COMPOSITIO MATHEMATICA

Cluster complexes via semi-invariants

Kiyoshi Igusa, Kent Orr, Gordana Todorov and Jerzy Weyman

Compositio Math. **145** (2009), 1001–1034.

doi:10.1112/S0010437X09004151

FOUNDATION
COMPOSITIO
MATHEMATICA

The London
Mathematical
Society

Compositio Math. 145 (2009) 1001–1034
doi:10.1112/S0010437X09004151

Cluster complexes via semi-invariants

Kiyoshi Igusa, Kent Orr, Gordana Todorov and Jerzy Weyman

Abstract

We define and study virtual representation spaces for vectors having both positive and negative dimensions at the vertices of a quiver without oriented cycles. We consider the natural semi-invariants on these spaces which we call *virtual semi-invariants* and prove that they satisfy the three basic theorems: the first fundamental theorem, the saturation theorem and the canonical decomposition theorem. In the special case of Dynkin quivers with n vertices, this gives the fundamental interrelationship between supports of the semi-invariants and the tilting triangulation of the $(n-1)$-sphere.

Contents

Introduction

This paper initiates a project to apply quiver representations and their semi-invariants to expose compatible combinatorial underpinnings for the tilting objects of cluster categories (and, hence, clusters for cluster algebras) and for the homology of nilpotent groups. Here we focus on semi-invariants and tilting objects in cluster categories; we extend the classical semi-invariant results of Kac, Schofield, Derksen and Weyman, and interpret the fundamental results from [BMRRT06] about cluster categories in this setting.

Modeled on K-theory, for an arbitrary quiver without oriented cycles we consider semi-invariants in the derived category by extending the definition of representation spaces to *virtual dimension vectors* of virtual modules over the path algebra of the quiver. Such virtual

Received 8 August 2007, first revision 14 October 2008, accepted in final form 14 January 2009.
2000 Mathematics Subject Classification 14L24, 16G20 (primary), 13A50, 16G70 (secondary).
Keywords: cluster complex, semi-invariants of quivers, tilting object, projective presentation, presentation spaces.
The second author was supported by NSF grant DMS-0406563. The third author was supported by the NSA. The fourth author was supported by NSF grant DMS-0600229.

dimension vectors have both positive and negative coordinates. Specifically, instead of working with representation spaces of the quiver acted upon by products of general linear groups, we work with *presentation spaces*, the spaces $\mathrm{Hom}_Q(P_1, P_0)$ for projective modules P_0 and P_1. The natural action of the group $\mathrm{Aut}(P_0) \times (\mathrm{Aut}(P_1))^{\mathrm{op}}$ replaces the action of the product of general linear groups, and we study the semi-invariants for these actions. We construct the *virtual representation space* for virtual dimension vectors $\alpha \in \mathbb{Z}^n$ as a direct limit:

$$R^{\mathrm{vir}}(\alpha) = \lim_{\substack{\rightarrow \\ P}} \mathrm{Hom}_Q(P_1 \coprod P, P_0 \coprod P),$$

where $\alpha = \underline{\dim}\, P_0 - \underline{\dim}\, P_1$ and the limit is taken over all projectives P. The natural semi-invariants are obtained as inverse limits of semi-invariants on the presentation spaces; we call them *virtual semi-invariants*.

We shall prove the three basic theorems in this virtual setting. The virtual first fundamental theorem, Theorem 6.4.1, relates virtual semi-invariants to quiver representations: all virtual semi-invariants are linear combinations of determinantal semi-invariants. The virtual saturation theorem, Theorem 6.5.11, describes the supports of semi-invariants, i.e. it describes when the determinantal semi-invariants are non-zero. The virtual generic decomposition theorem, Theorem 6.3.1, determines the dimension vectors of the indecomposable components of all generic representations of all dimension vectors.

Using the above results on virtual semi-invariants, to each quiver with n vertices we associate a simplicial complex $\mathcal{T}(Q)$ together with a mapping of its geometric realization to the $(n-1)$-dimensional sphere,

$$\lambda : |\mathcal{T}(Q)| \to S^{n-1}.$$

The simplices of $\mathcal{T}(Q)$ are virtual partial tilting sets of Schur roots and shifted projective roots. We call this complex the *complex of virtual tilting sets*. In general, it has infinitely many simplices. The continuous mapping λ maps each closed simplex σ of $|\mathcal{T}(Q)|$ to the geodesic simplex in the sphere with the same vertex set as σ. If we restrict to a certain subcomplex $|\mathcal{T}'(Q)|$ of $|\mathcal{T}(Q)|$ spanned by the 'minimal' Schur roots, we get a continuous monomorphism onto a dense subset of the sphere. Strictly speaking, this is not a triangulation of a subset of the sphere; however, it does express a subset of the sphere as a union of simplices with disjoint interiors.

The $(n-2)$-dimensional faces of our 'triangulation' are labeled by dimension vectors of indecomposable representations via semi-invariants or, more precisely, supports of semi-invariants with prescribed weights. This labelling depends on the orientation of the quiver. For the vectors with non-negative coordinates, we recover the simplicial complex corresponding to generic decompositions of dimension vectors obtained in [DW02].

In the case of a Dynkin quiver, the complex of virtual tilting sets gives a finite triangulation of the sphere, and it coincides with the cluster tilting triangulation. Its simplices correspond to tilting objects in a corresponding cluster category defined in [BMRRT06]. In addition to the cluster tilting triangulation, we get the labeling of codimension-one faces by dimension vectors. This depends on the orientation of the quiver, unlike the cluster tilting triangulation itself. One can show that this triangulation is Poincaré dual to the cluster associahedron of [CFZ02].

In a future paper we will study the presentation, given by semi-invariants, of the nilpotent group associated to a Dynkin quiver. This is almost the same as the Steinberg presentation using the Chevalley commutator relations [Ste64]. We will also examine the residually nilpotent groups associated to quivers of affine type (also called tame quivers). In prior work [IO01], two of the authors constructed an explicit chain resolution for torsion-free nilpotent groups and used this to study Milnor's $\overline{\mu}$-link invariants. We will show in our next paper how this is related to semi-invariants.

Owing to the diversity of our collaborators, we have written this paper to be readable by both topologists and algebraists.

The outline of the paper is as follows. We shall assume throughout that Q has no oriented cycles. In § 1, we review the basic definitions and properties of quiver representations and establish notation; in particular, we recall the definition of the canonical projective presentation of any representation that Schofield used in his original study of semi-invariants on quiver representations. In § 2 we discuss fundamental known results on semi-invariants of quivers and generic decompositions. These first two sections serve the reader as background. Section 3 defines presentation spaces and their semi-invariants which form directed systems and, as such, are used to define virtual representation spaces and virtual semi-invariants. We motivate our definition of virtual semi-invariants in § 4 by making precise, in the case of non-negative dimension vectors, the relationship between classical semi-invariants and semi-invariants on certain special presentation spaces; we show that these rings of semi-invariants are isomorphic. In § 5 we prove some properties of presentation spaces and their semi-invariants, including the description of the general elements in presentation spaces. In § 6 we introduce the virtual representation spaces; we build on the material of § 5 to prove the stability theorem, which shows that the general element in the stable representation space is chain homotopy equivalent to a minimal projective presentation and thus lies in the orbit of the image of an associated space of minimal presentations. We extend the generic decomposition and the first fundamental theorem to virtual dimension vectors, and in § 6.5 we prove the virtual saturation theorem. In § 7 we construct the simplicial complex of generalized cluster tilting sets and the mapping of the realization of this complex to the sphere. We derive the general properties of this mapping as a consequence of the generic decomposition theorem. In § 7.2 we also review the definition and properties of the cluster category for comparison. Finally, in § 8, we restrict to the special case where the quiver is of Dynkin type and prove Theorem 8.1.7, which says that the 'cluster tilting triangulation is given by supports of semi-invariants'.

1. Review of representation spaces

In this section, basic notions related to quiver representations and semi-invariants are reviewed, and we state some of the well-known results about semi-invariants from [DW00, Sch91].

1.1 Quiver representations

Let $\Bbbk$ be an algebraically closed field. A quiver Q is a directed graph; denote its set of vertices by Q_0 and its set of arrows by Q_1. The path algebra, $\Bbbk Q$, is the $\Bbbk$-algebra generated by the paths in Q, with the product given by composition of paths, where $\alpha \cdot \beta$ means 'traverse first the path β and then the path α'. For a given vertex $v \in Q_0$, the idempotent e_v is the constant path at v. Note that $\Bbbk Q e_v$ is the left $\Bbbk Q$-module of paths starting at v, and similarly for the right $\Bbbk Q$-module, $e_u \Bbbk Q$. The algebra $\Bbbk Q$ is easily seen to be hereditary. We assume Q is finite and has no oriented cycles, so that its path algebra $\Bbbk Q$ is finite-dimensional. The category of finitely generated $\Bbbk Q$-modules is equivalent to the category of finite-dimensional Q-representations over $\Bbbk$.

Recall that a *quiver representation* is a family consisting of vector spaces $\{M_v\}_{v \in Q_0}$ and linear maps $M_a : M_{ta} \to M_{ha}$ for each arrow $a \in Q_1$, where ta and ha denote, respectively, the tail and head of a. A map $f = (f_v)$ between two representations M and M' consists of $\Bbbk$-linear maps $f_v : M_v \to M'_v$ satisfying commutativity relations $f_{ha} M_a = M'_a f_{ta}$ for all arrows $a \in Q_1$. (We will sometimes refer to these representations as 'modules', so as not to confuse them with the generalized or virtual representations which will be introduced later in §§ 3.2 and 6.1). The *dimension vector* of M is the vector $\underline{\dim}\, M := (\dim M_v) \in \mathbb{N}^n$ where $n = \operatorname{card}(Q_0)$. The *radical* of a representation M is $\operatorname{rad} M$ where $\operatorname{rad} \subset \Bbbk Q$ is the ideal generated by all arrows, i.e. all elements of Q_1.

For a vertex v, we denote by $S(v)$ the *simple representation* supported at v, i.e. $S(v)_v = \Bbbk$ and $S(v)_u = 0$ for all $u \neq v$. Also, we denote by $P(v)$ the *canonical indecomposable projective* which maps onto $S(v)$:

$$P(v)_v = e_v \Bbbk, \quad P(v)_u = e_u \Bbbk Q e_v \otimes_{\Bbbk} P(v)_v \cong \coprod_{\text{paths } v \to u} P(v)_v;$$

in other words, $P(v)$ is the free $\Bbbk$-vector space with basis {all paths from v to u} and such that, for each arrow a, the linear maps $P(v)_a : P(v)_{ta} \to P(v)_{ha}$ are defined on the generating paths by $P(v)_a(p) := ap$. Similarly, we denote by $I(v)$ the *canonical indecomposable injective*, defined as follows: $I(v)_v = \Bbbk$ and $I(v)_u$ is the free $\Bbbk$-vector space with basis {all paths from u to v}; more precisely, it is the dual of this vector space, with the dual basis

$$I(v)_u = \operatorname{Hom}_{\Bbbk}(e_v \Bbbk Q e_u, I(v)_v) \cong \prod_{\text{paths } u \to v} I(v)_v.$$

An arrow $a : ta \to ha$ gives a linear map $e_v \Bbbk Q e_{ha} \to e_v \Bbbk Q e_{ta}$ which induces a map $I(v)_{ta} \to I(v)_{ha}$.

Every simple representation is isomorphic to one of the $S(v)$, indecomposable projective to one of the $P(v)$, and indecomposable injective to one of the $I(v)$ (see [ASS06, § III.2]).

We will use the following notation for projective representations having a prescribed number of indecomposable summands: for $\gamma \in \mathbb{N}^n$, let $P(\gamma)$ denote the projective representation

$$\coprod_{v \in Q_0} P(v)^{\gamma_v}.$$

Notice that with this notation, we have $P(\underline{\dim}\, S(v)) = P(v)$.

1.2 Representation space for $\alpha \in \mathbb{N}^n$

The representation space for a non-negative integral vector $\alpha = (\alpha_v)_{v \in Q_0}$ is the affine space

$$R(\alpha) = \prod_{(u \to v) \in Q_1} \operatorname{Hom}_{\Bbbk}(\Bbbk^{\alpha_u}, \Bbbk^{\alpha_v}).$$

Elements of $R(\alpha)$ will be called based representations with *dimension vector* α. Every element of $R(\alpha)$ can be viewed as a collection of $\alpha_v \times \alpha_u$ matrices, one for each arrow $a : u \to v$. The group

$$G(\alpha) = \prod_{v \in Q_0} Gl_{\alpha_v}(\Bbbk)$$

acts on $R(\alpha)$ by $g M_a := (g_{ha}) M_a (g_{ta})^{-1}$. We use the convention that $Gl_0(\Bbbk)$ is the trivial group. Two representations of dimension α are isomorphic if and only if they lie in the same orbit of the action of $G(\alpha)$.

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

1.3 Euler matrix and bilinear form

The vertices of the quiver Q are partially ordered by setting $u < v$ if there is a directed path from u to v. We choose a fixed extension of this partial ordering to a total ordering and denote it by $<$ also. The *Euler matrix* E is defined as the $n \times n$ matrix with rows and columns labeled by Q_0 (order as above) such that the diagonal entries are equal to 1 and the (u, v)th entry is given by $E_{u,v} = -$(the number of arrows from u to v) for $u \neq v$. The *Euler form* is the non-symmetric bilinear form on $\mathbb{Z}^n$ given by the matrix E; that is,

$$\langle \alpha, \beta \rangle := \alpha^t E \beta.$$

Example 1.3.1 (Quiver Q and the corresponding Euler matrix). The rows and columns of the inverse and transpose of the Euler matrix have interpretations as dimension vectors of projective modules; we illustrate this with an example here, and discuss the issue in Remark 1.3.2.

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \arrow[r, bend left] \arrow[r, bend right] & \underset{3}{\bullet} \end{tikzcd}$$

$$E = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad (E^t)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}.$$

Remark 1.3.2 (Useful facts about the Euler form and Euler matrix [ASS06, § III.3]).

(i) $\langle \alpha, \alpha' \rangle = \dim_{\Bbbk} \operatorname{Hom}_Q(M, M') - \dim_{\Bbbk} \operatorname{Ext}^1_Q(M, M')$ for all representations M and M' such that $\alpha = \underline{\dim}\, M$ and $\alpha' = \underline{\dim}\, M'$.

(ii) The row of E corresponding to the vertex v consists of coefficients of the dimension vector $\underline{\dim}\, S(v)$ written as a linear combination of the dimension vectors $\underline{\dim}\, P(w)$. In the example above, we have $\underline{\dim}\, S(2) = \underline{\dim}\, P(2) - 2\, \underline{\dim}\, P(3)$.

(iii) $E^t\, \underline{\dim}\, P(v) = \underline{\dim}\, S(v)$ or, equivalently, $(E^t)^{-1} \underline{\dim}\, S(v) = \underline{\dim}\, P(v)$. In other words, $\underline{\dim}\, P(v)$ is the column of $(E^t)^{-1}$ corresponding to vertex v.

(iv) The product $E^t \alpha$ gives the coefficients of a vector α written as a linear combination of the vectors $\underline{\dim}\, P(w)$. In particular, $E^t\, \underline{\dim}\, P(\gamma) = \gamma$ for any $\gamma \in \mathbb{N}^n$.

(v) $(E^t)^{-1}(\gamma) = \underline{\dim}\, P(\gamma)$ for all $\gamma \in \mathbb{N}^n$. In particular, for $\gamma \in \mathbb{N}^n$, $(E^t)^{-1}(\gamma) \in \mathbb{N}^n$.

(vi) $\alpha - E^t(\alpha)$ has non-negative coefficients for $\alpha \in \mathbb{N}^n$; for example, if $\alpha = \underline{\dim}\, S(v)$, then $\alpha - E^t(\alpha) = \underline{\dim}\, (\operatorname{rad} P(v)/\operatorname{rad}^2 P(v))$.

(vii) The dimension vectors $\underline{\dim}\, I(v)$ of the indecomposable injective vectors occupy the columns of E^{-1}. In particular, the entries of E^{-1} are all non-negative.

