, w_r$ form a basis of $\mathbb{Z}^r$. For any object M of $\mathcal{D}$, denote by $(\operatorname{ind}_\Gamma M)_0$ the vector containing the first r components of $\operatorname{ind}_\Gamma M$. In view of Proposition 3.6, it is sufficient to prove that the vectors $(\operatorname{ind}_\Gamma T_1)_0, \dots, (\operatorname{ind}_\Gamma T_r)_0$ form a basis of $\mathbb{Z}^r$.

We prove this by induction. The statement is trivially true for Γ. Now suppose that it is true for some reachable object T, as above. Let $1 \leqslant \ell \leqslant r$ be a vertex of Q, and let $T' = \mu_\ell(T)$. We can write $T' = \bigoplus_{j=1}^n T_j'$, where $T_j' = T_j$ if $j \neq \ell$, and there are triangles

$$T_\ell \longrightarrow \bigoplus_{\substack{\alpha \in Q_1^T \\ s(\alpha) = \ell}} T_{t(\alpha)} \longrightarrow T_\ell' \longrightarrow \Sigma T_\ell \quad \text{and} \quad T_\ell' \longrightarrow \bigoplus_{\substack{\alpha \in Q_1^T \\ t(\alpha) = \ell}} T_{s(\alpha)} \longrightarrow T_\ell \longrightarrow \Sigma T_\ell'$$

thanks to [KY11]. Moreover, the space $\operatorname{Hom}_\mathcal{C}(T_\ell', \Sigma T_\ell)$ is one dimensional; by applying [Pla11, Lemma 3.8] (with the T of the lemma being equal to our $\Sigma^{-1}\Gamma$), we get an isomorphism

$$(\Gamma)(T_\ell', \Sigma T_\ell) \longrightarrow D\operatorname{Hom}_\mathcal{C}(T_\ell, \Sigma T_\ell')/(\Gamma).$$

Therefore, one of the two morphisms $T_\ell' \to \Sigma T_\ell$ and $T_\ell \to \Sigma T_\ell'$ in the triangles above is in (Γ). Depending on which one is in (Γ), and applying [Pla11, Lemma 3.4(2)], we get that either

$$\operatorname{ind}_\Gamma T_j' = \begin{cases} \operatorname{ind}_\Gamma T_j & \text{if } j \neq \ell, \\ -\operatorname{ind}_\Gamma T_\ell + \displaystyle\sum_{\substack{\alpha \in Q_1^T \\ s(\alpha) = \ell}} \operatorname{ind}_\Gamma T_{t(\alpha)} & \text{if } j = \ell \end{cases}$$

or

$$\operatorname{ind}_\Gamma T_j' = \begin{cases} \operatorname{ind}_\Gamma T_j & \text{if } j \neq \ell, \\ -\operatorname{ind}_\Gamma T_\ell + \displaystyle\sum_{\substack{\alpha \in Q_1^T \\ t(\alpha) = \ell}} \operatorname{ind}_\Gamma T_{s(\alpha)} & \text{if } j = \ell \end{cases}$$

holds. Therefore, the $(\operatorname{ind}_\Gamma T_j')_0$ still form a basis of $\mathbb{Z}^r$. Conjecture 7.10 is proved.

Finally, let us now prove Conjecture 7.12. Let T and T' be reachable rigid objects related by a mutation at vertex ℓ, as above. Then we have two triangles

$$T_\ell \longrightarrow E \longrightarrow T_\ell' \longrightarrow \Sigma T_\ell \quad \text{and} \quad T_\ell' \longrightarrow E' \longrightarrow T_\ell \longrightarrow \Sigma T_\ell',$$

where $E = \bigoplus_{\substack{\alpha \in Q_1^T \\ s(\alpha) = \ell}} T_{t(\alpha)}$ and $E' = \bigoplus_{\substack{\alpha \in Q_1^T \\ t(\alpha) = \ell}} T_{s(\alpha)}$. Moreover, the dimension of $\operatorname{Hom}_\mathcal{C}(T, \Sigma T')$ is one. Thus, we can apply [Pla11, Proposition 2.7].

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Let M be a rigid object in $\mathrm{pr}_{\mathcal{C}}T$, and let $T_1^M \to T_0^M \to M \to \Sigma T_1^M$ be a minimal $(\mathrm{add}\,T)$-presentation. Then, by [Pla11, Proposition 2.7], M is in $\mathrm{pr}_{\mathcal{C}}T'$. Moreover, if $T_0^M = \overline{T}_0^M \oplus T_\ell^a$ and $T_1^M = \overline{T}_1^M \oplus T_\ell^b$, where T_ℓ is not a direct summand of $\overline{T}_0^M \oplus \overline{T}_1^M$, then the end of the proof of that proposition gives us a triangle

$$(E')^c \oplus E^{b-c} \oplus (T_\ell')^{a-c} \oplus \overline{T}_1^M \longrightarrow (T_\ell')^{b-c} \oplus \overline{T}_0^M \oplus (E')^a \longrightarrow M \longrightarrow \cdots,$$

for some integer c. Notice that $[\mathrm{ind}_T M : T_\ell] = (a - b)$, and that since T_0^M and T_0^M have no direct factor in common by Lemma 3.5, one of a and b must vanish; thus, c also vanishes, since $c \leqslant \min(a, b)$. Notice further that $b = -\min([\mathrm{ind}_T M : T_\ell], 0)$. Thus,

$$[\mathrm{ind}_{T'} M : T_j'] = \begin{cases} -[\mathrm{ind}_T M : T_\ell] & \text{if } j = \ell, \\ [\mathrm{ind}_T M : T_j] + [\mathrm{ind}_T M : T_\ell][b_{j\ell}^T]_+ - b_{j\ell}^T \min([\mathrm{ind}_T M : T_\ell], 0) & \text{if } j \neq \ell. \end{cases}$$

This proves the desired result on **g**-vectors. $\qquad\square$

Remark 3.8. Using the notation of the end of the proof of Theorem 3.7, we get that, if M is an object of $\mathcal{D}$ which is not necessarily rigid, then

$$[\mathrm{ind}_{T'} M : T_j'] = \begin{cases} -[\mathrm{ind}_T M : T_\ell] & \text{if } j = \ell, \\ [\mathrm{ind}_T M : T_j] + a[b_{j\ell}]_+ - b[-b_{j\ell}]_+ & \text{if } j \neq \ell. \end{cases}$$

Moreover, if the presentation $T_1^M \to T_0^M \to M \to \Sigma T_1^M$ is minimal, then the integer c vanishes. Indeed, in the proof of [Pla11, Proposition 2.7], c (or r in [Pla11]) is defined by means of the composition

$$T_1^M \longrightarrow \overline{T}_0^M \oplus T_\ell^a \longrightarrow \Sigma(T_\ell')^a.$$

The minimality of the presentation implies that this composition vanishes, and thus that $c = 0$.

Using Theorem 3.7, we get a refinement of [Pla11, Theorem 4.1].

COROLLARY 3.9. *The cluster character $X_?'$ associated to Γ induces a bijection between the set of isomorphism classes of indecomposable reachable rigid objects of $\mathcal{C}$ and the set of cluster variables of $\mathcal{A}$.*

Proof. It was proved in [Pla11, Theorem 4.1] that we have a surjection. We deduce from Theorem 3.7 that different indecomposable reachable rigid objects are sent to different cluster variables. Indeed, different such objects are sent to elements in $\mathcal{A}$ which are linearly independent, and thus different. $\qquad\square$

We also get that the mutation of rigid objects governs the mutation of tropical Y-variables, as shown in [Kel10, Corollary 6.9] in the Hom-finite case.

COROLLARY 3.10. *Let (Q, W) be a quiver with potential, and let $\mathcal{C}$ be the associated cluster category. Let $\underline{i} = (i_1, \ldots, i_m)$ be an admissible sequence of vertices, and let T' be the object $\mu_{\underline{i}}(\Gamma)$. Let $(Q, \mathbf{y})$ be a Y-seed, with $\mathbf{y} = (y_1, \ldots, y_n)$.*
Then $\mu_{\underline{i}}(Q, \mathbf{y})$ is given by $(\mu_{\underline{i}}(Q), \mathbf{y}')$, where

$$y_j' = \prod_{s=1}^{n} y_s^{-[\mathrm{ind}_{\Sigma^{-1}T'}\Gamma_s : \Sigma^{-1}T_j']}.$$

Proof. The result is proved by induction on m. It is trivially true for $m = 0$, that is, for empty sequences of mutations. Suppose that it is true for any sequence of m mutations.

