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CLUSTER CHARACTERS FOR 2-CALABI–YAU TRIANGULATED CATEGORIES

by Yann PALU

Abstract. — Starting from an arbitrary cluster-tilting object T in a 2-Calabi–Yau triangulated category over an algebraically closed field, as in the setting of Keller and Reiten, we define, for each object L, a fraction $X(T, L)$ using a formula proposed by Caldero and Keller. We show that the map taking L to $X(T, L)$ is a cluster character, i.e. that it satisfies a certain multiplication formula. We deduce that it induces a bijection, in the finite and the acyclic case, between the indecomposable rigid objects of the cluster category and the cluster variables, which confirms a conjecture of Caldero and Keller.

Résumé. — Etant donné un objet amas-basculant T quelconque dans une catégorie triangulée 2-Calabi–Yau sur un corps algébriquement clos (comme dans le cadre de Keller et Reiten), il est possible de définir, pour chaque objet L, une fraction rationnelle $X(T, L)$, en utilisant une formule proposée par Caldero et Keller. On montre, de plus, que l'application associant $X(T, L)$ à L est un caractère amassé ; c'est-à-dire qu'elle vérifie une certaine formule de multiplication. Cela permet de prouver qu'elle induit, dans les cas fini et acyclique, une bijection entre objets rigides indécomposables de la catégorie amassée et variables d'amas de l'algèbre amassée correspondante, confirmant ainsi une conjecture de Caldero et Keller.

Introduction

Cluster algebras were invented and studied by S. Fomin and A. Zelevinsky in [12], [13], [11] and in collaboration with A. Berenstein in [1]. They are commutative algebras endowed with a distinguished set of generators called the cluster variables. These generators are gathered into overlapping sets of fixed finite cardinality, called clusters, which are defined recursively from an initial one via an operation called mutation. A cluster algebra is

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said to be of finite type if it only has a finite number of cluster variables. The finite type cluster algebras were classified in [13].

It was recognized in [26] that the combinatorics of cluster mutation are closely related to those of tilting theory in the representation theory of quivers and finite dimensional algebras. This discovery was the main motivation for the invention of cluster categories (in [7] for the A_n-case and in [4] for the general case). These are certain triangulated categories [20] which, in many cases, allow one to 'categorify' cluster algebras: In the categorical setting, the cluster-tilting objects play the role of the clusters, and their indecomposable direct summands the one of the cluster variables.

In [17], [16], [15], the authors study another setting for the categorification of cluster algebras: The module categories of preprojective algebras of Dynkin type. They succeed in categorifying a different class of cluster algebras, which also contains many cluster algebras of infinite type.

Both cluster categories and module categories of preprojective algebras of Dynkin type are 2-Calabi–Yau categories in the sense that we have bifunctorial isomorphisms

$$\operatorname{Ext}^1(X,Y) \simeq D\operatorname{Ext}^1(Y,X),$$

which are highly relevant in establishing the link with cluster algebras. This motivates the study of more general 2-Calabi–Yau categories in [23], [22], [27], [24], [18], [19], [3]. In order to show that a given 2-Calabi–Yau category "categorifies" a given cluster algebra, a crucial point is

a) to construct an explicit map from the set of indecomposable factors of cluster-tilting objects to the set of cluster variables, and
b) to show that it is bijective.

Such a map was constructed for module categories of preprojective algebras of Dynkin type in [17] using Lusztig's work [25]. For cluster categories, it was defined by P. Caldero and F. Chapoton in [8]. More generally, for each object M of the cluster category, they defined a fraction X_M in $\mathbb{Q}(x_1,\ldots,x_n)$. The bijectivity property of the Caldero–Chapoton map was proved in [8] for finite type and in [10], cf. also [2], for acyclic type.

A crucial property of the Caldero–Chapoton map is the following. For any pair of indecomposable objects L and M of $\mathcal{C}$ whose extension space $\mathcal{C}(L,\Sigma M)$ is one-dimensional, we have

$$X_L X_M = X_B + X_{B'},$$

where Σ denotes the suspension in $\mathcal{C}$ and where B and B' are the middle terms of "the" two non-split triangles with outer terms L and M. We define, in defintion 1.2 a cluster character to be a map satisfying this multiplication formula.

This property has been proved in [9] in the finite case, in [15] for the analogue of the Caldero–Chapoton map in the preprojective case, and in [10] in the acyclic case.

The main result of this article is a generalisation of this multiplication formula. Starting from an arbitrary cluster-tilting object T in an arbitrary 2-Calabi–Yau category $\mathcal{C}$ over an algebraically closed field (as in the setting of [23]), we define, for each object L of $\mathcal{C}$, a fraction X_L^T using a formula proposed in [9, 6.1]. We show that the map $L \mapsto X_L^T$ is a cluster character. We deduce that it has the bijectivity property in the finite and the acyclic case, which confirms conjecture 2 of [9]. Here, it yields a new way of expressing cluster variables as Laurent polynomials in the variables of a fixed cluster. Our theorem also applies to stable categories of preprojective algebras of Dynkin type and their Calabi–Yau reductions studied in [14] and [3].

Let k be an algebraically closed field, and let $\mathcal{C}$ be a 2-Calabi–Yau Hom-finite triangulated k-category with a cluster-tilting object T (see section 1).

The article is organised as follows: In the first section, the notations are given and the main result is stated. In the next two sections, we investigate the exponents appearing in the definition of X_L^T. In section 2, we define the index and the coindex of an object of $\mathcal{C}$ and show how they are related to the exponents. Section 3 is devoted to the study of the antisymmetric bilinear form

$$\langle\ ,\ \rangle_a$$

on $\mathsf{mod}\,\mathrm{End}_{\mathcal{C}}T$. We show that this form descends to the Grothendieck group $\mathrm{K}_0(\mathsf{mod}\,\mathrm{End}_{\mathcal{C}}T)$, confirming conjecture 1 of [9, 6.1]. In section 4, we prove that the same phenomenon of dichotomy as in [10, section 3] (see also [15]) still holds in our setting. The results of the first sections are used in section 5 to prove the multiplication formula. We draw some consequences in section 5.2. Two examples are given in section 6.

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1. Main result

Let k be an algebraically closed field, and let $\mathcal{C}$ be a k-linear triangulated category with split idempotents. Denote by Σ its suspension functor. Assume moreover that the category $\mathcal{C}$

a) is Hom-finite: For any two objects X and Y in $\mathcal{C}$, the space of morphisms $\mathcal{C}(X, Y)$ is finite-dimensional,
b) is 2-Calabi–Yau: There exist bifunctorial isomorphisms

$$\mathcal{C}(X, \Sigma Y) \simeq D\mathcal{C}(Y, \Sigma X),$$

where D denotes the duality functor $\mathsf{Hom}_k(?, k)$, and
c) admits a cluster-tilting object T, which means that
   i) $\mathcal{C}(T, \Sigma T) = 0$ and
   ii) for any X in $\mathcal{C}$, if $\mathcal{C}(X, \Sigma T) = 0$, then X belongs to the full subcategory $\operatorname{add} T$ formed by the direct summands of sums of copies of T.

For two objects X and Y of $\mathcal{C}$, we often write (X, Y) for the space of morphisms $\mathcal{C}(X, Y)$ and we denote its dimension by $[X, Y]$. Similarly, we write ${}^1(X, Y)$ for $\mathcal{C}(X, \Sigma Y)$ and ${}^1[X, Y]$ for its dimension. Let B be the endomorphism algebra of T in $\mathcal{C}$, and let $\operatorname{mod} B$ be the category of finite-dimensional right B-modules. As shown in [6], cf. also [23], the functor

$$F : \mathcal{C} \longrightarrow \operatorname{mod} B \quad , \quad X \longmapsto \mathcal{C}(T, X),$$

induces an equivalence of categories

$$\mathcal{C}/(\Sigma T) \xrightarrow{\simeq} \operatorname{mod} B,$$

where (ΣT) denotes the ideal of morphisms of $\mathcal{C}$ which factor through a direct sum of copies of ΣT.

The following useful proposition is proved in [23] and [24]:

Proposition 1.1. — *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X$ be a triangle in $\mathcal{C}$. Then*

- *The morphism g induces a monomorphism in $\operatorname{mod} B$ if and only if $f \in (\Sigma T)$.*
- *The morphism f induces an epimorphism in $\operatorname{mod} B$ if and only if $g \in (\Sigma T)$*

Moreover, if X has no direct summands in $\operatorname{add} \Sigma T$, then FX is projective (resp. injective) if and only if X lies in $\operatorname{add}(T)$ (resp. in $\operatorname{add}(\Sigma^2 T)$).

Definition 1.2. — *A cluster character on $\mathcal{C}$ with values in a commutative ring A is a map*

$$\chi : \ obj(\mathcal{C}) \longrightarrow A$$

such that

- *for all isomorphic objects L and M, we have $\chi(L) = \chi(M)$,*
- *for all objects L and M of $\mathcal{C}$, we have $\chi(L \oplus M) = \chi(L)\chi(M)$,*
- *for all objects L and M of $\mathcal{C}$ such that $\dim \mathrm{Ext}^1_{\mathcal{C}}(L, M) = 1$, we have*

$$\chi(L)\chi(M) = \chi(B) + \chi(B'),$$

where B and B' are the middle terms of 'the' non-split triangles

$$L \to B \to M \to \Sigma L \ \text{ and } \ M \to B' \to L \to \Sigma M$$

with end terms L and M.