1.4 Canonical projective presentations

Recall that the *canonical projective presentation* of a representation M with $\underline{\dim}\, M = \alpha$ is

$$0 \to P_1 \xrightarrow{p_M} P_0 \to M \to 0$$
$$\text{where} \quad P_1 = \coprod_{(u \to v) \in Q_1} P(v)^{\alpha_u} \quad \text{and} \quad P_0 = P(\alpha) = \coprod_{v \in Q_0} P(v)^{\alpha_v}.$$

The mapping p_M can be described as follows. For each arrow $a = (u \to v) \in Q_1$, the restriction of p_M to $P(v)^{\alpha_u}$ is given by

$$P(v)^{\alpha_u} \xrightarrow{((-\mathrm{incl}_a)^{\alpha_u}, M_a)} P(u)^{\alpha_u} \coprod P(v)^{\alpha_v},$$

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

where $M_a : M_{ta} \to M_{ha}$ is the linear map associated to the arrow a in the definition of the representation M (as in § 1.1) and $\mathrm{incl}_a : P(v) \to P(u)$ is the inclusion map corresponding to the arrow $a : u \to v$. The representation P_1 can also be rewritten as

$$P_1 = \coprod_{(u\to v)\in Q_1} P(v)^{\alpha_u} = \coprod_{v\in Q_0} P(v)^{(\sum_{u\to v}\alpha_u)} = P(\alpha - E^t\alpha).$$

Consequently, $p_M \in \mathrm{Hom}_Q(P(\alpha - E^t\alpha), P(\alpha))$.

One constructs the canonical injective resolution in a similar way.

2. Review of classical results on semi-invariants of quivers

We recall now the notion of semi-invariants of a group acting on a variety and state the classical results about semi-invariants on the representation spaces of quivers due to Kac, Schofield, and Derksen and Weyman.

2.1 Definition of semi-invariants

For an algebraic group G acting on a variety X, an element f of the coordinate ring of X is called a *semi-invariant* if there exists a character χ of G such that for all $g \in G$ and all $v \in X$,

$$f(g \cdot v) = \chi(g)f(v).$$

We will refer to χ as *the character of the semi-invariant* f.

Remark 2.1.1. The rational characters (characters which are rational functions) on $Gl_n(\Bbbk)$ are $\det(g)^s$ where $s \in \mathbb{Z}$, provided that $\Bbbk$ has at least three elements and $n \geq 1$.

2.2 Semi-invariants of quivers

Let Q be a quiver with n vertices and let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. The group $G(\alpha)$ acts on the representation space $R(\alpha)$ (described in § 1.2). Since the group $G(\alpha)$ is the product of general linear groups, the character χ at g is the product $\chi(g) = (\det(g_1))^{\sigma_1} \cdots (\det(g_n))^{\sigma_n}$, where $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{Z}^n$. The following are some basic facts about semi-invariants, their characters and their weights.

Definition 2.2.1. Let Q be a quiver, $n = |Q_0|$ and $\alpha \in \mathbb{N}^n$. Let f be a semi-invariant on $R(\alpha)$ and χ the uniquely determined character of f. *A weight of the semi-invariant* f is any vector $\sigma = (\sigma_1, \ldots, \sigma_n)$ for which the character χ of f can be written as

$$\chi(g) = (\det(g_1))^{\sigma_1} \cdots (\det(g_n))^{\sigma_n}.$$

Furthermore, a vector $\sigma \in \mathbb{Z}^n$ will be called a *weight* if it is a weight for some semi-invariant.

Remark 2.2.2. Let $\alpha \in \mathbb{N}^n$ be fixed.

(i) Each vector $\sigma \in \mathbb{Z}^n$ determines a unique character, which we denote by χ_σ.

(ii) On the other hand, a character χ for some semi-invariant may not uniquely determine the weight of the semi-invariant; this occurs in the important non-sincere case: if $\alpha_i = 0$, then g_i is a 0×0 matrix, in which case $\det(g_i) = 1$ and therefore, for any $\sigma_i \in \mathbb{Z}$, $\det(g_i)^{\sigma_i} = \det(g_i) = 1$.

(iii) A character χ for a semi-invariant f determines uniquely a coset of a free abelian subgroup of $\mathbb{Z}^n$. The coset consists of all weights of f: $\Sigma_\chi = \Sigma_f = \{\sigma \mid \sigma \text{ is a weight of } f\}$.

Definition 2.2.3. Let $\alpha \in \mathbb{N}^n$. Define as in [DW00] the *cone of weights*,

$$\Sigma(Q, \alpha) := \{\sigma \mid \sigma \text{ is a weight of some semi-invariant on } R(\alpha)\}.$$

We now define rings of semi-invariants as graded rings, where the grading is given by the characters of the group $G(\alpha)$.

Definition 2.2.4. For $\alpha \in \mathbb{N}^n$, we denote by $\mathrm{SI}(Q, \alpha)_\chi$ the $\Bbbk$-vector space of semi-invariants on $R(\alpha)$ with character χ. We let $\mathrm{SI}(Q, \alpha)$ denote the graded ring

$$\mathrm{SI}(Q, \alpha) = \bigoplus_{\chi \in \mathrm{Char}\, G(\alpha)} \mathrm{SI}(Q, \alpha)_\chi,$$

called the *ring of semi-invariants* for the action of $G(\alpha)$ on $R(\alpha)$.

Remark 2.2.5. We point out the following facts and conventions.

(i) Our convention differs from [DW00] in that the weights are negated.

(ii) The polynomial 0 appears as a semi-invariant in each grading, i.e. of all possible weights.

2.3 Fundamental theorems for semi-invariants of quivers

The *first fundamental theorem* states that rings of semi-invariants of quivers are spanned by determinants. The *saturation theorem* describes all non-negative vectors with semi-invariants of a given weight. The third fundamental theorem, the *generic decomposition theorem* (in Kac's terminology), describes the decomposition of a general representation of any non-negative integral vector $\alpha \in \mathbb{N}^n$.

We recall the definitions of general representations and also of the fundamentally important polynomial functions c_V.

Definition 2.3.1. Let Q be a quiver and $\Bbbk$ a field. A *generic representation* of Q of dimension α is a representation over a transcendental extension (i.e. field of rational functions) of $\Bbbk$ with one variable for each entry of the matrix representation of the representation. Alternatively stated, *a general representation* is a representation from some non-empty Zariski open set in $R(\alpha)$.

Every semi-invariant vanishing on a generic (or general) representation is identically zero.

Definition 2.3.2. Let Q be a quiver and let $\alpha \in \mathbb{N}^n$. For a representation V of Q, define $(p_M, V) : \mathrm{Hom}\,(P_0, V) \to \mathrm{Hom}\,(P_1, V)$ to be the vector space homomorphism induced by the canonical presentation p_M of $M \in R(\alpha)$, as defined in § 1.4. Given V such that (p_M, V) is square (i.e., by Remark 1.3.2(ii), such that $\langle \alpha, \underline{\dim}\, V\rangle = \alpha^t E \underline{\dim}\, V = 0$), define the polynomial function $c_V \in \Bbbk[R(\alpha)]$ by setting

$$c_V(M) := \det(p_M, V).$$

Note that a decomposition $V = V_1 \coprod V_2$ gives a factorization of $c_V(M)$ as $c_V(M) = c_{V_1}(M) c_{V_2}(M)$, provided that $\langle \alpha, \underline{\dim}\, V_1\rangle = \langle \alpha, \underline{\dim}\, V_2\rangle = 0$.

THEOREM 2.3.3 (First fundamental theorem [DW00, Sch91]; see Remark 2.2.5). *Let Q be a quiver and let $\alpha \in \mathbb{N}^n$. Then the ring of semi-invariants $\mathrm{SI}(Q, \alpha)$ is spanned as a $\Bbbk$-vector space by the functions c_V for representations V satisfying $\langle \alpha, \underline{\dim}\, V\rangle = \alpha^t E\, \underline{\dim}\, V = 0$. Furthermore, the character of the semi-invariant c_V is χ_σ where $\sigma = E\, \underline{\dim}\, V$.*

By simply restricting to the support of α, we get the following.

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

COROLLARY 2.3.4. *$\mathrm{SI}(Q,\alpha)$ is spanned by those c_V as above where $V_v=0$ whenever $\alpha_v=0$.*

Definition 2.3.5. Let $\sigma\in\mathbb{Z}^n$. The $\mathbb{N}$-*support* of σ is defined as

$$\mathrm{supp}_{\mathbb{N}}(\sigma):=\{\alpha\in\mathbb{N}^n\mid \mathrm{SI}(Q,\alpha)_{\chi_\sigma}\neq 0\}.$$

Remark 2.3.6. It follows that $\alpha\in\mathrm{supp}_{\mathbb{N}}(\sigma)$ if and only if $\sigma\in\Sigma(Q,\alpha)$, as in the reciprocity theorem of [DW00] or Definition 2.2.3.

The next theorem is clearly equivalent to [DW00, Saturation Lemma], in view of the reciprocity property in that paper. In order to state the theorem, we need to define the sets $D(\beta)$, which will be used throughout this paper.

Definition 2.3.7. Let $\beta\in\mathbb{N}^n$. Define the subset $D(\beta)\subset\mathbb{R}^n$ as

$$D(\beta):=\{\alpha\in\mathbb{R}^n\mid\langle\alpha,\beta\rangle=0\}\cap\left(\bigcap_{\beta'\hookrightarrow\beta}\{\alpha\in\mathbb{R}^n\mid\langle\alpha,\beta'\rangle\leq 0\}\right);$$

here $\beta'\hookrightarrow\beta$ means that the general representation of dimension β has a subrepresentation of dimension β'.

THEOREM 2.3.8 (Saturation theorem [DW00]). *Let $\beta\in\mathbb{N}^n$. Then*

$$\mathrm{supp}_{\mathbb{N}}(E\beta)=\mathbb{N}^n\cap D(\beta).$$

Before stating the generic decomposition theorem, we need to recall the definition of 'Schur root' and also of 'hom' and 'ext' on vectors in $\mathbb{N}^n$.

Definition 2.3.9. Let Q be a quiver and let $\alpha\in\mathbb{N}^n$. Then α is called a *Schur root* if the general representation in $R(\alpha)$ is indecomposable.

Definition 2.3.10. Let Q be a quiver and let $\alpha,\beta\in\mathbb{N}^n$. Define

$$\hom_Q(\alpha,\beta):=\min\{\dim_{\Bbbk}\mathrm{Hom}_Q(A,B)\mid \underline{\dim}\,A=\alpha,\underline{\dim}\,B=\beta\},$$
$$\mathrm{ext}_Q(\alpha,\beta):=\min\{\dim_{\Bbbk}\mathrm{Ext}_Q(A,B)\mid \underline{\dim}\,A=\alpha,\underline{\dim}\,B=\beta\}.$$

Since $\dim_{\Bbbk}\mathrm{Hom}$ and $\dim_{\Bbbk}\mathrm{Ext}$ are upper semicontinuous and $\Bbbk$ is algebraically closed, these minima are attained for general modules of these dimension vectors. So this definition agrees with the usual definition, i.e. $\hom_Q(\alpha,\beta)=\dim_{\Bbbk}\mathrm{Hom}_Q(A,B)$ and $\mathrm{ext}_Q(\alpha,\beta)=\dim_{\Bbbk}\mathrm{Ext}_Q(A,B)$, where A and B are general representations with $\underline{\dim}\,A=\alpha$ and $\underline{\dim}\,B=\beta$.

THEOREM 2.3.11 (Generic decomposition theorem [DW02]). *Any $\alpha\in\mathbb{N}^n$ has a unique decomposition of the form $\alpha=\sum\alpha_i$, where $\mathrm{ext}_Q(\alpha_i,\alpha_j)=0$ for all $i\neq j$ and each α_i is a Schur root. Furthermore, the general representation M with $\underline{\dim}\,M=\alpha$ decomposes as $M\cong\coprod M_i$ with $\underline{\dim}\,M_i=\alpha_i$, where the M_i are indecomposable representations such that $\mathrm{Ext}_Q(M_i,M_j)=0$ for all $i\neq j$.*

3. Definition of presentation spaces and their semi-invariants

In this section we deal with integral vectors (not necessarily non-negative), define presentation spaces associated to these vectors, and consider semi-invariants with respect to the actions of certain non-reductive algebraic groups. To justify this, we shall prove in Corollary 4.2.7 that for non-negative vectors, the rings of semi-invariants on certain special presentation spaces are isomorphic to the classical rings of semi-invariants on the quivers as in [DW00]. In later sections we will prove results analogous to the three fundamental theorems.

3.1 Projective decompositions of integral vectors

Let $\alpha \in \mathbb{Z}^n$ and let

$$E^t\alpha = \gamma_0 - \gamma_1 \quad \text{with } \gamma_0, \gamma_1 \in \mathbb{N}^n.$$

We refer to (γ_0, γ_1) as a *projective decomposition* of α since, by Remark 1.3.2(v), we have $\alpha = \underline{\dim}\, P(\gamma_0) - \underline{\dim}\, P(\gamma_1)$. The projective decompositions (γ_0, γ_1) of α form a directed partially ordered set $\mathrm{PD}(\alpha)$, with partial ordering given by

$$(\gamma_0, \gamma_1) \leq (\gamma_0', \gamma_1') \text{ if } (\gamma_0', \gamma_1') = (\gamma_0 + \gamma, \gamma_1 + \gamma) \text{ for some } \gamma \in \mathbb{N}^n.$$

Note that there is a unique *minimal projective decomposition* where γ_0 and γ_1 have disjoint supports (with γ_0 being the positive and $-\gamma_1$ the negative part of $E^t\alpha$).

3.2 Presentation spaces

Let $\alpha \in \mathbb{Z}^n$. For each projective decomposition (γ_0, γ_1) of α, we define a *presentation space*

$$R(\gamma_0, \gamma_1) := \mathrm{Hom}_Q(P(\gamma_1), P(\gamma_0)).$$

The following are definitions of and references for some of the special presentation spaces that we will use in this paper.

- The *minimal presentation space* is $R^{\min}(\alpha) := R(\gamma_0, \gamma_1)$ for $\alpha \in \mathbb{Z}^n$, where (γ_0, γ_1) is the minimal projective decomposition (see the stability theorem, Theorem 5.2.2).
- The *canonical presentation space* is $R^{\mathrm{can}}(\alpha) := R(\beta, \beta - E^t\beta + \gamma)$ for $\alpha \in \mathbb{Z}^n$; it will be precisely defined in § 5.3.3.
- The special case of $R(\alpha, \alpha - E^t\alpha)$ for non-negative $\alpha \in \mathbb{N}^n$ is particularly important for several reasons: the canonical projective presentation is an element of it (see § 1.4); also, we will show that it is a special case of the canonical presentation space $R^{\mathrm{can}}(\alpha)$, which is introduced in § 5.3 and is important for the virtual generic decomposition theorem, Theorem 6.3.

The space $R(\gamma_0, \gamma_1) = \mathrm{Hom}_Q(P(\gamma_1), P(\gamma_0))$ is an affine space with the natural action of the group $\mathrm{Aut}(P(\gamma_0)) \times (\mathrm{Aut}(P(\gamma_1)))^{\mathrm{op}}$, given by

$$(g^0, g^1)\varphi := g^0\varphi g^1$$

for $(g^0, g^1) \in \mathrm{Aut}(P(\gamma_0)) \times \mathrm{Aut}(P(\gamma_1))^{\mathrm{op}}$ and each $\varphi \in \mathrm{Hom}_Q(P(\gamma_1), P(\gamma_0))$.

Definition 3.2.1. Two elements of the presentation space $R(\gamma_0, \gamma_1)$ are said to be *isomorphic* if they lie in the same orbit of the action of this group of automorphisms.

Remark 3.2.2. Similarly to the general representations in Definition 2.3.1, we have that:

(i) the *general presentation* or *general element* of the presentation space $R(\gamma_0, \gamma_1)$ is any element of a non-empty Zariski open subset of $R(\gamma_0, \gamma_1)$;

(ii) the rank of the general element in $R(\gamma_0, \gamma_1)$, i.e. the general presentation $P(\gamma_1) \xrightarrow{\phi} P(\gamma_0)$, is the maximum of all ranks of all presentations in $R(\gamma_0, \gamma_1)$.