Let $\underline{i}' = (i_1, \ldots, i_m, \ell)$ be an admissible sequence of $m+1$ mutations. Let $T'' = \mu_{\underline{i}'}(\Gamma)$ and $(\mu_{\underline{i}'}(Q), \mathbf{y}'') = \mu_{\underline{i}'}(Q, \mathbf{y})$.

Using the mutation rule for Y-seeds (see § 2.1.4) and the induction hypothesis, we get that

$$y_\ell'' = \prod_{s=1}^{n} y_s^{[\operatorname{ind}_{\Sigma^{-1}T'}\Gamma_s : \Sigma^{-1}T_j']}$$

and that, for any vertex j different from ℓ,

$$y_j'' = \prod_{s=1}^{n} y_s^{-[\operatorname{ind}_{\Sigma^{-1}T'}\Gamma_s : \Sigma^{-1}T_j'] - [\operatorname{ind}_{\Sigma^{-1}T'}\Gamma_s : \Sigma^{-1}T_\ell'][b_{\ell j}^{T'}]_+ - b_{\ell j}^{T'}\min(-[\operatorname{ind}_{\Sigma^{-1}T'}\Gamma_s : \Sigma^{-1}T_\ell'], 0)}.$$

Now, recall from the end of the proof of Theorem 3.7 that for any object M of $\operatorname{pr}_{\mathcal{C}}T'$, we have an $(\operatorname{add} T'')$-presentation

$$(E')^c \oplus E^{b-c} \oplus (T_\ell'')^{a-c} \oplus \overline{T}_1'^M \longrightarrow (T_\ell'')^{b-c} \oplus \overline{T}_0'^M \oplus (E')^a \longrightarrow M \longrightarrow \Sigma(E^b \oplus (T_\ell'')^{a-c} \oplus \overline{T}_1'^M),$$

and that $[\operatorname{ind}_{T'}M : T_\ell'] = (a - b)$. Notice also that $a = -\min([-\operatorname{ind}_{T'}M : T_\ell'], 0)$. Thus,

$$[\operatorname{ind}_{T''}M : T_j''] = \begin{cases} -[\operatorname{ind}_{T'}M : T_\ell'] & \text{if } j = \ell, \\ [\operatorname{ind}_{T'}M : T_j'] + [\operatorname{ind}_{T'}M : T_\ell'][b_{\ell j}^{T'}]_+ + b_{\ell j}^{T'}\min(-[\operatorname{ind}_{T'}M : T_\ell'], 0) & \text{if } j \neq \ell. \end{cases}$$

Replacing M by $\Sigma\Gamma_s$, and using the above computation of y_j'', we get exactly the desired equality. $\qquad\square$

Remark 3.11. The opposite category $\mathcal{C}^{\mathrm{op}}$ is triangulated with suspension functor $\Sigma_{\mathrm{op}} = \Sigma^{-1}$. If T is a rigid object of $\mathcal{C}$, then it is rigid in $\mathcal{C}^{\mathrm{op}}$, and any object X admitting an $(\operatorname{add}\Sigma^{-1}T)$-presentation in $\mathcal{C}$ admits an $(\operatorname{add} T)$-presentation in $\mathcal{C}^{\mathrm{op}}$. If we denote by $\operatorname{ind}_T^{\mathrm{op}}X$ the index of X with respect to T in $\mathcal{C}^{\mathrm{op}}$, then we have the equality $\operatorname{ind}_T^{\mathrm{op}}X = -\operatorname{ind}_{\Sigma^{-1}T}X$. Thus, the equality of Corollary 3.10 can be written as

$$y_j' = \prod_{s=1}^{n} y_s^{[\operatorname{ind}_{T'}^{\mathrm{op}}\Gamma_s : T_j']}.$$

This corresponds to the notation and point of view adopted in [Kel10, Corollary 6.9].

3.3 Cluster characters and F-polynomials

Let $\mathcal{A}$ be a cluster algebra with principal coefficients at a seed $((Q, F), \mathbf{x})$. In particular, $n = 2r$, and the matrix B associated to (Q, F) has full rank r.

Let W be a potential on Q, and let $\mathcal{C} = \mathcal{C}_{Q,W}$ be the cluster category associated to (Q, W). Let T be a rigid object of $\mathcal{C}$ reachable from Γ by mutation at an admissible sequence of vertices $(i_1, \ldots, i_s)$ not in F. Write T as $\bigoplus_{j=1}^{2r} T_j$, where $T_\ell = \Gamma_\ell$ for $r < \ell \leqslant 2r$.

For any vertex j not in F, X'_{T_j} is a cluster variable in $\mathcal{A}$. Specializing at $x_1 = \cdots = x_r = 1$, we obtain the corresponding F-polynomial (see § 2.1.3), which we will denote by F_{T_j}.

We thus have the equality

$$F_{T_j} = \prod_{i=r+1}^{2r} x_i^{[\operatorname{ind}_{\Gamma}T_j : \Gamma_i]} \sum_e \chi(\operatorname{Gr}_e(\operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, T_j))) \prod_{i=r+1}^{2r} x_i^{-\iota(e)_i},$$

where $\iota(e)$ was defined in § 2.5.2 and $\iota(e)_i$ is the ith component of $\iota(e)$.

Remark 3.12. The element X'_{T_j} of $\mathcal{A}$ is the jth cluster variable of the cluster obtained from the initial cluster at the sequence of vertices $(i_1, \ldots, i_s)$ by [Pla11, Theorem 4.1]. Therefore, the polynomial F_{T_j} is the corresponding F-polynomial.

It follows from our computation in §3.2 that for $r < i \leqslant 2r$, there is an equality $-\iota(e)_i = \sum_{j=1}^{r} e_j b_{ij}$ and, since our cluster algebra has principal coefficients, this number is e_{i-r}. Thus, we get the equality

$$F_{T_j} = \prod_{i=r+1}^{2r} x_i^{[\mathrm{ind}_\Gamma T_j : \Gamma_i]} \sum_e \chi(\mathrm{Gr}_e(\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, T_j))) \prod_{i=r+1}^{2r} x_i^{e_{i-r}}.$$

From this, we can prove the following theorem, using methods very similar to those found in [FK10], in which the theorem was proved in the Hom-finite case. Note that the theorem was shown in [DWZ10] using decorated representations and in [Nag10] using Donaldson–Thomas theory.

THEOREM 3.13. *Conjecture 5.6 of [FZ07] holds for $\mathcal{A}$, that is, any F-polynomial has constant term* 1.

Proof. It suffices to show that the polynomial F_{T_j} defined above has constant term 1. In order to do so, we will prove that, for any $r < i \leqslant 2r$, the number $[\mathrm{ind}_\Gamma T_j : \Gamma_i]$ vanishes.

We know that T_j lies in the subcategory $\mathcal{U}$ defined in §3.2, that is, for any $r < i \leqslant 2r$, the space $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma_i, T_j)$ vanishes. Using [Pla11, Proposition 2.15], we get that $\mathrm{Hom}_{\mathcal{C}}(T_j, \Sigma\Gamma_i)$ also vanishes.

Let $\overline{T}_1 \to \overline{T}_0 \to T_j \to \Sigma\overline{T}_1$ be a minimal (add Γ)-presentation of T_j. Let $r < i \leqslant 2r$ be a vertex of Q. Suppose that Γ_i is a direct summand of $\overline{T}_1$. Since $\mathrm{Hom}_{\mathcal{C}}(T_j, \Sigma\Gamma_i)$ is zero, this implies that the presentation has the triangle

$$\Gamma_i \xrightarrow{1_{\Gamma_i}} \Gamma_i \longrightarrow 0 \longrightarrow \Sigma\Gamma_i$$

as a direct summand, contradicting the minimality of the presentation. Thus, Γ_i is not a direct summand of $\overline{T}_1$.