Let N be a finite-dimensional B-module and e an element of $\mathrm{K}_0(\mathrm{mod}\, B)$. We write $\mathrm{Gr}_e(N)$ for the variety of submodules N' of N whose class in $\mathrm{K}_0(\mathrm{mod}\, B)$ is e. It is a closed, hence projective, subvariety of the classical Grassmannian of subspaces of N. Let $\chi(\mathrm{Gr}_e\, N)$ denote its Euler–Poincaré characteristic with respect to the étale cohomology with proper support. Let $\mathrm{K}_0^{\mathrm{sp}}(\mathrm{mod}\, B)$ denote the 'split' Grothendieck group of $\mathrm{mod}\, B$, i.e. the quotient of the free abelian group on the set of isomorphism classes $[N]$ of finite-dimensional B-modules N, modulo the subgroup generated by all elements

$$[N_1 \oplus N_2] - [N_1] - [N_2].$$

We define a bilinear form

$$\langle\ ,\ \rangle : \mathrm{K}_0^{\mathrm{sp}}(\mathrm{mod}\, B) \times \mathrm{K}_0^{\mathrm{sp}}(\mathrm{mod}\, B) \longrightarrow \mathbb{Z}$$

by setting

$$\langle N, N' \rangle = [N, N'] \ - \ {}^1[N, N']$$

for all finite-dimensional B-modules N and N'. We define an antisymmetric bilinear form on $\mathrm{K}_0^{\mathrm{sp}}(\mathrm{mod}\, B)$ by setting

$$\langle N, N' \rangle_a \ = \ \langle N, N' \rangle - \langle N', N \rangle$$

for all finite-dimensional B-modules N and N'. Let $T_1, \ldots, T_n$ be the pairwise non-isomorphic indecomposable direct summands of T and, for $i = 1, \ldots, n$, let S_i be the top of the projective B-module $P_i = FT_i$. The set $\{S_i, i = 1, \ldots, n\}$ is a set of representatives for the isoclasses of simple B-modules.

We need a lemma, the proof of which will be given in section 3.1.

LEMMA 1.3. — *For any $i = 1, \ldots, n$, the linear form $\langle S_i, ?\rangle_a$: $\mathrm{K}_0^{sp}(\mathrm{mod}\, B) \to \mathbb{Z}$ induces a well-defined form*

$$\langle S_i, ?\rangle_a \,:\, \mathrm{K}_0(\mathrm{mod}\, B) \to \mathbb{Z}.$$

Let $\mathrm{ind}\, \mathcal{C}$ be a set of representatives for the isoclasses of indecomposable objects of $\mathcal{C}$. Define, as in [9, 6.1], a Caldero–Chapoton map, $X_?^T$: $\mathrm{ind}\, \mathcal{C} \to \mathbb{Q}(x_1, \ldots, x_n)$ by

$$X_M^T = \begin{cases} x_i \text{ if } M \simeq \Sigma T_i \\ \sum_e \chi(\mathrm{Gr}_e\, FM) \prod_{i=1}^n x_i^{\langle S_i, e\rangle_a - \langle S_i, FM\rangle} \text{ else.} \end{cases}$$

Extend it to a map $X_?^T : \mathcal{C} \to \mathbb{Q}(x_1, \ldots, x_n)$ by requiring that $X_{M\oplus N}^T = X_M^T X_N^T$. When there are no possible confusions, we often denote X_M^T by X_M. The main result of this article is the following

THEOREM 1.4. — *The map $X_?^T : \mathcal{C} \to \mathbb{Q}(x_1, \ldots, x_n)$ is a cluster character.*

We will prove the theorem in section 5.1, illustrate it by examples in section 6 and draw some consequences in section 5.2.

2. Index, coindex and Euler form

In the next two sections, our aim is to understand the exponents appearing in the definition of X_M. More precisely, for two objects L and M of $\mathcal{C}$, we want to know how the exponents in X_B depend on the choice of the middle term B of a triangle with outer terms L and M.

2.1. Index and coindex

Let X be an object of $\mathcal{C}$. Define its index $\mathrm{ind}\, X \in \mathrm{K}_0(\mathrm{proj}\, B)$ as follows. There exists a triangle (see [KR1])

$$T_1^X \to T_0^X \to X \to \Sigma T_1^X$$

with T_0^X and T_1^X in $\mathrm{add}\, T$. Define $\mathrm{ind}\, X$ to be the class $[FT_0^X] - [FT_1^X]$ in $\mathrm{K}_0(\mathrm{proj}\, B)$. Similarly, define the coindex of X, denoted by $\mathrm{coind}\, X$, to be the class $[FT_X^0] - [FT_X^1]$ in $\mathrm{K}_0(\mathrm{proj}\, B)$, where

$$X \to \Sigma^2 T_X^0 \to \Sigma^2 T_X^1 \to \Sigma X$$

is a triangle in $\mathcal{C}$ with $T_X^0, T_X^1 \in \mathrm{add}\, T$.

Lemma 2.1. — *We have the following properties:*

1. *The index and coindex are well defined.*
2. $\operatorname{ind} X = -\operatorname{coind} \Sigma X$.
3. $\operatorname{ind} T_i = [P_i]$ *and* $\operatorname{ind} \Sigma T_i = -[P_i]$ *where* $P_i = FT_i$.
4. $\operatorname{ind} X - \operatorname{coind} X$ *only depends on* $FX \in \operatorname{mod} B$.

Proof. — A right $\operatorname{add} T$-approximation of an object X of $\mathcal{C}$ is a morphism $T' \xrightarrow{f} X$ with $T' \in \operatorname{add} T$ such that any morphism $T'' \longrightarrow X$ with $T'' \in \operatorname{add} T$ factors through f. It is called minimal if, moreover, any morphism $T' \xrightarrow{g} T'$ such that $fg = f$ is an isomorphism. A minimal approximation is unique up to isomorphism.
Assertions (2) and (3) are left to the reader.

(1) In any triangle of the form

$$T_1^X \to T_0^X \xrightarrow{f} X \to \Sigma T_1^X,$$

the morphism f is a right $\operatorname{add} T$-approximation. Therefore, any such triangle is obtained from one where f is minimal by adding a trivial triangle

$$T' \to T' \to 0 \to \Sigma T'$$

with $T' \in \operatorname{add} T$. The index is thus well-defined. Dually, one can define left approximations and show that the coindex is well-defined.

(4) Let T' be an object in $\operatorname{add} T$. Take two triangles

$$T_1^X \to T_0^X \to X \to \Sigma T_1^X \text{ and}$$

$$X \to \Sigma^2 T_X^0 \to \Sigma^2 T_X^1 \to \Sigma X$$

with T_0^X, T_1^X, T_X^0 and T_X^1 in $\operatorname{add} T$. Then, we have two triangles

$$T_1^X \oplus T' \to T_0^X \to X \oplus \Sigma T' \to \Sigma(T_1^X \oplus T') \text{ and}$$

$$X \oplus \Sigma T' \to \Sigma^2 T_X^0 \to \Sigma^2 (T_X^1 \oplus T') \to \Sigma X \oplus \Sigma^2 T'.$$

We thus have the equality:

$$\operatorname{ind}(X \oplus \Sigma T') - \operatorname{coind}(X \oplus \Sigma T') = \operatorname{ind} X - \operatorname{coind} X.$$

□

Proposition 2.2. — *Let* $X \xrightarrow{f} Z \xrightarrow{g} Y \xrightarrow{\varepsilon} \Sigma X$ *be a triangle in* $\mathcal{C}$. *Take* $C \in \mathcal{C}$ *(resp.* $K \in \mathcal{C}$*) to be any lift of* $\operatorname{Coker} Fg$ *(resp.* $\operatorname{Ker} Ff$*). Then*

$$\begin{aligned} \operatorname{ind} Z &= \operatorname{ind} X + \operatorname{ind} Y - \operatorname{ind} C - \operatorname{ind} \Sigma^{-1} C \text{ and} \\ \operatorname{coind} Z &= \operatorname{coind} X + \operatorname{coind} Y - \operatorname{coind} K - \operatorname{coind} \Sigma K. \end{aligned}$$

Proof. — Let us begin with the equality for the indices. First, consider the case where $FC = 0$. This means that the morphism ε belongs to the ideal (ΣT). Take two triangles

$$T_1^X \longrightarrow T_0^X \longrightarrow X \longrightarrow \Sigma T_1^X \text{ and } \; T_1^Y \longrightarrow T_0^Y \longrightarrow Y \longrightarrow \Sigma T_1^Y$$

in $\mathcal{C}$, where the objects $T_0^X, T_1^X, T_0^Y, T_1^Y$ belong to the subcategory $\operatorname{add} T$. Since the morphism ε belongs to the ideal (ΣT), the composition $T_0^Y \to Y \xrightarrow{\varepsilon} \Sigma X$ vanishes. The morphism $T_0^Y \to Y$ thus factors through g. This gives a commutative square

$$\begin{array}{ccc} T_0^X \oplus T_0^Y & \longrightarrow & T_0^Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y. \end{array}$$