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

3.3 Semi-invariants on presentation spaces

Since $R(\gamma_0,\gamma_1)$ is an affine space, its coordinate ring is a polynomial ring and we consider the semi-invariants for the action of $\mathrm{Aut}(P(\gamma_0))\times(\mathrm{Aut}(P(\gamma_1)))^{\mathrm{op}}$ on the coordinate ring $\Bbbk[R(\gamma_0,\gamma_1)]$. Recall that a semi-invariant on $R(\gamma_0,\gamma_1)=\mathrm{Hom}_Q(P(\gamma_1),P(\gamma_0))$ is a polynomial function f such that for some character χ,

$$f((g^0,g^1)\varphi)=\chi(g^0,g^1)f(\varphi)$$

for all $(g^0,g^1)\in\mathrm{Aut}(P(\gamma_0))\times(\mathrm{Aut}(P(\gamma_1)))^{\mathrm{op}}$ and all $\varphi\in\mathrm{Hom}_Q(P(\gamma_1),P(\gamma_0))$.

Proposition 3.3.1. *The following properties hold for the characters of the semi-invariants on the presentation spaces $R(\gamma_0,\gamma_1)$.*

(i) *Since the group is a product of two groups, we have*

$$\chi(g^0,g^1)=\chi^0(g^0)\chi^1(g^1)$$

where χ^0 and χ^1 are characters of $\mathrm{Aut}(P(\gamma_0))$ and $(\mathrm{Aut}(P(\gamma_1)))^{\mathrm{op}}$, respectively.

(ii) *For $P(\gamma)=\coprod_v P(v)^{\gamma_v}$, each element of $\mathrm{Aut}(P(\gamma))$ can be written as an $n\times n$ block triangular matrix $g=(g_{uv})$ with*

$$g_{uv}\in\mathrm{Hom}_Q(P(v)^{\gamma_v},P(u)^{\gamma_u})\quad\text{and}\quad g_{vv}\in\mathrm{Aut}(P(v)^{\gamma_v})\cong Gl_{\gamma_v}(\Bbbk).$$

Using the isomorphism above, we can identify these two groups and also the groups

$$\prod_v\mathrm{Aut}(P(v)^{\gamma_v})\quad\text{and}\quad G(\gamma)=\prod_v Gl_{\gamma_v}(\Bbbk).$$

We use the total order $u<v$ defined in § 1.3 to write the matrix.

(iii) *For a projective $P(\gamma)$, any character of the group $\mathrm{Aut}(P(\gamma))$ has the form*

$$\chi_\sigma(g)=\prod_{v\in Q_0}\det(g_{vv})^{\sigma_v}$$

with weight vector $\sigma\in\mathbb{Z}^n$; for the semi-invariants on presentation spaces, $\sigma\in\mathbb{N}^n$. As in § 2.2, if $\gamma_v=0$, i.e. $g_{vv}\in Gl_0(\Bbbk)$, then $\det(g_{vv})=1$ and σ_v is indeterminate.

From the above, we see that the semi-invariants on presentation spaces have pairs of characters, as well as pairs of weights, associated to them.

Definition 3.3.2. Denote by $\mathrm{SI}^{(\gamma_0,\gamma_1)}(Q,\alpha)_{(\chi^0,\chi^1)}$ the set of semi-invariants on $R(\gamma_0,\gamma_1)$ with character (χ^0,χ^1), and define the associated graded ring of semi-invariants by

$$\mathrm{SI}^{(\gamma_0,\gamma_1)}(Q,\alpha):=\bigoplus_{(\chi^0,\chi^1)}\mathrm{SI}^{(\gamma_0,\gamma_1)}(Q,\alpha)_{(\chi^0,\chi^1)}.$$

Proposition 3.3.3. *Let $\alpha\in\mathbb{Z}^n$, let (γ_0,γ_1) be a projective decomposition of α, and let $R(\gamma_0,\gamma_1)$ be the corresponding presentation space. Let f be a semi-invariant on $R(\gamma_0,\gamma_1)$ with character $(\chi^0_{\sigma^0},\chi^1_{\sigma^1})$. Then $\sigma^0_v=\sigma^1_v$ if both $\gamma_{0,v}\neq 0$ and $\gamma_{1,v}\neq 0$.*

Proof. Let f be a semi-invariant on $R(\gamma_0,\gamma_1)$ such that $f((g^0,g^1)\varphi)=\chi^0_{\sigma^0}(g^0)\chi^1_{\sigma^1}(g^1)f(\varphi)$ for all $(g^0,g^1)\in\mathrm{Aut}\,P(\gamma^0)\times\mathrm{Aut}\,P(\gamma^1)^{\mathrm{op}}$ and all $\varphi\in R(\gamma^0,\gamma^1)$. We need to show that $\sigma^0_v=\sigma^1_v$ for all $v\in Q_0$ for which both $\gamma_{0,v}\neq 0$ and $\gamma_{1,v}\neq 0$.

If (γ_0,γ_1) is the minimal projective decomposition of α, then there is no $v\in Q_0$ such that both $\gamma_{0,v}\neq 0$ and $\gamma_{1,v}\neq 0$, so there is nothing to prove.

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

In order to deal with any projective decomposition, define $\gamma_v := \min\{\gamma_{0,v}, \gamma_{1,v}\}$. Then $\gamma_v \neq 0$ precisely at the vertices where both $\gamma_{0,v} \neq 0$ and $\gamma_{1,v} \neq 0$. Let $\gamma \in \mathbb{N}^n$ be defined as $\gamma = (\gamma_v)$. Then $(\gamma_0 - \gamma), (\gamma_1 - \gamma) \in \mathbb{N}^n$ and, in fact, $(\gamma_0 - \gamma, \gamma_1 - \gamma)$ is the minimal projective decomposition of α.

Since we need to check the weights of the semi-invariant f only at those vertices v where $\gamma_v \neq 0$, we will consider the following presentations and group elements:

$$\begin{aligned} \varphi = \varphi' \coprod 1_{P(\gamma)} &\in \mathrm{Hom}_Q\left(P(\gamma_1 - \gamma) \coprod P(\gamma), P(\gamma_0 - \gamma) \coprod P(\gamma)\right) \\ \text{and } \ (1_{P(\gamma_1-\gamma)} \coprod g) \times (1_{P(\gamma_0-\gamma)} \coprod g^{-1}) &\in \mathrm{Aut}(P(\gamma_0)) \times \mathrm{Aut}(P(\gamma_1))^{\mathrm{op}} \end{aligned}$$

for $g \in \mathrm{Aut}(P(\gamma))$. Then we have

$$\begin{aligned} f(\varphi' \coprod 1_{P(\gamma)}) &= f((1_{P(\gamma_0-\gamma)} \coprod g) \cdot (\varphi' \coprod 1_{P(\gamma)}) \cdot (1_{P(\gamma_1-\gamma)} \coprod g^{-1})) \\ &= \chi^0_{\sigma^0}(g)\chi^1_{\sigma^1}(g^{-1}) f(\varphi' \coprod 1_{P(\gamma)}) \end{aligned}$$

and so $\chi^0_{\sigma^0}(g) = \chi^1_{\sigma^1}(g)$. Therefore, $\sigma^0_v = \sigma^1_v$ for all v such that $\gamma_v \neq 0$. □

Definition 3.3.4. Let f be a semi-invariant on $R(\gamma_0, \gamma_1)$. The *combined weight* $\sigma = \sigma^{\mathrm{comb}}$ of f is defined as $\sigma^{\mathrm{comb}}_v := \max\{\sigma^0_v, \sigma^1_v\}$ for all $v \in Q_0$, and the *combined character* χ_σ is defined to be $(\chi_\sigma, \chi_\sigma)$.

Definition 3.3.5. Denote by $\mathrm{SI}^{(\gamma_0,\gamma_1)}(Q, \alpha)_\chi$ the set of semi-invariants on $R(\gamma_0, \gamma_1)$ with combined character χ.

4. Relationships between representation and presentation spaces and their semi-invariants for $\alpha \in \mathbb{N}^n$

Now consider only non-negative integral vectors $\alpha \in \mathbb{N}^n$. We shall compare the classical representation space $R(\alpha)$ and the special presentation space $R(\alpha, \alpha - E^t\alpha)$ together with the natural group actions of $Gl(\alpha)$ and $\mathrm{Aut}\, P(\alpha) \times \mathrm{Aut}\, P(\alpha - E^t\alpha)^{\mathrm{op}}$. We give relations between these spaces and prove (in Corollary 4.2.7) that their rings of semi-invariants are isomorphic.

4.1 Representation and presentation spaces for $\alpha \in \mathbb{N}^n$

In order to compare these two spaces, we define the mapping

$$\begin{gathered} \zeta : R(\alpha) \to R(\alpha, \alpha - E^t\alpha), \\ \zeta(M) = p_M, \end{gathered}$$

where p_M is the canonical projective presentation of M (see § 1.4). Consider the subspace $\mathrm{Im}\,\zeta \subset R(\alpha, \alpha - E^t\alpha)$ and orbits of this subspace under the action of the groups $\mathrm{Aut}\, P(\alpha)$, $\mathrm{Aut}\, P(\alpha - E^t\alpha)^{\mathrm{op}}$ and $\mathrm{Aut}\, P(\alpha) \times \mathrm{Aut}\, P(\alpha - E^t\alpha)^{\mathrm{op}}$.

For each projective module $P(\alpha)$, define $T(\alpha)$ in the following way: if $\alpha = e_v$ (the unit vector at v), then set $P(\alpha) := P(v)$ as in § 1.1; that is, it is a vector space generated by all paths starting at v. Let $T(e_v) := \Bbbk e_v$ be the linear subspace generated by the constant path at v. For any $\alpha \in \mathbb{N}$, we have a decomposition of α as a sum of unit vectors e_v. In this way, we have chosen an internal direct sum decomposition $P(\alpha) = \sum P(e_v)$. Let $T(\alpha) = \sum T(e_v)$.

Definition 4.1.1. Let $U(\alpha, \alpha - E^t\alpha) \subset R(\alpha, \alpha - E^t\alpha)$ be the open subspace defined as the set of monomorphisms $\psi : P(\alpha - E^t\alpha) \to P(\alpha)$ for which $\mathrm{Im}(\psi)$ is complementary to $T(\alpha)$.

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

Example 4.1.2. If $P(\alpha)$ is indecomposable, then $\operatorname{rad} P(\alpha)$ is a submodule of $P(\alpha)$ which is complementary to $T(\alpha)$, since $P(\alpha) = T(\alpha) \oplus \operatorname{rad} P(\alpha)$ as a vector space over $\Bbbk$; however, when $P(\alpha)$ is not indecomposable, there may be other such submodules which are not equal to $\operatorname{rad} P(\alpha)$, even though each of them is isomorphic to $\operatorname{rad} P(\alpha)$ as a representation.

Lemma 4.1.3. *Let $\alpha \in \mathbb{N}^n$ and let $\zeta : R(\alpha) \to R(\alpha, \alpha - E^t\alpha)$ be defined by $\zeta(M) = p_M$, the canonical projective presentation. Then the orbit of* $\operatorname{Im}(\zeta)$ *under the action of* $\operatorname{Aut} P(\alpha - E^t\alpha)^{\mathrm{op}}$ *is $U(\alpha, \alpha - E^t\alpha)$ and is thus open.*

Proof. We shall show that $\operatorname{Aut}(P(\alpha - E^t\alpha))^{\mathrm{op}}\operatorname{Im}(\zeta) = \operatorname{Im}(\zeta) \operatorname{Aut} P(\alpha - E^t\alpha)$ is equal to $U(\alpha, \alpha - E^t\alpha)$ and, thus, is open. Let $\psi : P(\alpha - E^t\alpha) \to P(\alpha)$ be an element of $U(\alpha, \alpha - E^t\alpha)$ and let $M = \operatorname{coker} \psi$. Then, by definition, the quotient map $P(\alpha) \to M$ is the same as the map π in the canonical projective presentation

$$P(\alpha - E^t\alpha) \xrightarrow{\zeta(M)} P(\alpha) \xrightarrow{\pi} M.$$

Therefore the image of $\zeta(M)$ is the same as the image of ψ, and ψ and $\zeta(M)$ differ by an automorphism of $P(\alpha - E^t\alpha)$. □

Proposition 4.1.4. *Let $\alpha \in \mathbb{N}^n$ and let $\zeta : R(\alpha) \to R(\alpha, \alpha - E^t\alpha)$, $\zeta(M) = p_M$, be the canonical projective presentation. Then the orbit of* $\operatorname{Im}(\zeta)$ *under the action of* $\operatorname{Aut} P(\alpha) \times \operatorname{Aut} P(\alpha - E^t\alpha)^{\mathrm{op}}$ *is an open and dense subset of $R(\alpha, \alpha - E^t\alpha)$.*

Proof. Since $\operatorname{Aut} P(\alpha - E^t\alpha)^{\mathrm{op}}$ is a subgroup of $\operatorname{Aut} P(\alpha) \times \operatorname{Aut} P(\alpha - E^t\alpha)^{\mathrm{op}}$ and the $\operatorname{Aut} P(\alpha - E^t\alpha)^{\mathrm{op}}$ orbit of $\operatorname{Im}(\zeta)$ is open in $R(\alpha, \alpha - E^t\alpha)$, the result follows. □

Remark 4.1.5. General properties of representations (properties that hold on an open subset of $R(\alpha)$) are also general properties of elements of $R(\alpha, \alpha - E^t\alpha)$. This proposition tells us that, conversely, the general intrinsic (i.e. invariant under isomorphism) properties of elements of $R(\alpha, \alpha - E^t\alpha)$ are also general properties of elements of $R(\alpha)$.

Lemma 4.1.6. *There is a one-to-one correspondence, given by quotients, between the submodules of $P(\alpha)$ which are complementary to $T(\alpha)$ and the elements of the representation space $R(\alpha)$. Furthermore, all such submodules are isomorphic to $P(\alpha - E^t\alpha)$.*

Proof. Given a submodule $L \subset P(\alpha)$ which is complementary to $T(\alpha)$, we take the quotient module $P(\alpha)/L$. Since this is a vector space isomorphic to $T(\alpha)$, the structure maps are matrices and we get an explicit element of $R(\alpha)$.

Given any $M \in R(\alpha)$, the corresponding submodule of $P(\alpha)$ is the kernel of the canonical projection map $\pi : P(\alpha) \to M$. This is also the image of the canonical presentation map $p_M : P(\alpha - E^t\alpha) \to P(\alpha)$, which is always a monomorphism with image complementary to $T(\alpha)$.

These constructions are clearly inverse to each other. □

Proposition 4.1.7. *Cokernels of homomorphisms define a mapping, which we denote by* $\operatorname{coker} : U(\alpha, \alpha - E^t\alpha) \to R(\alpha)$. *Furthermore:*

(i) coker *is a rational map;*

(ii) $\operatorname{coker} \circ \zeta = \operatorname{Id}_{R(\alpha)}$ *and hence ζ is a monomorphism.*

Proof. For each $\psi \in U(\alpha, \alpha - E^t\alpha)$, the representation $L = \operatorname{im} \psi$ is complementary to $T(\alpha)$ by definition. Therefore $\operatorname{coker} \psi = P(\alpha)/L$ is an element of $R(\alpha)$ by the above lemma.

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

Straightforward linear algebra shows that this is a rational map. The canonical presentation of any element of $R(\alpha)$ lies in $U(\alpha, \alpha - E^t\alpha)$, and the following composition is the identity map: $R(\alpha) \xrightarrow{\zeta} U(\alpha, \alpha - E^t\alpha) \xrightarrow{\text{coker}} R(\alpha)$. $\square$

4.2 Semi-invariants on representation and presentation spaces for $\alpha \in \mathbb{N}^n$

First, we show that the weights of semi-invariants on the classical representation space and the new presentation space are related by the Euler matrix. We then use Proposition 4.1.4 to show that the ring of semi-invariants on $R(\alpha, \alpha - E^t\alpha)$ is generated by the maps C_V which are the classical semi-invariants c_V (i.e. the $\det \operatorname{Hom}_Q(p_M, V)$, but now evaluated on all elements of $R(\alpha, \alpha - E^t\alpha)$).