Suppose that Γ_i is a direct summand of $\overline{T}_0$. Since i is a sink in Q, and since Γ_i is not a direct summand of $\overline{T}_1$, we get that $\mathrm{Hom}_{\mathcal{C}}(\overline{T}_1, \Gamma_i)$ is zero. This implies that Γ_i is a direct summand of T_j and, since the latter is indecomposable, we get that it is isomorphic to the former. This is a contradiction, since T must be basic. $\square$

DEFINITION 3.14. For any object M of $\mathcal{D}$, the F-*polynomial of M* is the polynomial

$$F_M = \sum_e \chi(\mathrm{Gr}_e(\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, M))) \prod_{i=r+1}^{2r} x_i^{e_{i-r}}$$

in $\mathbb{Z}[x_{r+1}, \ldots, x_{2r}]$.

Thanks to Theorem 3.13, this definition is in accordance with the F_{T_i} used above. Note that we have the equality

$$X'_M|_{x_1 = \cdots = x_r = 1} = \prod_{i=r+1}^{2r} x_i^{[\mathrm{ind}_\Gamma M : \Gamma_i]} F_M.$$

We can deduce from the multiplication formula of [Pla11, Proposition 3.16] an equality for the polynomials F_M. This was first proved implicitly in [Pal08, §5.1]; see also [Kel10, Theorem 6.12].

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PROPOSITION 3.15. *Let M and N be objects of $\mathcal{D}$ such that the space $\mathrm{Hom}_{\mathcal{C}}(M, \Sigma N)$ is one dimensional. Let*

$$M \longrightarrow E \longrightarrow N \longrightarrow \Sigma M \quad and \quad N \longrightarrow E' \longrightarrow M \longrightarrow \Sigma N$$

be non-split triangles. Then

$$F_M F_N = \prod_{i=r+1}^{2r} x_i^{d_{i-r}} F_E + \prod_{i=r+1}^{2r} x_i^{d'_{i-r}} F_{E'},$$

where $d = (d_1, \ldots, d_{2r})$ is the dimension vector of the kernel K of the induced morphism $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, E)$ and $d' = (d'_1, \ldots, d'_{2r})$ is the dimension vector of the kernel K' of $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, N) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, E')$.

Proof. We know from [Pla11, Proposition 3.16] that $X'_M X'_N = X'_E + X'_{E'}$. Specializing at $x_1 = \cdots = x_r = 1$, we get the equality

$$\prod_{i=r+1}^{2r} x_i^{[\mathrm{ind}_\Gamma M : \Gamma_i] + [\mathrm{ind}_\Gamma N : \Gamma_i]} F_M F_N = \prod_{i=r+1}^{2r} x_i^{[\mathrm{ind}_\Gamma E : \Gamma_i]} F_E + \prod_{i=r+1}^{2r} x_i^{[\mathrm{ind}_\Gamma E' : \Gamma_i]} F_{E'}.$$

It follows from [Pla11, Lemma 3.5] (applied to the above triangles shifted by Σ^{-1}, and with $T = \Sigma^{-1}\Gamma$) that

$$\mathrm{ind}_\Gamma M + \mathrm{ind}_\Gamma N = \mathrm{ind}_\Gamma E + \mathrm{ind}_\Gamma K + \mathrm{ind}_\Gamma \Sigma K$$
$$= \mathrm{ind}_\Gamma E' + \mathrm{ind}_\Gamma K' + \mathrm{ind}_\Gamma \Sigma K',$$

where K and K' are as in the statement of the proposition. But $\mathrm{ind}_\Gamma K + \mathrm{ind}_\Gamma \Sigma K = \iota(d)$ and, using our computation of $\iota(e)$ of § 3.2, we get that $-\iota(d)_i = d_{i-r}$ for $r < i \leqslant 2r$.

Similarly, we get that $\mathrm{ind}_\Gamma K' + \mathrm{ind}_\Gamma \Sigma K' = \iota(d')$, and that $-\iota(d')_i = d'_{i-r}$ for $r < i \leqslant 2r$. The desired equality follows. $\square$

4. Link with decorated representations

In this section, an explicit link between cluster categories and the decorated representations of [DWZ08] is established. We show that the mutation of decorated representations of [DWZ08] corresponds to the derived equivalence of [KY11], and we give an interpretation of the E-invariant of [DWZ10] as half the dimension of the space of self-extensions of an object in the cluster category.

4.1 Mutations

Let (Q, W) be a quiver with potential. Let $\Gamma = \Gamma_{Q,W}$ be the associated complete Ginzburg dg algebra, and $\mathcal{C} = \mathcal{C}_{Q,W}$ be the associated cluster category. Let $B = B_{Q,W}$ be the endomorphism algebra of Γ in $\mathcal{C}$. Recall from [KY11, Lemma 2.8] that B is the Jacobian algebra of (Q, W). Denote by F the functor $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, ?)$ from $\mathcal{C}$ to $\mathrm{Mod}\, B$. Let $\mathcal{D} = \mathcal{D}_{Q,W}$ be the full subcategory of $\mathrm{pr}_{\mathcal{C}}\Gamma \cap \mathrm{pr}_{\mathcal{C}}\Sigma^{-1}\Gamma$ whose objects are those X such that FX is finite dimensional.

Consider the map $\Phi = \Phi_{Q,W}$ from the set of isomorphism classes of objects in $\mathcal{D}$ to the set of isomorphism classes of decorated representations of (Q, W) defined as follows. For any object X of $\mathcal{D}$, write $X = X' \oplus \bigoplus_{i \in Q_0} (e_i \Gamma)^{m_i}$, where X' has no direct summands in $\mathrm{add}\,\Gamma$. Such a decomposition of X is unique up to isomorphism, since $\mathrm{pr}_{\mathcal{C}}\Gamma$ is a Krull–Schmidt category, as shown in [Pla11]. Define $\Phi(X)$ to be the decorated representation $(F(X'), \bigoplus_{i \in Q_0} S_i^{m_i})$, where $(0, S_i)$ is the negative simple representation at the vertex i, for any i in Q_0.

Consider also the map $\Psi = \Psi_{Q,W}$ from the set of isomorphism classes of decorated representations of (Q, W) to the set of isomorphism classes of objects in $\mathcal{D}$ defined as follows. Recall from [Pla11] that F induces an equivalence $\operatorname{pr}_{\mathcal{C}} \Sigma^{-1} \Gamma / (\Gamma) \to \operatorname{mod} B$, where $\operatorname{mod} B$ is the category of finitely presented B-modules. Let G be a quasi-inverse equivalence. For any decorated representation $(M, \bigoplus_{i \in Q_0} S_i^{m_i})$, choose a representative $\overline{M}$ of $G(M)$ in $\mathcal{D}$ which has no direct summands in $\operatorname{add} \Gamma$ (the representative can be chosen to be in $\mathcal{D}$ thanks to [Pla11, Lemma 3.2]). Such a representative is unique up to (non-unique) isomorphism. The map Ψ then sends $(M, \bigoplus_{i \in Q_0} S_i^{m_i})$ to the object $\overline{M} \oplus \bigoplus_{i \in Q_0} (e_i \Gamma_i)^{m_i}$.

The diagram below summarizes the definitions of Φ and Ψ.

$$\left\{ \begin{matrix} \text{isoclasses of} \\ \text{objects of } \mathcal{D} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{isoclasses of decorated} \\ \text{representations of } (Q, W) \end{matrix} \right\},$$

$$X = X' \oplus \bigoplus_{i=1}^{n} (e_i \Gamma)^{m_i} \longmapsto \Phi(X) = \left(FX', \bigoplus_{i=1}^{n} (S_i)^{m_i} \right),$$

$$\Psi(\mathcal{M}) = \overline{M} \oplus \bigoplus_{i=1}^{n} (e_i \Gamma)^{m_i} \longleftarrow \mathcal{M} = \left(M, \bigoplus_{i=1}^{n} S_i^{m_i} \right).$$

The main result of this subsection states that the maps Φ and Ψ are mutually inverse bijections, on the one hand, and that, via these maps, the derived equivalences of [KY11] are compatible with the mutations of decorated representations of [DWZ08], on the other hand.

PROPOSITION 4.1. *With the above notation, Φ and Ψ are mutually inverse maps. Moreover, if $i \in Q_0$ is not on any cycle of length $\leqslant 2$, and if $(Q', W') = \widetilde{\mu}_i(Q, W)$, then for any object X of $\mathcal{D}$, we have that*

$$\Phi_{Q',W'}(\widetilde{\mu}_i^-(X)) = \widetilde{\mu}_i(\Phi_{Q,W}(X)),$$

where the functor $\widetilde{\mu}_i^-$ is as defined after Theorem 2.3

The rest of this section is devoted to the proof of the proposition.