Fit it into a nine-diagram

$$\begin{array}{ccccccc} T_1^X & \longrightarrow & Z' & \longrightarrow & T_1^Y & \longrightarrow & \Sigma T_1^X \\ \downarrow & & \downarrow & & \downarrow & & \\ T_0^X & \longrightarrow & T_0^X \oplus T_0^Y & \longrightarrow & T_0^Y & \xrightarrow{0} & \Sigma T_0^X \\ \downarrow & & \downarrow & \swarrow & \downarrow & \searrow^{0} & \\ X & \longrightarrow & Z & \xrightarrow{g} & Y & \xrightarrow{\varepsilon} & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \\ \Sigma T_1^X & & \Sigma Z' & & \Sigma T_1^Y, & & \end{array}$$

whose rows and columns are triangles. Since the morphism $T_1^Y \to \Sigma T_1^X$ vanishes, the triangle in the first row splits, so that we have

$$Z' \simeq T_1^X \oplus T_1^Y \text{ and } \; \operatorname{ind} Z = \operatorname{ind} X + \operatorname{ind} Y.$$

Now, let us prove the formula in the general case. Let $FY \xrightarrow{a} M$ be a cokernel for Fg. Since the composition $F\varepsilon\, Fg$ vanishes, the morphism $F\varepsilon$ factors through a:

$$\begin{array}{ccc} FY & \xrightarrow{F\varepsilon} & F\Sigma X. \\ & {}_{a}\searrow \quad \nearrow_{b} & \\ & M & \end{array}$$

Let $Y \xrightarrow{\alpha} C'$ be a lift of a in $\mathcal{C}$, and let β be a lift of b. The images under F of the morphisms ε and $\beta\,\alpha$ coincide, therefore the morphism $\beta\,\alpha - \varepsilon$

belongs to the ideal (ΣT). Thus there exist an object T' in $\operatorname{add} T$ and two morphisms α' and β' such that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\varepsilon} & \Sigma X. \\ & {\scriptstyle \left[\begin{smallmatrix}\alpha\\ \alpha'\end{smallmatrix}\right]}\searrow \quad \nearrow{\scriptstyle [\beta\ \beta']} & \\ & C' \oplus \Sigma T' & \end{array}$$

Let C be the direct sum $C' \oplus \Sigma T'$.
The octahedral axiom yields a commutative diagram

$$\begin{array}{ccccccc} & & U & = & U & & \\ & & \downarrow & & \downarrow & & \\ X & \longrightarrow & Z & \longrightarrow & Y & \xrightarrow{\varepsilon} & \Sigma X \\ \| & & \downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix}\alpha\\ \alpha'\end{smallmatrix}\right]} & & \| \\ X & \longrightarrow & V & \xrightarrow{\gamma'} & C & \xrightarrow{[\beta\ \beta']} & \Sigma X \\ & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma''} & & \\ & & \Sigma U & = & \Sigma U, & & \end{array}$$

whose two central rows and columns are triangles. Due to the choice of C, the morphisms γ', γ'', hence γ belong to the ideal (ΣT). We thus have the equalities:

$$\begin{aligned} \operatorname{ind} Y &= \operatorname{ind} C + \operatorname{ind} U, \\ \operatorname{ind} X &= \operatorname{ind} V + \operatorname{ind} \Sigma^{-1} C, \\ \operatorname{ind} Z &= \operatorname{ind} V + \operatorname{ind} U, \end{aligned}$$

giving the desired formula. Moreover, as seen in lemma 2.1 (4), the sum $\operatorname{ind} C + \operatorname{ind} \Sigma^{-1} C = \operatorname{ind} C - \operatorname{coind} C$ does not depend on the particular choice of C. Apply this formula to the triangle

$$\Sigma^{-1} X \longrightarrow \Sigma^{-1} Z \longrightarrow \Sigma^{-1} Y \longrightarrow X$$

and use lemma 2.1(2) to obtain the formula for the coindices. Remark that the long exact sequence yields the equality of $\operatorname{Coker}(-F\Sigma^{-1} g)$ and $\operatorname{Ker} Ff$. □

2.2. Exponents.

We now compute the index and coindex in terms of the Euler form.

Lemma 2.3. — *Let $X \in \mathcal{C}$ be indecomposable. Then*

$$\operatorname{ind} X = \begin{cases} -[P_i] \text{ if } X \simeq \Sigma T_i \\ \sum_{i=1}^n \langle FX, S_i\rangle [P_i] \text{ else,} \end{cases}$$

$$\operatorname{coind} X = \begin{cases} -[P_i] \text{ if } X \simeq \Sigma T_i \\ \sum_{i=1}^n \langle S_i, FX\rangle [P_i] \text{ else.} \end{cases}$$

Proof. — Let X be an indecomposable object in $\mathcal{C}$, non-isomorphic to any of the ΣT_i's. Take a triangle

$$T_1^X \xrightarrow{f} T_0^X \xrightarrow{g} X \xrightarrow{\varepsilon} \Sigma T_1^X$$

with the morphism g being a minimal right add T-approximation, as defined in the proof of lemma 2.1. We thus get a minimal projective presentation

$$P_1^X \longrightarrow P_0^X \longrightarrow FX \longrightarrow 0$$

where $P_i^X = FT_i^X, i = 0, 1$. For any i, the differential in the complex

$$0 \longrightarrow (P_0^X, S_i) \longrightarrow (P_1^X, S_i) \longrightarrow \cdots$$

vanishes. Therefore, we have

$$\begin{aligned} [FX, S_i] &= [P_0^X, S_i] = [P_0^X : P_i], \\ {}^1[FX, S_i] &= [P_1^X, S_i] = [P_1^X : P_i], \\ \langle FX, S_i\rangle &= [\operatorname{ind} X : P_i]. \end{aligned}$$

The proof for the coindex is analogous: We use a minimal injective copresentation of FX induced by a triangle

$$X \longrightarrow \Sigma^2 T_X^0 \longrightarrow \Sigma^2 T_X^1 \longrightarrow \Sigma X.$$

□

Let us write $\underline{x}^e$ for $\prod_{i=1}^n x_i^{[e:P_i]}$ where $e \in \mathrm{K}_0(\operatorname{proj} B)$ and $[e : P_i]$ is the ith coefficient of e in the basis $[P_1], \ldots, [P_n]$. Then, by lemma 2.3, for any indecomposable object M in $\mathcal{C}$, we have

$$X_M = \underline{x}^{-\operatorname{coind} M} \sum_e \chi(\mathrm{Gr}_e\, FM) \prod_{i=1}^n x_i^{<S_i, e>_a}.$$

3. The antisymmetric bilinear form

In this part, we give a positive answer to the first conjecture of [9, 6.1] and prove that the exponents in X_M are well defined. The first lemma is sufficient for this latter purpose, but is not very enlightening, whereas the second proof of theorem 3.4 gives us a better understanding of the antisymetric bilinear form. When the category $\mathcal{C}$ is algebraic, this form is, in fact, the usual Euler form on the Grothendieck group of a triangulated category together with a t-structure whose heart is the abelian category $\operatorname{mod} B$ itself.

3.1. The map X^T is well defined

Let us first show that any short exact sequence in $\operatorname{mod} B$ can be lifted to a triangle in $\mathcal{C}$.

Lemma 3.1. — *Let $0 \to x \to y \to z \to 0$ be a short exact sequence in $\operatorname{mod} B$. Then there exists a triangle in $\mathcal{C}$*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

whose image under F is isomorphic to the given short exact sequence.

Proof. — Let

$$0 \longrightarrow x \xrightarrow{i} y \xrightarrow{p} z \longrightarrow 0$$

be a short exact sequence in $\operatorname{mod} B$. Let $X \xrightarrow{f} Y$ be a lift of the monomorphism $x \xrightarrow{i} y$ in $\mathcal{C}$. Fix a triangle

$$T_1^X \longrightarrow T_0^X \longrightarrow X \longrightarrow \Sigma T_1^X$$

and form a triangle

$$X \longrightarrow Y \oplus \Sigma T_1^X \longrightarrow Z \xrightarrow{\varepsilon} \Sigma X \ .$$

The commutative left square extends to a morphism of triangles

$$\begin{array}{ccccccc}
X & \longrightarrow & Y \oplus \Sigma T_1^X & \longrightarrow & Z & \xrightarrow{\varepsilon} & \Sigma X \\
\| & & \downarrow{\scriptstyle [0\ 1]} & & \downarrow & & \| \\
X & \longrightarrow & \Sigma T_1^X & \longrightarrow & \Sigma T_0^X & \longrightarrow & \Sigma X.
\end{array}$$

so that the morphism ε lies in the ideal (ΣT). Therefore, the sequence

$$0 \longrightarrow x \xrightarrow{i} y \longrightarrow FZ \longrightarrow 0$$

is exact, and the modules FZ and z are isomorphic. □

Proof of lemma 1.3. — Let X be an object of the category $\mathcal{C}$. Using section 2.2 we have

$$\operatorname{coind} X - \operatorname{ind} X = \sum_{i=1}^{n} \langle S_i, FX \rangle_a \, [P_i] \, .$$

Therefore, it is sufficient to show that the form

$$\begin{array}{rcl} \mathrm{K}_0(\operatorname{mod} B) & \longrightarrow & \mathbb{Z} \\ [FX] & \longmapsto & \operatorname{coind} X - \operatorname{ind} X \end{array}$$

is well defined. We already know that $\operatorname{coind} X - \operatorname{ind} X$ only depends on FX. Take $0 \to x \to y \to z \to 0$ to be a short exact sequence in $\operatorname{mod} B$. Lift it, as in lemma 3.1, to a triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \text{ in } \mathcal{C}.$$

By proposition 2.2, we have

$$\operatorname{ind} Y - \operatorname{coind} Y = (\operatorname{ind} X + \operatorname{ind} Z) - (\operatorname{coind} X + \operatorname{coind} Z)$$

which is the required equality. □

Corollary 3.2. — *The map*

$$X_?^T \,: \mathcal{C} \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$$

is well defined.