Proposition 4.2.1. *Let $\alpha \in \mathbb{N}^n$ and let f be a semi-invariant on $R(\alpha, \alpha - E^t\alpha)$ with combined character χ_σ. Then $f \circ \zeta$ is a semi-invariant on $R(\alpha)$ with character $\chi_{E\sigma}$.*

Proof. By the assumption on f, we know that $f((g^0, g^1)p) = \chi_\sigma(g^0)\chi_\sigma(g^1)f(p)$ for all $(g^0, g^1) \in \operatorname{Aut}(P(\alpha)) \times \operatorname{Aut}(P(\alpha - E^t\alpha))^{\text{op}}$, all $p \in R(\alpha, \alpha - E^t\alpha) = \operatorname{Hom}_Q(P(\alpha - E^t\alpha), P(\alpha))$ and some combined character χ_σ. We want to show that

$$(f \circ \zeta)(gM) = \chi_{E\sigma}(g)(f \circ \zeta)(M)$$

for all $g = (g_v) \in Gl(\alpha) = \prod Gl_{\alpha_v}(\mathbb{k})$, all $M \in R(\alpha)$ and character $\chi_{E\sigma}$.

By definition, the representation gM consists of vector spaces $(gM)_v = M_v$ for $v \in Q_0$, and $(gM)_{u\to v} = g_v \circ M_{u\to v} \circ g_u^{-1}$ for all $(u \to v) \in Q_1$ (see § 1.2).

Then $\zeta(gM) = p_{gM}$, the canonical projective presentation of gM, fits in the commutative diagram

$$\begin{tikzcd}
P(\alpha - E^t\alpha) = \coprod_{u\to v} P(v)^{\alpha_u} \arrow[r, "\zeta(M)=p_M"] \arrow[d, "\varphi_1(g)"] & P(\alpha) = \coprod_v P(v)^{\alpha_v} \arrow[r] \arrow[d, "\varphi_0(g)"] & M \arrow[r] \arrow[d, "g"] & 0 \\
P(\alpha - E^t\alpha) = \coprod_{u\to v} P(v)^{\alpha_u} \arrow[r, "\zeta(gM)=p_{gM}"] & P(\alpha) = \coprod_v P(v)^{\alpha_v} \arrow[r] & gM \arrow[r] & 0
\end{tikzcd}$$

of Q-representations, where φ_0 and φ_1 are defined in the following way: after identifying $\prod_v Gl_{\alpha_v}(\mathbb{k})$ and $\prod_v \operatorname{Aut}(P(v)^{\alpha_v})$ as in Proposition 3.3.1(ii), $\varphi_0(g) = g$ and $\varphi_1(g) = h$ where $h_w = \prod_{v\to w} g_v$.

Now, it follows from the above diagram that

$$(f \circ \zeta)(gM) = f(\zeta(gM)) = \varphi_0(g) \circ \zeta(M) \circ (\varphi_1(g))^{-1}. \tag{4.1}$$

By the definition of the group action on $R(\alpha, \alpha - E^t\alpha)$, this equals

$$f((\varphi_0(g), (\varphi_1(g))^{-1})\zeta(M)).$$

By the assumption that f is a semi-invariant of combined character χ_σ, this is the same as

$$\chi_\sigma(\varphi_0(g))\chi_\sigma(\varphi_1(g))^{-1} f(\zeta(M)).$$

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

Finally, by the definitions of φ_0 and φ_1 and the fact that these characters are given by determinants, the above expression is further equal to

$$\begin{aligned}
&\prod_v \det(g_v)^{\sigma_v} \cdot \prod_w \det(h_w)^{-\sigma_w} \cdot f(\zeta(M)) \\
&\quad = \prod_v \det(g_v)^{\sigma_v} \cdot \prod_{v \to w} \det(g_v)^{-\sigma_w} \cdot f(\zeta(M)) \\
&\quad = \prod_v \det(g_v)^{\sigma_v - \Sigma_{v\to w}\sigma_w} (f \circ \zeta)(M) = \prod_v \det(g_v)^{(E\sigma)_v} (f \circ \zeta)(M) \\
&\quad = \chi_{E\sigma}(g)(f \circ \zeta)(M).
\end{aligned}$$

Thus, $f \circ \zeta$ is a semi-invariant on $R(\alpha)$ with character $\chi_{E\sigma}$. □

Remark 4.2.2. We note that $E\sigma$ may not determine σ even though E is invertible. The reason is that the weight $E\sigma$ of $f \cdot \zeta$ may have more indeterminacy than the weight σ of f. In fact, $E\sigma$ and σ have the same indeterminacy, i.e. $\sigma \to E\sigma$ maps the weight coset of f onto the weight coset of $f \cdot \zeta$, if and only if the support of $\alpha - E^t\alpha$ is contained in the support of α.

Now let us define maps $C_V : R(\alpha, \alpha - E^t\alpha) \to \Bbbk$ which extend the semi-invariants $c_V : R(\alpha) \to \Bbbk$.

Definition 4.2.3. Let $\alpha \in \mathbb{N}^n$ and let V be a representation such that $\langle \alpha, \underline{\dim}\, V\rangle = 0$. Define $C_V(\psi) := \det \operatorname{Hom}_Q(\psi, V)$ for $\psi \in R(\alpha, \alpha - E^t\alpha)$.

Remark 4.2.4. Notice that $C_V(\zeta M) = C_V(p_M) = \det \operatorname{Hom}_Q(p_M, V)$, which is equal to $c_V(M)$ by the definition of c_V. In other words, the composition

$$R(\alpha) \xrightarrow{\zeta} R(\alpha, \alpha - E^t\alpha) \xrightarrow{C_V} \Bbbk$$

coincides with the classical semi-invariant c_V on $R(\alpha)$ as in Theorem 2.3.3.

LEMMA 4.2.5. *Let $\alpha \in \mathbb{N}^n$ and let V be a representation such that $\langle \alpha, \underline{\dim}\, V \rangle = 0$. Then C_V is a semi-invariant on $R(\alpha, \alpha - E^t\alpha)$ of combined character $\chi_{\underline{\dim} V}$.*

To avoid repetition, we skip the proof of this lemma since the same statement is proved later in a more general setting for $\alpha \in \mathbb{Z}^n$ (see Proposition 5.1.3).

THEOREM 4.2.6. *If $\alpha \in \mathbb{N}^n$, then the space $\mathrm{SI}(\alpha, \alpha - E^t\alpha)_{\chi_\sigma}$ of semi-invariants on $R(\alpha, \alpha - E^t\alpha)$ of combined character χ_σ is spanned by the semi-invariants C_V for all modules V such that $\langle \alpha, \underline{\dim}\, V\rangle = 0$ and $\underline{\dim}\, V = \sigma$.*

Proof. Let f be a semi-invariant on $R(\alpha, \alpha - E^t\alpha)$ of weight $\sigma \in \mathbb{N}^n$. Then

$$f((g^0, g^1)\zeta(M)) = \chi_\sigma(g^0)\chi_\sigma(g^1)(f \circ \zeta)(M).$$

By Proposition 4.2.1, $f \circ \zeta$ is a semi-invariant on $R(\alpha)$ of weight $E\sigma$. Proposition 4.1.4 implies that the general element of $R(\alpha, \alpha - E^t\alpha)$ has the form $(g^0, g^1)\zeta(M)$ where $M \in R(\alpha)$. Therefore, the above formula shows that f is determined by $f \circ \zeta \in \mathrm{SI}(Q, \alpha)$ and the weight σ. Hence, it suffices to find a linear combination of semi-invariants C_V of weight σ so that the corresponding linear combination of classical semi-invariants c_V is equal to $f \circ \zeta$.

By the first fundamental theorem (Theorem 2.3.3), $f \circ \zeta$ is a linear combination of semi-invariants c_{V_i} of weight $E\sigma$, where we may assume that each V_i has support contained in the

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

support of α. Since $\chi_{E\sigma} = \chi_{E\underline{\dim}V_i}$, we have that:

(a) $(E\sigma)_v = (E\,\underline{\dim}V_i)_v$ for all v in the support of α; and

(b) $(E\,\underline{\dim}V_i)_v = -\sum_{v\to w} \dim(V_i)_w \leq 0$ if $\alpha_v = 0$.

Therefore,

$$\gamma_i = E\sigma - E\,\underline{\dim}V_i \in \mathbb{N}^n$$

for each V_i. Let $I(\gamma_i)$ be the injective module with socle $S(\gamma_i)$; then $\underline{\dim}\,I(\gamma_i) = E^{-1}\gamma_i$. So $C_{V_i\oplus I(\gamma_i)}$ is a semi-invariant of $R(\alpha, \alpha - E^t\alpha)$ of weight

$$\underline{\dim}V_i + \underline{\dim}\,I(\gamma_i) = \sigma.$$

Furthermore, $C_{V_i\oplus I(\gamma_i)} = c_{V_i\oplus I(\gamma_i)} = c_{V_i}$ since $c_{I(\gamma_i)} = 1$ on $R(Q, \alpha)$. Therefore, f is a linear combination of these determinantal semi-invariants. □

Corollary 4.2.7. *Let $\alpha \in \mathbb{N}^n$. There is an isomorphism of rings of semi-invariants,*

$$\mathrm{SI}^{(\alpha,\alpha-E^t\alpha)}(Q, \alpha) \cong \mathrm{SI}(Q, \alpha),$$

which sends C_V to c_V if and only if the support of $\alpha - E^t\alpha$ is contained in the support of α.

Proof. The mapping of rings is given by the mapping

$$\zeta : R(\alpha) \to R(\alpha - E^t\alpha),$$

which is equivariant with respect to the group homomorphism

$$(\phi_0, \phi_1^{-1}) : G(\alpha) \to \mathrm{Aut}(P(\alpha)) \times \mathrm{Aut}(P(\alpha - E^t\alpha))^{\mathrm{op}}$$

by (4.1) in the proof of Proposition 4.2.1. Therefore, ζ induces a homomorphism

$$\zeta^* : \mathrm{SI}^{(\alpha,\alpha-E^t\alpha)}(Q, \alpha) \to \mathrm{SI}(Q, \alpha)$$

of rings of semi-invariants. By Proposition 4.2.1 and Remark 4.2.2, this ring homomorphism is graded, sending semi-invariants of weight σ to semi-invariants of weight $E\sigma$, and this is a one-to-one correspondence of weight cosets when the support of $\alpha - E^t\alpha$ is contained in the support of α. Therefore, it suffices to show that ζ^* induces an isomorphism

$$\zeta^* : \mathrm{SI}^{(\alpha,\alpha-E^t\alpha)}(Q, \alpha)_\sigma \cong \mathrm{SI}(Q, \alpha)_{E\sigma}.$$

By Remark 4.2.4, ζ^* sends C_V to c_V. By the first fundamental theorem (Theorem 2.3.3), $\mathrm{SI}(Q, \alpha)$ is spanned by the functions c_V for all representations V with $\langle \alpha, \underline{\dim}\,V\rangle = 0$. Theorem 4.2.6 above tells us that $\mathrm{SI}^{(\alpha,\alpha-E^t\alpha)}(Q, \alpha)_\sigma$ is spanned by the corresponding semi-invariants C_V. Remark 4.2.2 assures us that C_V has weight σ. Therefore ζ^* is onto.

To show that ζ^* is one-to-one, take any element $f \in SI^{(\alpha,\alpha-E^t\alpha)}(Q, \alpha)_\sigma$ in the kernel of ζ^*; then f is a semi-invariant which is trivial on $R(\alpha)$. But the orbit of $\zeta(R(\alpha))$ is open, by Lemma 4.1.3. Therefore f is zero on an open set, and so f must be identically zero. Thus, ζ^* is an isomorphism as claimed. Conversely, suppose there is a vertex v in the support of $\alpha - E^t\alpha$ such that $\alpha_v = 0$. In that case, we take $V = I(v)$, the injective envelope of the simple representation supported at v; then $c_V = 1$ but C_V is not constant. So the rings of semi-invariants are not isomorphic in this case. □

5. Presentation spaces and their semi-invariants for vectors $\alpha \in \mathbb{Z}^n$

In this section, we return to the study of presentation spaces of arbitrary dimension vectors. First, we prove existence of determinantal semi-invariants for all presentation spaces. In preparation for the virtual generic decomposition theorem, it is instructive to define and prove existence of the particular projective decomposition of $\alpha \in \mathbb{Z}^n$ called the *canonical projective decomposition* (see Definition 5.3.3).

5.1 Determinantal semi-invariants

We now concentrate on the semi-invariants on presentation spaces which are defined using determinants and determine their weights. Only later will we show that the rings of all semi-invariants on presentation spaces are spanned by the determinants.

The following lemma is clear for the non-negative integral vectors $\alpha \in \mathbb{N}^n$ considered in Remark 1.3.2. It is, however, true for all integral vectors $\alpha \in \mathbb{Z}^n$.

Lemma 5.1.1. *Let $\alpha \in \mathbb{Z}^n$ and let V be a Q-representation. Then $\langle \alpha, \underline{\dim} V\rangle = 0$ if and only if $\mathrm{Hom}_Q(\varphi, V): \mathrm{Hom}_Q(P(\gamma_0), V) \to \mathrm{Hom}_Q(P(\gamma_1), V)$ is a square matrix for any presentation $\varphi \in R(\gamma_0, \gamma_1)$ and for any projective decomposition $E^t\alpha = \gamma_0 - \gamma_1$ of α.*

Proof. Note that

$$\begin{aligned}\langle \alpha, \underline{\dim} V\rangle &= \alpha^t E\, \underline{\dim} V = (\gamma_0 - \gamma_1)^t\, \underline{\dim} V = (\gamma_0)^t\, \underline{\dim} V - (\gamma_1)^t\, \underline{\dim} V \\ &= \dim_{\Bbbk} \mathrm{Hom}_Q(P(\gamma_0), V) - \dim_{\Bbbk} \mathrm{Hom}_Q(P(\gamma_1), V).\end{aligned}$$

It follows that $\langle \alpha, \underline{\dim} V\rangle = 0$ if and only if the matrix $\mathrm{Hom}_Q(\varphi, V)$ is square (not necessarily invertible). The dimensions of the matrix are $(\sum_{v \in Q_0} \dim V_v \cdot \gamma_{1,v}) \times (\sum_{v \in Q_0} \dim V_v \cdot \gamma_{0,v})$, since $\dim \mathrm{Hom}_Q(P(\gamma_i), V) = \sum_{v \in Q_0} \dim V_v \cdot \gamma_{i,v}$ for $i = 0, 1$. (For a more detailed description of this matrix, see the proof of Proposition 5.1.3.) □

Definition 5.1.2. Let $\alpha \in \mathbb{Z}^n$ and let V be a Q-representation such that $\langle \alpha, \underline{\dim} V\rangle = 0$. For any projective decomposition $E^t\alpha = \gamma_0 - \gamma_1$ of α we define, on the presentation space $R(\gamma_0, \gamma_1)$, the function

$$C_V^{(\gamma_0,\gamma_1)} := \det(\mathrm{Hom}_Q(\ , V)).$$

Proposition 5.1.3. *Let $\alpha \in \mathbb{Z}^n$ and let V be a Q-representation with $\langle \alpha, \underline{\dim} V\rangle = 0$.*

(i) *The functions $C_V^{(\gamma_0,\gamma_1)}$ are semi-invariants for all projective decompositions (γ_0, γ_1) of α.*

(ii) *The weight of $C_V^{(\gamma_0,\gamma_1)}$ is $(\chi_{\underline{\dim} V}, \chi_{\underline{\dim} V})$.*

Proof. Consider a presentation $\varphi \in R(\gamma_0, \gamma_1) = \mathrm{Hom}_Q(P(\gamma_1), P(\gamma_0))$,

$$P(\gamma_1) = \coprod_{v \in Q_0} P(v)^{\gamma_{1,v}} \xrightarrow{\varphi} P(\gamma_0) = \coprod_{v \in Q_0} P(v)^{\gamma_{0,v}}.$$

The map φ is given by a matrix of size $(\sum_{v \in Q_0} \gamma_{0,v}) \times (\sum_{v \in Q_0} \gamma_{1,v})$ with entries in $\mathrm{Hom}_Q(P(v), P(u)) = \prod_{p:u \to v} \Bbbk$, the vector space generated by all directed paths $p : u \to v$. For each pair of vertices u, v of Q and all paths $p : u \to v$, let $\varphi_{uv,p} = \varphi_p$ denote the $\gamma_{0,u} \times \gamma_{1,v}$ matrix with coefficients in $\Bbbk$ that corresponds to the p-coordinate of the composition

$$P(v)^{\gamma_{1,v}} \xrightarrow{\text{incl}} P(\gamma_1) \xrightarrow{\varphi} P(\gamma_0) \xrightarrow{\text{proj}} P(u)^{\gamma_{0,u}}.$$

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

Then the matrix representing the map

$$\mathrm{Hom}_Q(\varphi, V) : \mathrm{Hom}_Q(P(\gamma_0), V) \to \mathrm{Hom}_Q(P(\gamma_1), V)$$

is a block matrix with blocks of size $(\dim V_v \cdot \gamma_{1,v}) \times (\dim V_u \cdot \gamma_{0,u})$ with coefficients in $\Bbbk$:

$$(\mathrm{Hom}_Q(\varphi, V))_{vu} = \sum_{p:u\to v} \mathrm{Hom}_{\Bbbk}(\varphi_p, \Bbbk) \otimes_{\Bbbk} V_p = \sum_{p:u\to v} \varphi_p^* \otimes_{\Bbbk} V_p,$$

where $V_p : V_u \to V_v$ is the map induced by the representation V.