It is obvious from the definitions of Φ and Ψ that the two maps are mutual inverses. Thus, we only need to show that the two mutations agree.

Let Γ' be the complete Ginzburg dg algebra of (Q', W'). Note that $\operatorname{End}_{\mathcal{C}'}(\Gamma')$ is the Jacobian algebra $J(Q', W')$, by [KY11, Lemma 2.8]. Let $\mathcal{C}'$ be the cluster category associated to (Q', W'). We know from [DWZ10] that $\widetilde{\mu}_i(\Phi_{Q,W}(X))$ is a decorated representation of $(Q', W') = \widetilde{\mu}_i(Q, W)$. We need to show that it is isomorphic to $\Phi_{Q',W'}(\widetilde{\mu}_i^-(X))$.

We can (and will) assume for the rest of the proof that X is indecomposable, as all the maps and functors considered commute with finite direct sums.

We first prove the proposition for some special cases.

LEMMA 4.2. *Assume that X is an indecomposable object of $\mathcal{D}$ such that either:*

- *X is of the form $e_j \Gamma$ for $j \neq i$; or*
- *X is the cone Γ_i^* of the morphism*

$$\Gamma_i \longrightarrow \bigoplus_{\alpha} \Gamma_{t(\alpha)}$$

whose components are given by left multiplication by α.

Then the equality $\Phi_{Q',W'}(\widetilde{\mu}_i^-(X)) = \widetilde{\mu}_i(\Phi_{Q,W}(X))$ holds.

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Proof. Suppose that $X = e_j\Gamma$ for some vertex $i \neq j$. Then $\widetilde{\mu}_i(\Phi_{Q,W}(X)) = \widetilde{\mu}_i(0, S_j) = (0, S_j)$, and $\Phi_{Q',W'}(\widetilde{\mu}_i^-(X)) = \Phi_{Q',W'}(e_j\Gamma') = (0, S_j)$, so the desired equality holds.

Suppose now that X is the cone Γ_i^* of the morphism

$$\Gamma_i \longrightarrow \bigoplus_\alpha \Gamma_{t(\alpha)}$$

whose components are given by left multiplication by α. In that case, $\widetilde{\mu}_i^-(X) = e_i\Gamma'$ and $\Phi(X) = (S_i, 0)$, so the desired equality is also satisfied. $\qquad\square$

Now suppose that X is not of the above form. Using the definition of $\widetilde{\mu}_i^-$, we get that $\Phi(\widetilde{\mu}_i^-(X))$ is equal to $\Phi(\mathcal{H}om_\Gamma(T, X))$, where T is as defined in § 2.6. Because of our assumptions on X, this decorated representation is given by $(\mathrm{Hom}_{\mathcal{C}'}(\Sigma^{-1}\Gamma', \mathcal{H}om_\Gamma(T, X)), 0)$.

We have the isomorphisms of $\mathrm{End}_{\mathcal{C}'}(\Gamma')$-modules

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}'}(\Sigma^{-1}\Gamma', \mathcal{H}om_\Gamma(T, X)) &= \mathrm{Hom}_{\mathcal{D}\Gamma'}(\Sigma^{-1}\Gamma', \mathcal{H}om_\Gamma(T, \overline{X})) \\
&= \mathrm{Hom}_{\mathcal{D}\Gamma}(\Sigma^{-1}\Gamma' \otimes_{\Gamma'}^L T, \overline{X}) \\
&= \mathrm{Hom}_{\mathcal{D}\Gamma}(\Sigma^{-1}T, \overline{X}) \\
&= \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X),
\end{aligned}$$

where $\overline{X}$ is a lift of X in $\mathrm{pr}_{\mathcal{D}\Gamma}\Sigma^{-1}\Gamma$. Using this, we prove the proposition for another special case.

LEMMA 4.3. *If $X = e_i\Gamma$, then $\Phi_{Q',W'}(\widetilde{\mu}_i^-(X)) = \widetilde{\mu}_i(\Phi_{Q,W}(X))$.*

Proof. We have that $\widetilde{\mu}_i(\Phi_{Q,W}(e_i\Gamma)) = (S_i, 0)$. Moreover, the above calculation gives that $\Phi_{Q',W'}(\widetilde{\mu}_i^-(e_i\Gamma)) = (\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, e_i\Gamma), 0)$.

For any vertex $j \neq i$, we have isomorphisms $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, e_i\Gamma)e_j = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}(e_jT), e_i\Gamma) = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}(e_j\Gamma), e_i\Gamma)$, and this last space is zero.

For the vertex i, we have $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, e_i\Gamma)e_i = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}(e_iT), e_i\Gamma) = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma_i^*, e_i\Gamma)$, and this space is one dimensional.

Therefore, $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, e_i\Gamma)$ is the simple module at the vertex i, and this proves the desired equality. $\qquad\square$

We now treat the remaining cases, that is, those where X is not in $\mathrm{add}\,\Gamma$ and is not Γ_i^*. Then $\Phi(X) = (FX, 0)$, and $\widetilde{\mu}_i(\Phi_{Q,W}(X)) = \widetilde{\mu}_i(FX, 0) = (M', 0)$ is computed using § 2.3. We will show that $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ is isomorphic to M' as a $J(Q', W')$-module, using heavily the definition of T given in § 2.6.

LEMMA 4.4. *For any vertex j, the vector spaces $M'e_j$ and $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)e_j$ are isomorphic.*

Proof. If j is a vertex different from i, then we have the isomorphisms of vector spaces $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)e_j = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}(e_jT), X) = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}(e_j\Gamma), X) = (FX)e_j = M'e_j$.

For the vertex i, we have that $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)e_i = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}(e_iT), X) = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma_i^*, X)$. Let us show that this space is isomorphic to $M'e_i$.

We have triangles in $\mathcal{C}$:

$$e_i\Gamma \longrightarrow \bigoplus_{s(a)=i} e_{t(a)}\Gamma \longrightarrow \Gamma_i^* \longrightarrow \Sigma(e_i\Gamma)$$

and

$$\overline{\Gamma}_i^* \longrightarrow \bigoplus_{t(a)=i} e_{s(a)}\Gamma \longrightarrow e_i\Gamma \longrightarrow \Sigma\Gamma_i^*.$$

These triangles yield a diagram with exact rows,

$$
\begin{array}{ccccccc}
(\Sigma^{-1}\overline{\Gamma}_i^*, X) & \xleftarrow{\ h\ } & (\Sigma^{-1}\bigoplus_{t(a)=i} e_{s(a)}\Gamma, X) & \xleftarrow{\ \beta\ } & (\Sigma^{-1}(e_i\Gamma), X) & \longleftarrow & (\Gamma_i^*, X) \\
\downarrow{\varphi_i^*} & & \downarrow{-\gamma} & & \| & & \\
(\Sigma^{-1}\Gamma_i^*, X) & \xrightarrow{\ g\ } & (\Sigma^{-1}\bigoplus_{s(a)=i} e_{t(a)}\Gamma, X) & \xrightarrow{\ \alpha\ } & (\Sigma^{-1}(e_i\Gamma), X) & \longrightarrow & (\Sigma^{-2}\Gamma_i^*, X)
\end{array}
$$

where we write (Y_1, Y_2) for $\mathrm{Hom}_{\mathcal{C}}(Y_1, Y_2)$, where $-\gamma = g\varphi_i^* h$, and where φ_i was defined in § 2.6. Note that φ_i^* is an isomorphism.

Notice that, in the notation of § 2.3, we have $(\Sigma^{-1}\bigoplus_{t(a)=i} e_{s(a)}\Gamma, X) = (FX)_{\mathrm{out}}$ and $(\Sigma^{-1}\bigoplus_{s(a)=i} e_{t(a)}\Gamma, X) = (FX)_{\mathrm{in}}$. Moreover, the maps α and β in the diagram above correspond to the maps α and β of § 2.3.

The map γ above also corresponds to the map γ defined in § 2.3. This follows from the computation we made in Remark 2.2.