3.2. The antisymmetric bilinear form descends to the Grothendieck group.

In this subsection, we prove a stronger result than in the previous one. This gives a positive answer to the first conjecture in [9, 6.1].

Lemma 3.3. — *Let T' be any cluster-tilting object in $\mathcal{C}$. We have bifunctorial isomorphisms*

$$\mathcal{C}/_{(T')}(\Sigma^{-1} X, Y) \simeq D(T')(\Sigma^{-1} Y, X).$$

Proof. — Let X and Y be two objects of $\mathcal{C}$, and let $T_1' \longrightarrow T_0' \longrightarrow X \xrightarrow{\eta} \Sigma T_1'$ be a triangle in $\mathcal{C}$, with T_0' and T_1' in $\operatorname{add} T'$. Consider the morphism

$$\begin{array}{rrcl} \alpha : & (T_1', Y) & \longrightarrow & (\Sigma^{-1} X, Y) \\ & f & \longmapsto & f \circ \Sigma^{-1} \eta. \end{array}$$

We have

$$D(T')(\Sigma^{-1} X, Y) \simeq D \operatorname{Im} \alpha \simeq \operatorname{Im} D\alpha.$$

Since the category $\mathcal{C}$ is 2-Calabi–Yau, the dual of α, $D\alpha$, is isomorphic to

$$\begin{array}{rcl} \alpha' : \ (\Sigma^{-1}Y, X) & \longrightarrow & (\Sigma^{-1}Y, \Sigma T'_1) \\ g & \longmapsto & \eta \circ g. \end{array}$$

We thus have isomorphisms

$$\begin{array}{rcl} D(T')(\Sigma^{-1}X, Y) & \simeq & \operatorname{Im} \alpha' \\ & \simeq & (\Sigma^{-1}Y, X) / \operatorname{Ker} \alpha' \\ & \simeq & \mathcal{C}/_{(T')}(\Sigma^{-1}Y, X). \end{array}$$

□

THEOREM 3.4. — *The antisymmetric bilinear form* $\langle\ ,\ \rangle_a$ *descends to the Grothendieck group* $\mathrm{K}_0(\operatorname{mod} B)$.

Proof. — Let X and Y be two objects in the category $\mathcal{C}$. In order to compute $\langle FX, FY\rangle = [FX, FY] - {}^1[FX, FY]$, let us construct a projective presentation in the following way. Let

$$\Sigma^{-1}X \xrightarrow{g} T_1^X \xrightarrow{f} T_0^X \longrightarrow X$$

be a triangle in $\mathcal{C}$ with T_0^X and T_1^X being two objects in the subcategory $\operatorname{add} T$. This triangle induces an exact sequence in $\operatorname{mod} B$

$$F\Sigma^{-1}X \xrightarrow{Fg} FT_1^X \xrightarrow{Ff} FT_0^X \longrightarrow FX \longrightarrow 0,$$

where FT_0^X and FT_1^X are finite-dimensional projective B-modules. Form the complex

$$(*) \qquad 0 \longrightarrow \operatorname{Hom}_B(FT_0^X, FY) \longrightarrow \operatorname{Hom}_B(FT_1^X, FY) \longrightarrow \operatorname{Hom}_B(F\Sigma^{-1}X, FY).$$

Since the object T is cluster-tilting in $\mathcal{C}$, there are no morphisms from any object in $\operatorname{add} T$ to any object in $\operatorname{add} \Sigma T$. The complex $(*)$ is thus isomorphic to the following one :

$$0 \longrightarrow \mathcal{C}(T_0^X, Y) \xrightarrow{f^*} \mathcal{C}(T_1^X, Y) \xrightarrow{g^*} \mathcal{C}/_{(\Sigma T)}(\Sigma^{-1}X, Y),$$

where f^* (resp. g^*) denotes the composition by f (resp. g). Therefore, we have

$$\begin{array}{rcl} \operatorname{Hom}_B(FX, FY) & \simeq & \operatorname{Ker} f^* \\ \operatorname{Ext}^1_B(FX, FY) & \simeq & \operatorname{Ker} g^* / \operatorname{Im} f^*. \end{array}$$

We can now express the bilinear form as

$$\begin{array}{rcl} \langle FX, FY\rangle & = & \dim \operatorname{Ker} f^* - \dim \operatorname{Ker} g^* + \mathsf{rk}\, f^* \\ & = & [T_0^X, Y] - [T_1^X, Y] + \mathsf{rk}\, g^*, \end{array}$$

with the image of the morphism g^* being the quotient by the ideal (ΣT) of the space of morphisms from $\Sigma^{-1}X$ to Y, in $\mathcal{C}$, which belong to the ideal (T):

$$\operatorname{Im} g^* = (T)/_{(\Sigma T)}(\Sigma^{-1}X, Y).$$

Similarily, using an injective copresentation given by a triangle of the form

$$X \longrightarrow \Sigma^2 T_X^0 \longrightarrow \Sigma^2 T_X^1 \xrightarrow{\beta} \Sigma X,$$

we obtain

$$\langle FY, FX \rangle = [Y, \Sigma^2 T_X^0] - [Y, \Sigma^2 T_X^1] + \mathsf{rk}\, \beta_*,$$

and $\operatorname{Im} \beta_* = (\Sigma^2 T)/_{(\Sigma T)}(Y, \Sigma X)$. By lemma 3.3, we have bifunctorial isomorphisms

$$(T)/_{(\Sigma T)}(\Sigma^{-1}X, Y) \simeq D(\Sigma T)/_{(T)}(\Sigma^{-1}Y, X) \simeq D(\Sigma^2 T)/_{(\Sigma T)}(Y, \Sigma X).$$

Therefore, we have the equality

$$\begin{aligned}\langle FX, FY \rangle_a &= [T_0^X, Y] - [T_1^X, Y] - [Y, \Sigma^2 T_X^0] + [Y, \Sigma^2 T_X^1] \\ &= [FT_0^X, FY] - [FT_1^X, FY] - [FY, F\Sigma^2 T_X^0] + [FY, F\Sigma^2 T_X^1].\end{aligned}$$

Since FT is projective and $F\Sigma^2 T$ in injective, this formula shows that $\langle\ ,\ \rangle_a$ descends to a bilinear form on the Grothendieck group $\mathrm{K}_0(\operatorname{mod} B)$. □

3.3. The antisymmetric bilinear form and the Euler form

In this subsection, assume moreover that the category $\mathcal{C}$ is algebraic, as in [23, section 4]: There exists a k-linear Frobenius category with split idempotents $\mathcal{E}$ whose stable category is $\mathcal{C}$. Denote by $\mathcal{M}$ the preimage, in $\mathcal{E}$, of $\operatorname{add} T$ via the canonical projection functor. The category $\mathcal{M}$ thus contains the full subcategory $\mathcal{P}$ of $\mathcal{E}$ whose objects are the projective objects in $\mathcal{E}$, and we have $\underline{\mathcal{M}} = \operatorname{add} T$. Let $\operatorname{Mod} \mathcal{M}$ be the category of $\mathcal{M}$-modules, i.e. of k-linear contravariant functors from $\mathcal{M}$ to the category of k-vector spaces. The category $\operatorname{mod} \underline{\mathcal{M}}$ of finitely presented $\underline{\mathcal{M}}$-modules is identified with the full subcategory of $\operatorname{Mod} \mathcal{M}$ of finitely presented $\mathcal{M}$-modules vanishing on $\mathcal{P}$. This last category is equivalent to the abelian category $\operatorname{mod} B$ of finitely generated B-modules. Recall that the perfect derived category $\operatorname{per} \mathcal{M}$ is the full triangulated subcategory of the derived category of $\mathcal{D} \operatorname{Mod} \mathcal{M}$ generated by the finitely generated projective $\mathcal{M}$-modules. Define $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$ to be the full subcategory of $\operatorname{per} \mathcal{M}$ whose objects X satisfy the following conditions:

- for each integer n, the finitely presented $\mathcal{M}$-module $\mathsf{H}^n X$ belongs to $\operatorname{mod} \underline{\mathcal{M}}$,
- the module $\mathsf{H}^n X$ vanishes for all but finitely many $n \in \mathbb{Z}$.