The fact that $\langle \alpha, \underline{\dim} V\rangle = 0$ implies, by Lemma 5.1.1, that the matrix $\mathrm{Hom}_Q(\varphi, V)$ is a square matrix. Hence the determinant $\det(\mathrm{Hom}_Q(\varphi, V))$ is defined, and therefore $C_V^{(\gamma_0,\gamma_1)}$ is a polynomial function on $R(\gamma_0, \gamma_1)$. To show that $C_V^{(\gamma_0,\gamma_1)}$ is a semi-invariant, we need to show that

$$C_V^{(\gamma_0,\gamma_1)}((g^0, g^1)\varphi) = \chi(g^0, g^1) C_V^{(\gamma_0,\gamma_1)}(\varphi)$$

for some character χ. Using the properties of characters from § 3.3, we have

$$\begin{aligned} C_V^{(\gamma_0,\gamma_1)}((g^0, g^1)\varphi) &= \det \mathrm{Hom}_Q((g^0, g^1)\varphi, V) = \det \mathrm{Hom}_Q(g^0\varphi g^1, V) \\ &= (\det \mathrm{Hom}_Q(g^0, V)) \cdot (\det \mathrm{Hom}_Q(\varphi, V)) \cdot (\det \mathrm{Hom}_Q(g^1, V)). \end{aligned}$$

Note that in the matrix $\mathrm{Hom}_Q(g^i, V)$, we have that g^i_{vv} is a $\gamma_{i,v} \times \gamma_{i,v}$ matrix which occurs $\dim V_v$ times, for $i = 0, 1$. So the above is equal to

$$\begin{aligned} &\left(\prod_{v\in Q_0} \det(g^0_{vv})^{\dim V_v}\right) \cdot (C_V^{(\gamma_0,\gamma_1)}(\varphi)) \cdot \left(\prod_{v\in Q_0} \det(g^1_{vv})^{\dim V_v}\right) \\ &\quad = \chi^0_{\underline{\dim} V}(g^0) \cdot C_V^{(\gamma_0,\gamma_1)}(\varphi) \cdot \chi^1_{\underline{\dim} V}(g^1) = \chi_{(\underline{\dim} V, \underline{\dim} V)}(g^0, g^1) \cdot C_V^{(\gamma_0,\gamma_1)}(\varphi). \end{aligned}$$ □

We now consider all projective decompositions (γ_0, γ_1) of α in the directed poset $\mathrm{PD}(\alpha)$ and give conditions under which the determinantal semi-invariants will be non-zero.

PROPOSITION 5.1.4. *Let $\alpha \in \mathbb{Z}^n$ and let V be a representation. Then the following statements are equivalent.*

(i) *There exists a projective decomposition $(\gamma_0, \gamma_1) \in \mathrm{PD}(\alpha)$ of α such that $C_V^{(\gamma_0,\gamma_1)}$ is a non-zero semi-invariant on $R(\gamma_0, \gamma_1)$.*

(ii) *There exists a module M and a projective module P such that:*
   (a) $\alpha = \underline{\dim}\, M - \underline{\dim}\, P$;
   (b) $\mathrm{Hom}_Q(P, V) = 0$;
   (c) $\mathrm{Hom}_Q(M, V) = 0$;
   (d) $\mathrm{Ext}_Q(M, V) = 0$.

Proof. (i) $\Rightarrow$ (ii). Let (γ_0, γ_1) be a projective decomposition of α such that $C_V^{(\gamma_0,\gamma_1)} = \det \mathrm{Hom}_Q(\ \ , V)$ is a non-zero semi-invariant on $R(\gamma_0, \gamma_1)$, and let φ be a general element of $R(\gamma_0, \gamma_1)$. Consider the exact sequence

$$0 \to \mathrm{Ker}(\varphi) \to P(\gamma_1) \xrightarrow{\varphi} P(\gamma_0) \to \mathrm{Coker}(\varphi) \to 0,$$

and let $M := \mathrm{Coker}(\varphi)$ and $P := \mathrm{Ker}(\varphi)$. It is easy to check that M and P satisfy conditions (a)–(d).

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

(ii) $\Rightarrow$ (i) Given P and M that satisfy conditions (a)–(d), let $P_1 \xrightarrow{\psi} P_0 \to M \to 0$ be a projective resolution of M, and let

$$\gamma_0 = \underline{\dim}(P_0/\operatorname{rad} P_0) \quad \text{and} \quad \gamma_1 = \underline{\dim}((P_1 \coprod P)/\operatorname{rad}(P_1 \coprod P)).$$

Consider the presentation space

$$R(\gamma_0, \gamma_1) = \operatorname{Hom}_Q(P \coprod P_1, P_0) = \operatorname{Hom}_Q(P, P_0) \times \operatorname{Hom}_Q(P_1, P_0),$$

and let $\varphi := (0, \psi) \in R(\gamma_0, \gamma_1)$. Then the mapping

$$\operatorname{Hom}_Q(\varphi, V) = \operatorname{Hom}_Q((0, \psi), V) : \operatorname{Hom}_Q(P_0, V) \to \operatorname{Hom}_Q(P \coprod P_1, V)$$

is a monomorphism by (c) and an epimorphism by (b) and (d). Consequently, $\det \operatorname{Hom}_Q(\varphi, V) \neq 0$, i.e. $C_V^{(\gamma_0,\gamma_1)} \neq 0$. □

5.2 Stability in presentation spaces

This subsection is devoted to investigating the general elements in the presentation spaces. We prove the stability theorem, which asserts that the general element in the presentation space is homotopically equivalent to an element in the space corresponding to a minimal decomposition of α.

We recall that the direct sum of homomorphisms gives a mapping

$$\coprod : R(\gamma_0, \gamma_1) \coprod R(\gamma_0', \gamma_1') \to R(\gamma_0 + \gamma_0', \gamma_1 + \gamma_1').$$

Definition 5.2.1. For any $\gamma_0, \gamma_1, \gamma \in \mathbb{N}^n$ we define the *stabilization maps*

$$\operatorname{St}_\gamma^{(\gamma_0,\gamma_1)} : R(\gamma_0, \gamma_1) \to R(\gamma_0 + \gamma, \gamma_1 + \gamma)$$

by $\operatorname{St}_\gamma^{(\gamma_0,\gamma_1)}(\varphi) := \varphi \coprod 1_{P(\gamma)}$ for each $\varphi \in R(\gamma_0, \gamma_1) = \operatorname{Hom}_Q(P(\gamma_1), P(\gamma_0))$.

Theorem 5.2.2 (Stability theorem). *Given any projective decomposition $E^t\alpha = \gamma_0 - \gamma_1$ of $\alpha \in \mathbb{Z}^n$, the general element of $R(\gamma_0, \gamma_1)$ is isomorphic to an element in the image of the stabilization map $R^{\min}(\alpha) \to R(\gamma_0, \gamma_1)$.*

Proof. Let $\alpha \in \mathbb{Z}^n$ and let $E^t\alpha = \gamma_0 - \gamma_1$ be a projective decomposition of α. Suppose it is not minimal; then $(\gamma_0, \gamma_1) = (\gamma_0^{\min} + \gamma, \gamma_1^{\min} + \gamma)$, where $(\gamma_0^{\min}, \gamma_1^{\min})$ is the minimal projective decomposition of α.

Let $\varphi \in R(\gamma_0, \gamma_1) = R(\gamma_0^{\min} + \gamma, \gamma_1^{\min} + \gamma)$ be a general element. We will show that

$$\varphi = (g^0, g^1)(\varphi^{\min} \coprod 1_{P(\gamma)}) = (g^0, g^1)(\operatorname{St}_\gamma^{(\gamma_0^{\min},\gamma_1^{\min})}(\varphi^{\min})) = (g^0, g^1) \operatorname{St}_\gamma^{\min}(\varphi^{\min}),$$

where $\varphi^{\min}$ is an element in $R(\gamma_0^{\min}, \gamma_1^{\min}) = R^{\min}(\alpha)$.

By the above projective decomposition of α, we have

$$\operatorname{Hom}_Q(P(\gamma_1) \coprod P(\gamma)) \xrightarrow{\varphi} \operatorname{hom}_Q(P(\gamma_0) \coprod P(\gamma)).$$

So φ can be viewed as a matrix

$$\varphi = \begin{pmatrix} f & h \\ g & r \end{pmatrix} : P(\gamma_1) \coprod P(\gamma) \to P(\gamma_0) \coprod P(\gamma),$$

where $r : P(\gamma) \to P(\gamma)$ is an isomorphism since φ is a general element (see Remark 3.2.2). From

$$\begin{pmatrix} f & h \\ g & r \end{pmatrix} = \begin{pmatrix} 1_{P(\gamma_0)} & hr^{-1} \\ 0 & 1_{P(\gamma)} \end{pmatrix} \begin{pmatrix} f - hr^{-1}g & 0 \\ 0 & 1_{P(\gamma)} \end{pmatrix} \begin{pmatrix} 1_{P(\gamma_1)} & 0 \\ g & r \end{pmatrix}$$

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

it follows that

$$\varphi = (g^0, g^1)\begin{pmatrix} f - hr^{-1}g & 0 \\ 0 & 1_{P(\gamma)} \end{pmatrix} = (g^0, g^1)\begin{pmatrix} \varphi^{\min} & 0 \\ 0 & 1_{P(\gamma)} \end{pmatrix} = (g^0, g^1)\mathrm{St}_\gamma^{\min}(\varphi^{\min}),$$

where

$$g^0 = \begin{pmatrix} 1_{P(\gamma_0)} & hr^{-1} \\ 0 & 1_{P(\gamma)} \end{pmatrix} \in \mathrm{Aut}(P(\gamma_0) \textstyle\coprod P(\gamma)),$$

$$g^1 = \begin{pmatrix} 1_{P(\gamma_1)} & 0 \\ g & r \end{pmatrix} \in \mathrm{Aut}(P(\gamma_1) \textstyle\coprod P(\gamma))^{\mathrm{op}}$$

and $\varphi^{\min} = f - hr^{-1}g \in R(\gamma_0^{\min}, \gamma_1^{\min}) = R^{\min}(\alpha)$. □

Remark 5.2.3. For $\alpha \in \mathbb{N}^n$, this says that for the minimal projective resolution $0 \to P_1 \to P_0 \to M \to 0$ of a general module M of dimension α, P_0 and P_1 have no summands in common. (Apply the above theorem to $R(\alpha, \alpha - E^t\alpha)$ and use Remark 4.1.5 to pass from general properties of elements of $R(\alpha, \alpha - E^t\alpha)$ to general properties of modules.)

5.3 Canonical presentation spaces for $\alpha \in \mathbb{Z}^n$

It was observed in § 3.2 that for $\alpha \in \mathbb{N}^n$, the canonical presentation is an element of the presentation space $R(\alpha, \alpha - E^t\alpha)$ and there is a close relationship between the classical representation space $R(\alpha)$ and the presentation space $R(\alpha, \alpha - E^t\alpha)$ (see § 4). For $\alpha \in \mathbb{Z}^n$, we generalize this special presentation space to the canonical presentation space $R^{\mathrm{can}}(\alpha)$, which, in the case of $\alpha \in \mathbb{N}^n$, turns out to be the same as $R(\alpha, \alpha - E^t\alpha)$.

Lemma 5.3.1. *Let $\alpha \in \mathbb{Z}^n$ be fixed and let $E^t\alpha = \gamma_0 - \gamma_1$ be the minimal decomposition. Let $\phi : P(\gamma_1) \to P(\gamma_0)$ be a general element of $R^{\min}(\alpha) = \mathrm{Hom}_Q(P(\gamma_1), P(\gamma_0))$ with kernel P and cokernel M, so that*

$$0 \to P \to P(\gamma_1) \xrightarrow{\phi} P(\gamma_0) \to M \to 0.$$

Then:

(a) *P must be a direct summand of $P(\gamma_1)$;*

(b) $\mathrm{Hom}_Q(P, M) = 0$.

Proof. (a) Note that P must be a direct summand of $P(\gamma_1)$, since $\mathrm{Im}\,\phi \subset P(\gamma_0)$ is projective. Let $P(\gamma_1) = P' \coprod P$.

(b) To see that $\mathrm{Hom}_Q(P, M) = 0$, let $f : P \to M$ be any non-zero homomorphism. Then f lifts to a homomorphism $\psi : P \to P(\gamma_0)$ whose image is not contained in the image of ϕ. This implies that the homomorphism $\phi + \psi : P' \coprod P \to P(\gamma_0)$ has image strictly containing the image of ϕ and, therefore, has rank greater than the rank of ϕ; but this gives a contradiction, since $\phi + \psi$ is a specialization of the general map ϕ. □

Let $\gamma = \underline{\dim}(P/\mathrm{rad}\, P)$ so that $P \cong P(\gamma)$. Let $\mu = \underline{\dim}\, M$.

Lemma 5.3.2. *Let $\alpha \in \mathbb{Z}^n$. Then the vectors $\mu = \underline{\dim}\, M$ and $\gamma = \underline{\dim}(P/\mathrm{rad}\, P)$ satisfy the following properties. These properties determine $\mu, \gamma \in \mathbb{N}^n$ uniquely.*

(i) *μ and γ have disjoint support.*

(ii) $\alpha = \mu - (E^t)^{-1}\gamma$.

Furthermore, in the special case where $\alpha \in \mathbb{N}^n$, we have $\mu = \alpha$ and $\gamma = 0$.

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

Proof. Property (i) follows from the fact that $\mathrm{Hom}_Q(P, M) = 0$, and property (ii) follows from a dimension counting argument. Therefore it remains to prove the uniqueness of μ and γ.

Let $\alpha \in \mathbb{N}^n$ and suppose that we are given a decomposition $\alpha = \mu - (E^t)^{-1}\gamma$. Let v be a vertex in the support of γ which is minimal with respect to the partial ordering of the vertices of Q. Then $\alpha_v < 0$ from Remark 1.3.2(iv), for instance. Therefore $\gamma = 0$, which proves the uniqueness.

Now we proceed by induction on the number of negative coordinates of α. If α has negative coordinates, let v be the minimal vertex such that $\alpha_v < 0$. Then $\alpha' = \alpha + |\alpha_v|(E^t)^{-1}e_v$ has fewer negative coordinates than does α, so we have a unique decomposition $\alpha' = \mu - (E^t)^{-1}\gamma$. It follows that we must have $\mu_v = \gamma_v = 0$ and that $\alpha = \mu - (E^t)^{-1}(\gamma + |\alpha_v|e_v)$ is the unique admissible decomposition of α. □

This lemma motivates the following.

Definition 5.3.3. Let $\alpha \in \mathbb{Z}^n$. The *canonical projective decomposition of* α is $(\mu, \mu - E^t\mu + \gamma)$ where $\mu, \gamma \in \mathbb{N}^n$ are uniquely defined vectors as in Lemma 5.3.2. Note that $\mu - E^t\mu \in \mathbb{N}^n$ by Remark 1.3.2(vi). We also define the *canonical presentation space*

$$R^{\mathrm{can}}(\alpha) := R(\mu, \mu - E^t\mu + \gamma).$$

Remark 5.3.4. For $\alpha \in \mathbb{N}^n$ we have $R^{\mathrm{can}}(\alpha) = R(\alpha, \alpha - E^t\alpha)$, which is the special case that we considered in the previous section.