Using the above diagram, we get isomorphisms

$$(\Sigma^{-1}\Gamma_i^*, X) \cong \mathrm{Im}\, g \oplus \mathrm{Ker}\, g$$
$$\cong \mathrm{Ker}\, \alpha \oplus \mathrm{Ker}\, g$$

and

$$\mathrm{Ker}\, \gamma \cong h^{-1}(\varphi_i^{*-1}(\mathrm{Ker}\, g))$$
$$\cong \mathrm{Ker}\, h \oplus \mathrm{Ker}\, g$$
$$\cong \mathrm{Im}\, \beta \oplus \mathrm{Ker}\, g.$$

Thus, $(\Sigma^{-1}\Gamma_i^*, X)$ is (non-canonically) isomorphic to $\mathrm{Ker}\, \alpha \oplus \mathrm{Ker}\, \gamma / \mathrm{Im}\, \beta$, which is in turn isomorphic to $\mathrm{Ker}\, \gamma / \mathrm{Im}\, \beta \oplus \mathrm{Im}\, \gamma \oplus \mathrm{Ker}\, \alpha / \mathrm{Im}\, \gamma$. But this is precisely $M'e_i$. □

It remains to be shown that the action of the arrows of Q' on $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ is the same as on M' in order to get the following lemma.

LEMMA 4.5. *As a $J(Q', W')$-module, $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ is isomorphic to M'.*

Proof. We know from Lemma 4.4 that the two modules considered are isomorphic as Λ-modules, where Λ is as in § 2.2.

Now let a be an arrow of Q not incident with i. Then a is an arrow of Q', and its action on $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ is obviously the same as its action on M'.

Consider now an arrow of Q' of the form $[ba]$, where $t(a) = i = s(b)$ in Q. By the definition of M' given in § 2.3, $[ba]$ acts as ba on M', that is, $M'_{[ba]} = (FX)_{ba}$.

According to the definition of T given in § 2.6, $[ba]$ acts on T as the map

$$T_{s(a)} \longrightarrow T_{t(b)},$$
$$x \longmapsto bax.$$

Hence, the action of $[ba]$ on $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ is also given by multiplication by ba. Thus, the actions of $[ba]$ on M' and on $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ coincide.

There remains to be considered the action of the arrows incident with i.

Keep the notation introduced in the proof of Lemma 4.4. We assert that the maps $\varphi_i^* h$ and g encode the action of the arrows incident with i.

Recall that in $\mathcal{D}\Gamma$, the object Γ_i^* is isomorphic as a graded module to

$$\Sigma(e_i\Gamma) \oplus \bigoplus_{\substack{a \in Q_1 \\ s(a)=i}} e_{t(a)}\Gamma,$$

and that the map $\bigoplus_{\substack{a \in Q_1 \\ s(a)=i}} e_{t(a)}\Gamma \longrightarrow \Gamma_i^*$ is the canonical inclusion. Thus, its components are given by

$$e_{t(a)}\Gamma \longrightarrow \Gamma_i^*,$$
$$x \longmapsto e_a x$$

for any arrow a of Q such that $s(a) = i$. By the definition of T, this is multiplication by $a^\star$. Therefore, g encodes the action of the arrows $a^\star$ of Q', where $s(a) = i$ in Q.

Similarly, recall that in $\mathcal{D}\Gamma$, the object $\overline{\Gamma}_i^*$ is isomorphic as a graded module to

$$\left(\bigoplus_{\substack{b \in Q_1 \\ t(b)=i}} e_{t(b)}\Gamma \right) \oplus \Sigma^{-1}(e_i\Gamma)$$

and that the map $\overline{\Gamma}_i^* \longrightarrow \bigoplus_{\substack{b \in Q_1 \\ t(b)=i}} e_{t(b)}\Gamma$ is given by the canonical projection. Thus, its composition with φ_i^* is given by the matrix $\begin{pmatrix} -b^* & -\partial_{ab}W \end{pmatrix}$. Its components are the maps

$$\Gamma_i^* \longrightarrow e_{s(b)}\Gamma,$$
$$e_{\Sigma i}x_i + \sum_{s(a)=i} e_a x_a \longmapsto -b^* x_i + \sum_{s(a)=i} (\partial_{ab}W)x_a$$

for any arrow b of Q such that $t(b) = i$. By the definition of T, this is multiplication by $b^\star$. Thus, $\varphi_i^* h$ encodes the action of the arrows $b^\star$ of Q', where $t(b) = i$ in Q.

Finally, recall from Lemma 4.4 that $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma_i^*, X)$ is isomorphic to $\mathrm{Ker}\,\gamma/\mathrm{Im}\,\beta \oplus \mathrm{Im}\,\gamma \oplus \mathrm{Ker}\,\alpha/\mathrm{Im}\,\gamma$. Recall that the summand $\mathrm{Ker}\,\gamma/\mathrm{Im}\,\beta$ corresponds to $\mathrm{Ker}\,g$, while the summand $\mathrm{Im}\,\gamma \oplus \mathrm{Ker}\,\alpha/\mathrm{Im}\,\gamma$ corresponds to $\mathrm{Im}\,g$.

We choose a splitting $\mathrm{Im}\,\gamma \oplus \mathrm{Ker}\,\alpha/\mathrm{Im}\,\gamma$ in such a way that $\mathrm{Im}\,\varphi_i^* h \cap \mathrm{Ker}\,\alpha/\mathrm{Im}\,\gamma = 0$. In that case, g is given in matrix form by $(0 \ \iota \ \iota\sigma)$ and $\varphi_i^* h$, by $\begin{pmatrix} -\pi\rho \\ -\gamma \\ 0 \end{pmatrix}$, in the notation of §2.3. This proves that the actions of the arrows of Q' on M' and on $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ coincide, finishing the proof of the lemma. $\square$

We have proved Proposition 4.1.

4.2 Interpretation of F-polynomials, g-vectors and h-vectors

In this section, we study the relation between the F-polynomials of objects of $\mathcal{D}$ and of decorated representations, and between the index of objects in $\mathcal{D}$ and the **g**-vectors of decorated representations. We also give an interpretation of the **h**-vector.

Let (Q, W) be a quiver with potential, and let $\mathcal{C}$ be the associated cluster category. We keep the notation of the previous section for the maps Φ and Ψ.

We first prove a result regarding F-polynomials.

PROPOSITION 4.6. *Let X be an object of $\mathcal{D}$. Then we have the equality*

$$F_X(x_{r+1}, \ldots, x_n) = F_{\Phi(X)}(x_{r+1}, \ldots, x_n).$$

Proof. This is immediate from the definitions of F_X, Φ and $F_{\Phi(X)}$, given in Definition 3.14 and §§ 4.1 and 2.4, respectively. □

We now prove that **g**-vectors of decorated representations and indices of objects in the cluster category are closely related. We will need the following Hom-infinite extension of [Pal08, Lemma 7].

LEMMA 4.7. *Let M be an indecomposable object of $\mathcal{D}$. Then*

$$[\mathrm{ind}_\Gamma M : e_i\Gamma] = \begin{cases} \delta_{ij} & \text{if } M \cong e_i\Gamma, \\ \dim \mathrm{Ext}^1_B(S_i, FM) - \dim \mathrm{Hom}_B(S_i, FM) & \text{otherwise,} \end{cases}$$

where $B = \mathrm{End}_\mathcal{C}(\Gamma)$.

Proof. The result is obvious if M lies in $\mathrm{add}\,\Gamma$. Suppose that it does not. Let $T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow \Sigma T_1$ be an $(\mathrm{add}\,\Gamma)$-presentation of M.

The opposite category $\mathcal{C}^{\mathrm{op}}$ is triangulated, with suspension functor $\Sigma_{\mathrm{op}} = \Sigma^{-1}$. Thus, in $\mathcal{C}^{\mathrm{op}}$, we have a triangle $\Sigma_{\mathrm{op}}^{-1}T_0 \longrightarrow \Sigma_{\mathrm{op}}^{-1}T_1 \longrightarrow M \longrightarrow T_0$. Applying the functor $F' = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(\Sigma_{\mathrm{op}}^{-1}\Gamma, ?)$, we get a minimal projective resolution

$$F'\Sigma_{\mathrm{op}}^{-1}T_0 \longrightarrow F'\Sigma_{\mathrm{op}}^{-1}T_1 \longrightarrow F'M \longrightarrow 0$$

of $F'M$ as a B^{op}-module.

Letting S'_i be the simple B^{op}-module at the vertex i, we apply $\mathrm{Hom}_{B'}(?, S_i)$ to the above exact sequence and get a complex

$$0 \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}(F'\Sigma_{\mathrm{op}}^{-1}T_1, S'_i) \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}(F'\Sigma_{\mathrm{op}}^{-1}T_0, S'_i) \longrightarrow \cdots$$

whose differential vanishes, since the presentation is minimal.