It can easily be shown that $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$ is a triangulated subcategory of $\operatorname{per} \mathcal{M}$. Moreover, as shown in [27], the canonical t-structure on $\mathcal{D} \operatorname{Mod} \mathcal{M}$ induces a t-structure on $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$, whose heart is the abelian category $\operatorname{mod} \underline{\mathcal{M}}$. The following lemma shows that the Euler form

$$\begin{aligned} \mathsf{K}_0\left(\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}\right) \times \mathsf{K}_0\left(\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}\right) &\longrightarrow \mathbb{Z} \\ ([X],[Y]) \longmapsto \langle [X],[Y]\rangle &= \sum_{i\in\mathbb{Z}} (-1)^i \dim \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}\left(X, \Sigma^i Y\right) \end{aligned}$$

is well defined.

Lemma 3.5. — *Let X and Y belong to $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$. Then the vector spaces $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}\left(X, \Sigma^i Y\right)$ are finite dimensional and only finitely many of them are non-zero.*

Proof. — Since X belongs to $\operatorname{per} \mathcal{M}$, we may assume that it is representable: There exists M in $\mathcal{M}$ such that $X = M\hat{}$. Moreover, the module $\mathsf{H}^n Y$ vanishes for all but finitely many $n \in \mathbb{Z}$. We thus may assume Y to be concentrated in degree 0. Therefore, the space $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}\left(X, \Sigma^i Y\right) = \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}(M\hat{}, \Sigma^i \mathsf{H}^0 Y)$ vanishes for all non-zero i. For $i = 0$, it equals

$$\begin{aligned} \operatorname{Hom}_{\mathcal{M}}\left(M\hat{}, \mathsf{H}^0 Y\right) &= \mathsf{H}^0 Y(M) \\ &= \operatorname{Hom}_{\underline{\mathcal{M}}}\left(\underline{\mathcal{M}}(?, M), \mathsf{H}^0 Y\right). \end{aligned}$$

this last space being finite dimensional. □

This enables us to give another proof of theorem 3.4. This proof is less general than the previous one, but is nevertheless much more enlightening.

Proof of theorem 3.4. — Let X and Y be two finitely presented $\underline{\mathcal{M}}$-modules, lying in the heart of the t-structure on $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$. We have:

$$\begin{aligned} \langle [X],[Y]\rangle &= \sum_{i\in\mathbb{Z}} (-1)^i \dim \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}\left(X, \Sigma^i Y\right) \\ &= \sum_{i=0}^{3} (-1)^i \dim \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}\left(X, \Sigma^i Y\right) \qquad (3.1) \\ &= \dim \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}(X,Y) - \dim \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}(X,\Sigma Y) \\ &\qquad + \dim \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}(X,\Sigma^2 Y) - \dim \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}(X,\Sigma^3 Y) \\ &= \dim \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}(X,Y) - \dim \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}(X,\Sigma Y) \qquad (3.2) \\ &\qquad + \dim \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}(Y,X) - \dim \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}(Y,\Sigma X) \\ &= \dim \operatorname{Hom}_{\underline{\mathcal{M}}}(X,Y) - \dim \operatorname{Ext}^1_{\underline{\mathcal{M}}}(X,Y) \\ &\qquad + \dim \operatorname{Hom}_{\underline{\mathcal{M}}}(Y,X) - \dim \operatorname{Ext}^1_{\underline{\mathcal{M}}}(Y,X) \\ &= \langle [X],[Y]\rangle_a \end{aligned}$$

where the classes are now taken in $\mathrm{K}_0(\operatorname{mod} B)$. Equalities (3.1) and (3.2) are consequences of the 3-Calabi–Yau property of the category $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$, cf. [23]. □

4. Dichotomy

Our aim in this part is to study the coefficients appearing in the definition of X_M. In particular, we will prove that the phenomenon of dichotomy proved in [10] (see also [15]) remains true in this more general setting.

Recall that we write $\underline{x}^e$ for $\prod_{i=1}^n x_i^{[e:P_i]}$ where $e \in \mathrm{K}_0(\operatorname{proj} B)$ and $[e : P_i]$ is the ith coefficient of e in the basis $[P_1], \ldots, [P_n]$.

Lemma 4.1. — *For any $M \in \mathcal{C}$, we have*

$$X_M = \underline{x}^{-\operatorname{coind} M} \sum_e \chi(\operatorname{Gr}_e FM) \prod_{i=1}^n x_i^{\langle S_i, e\rangle_a}.$$

Proof. — We already know that this formula holds for indecomposable objects of $\mathcal{C}$, cf. section 2.2. Let us prove that it still holds for decomposable objects, by recursion on the number of indecomposable direct summands.

Let M and N be two objects in $\mathcal{C}$. As shown in [8], we have

$$\chi\left(\operatorname{Gr}_g F(M \oplus N)\right) = \sum_{e+f=g} \chi\left(\operatorname{Gr}_e FM\right) \chi\left(\operatorname{Gr}_f FN\right).$$

Therefore, we have $X_{M\oplus N} = X_M X_N =$

$$\left(\underline{x}^{-\operatorname{coind} M} \sum_e \chi(\operatorname{Gr}_e FM) \prod_{i=1}^n x_i^{<S_i, e>_a}\right)\left(\underline{x}^{-\operatorname{coind} N} \sum_f \chi(\operatorname{Gr}_e FN) \prod_{i=1}^n x_i^{<S_i, f>_a}\right)$$

$$= \underline{x}^{-(\operatorname{coind} M + \operatorname{coind} N)} \sum_g \sum_{e+f=g} \chi\left(\operatorname{Gr}_e FM\right) \chi\left(\operatorname{Gr}_f FN\right) \prod_{i=1}^n x_i^{<S_i, e+f>_a}$$

$$= \underline{x}^{-\operatorname{coind}(M\oplus N)} \sum_g \chi\left(\operatorname{Gr}_g F(M \oplus N)\right) \prod_{i=1}^n x_i^{<S_i, g>_a}$$

□

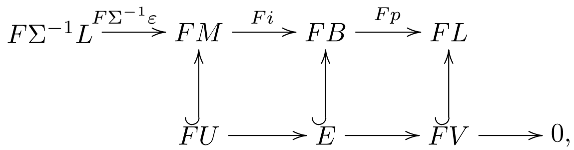
Lemma 4.2. — *Let $M \xrightarrow{i} B \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$ be a triangle in $\mathcal{C}$, and let $U \xrightarrow{i_U} M$ and $V \xrightarrow{i_V} L$ be two morphisms whose images under F are monomorphisms. Then the following conditions are equivalent:*

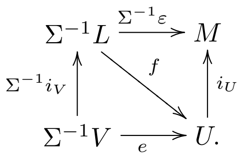
i) *There exists a submodule $E \subset FB$ such that*
$FV = (Fp)E$ and $FU = (Fi)^{-1}E$,

ii) *There exist two morphisms* $e : \Sigma^{-1}V \longrightarrow U$ *and* $f : \Sigma^{-1}L \longrightarrow U$ *such that*
  a) $(\Sigma^{-1}\varepsilon)(\Sigma^{-1}i_V) = i_U e$
  b) $e \in (T)$
  c) $i_U f - \Sigma^{-1}\varepsilon \in (\Sigma T)$.

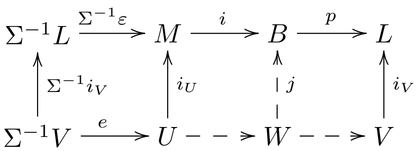
iii) *Condition ii) where, moreover,* $e = f\Sigma^{-1}i_V$.

The following diagrams will help the reader parse the conditions:

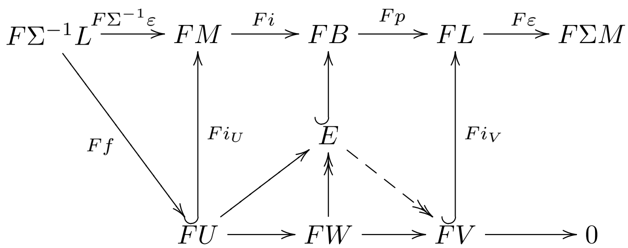




Proof. — Assume condition ii) holds. Then, by a), there exists a morphism of triangles



Take E to be the image of the morphism Fj. The morphism e factors through $\operatorname{add} T$, so that we have $F\Sigma e = 0$ and the functor F induces a commutative diagram



whose rows are exact sequences. It remains to show that $FU = (Fi)^{-1}E$.

We have $FU \subset (Fi)^{-1}E$ since $(Fi)(Fi_U)$ factors through the monomorphism $E \to FB$. The existence of the morphism Ff shows, via diagram chasing, the converse inclusion.