Example 5.3.5. We now illustrate Lemma 5.3.2 and Definition 5.3.3 with Example 1.3.1 from earlier. If $\alpha = (1, 2, -3)^t$, then

$$\begin{gathered} \gamma = (0, 0, 3)^t, \quad \mu = (1, 2, 0)^t, \\ R^{\mathrm{can}}(\alpha) = R^{\mathrm{can}}((1, 2, -3)^t) = R((1, 2, 0)^t, (0, 1, 7)^t), \\ R^{\mathrm{min}}(\alpha) = R^{\mathrm{min}}((1, 2, -3)^t) = R((1, 1, 0)^t, (0, 0, 7)^t). \end{gathered}$$

Proposition 5.3.6. *The general element of $R^{\mathrm{can}}(\alpha)$ is isomorphic to the direct sum of the canonical presentation p_M of the general element M of $R(\mu)$ and the unique element of $R(0, \gamma)$.*

Proof. By Theorem 5.2.2, the general element of $R^{\mathrm{can}}(\alpha)$ is isomorphic to a stabilized element of $R^{\mathrm{min}}(\alpha)$. Therefore the general element

$$P(\mu - E^t\mu + \gamma) \to P(\mu)$$

will have kernel $P(\gamma)$ which is a direct summand. Consequently, the general element of $R^{\mathrm{can}}(\alpha)$ is a direct sum of the unique element of $R(0, \gamma)$ and an element of $R^{\mathrm{can}}(\mu)$. Since $\mu \in \mathbb{N}^n$, Proposition 4.1.4 now applies. Therefore the general element of $R^{\mathrm{can}}(\mu)$ lies in the orbit of $\zeta(R(\mu))$, i.e. it is isomorphic to a canonical presentation of an element of $R(\mu)$. □

6. Virtual representation spaces and virtual semi-invariants

In this section we again deal with integral vectors $\alpha \in \mathbb{Z}^n$, and define the virtual representation space as the direct limit of presentation spaces. Similarly, we define the rings of virtual semi-invariants on the virtual representation spaces as the inverse limits of the rings of semi-invariants on the presentation spaces. Finally, we prove the virtual generic decomposition theorem (generalizing Proposition 5.3.6) and the virtual first fundamental theorem.

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

6.1 Virtual representation spaces

We recall that the stabilization maps

$$\mathrm{Hom}_Q(P(\gamma_1), P(\gamma_0)) \to \mathrm{Hom}_Q(P(\gamma_1) \coprod P(\gamma), P(\gamma_0) \coprod P(\gamma))$$

are the maps which send φ to $\varphi \coprod 1_{P(\gamma)}$.

Let $\alpha \in \mathbb{Z}^n$. Then the set of all representation spaces $\{R(\gamma_0, \gamma_1)\}_{(\gamma_0,\gamma_1)\in PD(\alpha)}$ together with the stabilization maps constitutes a directed system. We define the *virtual representation space* as the direct limit over $\mathrm{PD}(\alpha)$,

$$R^{\mathrm{vir}}(\alpha) = \varinjlim R(\gamma_0, \gamma_1).$$

(Notice that a given pair (γ_0, γ_1) of dimension vectors belongs to exactly one partially ordered set $\mathrm{PD}(\alpha)$, namely the one where $\alpha = (E^t)^{-1}(\gamma_0 - \gamma_1)$.)

6.2 Virtual semi-invariants

The rings $\mathrm{SI}^{(\gamma_0,\gamma_1)}(Q, \alpha)$ and the restriction maps induced by stabilizing define an inverse system of rings on the directed partially ordered set $\mathrm{PD}(\alpha)$. We define the ring of *virtual semi-invariants* as the inverse limit over $\mathrm{PD}(\alpha)$:

$$\mathrm{SI}^{vir}(Q, \alpha) := \varprojlim \mathrm{SI}^{(\gamma_0,\gamma_1)}(Q, \alpha).$$

In other words, a *virtual semi-invariant* on $R^{\mathrm{vir}}(\alpha)$ is a function f^{vir} induced by a family of stabilization-compatible semi-invariants $f^{(\gamma_0,\gamma_1)}$ on the representation spaces $R(\gamma_0, \gamma_1)$.

The definition of invariants C_V induced by the determinants given in Proposition 5.1.3 generalizes to virtual semi-invariants. More precisely, we have the following proposition.

Proposition 6.2.1. *Let $\alpha \in \mathbb{Z}^n$ and let V be a Q-representation such that $\langle \alpha, \underline{\dim} V\rangle = 0$. Then the following hold.*

(i) *The family of semi-invariants $\{C_V^{(\gamma_0,\gamma_1)}\}_{(\gamma_0,\gamma_1)\in \mathrm{PD}(\alpha)}$ is compatible with stabilizations and thus defines an element $C_V^{\mathrm{vir}} \in \mathrm{SI}^{\mathrm{vir}}(Q, \alpha)$.*

(ii) *The induced semi-invariant C_V^{vir} on the virtual representation space $R^{\mathrm{vir}}(\alpha)$ has combined character $\chi_{\underline{\dim} V}$.*

We proceed to analyze the general elements in the virtual representation spaces.

Corollary 6.2.2. *Let $\alpha \in \mathbb{Z}^n$ and let V be a representation of Q. Then the following are equivalent.*

(i) *$C_V \neq 0$ on $R^{\min}(\alpha)$.*

(ii) *$C_V^{(\gamma_0,\gamma_1)} \neq 0$ on $R(\gamma_0, \gamma_1)$ for all projective decompositions $(\gamma_0, \gamma_1) \in \mathrm{PD}(\alpha)$.*

(iii) *$C_V^{(\gamma_0,\gamma_1)} \neq 0$ on $R(\gamma_0, \gamma_1)$ for some projective decomposition $(\gamma_0, \gamma_1) \in \mathrm{PD}(\alpha)$.*

(iv) *$C_V^{\mathrm{vir}} \neq 0$ on $R^{\mathrm{vir}}(\alpha)$.*

Remark 6.2.3. The above corollary is an extension of Proposition 5.1.4.

Proof. If $C_V \neq 0$ on $R^{\min}(\alpha)$, then the composition $R^{\min}(\alpha) \to R(\gamma_0, \gamma_1) \xrightarrow{C_V} \mathbb{k}$ is non-zero; so (i) $\Rightarrow$ (ii).

Clearly, (ii) $\Rightarrow$ (iii) and (iii) $\Leftrightarrow$ (iv) by the definition of the virtual semi-invariant C_V^{vir}. Finally, (iii) $\Rightarrow$ (i) by the stability theorem: if $C_V^{(\gamma_0,\gamma_1)} \neq 0$ on $R(\gamma_0, \gamma_1)$, then $C_V^{(\gamma_0,\gamma_1)} \neq 0$ on the general element of $R(\gamma_0, \gamma_1)$, which is equivalent to an element of $R^{\min}(\alpha)$ by stability. □

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

Definition 6.2.4. We define the $\mathbb{Z}$-*support* of C_V to be the set of all $\alpha \in \mathbb{Z}^n$ such that any of the equivalent conditions in Corollary 6.2.2 hold; for instance,

$$\mathrm{supp}_{\mathbb{Z}}(C_V) := \{\alpha \in \mathbb{Z}^n \mid C_V^{\mathrm{vir}} \neq 0 \text{ on } R^{\mathrm{vir}}(\alpha)\}.$$

LEMMA 6.2.5. *If $\beta = \underline{\dim}\, V$ is sincere, i.e. $\beta_v \neq 0$ for all $v \in Q_0$, then*

$$\mathrm{supp}_{\mathbb{Z}}(C_V) = \{\alpha \in \mathbb{N}^n \mid \langle \alpha, \beta \rangle = 0 \text{ and } \exists M \in R(\alpha) \text{ such that } \mathrm{Hom}_Q(M, V) = 0\}.$$

Proof. This follows from Proposition 5.1.4, Corollary 6.2.2 and Remark 1.3.2(i). □

6.3 The virtual generic decomposition theorem

Let us make preparations to state the virtual generic decomposition theorem. We want to extend the notions of $\hom_Q$ and ext_Q to include shifted projective modules such as $P(\gamma)[1]$; this is the projective complex $P(\gamma) \to 0$ which is the unique element of $R(0, \gamma)$. We note that a shifted projective is uniquely determined up to isomorphism by its dimension vector, which is negative:

$$\underline{\dim}\, P(\gamma)[1] = -(E^t)^{-1}\gamma.$$

We use the notation $\hom_{D^b}(\alpha, \beta[1]) = \mathrm{ext}_Q(\alpha, \beta)$ and $\mathrm{ext}_{D^b}(\alpha[1], \beta) = \hom_Q(\alpha, \beta)$. In general,

$$\mathrm{ext}_{D^b}(\alpha[p], \beta[q]) = \hom_{D^b}(\alpha[p], \beta[q+1]) := \begin{cases} \mathrm{ext}_Q(\alpha, \beta) & \text{if } p = q, \\ \hom_Q(\alpha, \beta) & \text{if } p = q+1, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\alpha, \beta \in \mathbb{N}^n$. In particular, $\mathrm{ext}_{D^b}(\pi(\gamma)[1], \beta) = 0$ for $\beta \in \mathbb{N}^n$ if and only if β and γ have disjoint supports.

Let $\alpha \in \mathbb{Z}^n$. Consider the canonical representation space as defined in Definition 5.3.3.

$$R^{\mathrm{can}}(\alpha) := R(\mu, \mu - E^t\mu + \gamma).$$

The dimension vector $\mu = \underline{\dim}\, M$ has a generic decomposition

$$\mu = \sum \beta_i$$

where the β_i are Schur roots with the property that $\mathrm{ext}_Q(\beta_i, \beta_j) = 0$ for all $i \neq j$. We recall that *Schur roots* are dimension vectors β_i such that the general representation of dimension β_i is indecomposable; see Definition 2.3.9. Thus M decomposes as $M \cong \coprod M_i$ with $\underline{\dim}\, M_i = \beta_i$, where the M_i are indecomposable modules which do not extend each other.

THEOREM 6.3.1 (Virtual generic decomposition theorem). *Any $\alpha \in \mathbb{Z}^n$ has a unique decomposition of the form*

$$\alpha = \beta_1 + \beta_2 + \cdots + \beta_k - (E^t)^{-1}\gamma$$

where:

(a) $\beta_1, \ldots, \beta_k, \gamma \in \mathbb{N}^n$;

(b) *β_i, γ have disjoint support for all i;*

(c) *$\mathrm{ext}_{D^b}(\beta_i, \beta_j) = 0$ for all $i \neq j$;*

(d) *each β_j is a Schur root.*

Furthermore, the general element $f : P_1 \to P_0$ of the canonical presentation space $R^{\mathrm{can}}(\alpha)$ is homotopy equivalent to a direct sum of projective complexes which are of one of the following types:

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

(i) *minimal resolutions of indecomposable modules* M_j *with* $\underline{\dim}\, M_j = \beta_j$;

(ii) *complexes of the form* $P(v_i)[1] = (P(v_i) \to 0)$ *with* $\underline{\dim}\, P(v_i)[1] = -(E^t)^{-1} e_{v_i}$.

Proof. Proposition 5.3.6 tells us that the general element $f: P_1 \to P_0$ has $\ker f = P(\gamma)$. The classical generic decomposition theorem, Theorem 2.3.11, gives us the stated decomposition of $M = \operatorname{coker} f$. □

6.4 The virtual first fundamental theorem

Theorem 6.4.1 (Virtual first fundamental theorem). *For any* $\alpha \in \mathbb{Z}^n$, *the ring of virtual semi-invariants on* $R^{\mathrm{vir}}(\alpha)$ *is generated by the semi-invariants* C_V^{vir} *for all modules* V *such that* $\langle \alpha, \underline{\dim} V\rangle = 0$ *and* $\langle \beta_j, \underline{\dim} V\rangle = 0$ *for all* β_j *in the generic decomposition of* α. *Consequently, we have a graded decomposition of the ring of virtual semi-invariants,*

$$\mathrm{SI}^{\mathrm{vir}}(Q, \alpha) = \bigoplus_{\sigma} \mathrm{SI}^{\mathrm{vir}}(Q, \alpha)_{\chi_\sigma},$$

where the sum is over all $\sigma \in \mathbb{N}^n$ *with support disjoint from the support of* γ *such that* $\langle \alpha, \sigma\rangle = \langle \beta_i, \sigma\rangle = 0$ *for all* β_i *in the generic decomposition of* α.

Proof. By Proposition 5.3.6, the general element of $R^{\mathrm{can}}(\alpha)$ is a direct sum of a presentation of a module M with dimension vector μ and the shifted projective module $P(\gamma)[1]$. This shows that every semi-invariant on $R^{\mathrm{can}}(\alpha)$ restricts to a semi-invariant on $R(\mu, \mu - E^t\mu)$ which determines it uniquely. But, by Theorem 4.2.6, $\mathrm{SI}(\mu, \mu - E^t\mu)$ is spanned by semi-invariants C_V. Also, C_V extends to $R^{\mathrm{can}}(\alpha)$ if and only if $\langle \alpha, \underline{\dim}\, V\rangle = 0$. Furthermore, C_V will be trivial on the general element of $R^{\mathrm{can}}(\alpha)$ unless $\langle \beta_i, \underline{\dim}\, V\rangle = 0$ for each β_i in the generic decomposition of α.

For a general pair $(\gamma_0, \gamma_1) \in \mathrm{PD}(\alpha)$, the elements of some Zariski open set in $R(\gamma_0, \gamma_1)$ will be the direct sum of an identity map on some projective module and a map from $R^{\min}(\alpha)$. Since a semi-invariant is determined by its restriction to the open set, the result follows. □

6.5 The virtual saturation theorem

The original saturation theorem, Theorem 2.3.8, gives all non-negative integral vectors α such that the classical representation space $R(\alpha)$ has a non-zero semi-invariant of a prescribed weight. In this paper, we describe all integral vectors α such that the virtual representation space $R^{\mathrm{vir}}(\alpha)$ has a virtual semi-invariant of a prescribed weight.

Following the classical definition of the support (Definition 2.3.5) for the weights of semi-invariants, we give the following definition.

Definition 6.5.1. The *$\mathbb{Z}$-support* of a vector $\beta \in \mathbb{N}^n$ is defined to be

$$\mathrm{supp}_{\mathbb{Z}}(\beta) = \{\alpha \in \mathbb{Z}^n \mid \mathrm{SI}^{\mathrm{vir}}(Q, \alpha)_{\chi_\beta} \neq 0\}.$$

As a corollary to the virtual first fundamental theorem, we have the following description of the supports of semi-invariants.

Corollary 6.5.2. *The* $\mathbb{Z}$-support *of a vector* $\beta \in \mathbb{N}^n$ *is defined to be*

$$\mathrm{supp}_{\mathbb{Z}}(\beta) = \{\alpha \in \mathbb{Z}^n \mid C_V^{\mathrm{vir}} \neq 0 \text{ on } R^{\mathrm{vir}}(\alpha) \text{ for some module } V \text{ with } \underline{\dim}\, V = \beta\}.$$

Remark 6.5.3. By Corollary 6.2.2, $C_V^{\mathrm{vir}} \neq 0$ on $R^{\mathrm{vir}}(\alpha)$ if and only if $C_V \neq 0$ on $R(\gamma_0, \gamma_1)$ for some fixed $(\gamma_0, \gamma_1) \in \mathrm{PD}(\alpha)$. Note that this is an open condition on V; therefore, if it holds for some choice of V, then it will hold for a general representation of dimension $\beta = \underline{\dim}\, V$.

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

The saturation and generalized saturation theorems describe supports of semi-invariants and virtual semi-invariants as $\mathbb{N}^n \cap D(\beta)$ and $\mathbb{Z}^n \cap D(\beta)$, respectively. The sets $D(\beta) \subset \mathbb{R}^n$ were already defined in Definition 2.3.7, but now we give a more detailed description together with a description of a particular $D(\beta)$ for Example 1.3.1.