Therefore, we have the equalities

$$\begin{aligned}[\mathrm{ind}_\Gamma M : e_i\Gamma] &= \dim \mathrm{Ext}^1_{B^{\mathrm{op}}}(F'M, S'_i) - \dim \mathrm{Hom}_{B^{\mathrm{op}}}(F'M, S'_i) \\ &= \dim \mathrm{Ext}^1_B(S_i, DF'M) - \dim \mathrm{Hom}_B(S_i, DF'M),\end{aligned}$$

where S_i is the simple B-module at the vertex i.

Now, using [Pla11, Proposition 2.16], we get that

$$DF'M = D\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(\Sigma_{\mathrm{op}}^{-1}\Gamma, M) = D\mathrm{Hom}_\mathcal{C}(M, \Sigma\Gamma) \cong \mathrm{Hom}_\mathcal{C}(\Sigma^{-1}\Gamma, M) = FM.$$

Thus, $DF'M$ is isomorphic to FM as a B-module. This proves the lemma. □

We now prove the result on **g**-vectors of decorated representations.

PROPOSITION 4.8. *Let (Q, W) be a quiver with potential, and let $\mathcal{C}$ be the associated cluster category. Let X be an object of $\mathcal{D}$. Let $\mathbf{g}_{\Phi(X)} = (g_1, \ldots, g_n)$ be the **g**-vector of the decorated representation $\Phi(X)$. Then we have the equality*

$$g_i = [\mathrm{ind}_\Gamma X : \Gamma_i]$$

for any vertex i of Q.

Proof. We can assume that X is indecomposable. If X lies in add Γ, then the result is obviously true. Suppose that X does not lie in add Γ.

Using the two triangles in $\mathcal{C}$,

$$e_i\Gamma \longrightarrow \bigoplus_{s(a)=i} e_{t(a)}\Gamma \longrightarrow \Gamma_i^* \longrightarrow \Sigma(e_i\Gamma)$$

and

$$\overline{\Gamma}_i^* \longrightarrow \bigoplus_{t(a)=i} e_{s(a)}\Gamma \longrightarrow e_i\Gamma \longrightarrow \Sigma\Gamma_i^*$$

and applying the functor $F = \operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, ?)$, we get a projective resolution of the simple B-module S_i at the vertex i:

$$P_i \longrightarrow \bigoplus_{s(a)=i} P_{t(a)} \longrightarrow \bigoplus_{t(a)=i} P_{s(a)} \longrightarrow P_i \longrightarrow S_i \longrightarrow 0,$$

where P_j is the indecomposable projective B-module at the vertex j. Applying now the functor $\operatorname{Hom}_B(?, FM)$, we get the complex

$$0 \longrightarrow (FM)_i \xrightarrow{\beta_i} (FM)_{\text{out}} \xrightarrow{-\gamma_i} (FM)_{\text{in}} \xrightarrow{\alpha_i} (FM)_i.$$

From this complex, we see that $\operatorname{Hom}_B(S_i, M) = \operatorname{Ker}\beta_i$ and that $\operatorname{Ext}_B^1(S_i, M) = \operatorname{Ker}\gamma_i/\operatorname{Im}\beta_i$. We also deduce an exact sequence

$$0 \longrightarrow \operatorname{Ker}\beta_i \longrightarrow (FM)_i \xrightarrow{\beta_i} \operatorname{Ker}\gamma_i \longrightarrow \operatorname{Ker}\gamma_i/\operatorname{Im}\beta_i \longrightarrow 0.$$

Using the above arguments and Lemma 4.7, we get the equalities

$$\begin{aligned}
[\operatorname{ind}_\Gamma X : e_i\Gamma] &= \dim\operatorname{Ext}_B^1(S_i, M) - \dim\operatorname{Hom}_B(S_i, M) \\
&= \dim(\operatorname{Ker}\gamma_i/\operatorname{Im}\beta_i)\dim\operatorname{Ker}\beta_i \\
&= \dim\operatorname{Ker}\gamma_i - \dim(FM)_i \\
&= g_i.
\end{aligned}$$

This finishes the proof. $\qquad\qquad\square$

As a corollary of the proof of the above proposition, we get an interpretation of the **h**-vector of a decorated representation.

COROLLARY 4.9. *For any decorated representation* $\mathcal{M} = (M, V)$ *of a quiver with potential* (Q, W), *we have the equality*

$$h_i = -\dim\operatorname{Hom}_{J(Q,W)}(S_i, M)$$

for any vertex i *of* Q.

This provides us with a way of 'counting' the number of terms in a minimal presentation.

COROLLARY 4.10. *If* $\mathbf{g} = (g_1, \ldots, g_n)$ *and* $\mathbf{h} = (h_1, \ldots, h_n)$ *are the* **g**-vector *and* **h**-vector *of a decorated representation* $\mathcal{M} = (M, V)$, *if* $\mathbf{h}' = (h_1', \ldots, h_n')$ *is the* **h**-vector *of* $\mu_i(\mathcal{M})$ *and if*

$$T_1 \longrightarrow T_0 \longrightarrow \Psi(\mathcal{M}) \longrightarrow \Sigma T_1$$

is a minimal (add Γ)-*presentation of* $\Psi(\mathcal{M})$ *(see Proposition 4.1), then* $-h_i$ *and* $-h_i'$ *are the numbers of direct summands of* T_1 *and* T_0 *which are isomorphic to* Γ_i, *respectively.*

Proof. It follows from Corollary 4.9 that $-h_i = \dim \operatorname{Hom}_{J(Q,W)}(S_i, M)$.

Let T_i^* be an indecomposable object of $\mathcal{D}$ such that $\operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, T_i^*)$ is the simple $J(Q,W)$-module S_i. Then, by [Pla11, Lemma 3.2], we have that

$$\operatorname{Hom}_{J(Q,W)}(S_i, M) \cong \operatorname{Hom}_{\mathcal{C}}(T_i^*, \Psi(M))/(\Gamma).$$

Applying $\operatorname{Hom}_{\mathcal{C}}(T_i^*, ?)$ to the presentation, we get a long exact sequence

$$(T_i^*, T_0) \xrightarrow{\psi_*} (T_i^*, \Psi(\mathcal{M})) \xrightarrow{\phi_*} (T_i^*, \Sigma T_1) \longrightarrow (T_i^*, \Sigma T_0).$$

We see that the image of ψ_* is $(\Gamma)(T_i^*, \Psi(M))$, so that $\operatorname{Hom}_{\mathcal{C}}(T_i^*, \Psi(M))/(\Gamma)$ is isomorphic to the image of ϕ_*. Thus, $-h_i$ is the dimension of the image of ϕ_*.

Using [Pla11, Proposition 2.16], we get that the morphism $(T_i^*, \Sigma T_1) \longrightarrow (T_i^*, \Sigma T_0)$ is isomorphic to the morphism $D(\Sigma^{-1}T_1, T_i^*) \longrightarrow D(\Sigma^{-1}T_0, T_i^*)$, and this morphism is zero since the presentation is minimal. Thus, ϕ_* is surjective.

Therefore, $-h_i$ is equal to the dimension of $\operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}T_1, T_i^*)$, which is equal to the number of direct factors of T_1 isomorphic to Γ_i in any decomposition of T_1.

Furthermore, [DWZ10, Lemma 5.2] gives us that $g_i = h_i - h_i'$ and, by Proposition 4.8, $g_i = [\operatorname{ind}_\Gamma \Psi(\mathcal{M}) : \Gamma_i]$. This immediately implies that $-h_i'$ is equal to the number of direct factors of T_0 isomorphic to Γ_i, and finishes the proof. $\square$

Remark 4.11. Corollary 4.10 allows us to reformulate Remark 3.8 in the following way. If M is any object of $\mathcal{D}$, and if $\mathbf{h} = (h_1, \ldots, h_n)$ and $\mathbf{h}' = (h_1', \ldots, h_n')$ are the $\mathbf{h}$-vectors of $\Phi(M)$ and $\tilde{\mu}_i \Phi(M)$, respectively, then

$$[\operatorname{ind}_{T'} M : T_j'] = \begin{cases} -[\operatorname{ind}_T M : T_i] & \text{if } i = j, \\ [\operatorname{ind}_T M : T_j] - h_i'[b_{ji}]_+ + h_i[-b_{ji}]_+ & \text{if } i \neq j. \end{cases}$$

As a corollary, we get a proof of Conjecture 6.10 of [FZ07].