Conversely, let $E \subset FB$ be such that $FV = (Fp)E$ and $FU = (Fi)^{-1}E$. In particular, FU contains $\operatorname{Ker} Fi = \operatorname{Im} F\Sigma^{-1}\varepsilon$ so that $F\Sigma^{-1}\varepsilon$ factors through Fi_U. This gives us the morphism f, satisfying condition c). Define the morphism e as follows. There exists a triangle

$$T_1 \longrightarrow T_0 \longrightarrow V \longrightarrow \Sigma T_1,$$

where T_1, T_0 belong to $\operatorname{add} T$. Applying the functor F to this triangle, we get an epimorphism $FT_0 \to FV$ with FT_0 projective. This epimorphism thus factors through the surjection $E \to FV$, and composing it with $E \to FB$ gives a commutative square

$$\begin{array}{ccc} FT_0 & \longrightarrow & FV \\ \downarrow & & \downarrow \\ FB & \longrightarrow & FL. \end{array}$$

Since $\mathcal{C}(T, \Sigma T) = 0$, this commutative square lifts to a morphism of triangles

$$\begin{array}{ccccccc} \Sigma^{-1}V & \longrightarrow & T_1 & \longrightarrow & T_0 & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-1}L & \longrightarrow & M & \longrightarrow & B & \longrightarrow & L. \end{array}$$

The morphism $T_1 \to M$ thus induced, factors through the morphism $U \to M$. Indeed, we have $FU = (Fi)^{-1}E$ and the following diagram commutes :

$$\begin{array}{ccccc} & & FM & \longrightarrow & FB \\ & \nearrow & \uparrow & \nearrow & \uparrow \\ FT_1 & \longrightarrow & \longrightarrow & FT_0 & \\ & & \uparrow & \searrow & \uparrow \\ & & FU & \longrightarrow & E. \end{array}$$

The morphism e is then given by the composition $\Sigma^{-1}V \longrightarrow T_1 \longrightarrow U$.

Let us show that condition ii) implies condition iii). By hypothesis, we have

$$i_U e = (\Sigma^{-1}\varepsilon)(\Sigma^{-1}i_V)$$

and

$$i_U f \Sigma^{-1} i_V \equiv (\Sigma^{-1}\varepsilon)(\Sigma^{-1}i_V) \bmod (\Sigma T).$$

Therefore, the morphism $i_U\left(f\Sigma^{-1}i_V - e\right)$ belongs to the ideal (ΣT). The morphism Fi_U is a monomorphism, so that the morphism $h := f\Sigma^{-1}i_V - e$ lies in (ΣT). There exists a morphism $\Sigma^{-1}L \xrightarrow{l} U$ such that $h = l\Sigma^{-1}i_V$:

$$\begin{array}{ccccccc} \Sigma^{-1}C & \xrightarrow{\in(T)} & \Sigma^{-1}V & \xrightarrow{\Sigma^{-1}i_V} & \Sigma^{-1}L & \xrightarrow{c} & C \\ & {\scriptstyle 0}\searrow & {\scriptstyle h}\downarrow{\scriptstyle \in(\Sigma T)} & \swarrow{\scriptstyle l} & & & \\ & & U & & & & . \end{array}$$

Since the morphism $\Sigma^{-1}C \to \Sigma^{-1}V$ lies in the ideal (T), there exists a morphism of triangles

$$\begin{array}{ccccccc} \Sigma^{-1}C & \longrightarrow & \Sigma^{-1}V & \longrightarrow & \Sigma^{-1}L & \xrightarrow{c} & C \\ \downarrow & & \| & & \downarrow{\scriptstyle v} & & \downarrow \\ T_V^1 & \xrightarrow{u} & \Sigma^{-1}V & \longrightarrow & \Sigma T_V^0 & \longrightarrow & \Sigma T_V^1. \end{array}$$

The composition $l\Sigma^{-1}i_V$ belongs to the ideal (ΣT), so that the composition $l(\Sigma^{-1}i_V)u$ vanishes. We thus have a morphism of triangles

$$\begin{array}{ccccccc} T_V^1 & \xrightarrow{u} & \Sigma^{-1}V & \longrightarrow & \Sigma T_V^0 & \longrightarrow & \Sigma T_V^1 \\ \downarrow & & {\scriptstyle \Sigma^{-1}i_V}\downarrow & & {\scriptstyle w}\downarrow & & \downarrow \\ \Sigma^{-1}C' & \longrightarrow & \Sigma^{-1}L & \xrightarrow{l} & U & \longrightarrow & C'. \end{array}$$

Therefore, we have $(\Sigma^{-1}i_V)(l - wv) = 0$, and there exists a morphism $C \xrightarrow{l'} U$ such that $l - wv = l'c$. The morphism $l_0 = l - l'c$ thus factors through ΣT_1. Put $f_0 = f - l_0$. We have

$$f_0\Sigma^{-1}i_V = f\Sigma^{-1}i_V - l\Sigma^{-1}i_V + l'c\Sigma^{-1}i_V = e$$

and

$$\begin{aligned} i_U f_0 &= i_U f - i_U l_0 \\ &\equiv i_U f \mod (\Sigma T) \\ &\equiv \Sigma^{-1}\varepsilon \mod (\Sigma T). \end{aligned}$$

□

Proposition 4.3. — *Let* $L, M \in \mathcal{C}$ *be such that* $\dim \mathcal{C}(L, \Sigma M) = 1$. *Let*

$$\Delta : M \xrightarrow{i} B \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$$

$$\text{and } \Delta' : L \xrightarrow{i'} B' \xrightarrow{p'} M \xrightarrow{\varepsilon'} \Sigma L$$

be non-split triangles. Then conditions i) to iii) hold for the triangle Δ if and only if they do not for the triangle Δ'.

Proof. — Define maps

$$\begin{aligned}(\Sigma^{-1}L, U) \oplus (\Sigma^{-1}L, M) &\xrightarrow{\alpha} \mathcal{C}/(T)\big(\Sigma^{-1}V, U\big) \oplus (\Sigma^{-1}V, M) \\ &\qquad\qquad \oplus\, \mathcal{C}/(\Sigma T)\big(\Sigma^{-1}L, M\big) \\ (f, \eta) &\longmapsto (f\Sigma^{-1}i_V, i_U f\Sigma^{-1}i_V - \eta\Sigma^{-1}i_V, i_U f - \eta)\end{aligned}$$

and

$$\begin{aligned}(\Sigma^{-1}U, L) \oplus (\Sigma^{-1}M, L) &\xleftarrow{\alpha'} (T)(\Sigma^{-1}U, V) \oplus (\Sigma^{-1}M, V) \\ &\qquad\qquad \oplus\, (\Sigma T)(\Sigma^{-1}M, L) \\ (i_V e' + g'\Sigma^{-1}i_U + i_V f'\Sigma^{-1}i_U, -g' - i_V f') &\longleftarrow (e', f', g').\end{aligned}$$

Since the morphism space $\mathcal{C}(L, \Sigma M)$ is one-dimensional, the morphism ε satisfies condition iii) if and only if the composition

$$\beta : \operatorname{Ker}\alpha \hookrightarrow (\Sigma^{-1}L, U) \oplus (\Sigma^{-1}L, M) \twoheadrightarrow (\Sigma^{-1}L, M)$$

does not vanish. Assume condition iii) to be false for the triangle Δ. This happens if and only if the morphism β vanishes, if and only if its dual $D\beta$ vanishes. Since the category $\mathcal{C}$ is 2-Calabi–Yau, lemma 3.3 implies that the morphism $D\beta$ is isomorphic to the morphism:

$$\beta' : (\Sigma^{-1}M, L) \hookrightarrow (\Sigma^{-1}U, L) \oplus (\Sigma^{-1}M, L) \twoheadrightarrow \operatorname{Coker}\alpha'.$$

Therefore, $\beta'(\Sigma^{-1}\varepsilon) = 0$ is equivalent to $\Sigma^{-1}\varepsilon$ being in $\operatorname{Im}\alpha'$, which is equivalent to the existence of three mophisms e', f', g' as in the diagram

$$\begin{array}{ccc} \Sigma^{-1}M & \xrightarrow{g'} & L \\ {\scriptstyle \Sigma^{-1}i_U}\uparrow & \searrow^{f'} & \uparrow{\scriptstyle i_V} \\ \Sigma^{-1}U & \xrightarrow[e']{} & V \end{array}$$

such that

$$\begin{cases} e' \in (T) \\ g' \in (\Sigma T) \\ \Sigma^{-1}\varepsilon' = i_V f' + g' \\ i_V e' = (\Sigma^{-1}\varepsilon')(\Sigma^{-1}i_U). \end{cases}$$

We have thus shown that condition iii) does not hold for the triangle Δ if and only if condition ii) holds for the triangle Δ'. □

5. The multiplication formula

We use sections 2 and 4 to prove the multiplication formula, and apply it to prove conjecture 2 in [9].