Let $\beta \in \mathbb{N}^n$. Define $H(\beta)$ to be the hyperplane in the root space $\mathbb{R}^n$ given by

$$H(\beta) := \{\alpha \in \mathbb{R}^n \mid \langle \alpha, \beta \rangle = 0\}.$$

Let $H_+(\beta), H_-(\beta) \subseteq \mathbb{R}^n$ be the closed half-spaces given by

$$H_+(\beta) := \{\alpha \in \mathbb{R}^n \mid \langle \alpha, \beta\rangle \geq 0\}, \quad H_-(\beta) := \{\alpha \in \mathbb{R}^n \mid \langle \alpha, \beta\rangle \leq 0\}.$$

Then

$$D(\beta) = H(\beta) \cap \bigcap_{\beta' \hookrightarrow \beta} H_-(\beta').$$

Here we recall that $\beta' \hookrightarrow \beta$ means that the general representation of dimension β has a subrepresentation of dimension β'. It also has a quotient of dimension $\beta'' = \beta - \beta'$, and we write $\beta \twoheadrightarrow \beta''$. Since $\langle \alpha, \beta\rangle = \langle \alpha, \beta'\rangle + \langle \alpha, \beta''\rangle$, we see that

$$H(\beta) \cap H_-(\beta') = H(\beta) \cap H_+(\beta'').$$

Example 6.5.4. Again, we illustrate the above concepts using Example 1.3.1. Let $\beta = (0, 1, 2)^t$. Then

$$\begin{aligned} D((0,1,2)^t) &= H((0,1,2)^t) \cap H_-((0,0,1)^t) \cap H_-((0,0,2)^t) \\ &= H((0,1,2)^t) \cap H_+((0,1,0)^t) \cap H_+((0,1,1)^t). \end{aligned}$$

Therefore

$$D(\beta) = \{\alpha \in \mathbb{R}^3 \mid 2\alpha_3 = 3\alpha_2 + \alpha_1,\ \ \alpha_2 \geq \alpha_1\}.$$

The following proposition follows immediately from the definition.

Proposition 6.5.5. *The set $D(\beta)$ is a closed and convex subset of the hyperplane $H(\beta)$ for any non-zero $\beta \in \mathbb{N}^n$.* □

The saturation theorem for $\alpha \in \mathbb{Z}^n$ follows from the original saturation theorem of Derksen and Weyman (Theorem 2.3.8 in this paper) and the following lemmas.

Lemma 6.5.6. *Let $P(v)$ be an indecomposable projective and let $\beta \in \mathbb{N}^n$. Then the following conditions are equivalent.*

(i) $\beta_v = 0$.

(ii) $\underline{\dim}\, P(v) \in D(\beta)$.

(iii) $-\underline{\dim}\, P(v) \in D(\beta)$.

Proof. If $\beta_v = 0$, then $\beta'_v = 0$ for all $\beta' \hookrightarrow \beta$. The rest of the proof follows from the fact that for any indecomposable projective $P(v)$,

$$\langle \underline{\dim}\, P(v), \beta\rangle = (\underline{\dim}\, P(v))^t E\beta = (E^t \underline{\dim}\, P(v))^t\beta = (\underline{\dim}\, S(v))^t\beta = \beta_v.$$ □

Lemma 6.5.7. *If $\alpha \in D(\beta)$ and $\alpha_v < 0$, then there is a vertex w in the support of the injective envelope of $S(v)$ such that $\beta_w = 0$.*

Remark 6.5.8. We have $w \in \operatorname{supp}(I(v))$ if and only if there is a path from w to v. Since Q has no oriented cycles, this implies that $w \leq v$.

Proof. Suppose that the conclusion of the lemma does not hold; then there is a vertex v of Q where $\alpha_v < 0$ but $\beta_w > 0$ for all $w \leq v$ in the support of the injective envelope of $S(v)$, i.e. for all w having a path to v. Let v be the minimal vertex with this property. We have $\mathrm{Hom}_Q(V, I(v)) = V_v \neq 0$, so there are non-zero homomorphisms from V to $I(v)$. Let us choose such a homomorphism with image of maximal length and take L to be its image. Then $\beta \twoheadrightarrow \gamma$ where $\gamma = \underline{\dim}\, L$, and so $D(\beta) \subseteq H_+(\gamma)$. In other words, $\langle \alpha, \gamma \rangle \geq 0$. But an injective resolution of L is given by

$$0 \to L \to I(v) \to \coprod I(w_i)$$

where $w_i < v$. By minimality of v, we have $\alpha_{w_i} \geq 0$. Therefore,

$$\langle \alpha, \gamma \rangle = \alpha_v - \sum \alpha_{w_i} < 0,$$

which is a contradiction. □

Example 6.5.9. In Example 1.3.1 it can be seen that $\alpha = (-1, 0, -2)^t \in D((0, 1, 2)^t)$ but that $\alpha_1, \alpha_3 < 0$. This is possible since there is a path from $w = 1$ to $v = 3$ (and to $v = 1$). Also, $-\underline{\dim}\, P(v_1) = (-1, -1, -2)^t \in D((0, 1, 2)^t)$ since $\beta_1 = 0$.

In the following lemma, we compare the supports for semi-invariants on the classical representation space, as defined in Definition 2.3.5, with the supports of semi-invariants on presentation spaces; this will be used in the proof of the virtual saturation theorem.

Lemma 6.5.10. *Let $\beta \in \mathbb{N}^n$. Then* $\mathrm{supp}_{\mathbb{N}}(E\beta) = \mathbb{N}^n \cap \mathrm{supp}_{\mathbb{Z}}(\beta)$.

Proof. ($\mathrm{supp}_{\mathbb{N}}(E\beta) \subseteq \mathbb{N}^n \cap \mathrm{supp}_{\mathbb{Z}}(\beta)$.) Let $\alpha \in \mathrm{supp}_{\mathbb{N}}(E\beta)$; then $\alpha \in \mathbb{N}^n$ and there exists a non-zero semi-invariant p_V on $R(\alpha)$ of weight $\chi_{E\beta}$. Furthermore (by Theorem 2.3.3),

$$c_V = \det \mathrm{Hom}_Q(p_-, V) : R(\alpha) \to \Bbbk$$

for some module V with $\underline{\dim} V = \beta$, and the canonical projective presentation for $M \in R(\alpha)$ is

$$P_1 \xrightarrow{p_M} P_0 \to M \to 0.$$

Let $\gamma_0 = \underline{\dim}(P_0/\,\mathrm{rad}\, P_0)$ and $\gamma_1 = \underline{\dim}(P_1/\,\mathrm{rad}\, P_1)$. Then $p_M \in \mathrm{Hom}_Q(P(\gamma_1), P(\gamma_0)) = R(\gamma_0, \gamma_1)$ and $C_V = C_V^{(\gamma_0,\gamma_1)}$ can be viewed as a non-zero semi-invariant for the projective decomposition (γ_0, γ_1) of α. Hence $\alpha \in \mathbb{N}^n \cap \mathrm{supp}_{\mathbb{Z}}(\beta)$.

($\mathrm{supp}_{\mathbb{N}}(E\beta) \supseteq \mathbb{N}^n \cap \mathrm{supp}_{\mathbb{Z}}(\beta)$.) Conversely, let $\alpha \in \mathbb{N}^n \cap \mathrm{supp}_{\mathbb{Z}}(\beta)$ and let $C_V^{(\gamma_0,\gamma_1)}$ be a non-zero semi-invariant. Since $\alpha \in \mathbb{N}^n$, there is a projective presentation $0 \to P_1 \to P_0 \to M \to 0$ with $\underline{\dim}\, M = \alpha$, $P_1 = P(\gamma_1')$ and $P_0 = P(\gamma_0')$, where (γ_0', γ_1') is a projective decomposition of α. By stabilization, we may assume that $\gamma_0 = \gamma_0'$ and $\gamma_1 = \gamma_1'$. Since $C_V^{(\gamma_0,\gamma_1)}$ is non-zero, there exists a map $f : P_1 \to P_0$ so that $\mathrm{Hom}_Q(f, V)$ is an isomorphism. Also, $p_M : P_1 \to P_0$ is a monomorphism. Since both of these conditions are Zariski open, there exists a monomorphism $f : P_1 \to P_0$ so that $\mathrm{Hom}_Q(f, V)$ is an isomorphism. This implies that $\mathrm{Hom}_Q(M, V) = \mathrm{Ext}_Q(M, V) = 0$ for $M = P_0/f P_1$. So $\alpha = \underline{\dim}\, M \in \mathrm{supp}_{\mathbb{N}}(E\beta)$. □

Theorem 6.5.11 (Virtual saturation theorem). *Let $\beta \in \mathbb{N}^n$. Then*

$$\mathrm{supp}_{\mathbb{Z}}(\beta) = \mathbb{Z}^n \cap D(\beta).$$

Proof. We proceed in steps.

Claim 1. $\mathbb{N}^n \cap \mathrm{supp}_{\mathbb{Z}}(\beta) = \mathbb{N}^n \cap D(\beta)$.

This follows from Lemma 6.5.10 and the Derksen–Weyman saturation theorem, Theorem 2.3.8.

Claim 2. Suppose β is sincere. Then $\mathrm{supp}_{\mathbb{Z}}(\beta) = \mathbb{Z}^n \cap D(\beta)$.

Because β is sincere, Lemma 6.2.5 gives that $\mathrm{supp}_{\mathbb{Z}}(C_V) \subset \mathbb{N}^n$ whenever $\underline{\dim}\, V = \beta$. So

$$\mathrm{supp}_{\mathbb{Z}}(\beta) = \bigcup_{V \in R(\beta)} \mathrm{supp}_{\mathbb{Z}}(C_V) = \mathbb{N}^n \cap \mathrm{supp}_{\mathbb{Z}}(\beta).$$

By Lemma 6.5.10 and Theorem 2.3.8 (the classical saturation theorem),

$$\mathbb{N}^n \cap D(\beta) = \mathrm{supp}_{\mathbb{N}}(E\beta) = \mathbb{N}^n \cap D(\beta).$$

But $D(\beta) \subset \mathbb{N}^n$ by Lemma 6.5.7; so $\mathbb{N}^n \cap D(\beta) = \mathbb{Z}^n \cap D(\beta)$, which proves the claim.

For the remainder of the proof, we use the fact that $D(\beta)$ is closed under addition. If β is not sincere, let P be the sum of projective covers (cf. [ASS06, § I.5]) of all vertices not in the support of β, and let $\gamma = \underline{\dim}\, P$. Then $\gamma^t E\beta = 0$. For any α in the support of C_V, Proposition 5.1.4 implies there is an $m \geq 0$ with $\alpha + m\gamma \in \mathbb{N}^n$ which also lies in $\mathrm{supp}_{\mathbb{Z}}(C_V)$. Hence, $\alpha + m\gamma \in D(\beta)$. But Lemma 6.5.6 says that $-\gamma \in D(\beta)$. So $\alpha = (\alpha + m\gamma) + m(-\gamma) \in D(\beta)$.

Conversely, suppose that $\alpha \in D(\beta)$ with β not sincere. Then Lemmas 6.5.6 and 6.5.7 imply that $\alpha + m\gamma \in \mathbb{N}^n \cap D(\beta) = \mathrm{supp}_{\mathbb{N}}(E\beta)$ for some integer m. So there are modules $M \in R(\alpha + m\gamma)$ and $V \in R(\beta)$ such that $\mathrm{Hom}_Q(M, V) = 0 = \mathrm{Ext}^1_Q(M, V)$. Then P^m and M satisfy the conditions of Proposition 5.1.4, making $\alpha = (\alpha + m\gamma) - m\gamma$ an element of the support of C_V. $\square$

Corollary 6.5.12. *For any non-zero $\beta \in \mathbb{N}^n$, $\mathrm{supp}_{\mathbb{Z}}(\beta) = \mathrm{supp}_{\mathbb{Z}}(C_V)$ for V in a non-empty open subset of $R(\beta)$.*

Proof. By the virtual saturation theorem, $\mathrm{supp}_{\mathbb{Z}}(\beta) = \mathbb{Z} \cap D(\beta)$. By definition, $D(\beta) = H(\beta) \cap \bigcap_{\beta' \hookrightarrow \beta} H_-(\beta')$. This closed cone is the convex hull of a finite number of rays. These rays lie on intersections of transverse hyperplanes defined over $\mathbb{Q}$, so they contain elements of $\mathbb{Q}^n$ and therefore elements of $\mathbb{Z}^n$. By Remark 6.5.3, for each of these integral vectors α_i there is an open subset $U_i \subset R(\beta)$ such that $\alpha_i \in \mathrm{supp}_{\mathbb{Z}}(V)$ for all $V \in U_i$. Then, for any $V \in \bigcap U_i$, $\mathrm{supp}_{\mathbb{Z}}(V)$ contains all of the α_i and is therefore equal to $\mathbb{Z} \cap D(\beta) = \mathrm{supp}_{\mathbb{Z}}(\beta)$. $\square$

7. Simplicial complex of generalized cluster tilting sets

The collaboration leading to this paper began with the surprising discovery of combinatorial connections between cluster tilting objects and cluster categories [BMRRT06], the supports of semi-invariants [DW00], and chain resolutions of nilpotent groups [IO01]. In this section we begin to unveil these connections, after reminding the reader of some basic results and definitions concerning cluster categories in § 7.1. In § 7.2 we relate objects in cluster categories to Schur roots and negative projective roots. With these connections established, in § 7.4 we construct a continuous monomorphism from a subcomplex of virtual semi-tilting sets onto a dense subset of the $(n-1)$-sphere.

In the next section, at last, we restrict our considerations from general quivers to Dynkin quivers, and prove that in this special case the above continuous monomorphism is a homeomorphism, providing a simplicial decomposition of the sphere with codimension-one skeleta given by the domains of the virtual semi-invariants.

7.1 Cluster categories and cluster tilting objects

Let Q be a finite quiver with no oriented cycles. The associated cluster category $\mathcal{C}_Q$ was defined in [BMRRT06] as a special orbit category of the associated bounded derived category $\mathcal{D}^b_Q$ in the following way.

Let $\operatorname{mod} \Bbbk Q$ be the category of finitely generated modules over the path algebra $\Bbbk Q$, and let τ be the Auslander–Retien translation functor. Since $\Bbbk Q$ is hereditary, τ is a functor $\operatorname{mod} \Bbbk Q \to \operatorname{mod} \Bbbk Q$ which induces an equivalence of full subcategories:

$$\{\Bbbk Q\text{-modules without projective summands}\}$$
$$\xrightarrow{\tau} \{\Bbbk Q\text{-modules without injective summands}\}.$$

An important fact is that the Auslander–Reiten functor can be extended to an auto-equivalence of the associated derived category $\mathcal{D}^b_Q$, which we will describe now.

Let $\mathcal{D}^b_Q := \mathcal{D}^b(\operatorname{mod} \Bbbk Q)$ be the derived category of bounded complexes in $\operatorname{mod} \Bbbk Q$. Instead of recalling the general definition of the derived categories, we will describe objects and morphisms, which is quite easy since the algebra $\Bbbk Q$ is hereditary: the indecomposable complexes are isomorphic to stalk complexes, hence all indecomposable objects can be described as shifts of the indecomposable modules:

$$\operatorname{ind} \mathcal{D}^b_Q = \bigcup_{i\in\mathbb{Z}} (\operatorname{ind} \Bbbk Q)[i].$$

The morphisms in $\mathcal{D}^b_Q$ can also be easily described: for all $M, N \in \operatorname{mod} \Bbbk Q$,

$$\operatorname{Hom}_{\mathcal{D}^b_Q}(M, N) = \operatorname{Hom}_Q(M, N), \quad \operatorname{Hom}_{\mathcal{D}^b_Q}(M, N[1]) = \operatorname{Ext}^1_Q(M, N),$$
$$\operatorname{Hom}_{\mathcal{D}^b_Q}(M, N[i]) = 0 \quad \text{for } i \neq 0, 1,$$
$$\operatorname{Hom}_{\mathcal{D}^b_Q}(X, Y) = \operatorname{Hom}_{\mathcal{D}^b_Q}(X[i], Y[i]) \quad \text{for } i \in \mathbb{Z} \text{ and } X, Y \in \mathcal{D}^b_Q.$$

Let $\mathcal{D}^b_Q \xrightarrow{\tau} \mathcal{D}^b_Q$ be the automorphism of the category induced by the Auslander–Reiten translation functor, which we shall also call the Auslander–Reiten, or AR, functor. Then the composition functor $\mathcal{D}^b_Q \xrightarrow{[1]} \mathcal{D}^b_Q \xrightarrow{\tau^{-1}} \mathcal{D}^b_Q$ is an auto-equivalence of D^b_Q.

Definition 7.1.1 [BMRRT06]. *The cluster category* $\mathcal{C}_Q$ for a quiver Q is the orbit category

$$\mathcal{C}_Q := \mathcal{D}^b_Q/(\tau^{-1}[1])$$

of the derived category $\mathcal{D}^b_Q$, under the action of $\tau^{-1}[1]$.