COROLLARY 4.12. *Conjecture 6.10 of [FZ07] is true, that is, if $\mathbf{g} = (g_1, \ldots, g_n)$ and $\mathbf{g}' = (g_1', \ldots, g_n')$ are the $\mathbf{g}$-vectors of one cluster variable with respect to two clusters t and t' related by one mutation at vertex i, and if $\mathbf{h} = (h_1, \ldots, h_n)$ and $\mathbf{h}' = (h_1', \ldots, h_n')$ are its $\mathbf{h}$-vectors with respect to those clusters, then we have that*

$$h_i' = -[g_i]_+ \quad \text{and} \quad h_i = \min(0, g_i).$$

Proof. Let M be an indecomposable object of $\mathcal{D}$ such that X_M' is the cluster variable considered in the statement. In particular, M is reachable, and thus rigid. It follows from [DWZ10, Equation (5.5)] that the $\mathbf{h}$-vector of the cluster variable corresponds to the $\mathbf{h}$-vector of the associated decorated representation.

Since M is rigid, Lemma 3.5 tells us that any minimal (add Γ)-presentation of M has disjoint direct factors. The result follows directly from this and from Corollary 4.10. $\square$

Remark 4.13. Conjecture 6.10 of [FZ07] also follows directly from Conjecture 7.12 (see Theorem 3.7(4) above) and [FZ07, Equations (6.15) and (6.26)]. We give the above proof because it is an application of the results developed in this paper.

Finally, we get an interpretation of the substitution formula of [DWZ10, Lemma 5.2] in terms of cluster characters.

Corollary 4.14. *Let (Q, W) be a quiver with potential. Let i be an admissible vertex of Q, and let $\varphi_X : \mathbb{Q}(x'_1, \ldots, x'_n) \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$ be the field isomorphism sending x'_j to x_j if $i \neq j$ and to*

$$(x_i)^{-1}\left(\prod_{\ell=1}^n x_\ell^{[b_{\ell i}]_+} + \prod_{\ell=1}^n x_\ell^{[-b_{\ell i}]_+}\right)$$

if $i = j$. Let $\mathcal{C}$ and $\mathcal{C}'$ be the cluster categories of (Q, W) and $\widetilde{\mu}_i(Q, W)$, respectively, and let $\widetilde{\mu}_i^+ : \mathcal{C}' \longrightarrow \mathcal{C}$ be the associated functor (see [KY11, Theorem 3.2]).

Then, for any object M of the subcategory $\mathcal{D}'$ of $\mathcal{C}'$, we have that

$$X'_{\widetilde{\mu}_i^+(M)} = \varphi_X(X'_M).$$

Proof. Consider the field isomorphism $\varphi_Y : \mathbb{Q}(y'_1, \ldots, y'_n) \longrightarrow \mathbb{Q}(y_1, \ldots, y_n)$ whose action on y'_j is given by

$$\varphi_Y(y'_j) = \begin{cases} y_i^{-1} & \text{if } i = j, \\ y_j y_i^m (y_i + 1)^{-m} & \text{if there are } m \text{ arrows from } i \text{ to } j, \\ y_j(y_i + 1)^m & \text{if there are } m \text{ arrows from } j \text{ to } i. \end{cases}$$

Consider also the morphism $(\hat{-}) : \mathbb{Q}(y_1, \ldots, y_n) \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$ sending each y_j to

$$\hat{y}_j = \prod_{\ell=1}^n x_\ell^{b_{\ell j}}.$$

Denote by the same symbol the corresponding map from the field $\mathbb{Q}(y'_1, \ldots, y'_n)$ to $\mathbb{Q}(x'_1, \ldots, x'_n)$. Then [FZ07, Proposition 3.9] implies that $\varphi_X(\hat{z}) = \widehat{(\varphi_Y(z))}$ for any $z \in \mathbb{Q}(y'_1, \ldots, y'_n)$. In other words, the following diagram commutes.

$$\begin{array}{ccc} \mathbb{Q}(y_1, \ldots, y_n) & \xrightarrow{(\hat{-})} & \mathbb{Q}(x_1, \ldots, x_n) \\ \varphi_Y \uparrow & & \varphi_X \uparrow \\ \mathbb{Q}(y'_1, \ldots, y'_n) & \xrightarrow{(\hat{-})} & \mathbb{Q}(x'_1, \ldots, x'_n) \end{array}$$

Let us now compute $\varphi_X(X'_M)$. We have that

$$\varphi_X(X'_M) = \varphi_X(x'^{\mathrm{ind}_{\Gamma'} M}) F_M(\hat{y}'_1, \ldots, \hat{y}'_n)$$
$$= \varphi_X(x'^{\mathrm{ind}_{\Gamma'} M}) F_M(\widehat{(\varphi_Y(y'_1))}, \ldots, \widehat{(\varphi_Y(y'_n))}).$$

Now, using [DWZ10, Lemma 5.2], we can express the right-hand side of the equation in terms of the $\hat{y}_j$. The equalities thus continue:

$$\varphi_X(X'_M) = \varphi_X(x'^{\mathrm{ind}_{\Gamma'} M}) \varphi_X(1 + \hat{y}'_i)^{-h'_i}(1 + \hat{y}_i)^{h_i} F_M(\hat{y}_1, \ldots, \hat{y}_n)$$
$$= \varphi_X(x'^{\mathrm{ind}_{\Gamma'} M}) \varphi_X(1 + \hat{y}'_i)^{-h'_i}(1 + \hat{y}_i)^{h_i} x^{-\mathrm{ind}_\Gamma \widetilde{\mu}_i^+(M)} X'_{\widetilde{\mu}_i^+(M)}.$$

Thus, in order to prove the corollary, we must show that

$$\varphi_X(x'^{\mathrm{ind}_{\Gamma'} M}) \varphi_X(1 + \hat{y}'_i)^{-h'_i}(1 + \hat{y}_i)^{h_i} x^{-\mathrm{ind}_\Gamma \widetilde{\mu}_i^+(M)} = 1. \tag{1}$$

We do this in several steps. First, using the definitions of φ_X and φ_Y, we get

$$
\begin{aligned}
\varphi_X(1+\hat{y}_i')^{-h_i'}(1+\hat{y}_i)^{h_i} &= (1+\widehat{\varphi_Y(y_i')})^{-h_i'}(1+\hat{y}_i)^{h_i} \\
&= (1+\hat{y}_i^{-1})^{-h_i'}(1+\hat{y}_i)^{h_i} \\
&= \hat{y}_i^{h_i'}(1+\hat{y}_i)^{h_i-h_i'}.
\end{aligned}
$$

Now, using Proposition 4.8, we get the equalities

$$
\begin{aligned}
\varphi_X((x')^{\mathrm{ind}_{\Gamma'}M})x^{-\mathrm{ind}_\Gamma\widetilde{\mu}_i^+(M)} &= \varphi_X\bigg(\prod_{\ell=1}^{n}(x_\ell')^{g_\ell'}\bigg)\prod_{\ell=1}^{n}x_\ell^{-g_\ell} \\
&= x_i^{g_i}\bigg(\prod_{\ell=1}^{n}x_\ell^{[b_{\ell i}]_+}+\prod_{\ell=1}^{n}x_\ell^{[-b_{\ell i}]_+}\bigg)^{-g_i}\bigg(\prod_{\ell\neq i}x_\ell^{g_\ell'-g_\ell}\bigg)x_i^{-g_i} \\
&= \bigg(\prod_{\ell=1}^{n}x_\ell^{[b_{\ell i}]_+}+\prod_{\ell=1}^{n}x_\ell^{[-b_{\ell i}]_+}\bigg)^{-g_i}\bigg(\prod_{\ell\neq i}x_\ell^{g_\ell'-g_\ell}\bigg).
\end{aligned}
$$