5.1. Proof of theorem 1.4

We use the same notations as in the statement of theorem 1.4. Define, for any classes e, f, g in the Grothendieck group $\mathrm{K}_0(\operatorname{mod} B)$, the following varieties

$$\begin{array}{rcl} X_{e,f} & = & \{E \subset FB \text{ s.t. } [(Fi)^{-1}E] = e \text{ and } [(Fp)E] = f\} \\ Y_{e,f} & = & \{E \subset FB' \text{ s.t. } [(Fi')^{-1}E] = f \text{ and } [(Fp')E] = e\} \\ X^g_{e,f} & = & X_{e,f} \cap \mathrm{Gr}_g(FB) \\ Y^g_{e,f} & = & Y_{e,f} \cap \mathrm{Gr}_g(FB'). \end{array}$$

We thus have

$$\mathrm{Gr}_g(FB) = \coprod_{e,f} X^g_{e,f} \text{ and } \mathrm{Gr}_g(FB') = \coprod_{e,f} Y^g_{e,f}.$$

Moreover, we have

$$\begin{array}{rcl} \chi\left(\mathrm{Gr}_e(FM) \times \mathrm{Gr}_f(FL)\right) & = & \chi\left(X_{e,f} \sqcup Y_{e,f}\right) \\ & = & \chi\left(X_{e,f}\right) + \chi\left(Y_{e,f}\right) \\ & = & \sum_g \left(\chi\left(X^g_{e,f}\right) + \chi\left(Y^g_{e,f}\right)\right). \end{array}$$

where the first equality is a consequence of the dichotomy phenomenon as follows: Consider the map

$$X_{e,f} \sqcup Y_{e,f} \longrightarrow \mathrm{Gr}_e(FM) \times \mathrm{Gr}_f(FL)$$

which sends a submodule E of FB to the pair of submodules $\big((Fi)^{-1}E, (Fp)E\big)$. By proposition 4.3, it is surjective, and, as shown in [8], its fibers are affine spaces.

Lemma 5.1. — *Let e, f and g be classes in $\mathrm{K}_0(\operatorname{mod} \operatorname{End}_{\mathcal{C}}(T))$. Assume that $X^g_{e,f}$ is non-empty. Then, we have*

$$\sum \langle S_i, g\rangle_a [P_i] - \operatorname{coind} B = \sum \langle S_i, e+f\rangle_a [P_i] - \operatorname{coind} M - \operatorname{coind} L.$$

Proof. — Let E be a submodule of FB in $X^g_{e,f}$. Let $U \xrightarrow{i_U} M$ and $V \xrightarrow{i_V} L$ be two morphisms in the category $\mathcal{C}$ such that $FU \simeq (Fi)^{-1}E$, $FV \simeq (Fp)E$ and the images of i_U and i_V in $\operatorname{mod} B$ are isomorphic to the inclusions of FU in FM and FV in FL respectively. Let $K \in \mathcal{C}$ be a lift of the kernel of Fi. By proposition 2.2, the following equality holds:

$$\operatorname{coind} B = \operatorname{coind} M + \operatorname{coind} L - \operatorname{coind} K - \operatorname{coind}(\Sigma K). \tag{1}$$

By diagram chasing, the kernel of Fi is also a kernel of the induced morphism from FU to E. Therefore, in $\mathrm{K}_0(\operatorname{mod} B)$, we have

$$g = e + f - [FK]. \tag{2}$$

We have the following equalities:

$$\begin{aligned} \sum \langle S_i, FK\rangle_a [P_i] &= \operatorname{coind} K - \operatorname{ind} K \text{ (by lemma 2.3)} \\ &= \operatorname{coind} K + \operatorname{coind}(\Sigma K) \text{ (by lemma 2.1)}. \end{aligned}$$

Equality (2) thus yields

$$\sum \langle S_i, g\rangle_a [P_i] = \sum \langle S_i, e+f\rangle_a [P_i] - \operatorname{coind} K - \operatorname{coind}(\Sigma K). \tag{3}$$

It only remains to sum equalities (1) and (3) to finish the proof. □

Proof of theorem 1.4. — Using lemma 4.1, we have

$$\begin{aligned} X_M X_L &= \underline{x}^{-\operatorname{coind} M - \operatorname{coind} L} \sum_{e,f} \chi(\operatorname{Gr}_e FM)\chi(\operatorname{Gr}_f FL) \prod_{i=1}^{n} x_i^{\langle S_i, e+f\rangle_a}, \\ X_B &= \underline{x}^{-\operatorname{coind} B} \sum_{g} \chi(\operatorname{Gr}_g FB) \prod_{i=1}^{n} x_i^{\langle S_i, g\rangle_a} \text{ and} \\ X_{B'} &= \underline{x}^{-\operatorname{coind} B'} \sum_{g} \chi(\operatorname{Gr}_g FB') \prod_{i=1}^{n} x_i^{\langle S_i, g\rangle_a}. \end{aligned}$$

Therefore

$$\begin{aligned} X_M X_L &= \underline{x}^{-\operatorname{coind} M - \operatorname{coind} L} \sum_{e,f} \chi\left(\operatorname{Gr}_e(FM)\right) \chi\left(\operatorname{Gr}_f(FL)\right) \prod x_i^{\langle S_i, e+f\rangle_a} \\ &= \underline{x}^{-\operatorname{coind} M - \operatorname{coind} L} \sum_{e,f,g} \left(\chi\left(X_{e,f}^g\right) + \chi\left(Y_{e,f}^g\right)\right) \prod x_i^{\langle S_i, e+f\rangle_a} \\ &= \underline{x}^{-\operatorname{coind} B} \sum_{e,f,g} \chi\left(X_{e,f}^g\right) \prod x_i^{\langle S_i, g\rangle_a} \\ &\qquad + \underline{x}^{-\operatorname{coind} B'} \sum_{e,f,g} \chi\left(Y_{e,f}^g\right) \prod x_i^{\langle S_i, g\rangle_a} \\ &= \underline{x}^{-\operatorname{coind} B} \sum_{g} \chi\left(\operatorname{Gr}_g(FB)\right) \prod x_i^{\langle S_i, g\rangle_a} \\ &\qquad + \underline{x}^{-\operatorname{coind} B'} \sum_{g} \chi\left(\operatorname{Gr}_g(FB')\right) \prod x_i^{\langle S_i, g\rangle_a} \\ &= X_B + X_{B'}. \end{aligned}$$

□

5.2. Consequences

Let Q be a finite acyclic connected quiver, and let $\mathcal{C}$ be the cluster category associated to Q.

An object of $\mathcal{C}$ without self-extensions is called rigid. An object of $\mathcal{C}$ is called basic if its indecomposable direct summands are pairwise non-isomorphic. For a basic cluster-tilting object T of $\mathcal{C}$, let Q_T denote the quiver of $\operatorname{End}(T)$, and $\mathcal{A}_{Q_T}$ the associated cluster algebra.

PROPOSITION 5.2. — *A cluster character χ on $\mathcal{C}$ with values in $\mathbb{Q}(x_1, \ldots, x_n)$ which sends a basic cluster-tilting object T of $\mathcal{C}$ to a cluster of $\mathcal{A}_{Q_T}$ sends any cluster-tilting object T' of $\mathcal{C}$ to a cluster of $\mathcal{A}_{Q_T}$, and any rigid indecomposable object to a cluster variable.*

Proof. — Since the tilting graph of $\mathcal{C}$ is connected, cf. [4, proposition 3.5], we can prove the first part of the proposition by recursion on the minimal number of mutations linking T' to T. Let $T'' = T''_1 \oplus \cdots \oplus T''_n$ be a basic cluster-tilting object, whose image under χ is a cluster of $\mathcal{A}_{Q_T}$. Assume that $T' = T'_1 \oplus T''_2 \oplus \cdots \oplus T''_n$ is the mutation in direction 1 of T''. Since χ is a cluster character, it satisfies the multiplication formula, and theorem 6.1 of [5] shows that the mutation, in direction 1, of the cluster $(\chi(T''_1), \ldots, \chi(T''_n))$ is the cluster $(\chi(T'_1), \chi(T''_2), \ldots, \chi(T''_n))$. We have thus proved that the image under χ of any cluster-tilting object is a cluster. It is proved in [4, proposition 3.2] that any rigid indecomposable object of $\mathcal{C}$ is a direct summand of a basic cluster-tilting object. Therefore, the image under χ of any rigid indecomposable object is a cluster variable of $\mathcal{A}_{Q_T}$. □

Remark. — As a corollary of the proof of proposition 5.2, a cluster character is characterised, on a set of representatives for the isoclasses of indecomposable rigid objects of $\mathcal{C}$ by the image of each direct summand of any given cluster-tilting object. In fact, using [3, 1.10], this remains true in the more general context of [3]: Let $\mathcal{C}$ be a Hom-finite triangulated 2-Calabi–Yau category having maximal rigid objects without loops nor strong 2-cycles. Denote by n the number of non-isomorphic indecomposable direct summands of any maximal rigid object.

LEMMA 5.3. — *Let χ_1 and χ_2 be two cluster characters on $\mathcal{C}$ with values in $\mathbb{Q}(x_1, \ldots, x_n)$. Assume that χ_1 and χ_2 coincide on all indecomposable direct summands of a cluster-tilting object T in $\mathcal{C}$. Then χ_1 and χ_2 coincide on all direct summands of the cluster-tilting objects in $\mathcal{C}$ which are obtained from T by a finite sequence of mutations.*

The following corollary was conjectured for the finite case in [9]: Let $\mathcal{C}$ be the cluster category of the finite acyclic quiver Q.