Thus, we have, using the fact that $g_i = h_i - h_i'$ (see [DWZ10, Lemma 5.2]), that the left-hand side of equation (1) is equal to

$$
\hat{y}_i^{h_i'}(1+\hat{y}_i)^{g_i}\bigg(\prod_{\ell=1}^{n}x_\ell^{[b_{\ell i}]_+}+\prod_{\ell=1}^{n}x_\ell^{[-b_{\ell i}]_+}\bigg)^{-g_i}\bigg(\prod_{\ell\neq i}x_\ell^{g_\ell'-g_\ell}\bigg),
$$

which is in turn equal to (using Remark 4.11)

$$
\begin{aligned}
\hat{y}_i^{h_i'}\bigg(\prod_{\ell=1}^{n}x_\ell^{-[b_{\ell i}]_+}\bigg)^{g_i}\bigg(\prod_{\ell\neq i}x_\ell^{g_\ell'-g_\ell}\bigg) &= \hat{y}_i^{h_i'}\bigg(\prod_{\ell=1}^{n}x_\ell^{-[b_{\ell i}]_+}\bigg)^{g_i}\bigg(\prod_{\ell\neq i}x_\ell^{h_i[-b_{\ell i}]_++-h_i'[b_{\ell i}]_+}\bigg) \\
&= \hat{y}_i^{h_i'}\bigg(\prod_{\ell\neq i}x_\ell^{h_i'[-b_{\ell i}]_+-h_i[-b_{\ell i}]_++h_i[-b_{\ell i}]_+-h_i'[b_{\ell i}]_+}\bigg) \\
&= \hat{y}_i^{h_i'}\bigg(\prod_{\ell\neq i}x_\ell^{-h_i'b_{\ell i}}\bigg) \\
&= \prod_{\ell\neq i}x_\ell^{h_i'b_{\ell i}-h_i'b_{\ell i}} \\
&= 1.
\end{aligned}
$$

This finishes the proof. $\qquad\square$

4.3 Extensions and the E-invariant

In this section, we give an interpretation of the E-invariant of a decorated representation, as defined in [DWZ10] (its definition was recalled in §2.4), as the dimension of a space of extensions, using the map Φ of §4.1.

PROPOSITION 4.15. *Let (Q,W) be a quiver with potential, and let $\mathcal{C}$ be the associated cluster category. Let X and Y be objects of $\mathcal{D}$. Then we have the following equalities:*

(i) $E^{\mathrm{inj}}(\Phi(X),\Phi(Y)) = \dim(\Sigma\Gamma)(X,\Sigma Y)$;
(ii) $E^{\mathrm{sym}}(\Phi(X),\Phi(Y)) = \dim(\Sigma\Gamma)(X,\Sigma Y) + \dim(\Sigma\Gamma)(Y,\Sigma X)$;
(iii) $E(\Phi(X)) = (1/2)\dim\mathrm{Hom}_{\mathcal{C}}(X,\Sigma X)$,

where $(\Sigma\Gamma)(X,Y)$ is the subspace of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ containing all morphisms factoring through an object of $\mathrm{add}\,\Sigma\Gamma$.

Proof. The second equality follows immediately from the first one.

The third equality follows from the second one. Indeed, the second equality implies that $(\Sigma\Gamma)(X, \Sigma X)$ is finite dimensional. It then follows from [Pla11, Lemma 3.8] that we have an isomorphism

$$(\Sigma\Gamma)(X, \Sigma X) \cong D\mathrm{Hom}_{\mathcal{C}}(X, \Sigma X)/(\Sigma\Gamma).$$

Since $\dim \mathrm{Hom}_{\mathcal{C}}(X, \Sigma X) = \dim(\Sigma\Gamma)(X, \Sigma X) + \dim \mathrm{Hom}_{\mathcal{C}}(X, \Sigma X)/(\Sigma\Gamma)$, we get that

$$\dim \mathrm{Hom}_{\mathcal{C}}(X, \Sigma X) = 2 \dim(\Sigma\Gamma)(X, \Sigma X)$$
$$= E^{\mathrm{sym}}(\Phi(X), \Phi(X))$$
$$= 2E(\Phi(X)).$$

Let us now prove the first equality. Let

$$T_1^Y \longrightarrow T_0^Y \longrightarrow Y \longrightarrow \Sigma T_1^Y$$

be an (add Γ)-presentation of Y. This triangle yields an exact sequence

$$(X, Y) \xrightarrow{u} (X, \Sigma T_1^Y) \longrightarrow (X, \Sigma T_0^Y) \longrightarrow (X, \Sigma Y) \xrightarrow{v} (X, \Sigma^2 T_1^Y),$$

which in turn gives an exact sequence

$$0 \longrightarrow \mathrm{Im}\, u \longrightarrow (X, \Sigma T_1^Y) \longrightarrow (X, \Sigma T_0^Y) \longrightarrow \mathrm{Ker}\, v \longrightarrow 0.$$

Since X is in $\mathcal{D}$, the two middle terms of this exact sequence are isomorphic to $(T_i^Y, \Sigma X)$ (for $i = 1, 2$) thanks to [Pla11, Proposition 2.16], and these spaces are finite dimensional. Therefore, all of the terms of the exact sequence are finite dimensional.

Now, $\mathrm{Im}\, u$ is isomorphic to $(X, Y)/\mathrm{Ker}\, u$, and $\mathrm{Ker}\, u$ is exactly $(\Gamma)(X, Y)$. Therefore, by [Pla11, Lemma 3.2], $\mathrm{Im}\, u$ is isomorphic to the space $\mathrm{Hom}_{J(Q,W)}(FX, FY)$, where $F = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, ?)$.

Moreover, $\mathrm{Ker}\, v$ is exactly $(\Sigma\Gamma)(X, \Sigma Y)$.

Thus, using the above exact sequence and Proposition 4.8, we have the equalities

$$\dim(\Sigma\Gamma)(X, \Sigma Y) = \dim \mathrm{Hom}_{J(Q,W)}(FX, FY) - \dim(X, \Sigma T_1^Y) + \dim(X, \Sigma T_0^Y)$$
$$= \dim \mathrm{Hom}_{J(Q,W)}(FX, FY) - \dim(T_1^Y, \Sigma X) + \dim(T_0^Y, \Sigma X)$$
$$= \dim \mathrm{Hom}_{J(Q,W)}(FX, FY) - \sum_{i=1}^{n}[T_1^Y : T_i](\dim(FX)_i)$$
$$\quad + \sum_{i=1}^{n}[T_0^X : T_i](\dim(FX)_i)$$
$$= \dim \mathrm{Hom}_{J(Q,W)}(FX, FY) + \sum_{i=1}^{n}[\mathrm{ind}_\Gamma \Sigma Y : \Gamma_i](\dim(FX)_i)$$
$$= \dim \mathrm{Hom}_{J(Q,W)}(FX, FY) + \sum_{i=1}^{n} \mathbf{g}_i(\Phi(Y))(\dim(FX)_i)$$
$$= E^{\mathrm{inj}}(\Phi(X), \Phi(Y)),$$

where $[T_j^Y : T_i]$ is the number of direct summands of T_j^Y isomorphic to T_i in any decomposition of T_j^Y into indecomposable objects, and where the **g**-vector of $\Phi(Y)$ is $(\mathbf{g}_1(\Phi(Y)), \ldots, \mathbf{g}_n(\Phi(Y)))$. This finishes the proof. $\square$

As a corollary, we get the following stronger version of [DWZ10, Lemma 9.2].

COROLLARY 4.16. *Let $\mathcal{M}$ and $\mathcal{M}'$ be two decorated representations of a quiver with potential (Q, W). Assume that $E(\mathcal{M}') = 0$. Then the following conditions are equivalent:*

(i) *$\mathcal{M}$ and $\mathcal{M}'$ are isomorphic;*

(ii) *$E(\mathcal{M}) = 0$ and $\mathbf{g}_{\mathcal{M}} = \mathbf{g}_{\mathcal{M}'}$.*

Proof. Condition (i) obviously implies condition (ii). Now assume that condition (ii) is satisfied. Then Proposition 4.15 implies that $\Psi(\mathcal{M})$ and $\Psi(\mathcal{M}')$ are rigid objects of $\mathcal{D}$. By Proposition 4.8, the indices of $\Psi(\mathcal{M})$ and $\Psi(\mathcal{M}')$ are given by $\mathbf{g}_{\mathcal{M}}$ and $\mathbf{g}_{\mathcal{M}'}$. By hypothesis, their indices are the same. Thus, by Proposition 3.1, $\Psi(\mathcal{M})$ and $\Psi(\mathcal{M}')$ are isomorphic, and so are $\mathcal{M}$ and $\mathcal{M}'$. □

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