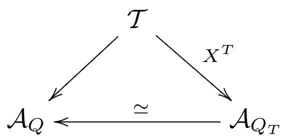
Corollary 5.4. — *Let T be any basic cluster-tilting object in $\mathcal{C}$, and let Q_T denote the quiver of End(T). Denote by $\mathcal{T}$ a set of representatives for the isoclasses of indecomposable rigid objects of $\mathcal{C}$. Then X^T induces a bijection from the set $\mathcal{T}$ to the set of cluster variables of the associated cluster algebra $\mathcal{A}_{Q_T}$, sending basic cluster-tilting objects to clusters.*

Proof. — In view of theorem 1.4, proposition 5.2 shows that the map X^T sends rigid indecomposable objects to cluster variables and cluster-tilting objects to clusters. It remains to show that it induces a bijection. This follows from [10, theorem 4], where it is proved for the Caldero-Chapoton map X^{kQ}.

As in the proof of proposition 5.2, we proceed by induction on the minimal number of mutations linking T to kQ.

Let T' be a basic cluster-tilting object such that the map $X^{T'}$ induces a bijection from the set $\mathcal{T}$ to the set of cluster variables. Assume that T is the mutation in direction 1 of T'. Denote by f the canonical isomorphism from $\mathcal{A}_{Q_{T'}}$ to $\mathcal{A}_{Q_T}$. Theorem 6.1 of [5] shows that the two cluster characters X^T and $f \circ X^{T'}$ coincide on the indecomposable direct summands of ΣT. Therefore, they coincide on all rigid objects and the map X^T also induces a bijection. □

Remark. — We have shown that, for any basic cluster-tilting object T, we have a commutative diagram

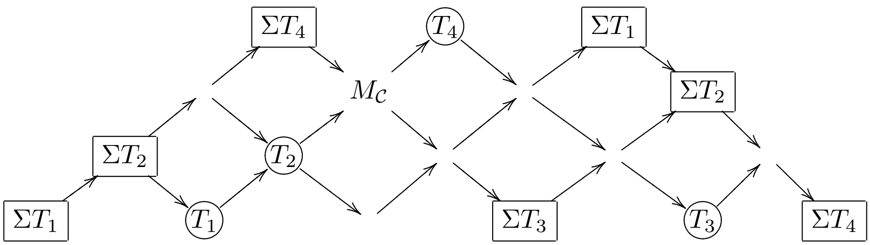


where the arrow on the left side is the Caldero–Chapoton map.

6. Examples

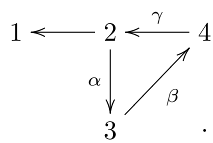
6.1. The cluster category $\mathcal{C}_{A_4}$

The Auslander–Reiten quiver of $\mathcal{C}_{A_4}$ is

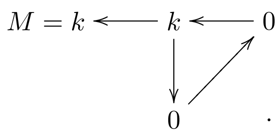


The object $T := T_1 \oplus T_2 \oplus T_3 \oplus T_4$ is cluster-tilting. Indeed, it is obtained from the image of the kQ-projective module kQ in $\mathcal{C}_{A_4}$ by the mutation of the third vertex.

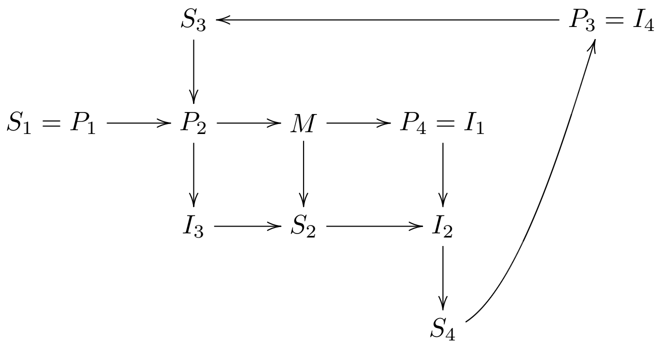
The quiver of $B = \mathrm{End}_{\mathcal{C}_{A_4}}(T)$ is



with relations $\beta\alpha = \gamma\beta = \alpha\gamma = 0$. For $i = 1, \ldots, n$, let P_i be the image of T_i in $\mathrm{mod}\, B$, let I_i be the image of $\Sigma^2 T_i$ and let S_i be the simple top of P_i. Let M be the finite-dimensional B-module given by:



The shape and the relations of the AR–quiver of B are obtained from the ones of $\mathcal{C}_{A_4}$ by deleting the vertices corresponding to the objects ΣT_i and all arrows ending to or starting from these vertices.



Let $M_{\mathcal{C}}$ be an indecomposable lift of M in $\mathcal{C}_{A_4}$. The triangles

$$T_3 \longrightarrow T_2 \longrightarrow M_{\mathcal{C}} \longrightarrow \Sigma T_3 \quad \text{and} \quad T_1 \longrightarrow T_4 \longrightarrow \Sigma^{-1} M_{\mathcal{C}} \longrightarrow \Sigma T_1$$

allows us to compute the index and coindex of $M_{\mathcal{C}}$:

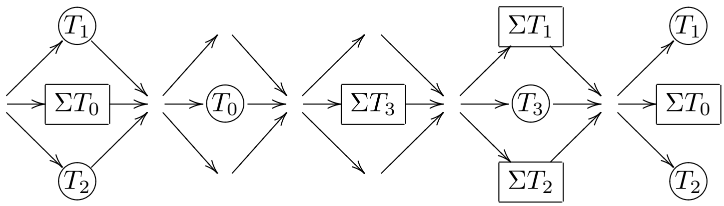
$$\begin{aligned} \mathrm{ind}\, M_{\mathcal{C}} &= [P_2] - [P_3] \\ \mathrm{coind}\, M_{\mathcal{C}} &= [P_1] - [P_4]. \end{aligned}$$

Up to isomorphism, the submodules of M are 0, the simple S_1, and M itself. We thus have

$$X_{M_{\mathcal{C}}} = \frac{x_4x_2 + x_4 + x_3x_1}{x_1x_2}.$$

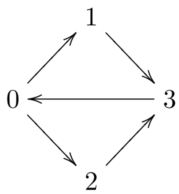
6.2. The cluster category $\mathcal{C}_{D_4}$

The Auslander–Reiten quiver of $\mathcal{C}_{D_4}$ is



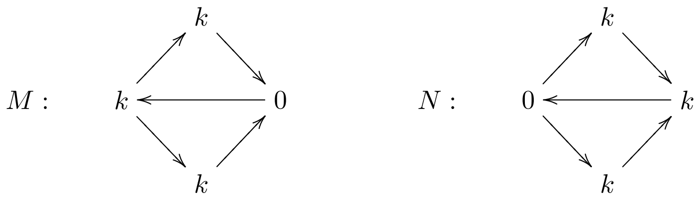
The object $T := T_1 \oplus T_2 \oplus T_3 \oplus T_4$ is cluster-tilting.

The quiver of $B = \operatorname{End}_{\mathcal{C}_{D_4}}(T)$ is

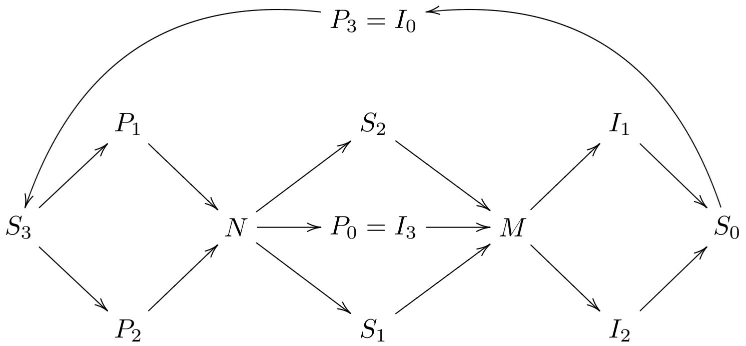


with the following relations: Any composition with the middle arrow vanishes, and the square is commutative.

For $i = 1, \ldots, n$, let P_i be the image of T_i in $\operatorname{mod} B$, let I_i be the image of $\Sigma^2 T_i$ and let S_i be the simple top of P_i. Let M and N be the finite-dimensional B-modules given by:



As in the previous example, one can easily compute the AR-quiver of B.



The submodules of M are, up to isomorphism, 0, S_1, S_2, $S_1 \oplus S_2$ and M. Let $M_{\mathcal{C}}$ be an indecomposable lift of M in $\mathcal{C}_{D_4}$. Either by using $\operatorname{add} T$-approximations and $\operatorname{add} \Sigma T$-approximations or by [21, section 5.2], one can compute the triangles

$$T_3 \longrightarrow T_0 \longrightarrow M_{\mathcal{C}} \longrightarrow \Sigma T_3 \quad \text{and} \quad T_1 \oplus T_2 \longrightarrow T_0 \longrightarrow \Sigma^{-1} M \longrightarrow \Sigma T_1 \oplus \Sigma T_2.$$

We thus have

$$\operatorname{ind} M_{\mathcal{C}} = [P_0] - [P_3], \ \operatorname{coind} M_{\mathcal{C}} = [P_1] + [P_2] - [P_0]$$

and

$$X_{M_{\mathcal{C}}} = \frac{(x_0 + x_3)^2 + x_1 x_2 x_3}{x_0 x_1 x_2}.$$

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