Remark 7.1.2. A set of representatives of the indecomposable $\mathcal{C}_Q$ objects, which are $(\tau^{-1}[1])$-orbits, may be chosen to be in

$$\operatorname{ind} \Bbbk Q \cup \{P(v)[1]\}_{v\in Q_0},$$

the set of indecomposable $\Bbbk Q$-modules and shifts $P(v)[1]$ of the indecomposable projective $\Bbbk Q$-modules $P(v)$.

Some particularly important objects in the cluster category are the *cluster tilting objects*, which are essential in the 'cluster algebra/cluster category' relations. Their definition extends the

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

classical definition [HU89, Ung96] of a *tilting module* as a module $T = \coprod_{i=1}^{n} T_i$ of non-isomorphic indecomposable modules T_i such that $\mathrm{Ext}(\coprod T_i, \coprod T_i) = 0$.

Definition 7.1.3. An object $T = \coprod_{i=1}^{n} T_i$ of the cluster category $\mathcal{C}_Q$ is called a *cluster tilting object* if $\mathrm{Ext}^1_{\mathcal{C}_Q}(T, T) = 0$, where the T_i are indecomposable and pairwise non-isomorphic.

7.2 A relation between objects of cluster categories and integral vectors

To each object of $\mathcal{C}_Q$ which has a representative in the module category mod $\Bbbk Q$ we associate the dimension vector $\underline{\dim}\, M \in \mathbb{N}^n \times 0 \subset \mathbb{Z}^n \times \mathbb{Z}$, and to each object of $\mathcal{C}_Q$ which has a representative shifted projective we associate the vector $(\underline{\dim}\, P)[1] \in \mathbb{N}^n \times 1 \subset \mathbb{Z}^n \times \mathbb{Z}$.

Definition 7.2.1. A *Schur representation* is a $\Bbbk Q$-module M such that $\mathrm{End}_Q(M) = \Bbbk$.

To translate from cluster category objects to integral vectors, first we make the translation from modules to Schur roots by using the following theorem.

THEOREM 7.2.2 [Kac82]. *A vector $\alpha \in \mathbb{N}^n$ is a Schur root if and only if there exists a Schur representation M with $\underline{\dim}\, M = \alpha$.*

Remark 7.2.3.

(i) The Auslander–Reiten functor τ on the level of dimension vectors, $\tau : \mathbb{Z}^n \cong \mathbb{Z}^n$, is given by $\tau = -E^{-1}E^t$.

(ii) $\langle \alpha, \beta \rangle = \beta^t E^t \alpha = -\langle \beta, \tau\alpha \rangle$.

(iii) $\langle \tau\alpha, \tau\beta \rangle = \langle \alpha, \beta \rangle$.

(iv) By the properties of the translation functor, this linear map sends Schur roots and negative projective roots to Schur roots and negative injective roots.

7.3 Virtual semi-tilting sets

In this section we consider only the dimension vectors of certain indecomposable modules and shifted indecomposable projective modules, which form a subset of the representatives of the indecomposable objects of cluster category $\mathcal{C}_Q$. Specifically, we consider Schur roots and shifted projective roots $p(v)[1] = (E^t)^{-1}(e_v)[1]$.

We prove the necessary corollaries to the generic decomposition theorem in order to construct the tilting triangulation of § 8 and exhibit its properties.

Definition 7.3.1. A *partial virtual semi-tilting set* for a quiver Q with n vertices is a collection of distinct Schur roots and shifted indecomposable projective roots,

$$\{\beta_1, \ldots, \beta_k, p(v_{k+1})[1], \ldots, p(v_m)[1]\},$$

with $\mathrm{ext}_Q(\beta_i, \beta_j) = 0$ and $\hom_Q(p(v_i), \beta_j) = (\beta_j)_{v_i} = 0$ for all $i \neq j$.

A *virtual semi-tilting set* is a partial virtual semi-tilting set with n elements.

Remark 7.3.2. We point out some similarities and differences between cluster tilting objects and virtual semi-tilting sets.

- Both cluster tilting objects and virtual semi-tilting sets may include shifted projectives (which is not the case for the classical tilting modules).
- Both cluster tilting objects and virtual semi-tilting sets must have all of their indecomposable components not extend each other.

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

- A module M and a shifted projective $P[1]$ do not extend each other if and only if $\mathrm{Hom}_Q(P, M) = 0$. This agrees with $\hom_Q(p(v_i), \beta_j) = 0$, the same condition as in the definition of partial virtual semi-tilting sets, Definition 7.3.1.
- The prefix *semi-* is used to emphasize that $\mathrm{ext}_Q(\beta_i, \beta_i)$ may be non-zero, whereas for any tilting or cluster tilting object we have $\mathrm{Ext}_{\mathcal{C}_Q}(T_i, T_i) = 0$.

Example 7.3.3. Some examples of virtual semi-tilting sets are:

- the set of indecomposable projective roots;
- the shifted projective roots;
- the roots corresponding to the indecomposable injective modules.

Also:

- each null root for an extended Dynkin diagram forms a partial virtual semi-tilting set, although no module of that dimension can be a partial cluster tilting object;
- in Example 1.3.1, the sets of roots $\{(-1, -1, -2), (0, 1, 1)\}$ and $\{\alpha = (1, 2, 2)\}$ form virtual semi-tilting sets, actually maximal virtual semi-tilting sets; note that $\mathrm{ext}(\alpha, \alpha) \neq 0$.

Remark 7.3.4. Note that: (a) every subset of a virtual semi-tilting set is a partial virtual semi-tilting set; thus (b) the partial virtual semi-tilting sets form a simplicial complex.

Recall that a *simplicial complex* is a collection K of finite non-empty sets δ, called *simplices*, such that any non-empty subset of a simplex is also a simplex. A simplex of K with $p+1$ elements is called a *p-simplex* of K, and the set of p-simplices of K is denoted by K_p.

7.4 Complex of virtual semi-tilting sets

We will construct a simplicial complex $\mathcal{T}(Q)$ and a subcomplex $\mathcal{T}'(Q)$. We will see that $\mathcal{T}(Q)$ is $(n-1)$-dimensional, where n is the number of vertices of Q, and that there is a continuous mapping of the geometric realization of $\mathcal{T}(Q)$ to the standard $(n-1)$-sphere S^{n-1}. When restricting to the subcomplex $\mathcal{T}'(Q)$ we will get a continuous monomorphism

$$\lambda : |\mathcal{T}'(Q)| \to S^{n-1}$$

whose image is dense. This implies that if $\mathcal{T}'(Q)$ is a finite simplicial complex, then λ is a homeomorphism.

Definition 7.4.1. Let Q be a quiver without oriented cycles. The *complex of virtual semi-tilting sets*, $\mathcal{T}(Q)$, is the simplicial complex whose simplices are the partial virtual semi-tilting sets of the Schur roots and the shifted indecomposable roots.

We will use Theorem 6.3.1 and the following result of Schofield to show that this simplicial complex is $(n-1)$-dimensional.

THEOREM 7.4.2 [Sch92]. *Let Q be a quiver with n vertices and no oriented cycles.*

(i) *Any multiple $m\alpha$ of a Schur root α either is a Schur root or decomposes generically as a sum of m copies of α.*

(ii) *If $\alpha = \sum \beta_i$ is a generic decomposition of $\alpha \in \mathbb{N}^n$, then $m\alpha = \sum (m\beta_i)$ is the generic decomposition of $m\alpha$, where $(m\beta_i)$ denotes either a single Schur root or a sum of m copies of β_i in the case where $m\beta_i$ is not a Schur root.*

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

Proposition 7.4.3. *If $\{\beta_1, \dots, \beta_k, p(v_{k+1})[1], \dots, p(v_m)[1]\}$ is a partial virtual semi-tilting set, then the corresponding subset $\{\beta_1, \dots, \beta_k, -p(v_{k+1}), \dots, -p(v_m)\} \subset \mathbb{Z}^n$ is linearly independent over $\mathbb{Q}$. In particular, it has at most n elements.*

Proof. Any rational linear relation on the vectors $p(v_i)$ and β_j gives an integral linear relation upon multiplication by the common denominators of the rational coefficients. Collecting terms with positive and negative coefficients, we get an equation of the form

$$\alpha = \sum (n_i\beta_i) - (E^t)^{-1}\gamma = \sum (m_j\beta_j) - (E^t)^{-1}\gamma'$$

where $n_i, m_j \geq 0$ and γ, γ' have support disjoint from any of the β_i, β_j. This gives two different generic decompositions of the same dimension vector, contradicting Theorem 6.3.1. To see that these are generic decompositions of α, we use Theorem 7.4.2 and the observation that $\text{ext}_Q(m_i\beta_i, m_j\beta_j) \leq m_i m_j \text{ext}_Q(\beta_i, \beta_j) = 0$. □

Corollary 7.4.4. *If Q has n vertices, then the simplicial complex $\mathcal{T}(Q)$ is $(n-1)$-dimensional.*

Let K be a simplicial complex. Let K_0 be the vertex set, that is, the set of 0-simplices. The *geometric realization* $|K|$ of K is defined to be the subspace of the infinite-dimensional vector space $\mathbb{R}^{K_0}$ consisting of all vectors $x = \sum t_i v_i$, $t_i \in [0, 1]$ and $v_i \in K_0$, with the property that $\sum t_i = 1$ and set of all vertices v_i having non-zero coefficient is a simplex $\delta \in K$. For a fixed $\delta \in K$, the set of all such x, i.e. all $x \in |K|$ which are positive linear combinations of the vertices of δ, is called the *open simplex* e^δ of δ. Note that $|K|$ is by definition a disjoint union of open simplices. The closure of an open simplex is the *closed simplex* Δ^δ of δ, which is the set of all $x \in |K|$ that are non-negative linear combinations of the vertices of δ.

If K_0 is finite, we take the usual Euclidean topology on $\mathbb{R}^{K_0}$; when it is infinite, we take the *weak topology* which is the direct limit of all $\mathbb{R}^S$ where S runs over all finite subsets of K_0. This is the weakest topology (i.e. the one having the fewest open sets) on $|K|$ with the property that a mapping $\lambda : |K| \to X$ is continuous if and only if it is continuous on every closed simplex Δ^δ.

Let $\Lambda : \mathcal{T}_0(Q) \to \mathbb{R}^n$ be the mapping which sends each Schur root β to itself and $p[1]$ to $-p$. Then Λ extends to a continuous mapping $|\Lambda| : |\mathcal{T}(Q)| \to \mathbb{R}^n$ given by

$$|\Lambda|\Big(\sum t_j\beta_j + \sum t_i p(v_i)[1]\Big) = \sum t_j\beta_j - \sum t_i p(v_i).$$

The proposition above implies that 0 is not in the image of this mapping. Therefore, we can normalize to get a continuous mapping $\lambda : |\mathcal{T}(Q)| \to S^{n-1}$,

$$\lambda\Big(\sum t_j\beta_j + \sum t_i p(v_i)[1]\Big) := \frac{\sum t_j\beta_j - \sum t_i p(v_i)}{\|\sum t_j\beta_j - \sum t_i p(v_i)\|}.$$

We would like this mapping to be a monomorphism. However, if there are Schur roots $k\beta$ and $m\beta$ which are multiples of the same $\beta \in \mathbb{N}^n$, then clearly $\lambda(k\beta) = \lambda(m\beta)$; so λ may not be a monomorphism. To remedy this, we restrict to a subcomplex $\mathcal{T}'(Q)$ of $\mathcal{T}(Q)$, which we now define.

We say that a Schur root β is *minimal* if its coefficients are relatively prime, i.e. if β is not a positive integer multiple of another vector in $\mathbb{N}^n$. By Schofield's theorem mentioned above, every Schur root is a multiple of a minimal Schur root, since any decomposition of β would result in a decomposition of $m\beta$.

The following corollary is a consequence of the virtual generic decomposition theorem.

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

Corollary 7.4.5. *Every non-zero vector $x \in \mathbb{Q}^n$ can be written uniquely as a linear combination*

$$x = \sum x_j \beta_j - \sum x_i p(v_i)$$

where $x_i, x_j > 0$ are positive rational numbers, the β_j are minimal Schur roots and the $p(v_i)[1]$ are shifted indecomposable projective roots forming a partial virtual semi-tilting set.

Proof. Multiply by a sufficiently large integer k to get $kx \in \mathbb{Z}^n$. Then apply Theorem 6.3.1 to kx to obtain a generic decomposition

$$kx = \sum (m_j \beta_j) - \sum m_i p(v_i),$$

where we have collected repeating factors β_j using Schofield's notation. Next, write each summand $(m_j \beta_j)$ as a multiple of a minimal Schur root. Then divide by k to get the desired rational linear decomposition of x.

To prove uniqueness, suppose we have two rational decompositions of x. Then we get integer decompositions of, say, kx and mx, giving two generic decompositions of mkx (using Theorem 7.4.2), which is a contradiction. □

Theorem 7.4.6. *The restriction of $\lambda : |\mathcal{T}(Q)| \to S^{n-1}$ to the subcomplex $\mathcal{T}'(Q)$ of $\mathcal{T}(Q)$ consisting of all simplices whose vertices are either minimal Schur roots or shifted indecomposable projective roots gives a continuous mapping*

$$\lambda' : |\mathcal{T}'(Q)| \to S^{n-1}$$

which is a monomorphism whose image is dense in the standard Euclidean topology on S^{n-1}.

Proof. The uniqueness statement of Corollary 7.4.5 implies that λ' is a monomorphism. The existence part of Corollary 7.4.5 implies that the image of λ' contains the image of $\mathbb{Q}^n - \{0\}$ under the normalization map $\cdot/||\cdot|| : \mathbb{Q}^n - \{0\} \to S^{n-1}$ whose image is dense. □

8. Semi-invariants and the cluster tilting triangulation associated to a Dynkin quiver

Until now, Q has been an arbitrary quiver without oriented cycles. We now assume that Q is a Dynkin quiver, i.e. a simply laced Dynkin diagram with any orientation of its edges. We define the *cluster tilting triangulation* associated to a Dynkin diagram, which triangulates the sphere via the complex of cluster tilting sets, and show that the supports of the semi-invariants of the quiver constitute the codimension-one skeleton.

8.1 Cluster tilting triangulation

Since Q is Dynkin, the set of Schur roots equals the set of positive roots Φ_+ of the Euler form. We also recall that if β is a positive root, then there is a unique indecomposable module of dimension β up to isomorphism, and the set of all elements of $R(\beta)$ isomorphic to this module is open. The Schur roots are all minimal; they and the shifted projectives form a finite set, which we denote by Φ'_+.

Definition 8.1.1. The *cluster tilting complex* of the Dynkin diagram Q is defined to be the simplicial complex $\mathcal{T}(Q) = \mathcal{T}'(Q)$ with vertex set Φ'_+ such that the faces of $\mathcal{T}(Q)$ are the virtual semi-tilting sets.

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press

Theorem 8.1.2. *For a Dynkin quiver, the geometric realization of the cluster tilting complex is homeomorphic to the $(n-1)$-sphere. Furthermore, a homeomorphism $\lambda : |\mathcal{T}(Q)| \to S^{n-1}$ is given by*

$$\lambda\Big(\sum t_j\beta_j + \sum t_i p(v_i)[1]\Big) := \frac{\sum t_j\beta_j - \sum t_i p(v_i)}{\|\sum t_j\beta_j - \sum t_i p(v_i)\|}.$$

Proof. Since all Schur roots are minimal, $\mathcal{T}'(Q) = \mathcal{T}(Q)$. Therefore, Corollary 7.4.6 applies to $\lambda = \lambda'$, showing that $\lambda : |\mathcal{T}(Q)| \to S^{n-1}$ is a continuous monomorphism with dense image. However, $\mathcal{T}(Q)$ is a finite complex; so $|\mathcal{T}(Q)|$ is compact. This means that λ is a homeomorphism onto its image, which must then also be compact and, therefore, equal to all of S^{n-1}. □

In any triangulation of a closed manifold, a codimension-one simplex is a face of exactly two simplices of maximal dimension. This gives the following corollary, which is a special case of a theorem from [BMRRT06].

Corollary 8.1.3. *In the Dynkin case, any almost complete generalized cluster tilting set of roots is contained in exactly two complete virtual tilting sets.*

In the finite case, the supports of the semi-invariants are easy to describe.

Lemma 8.1.4. *Let α and β be positive roots of the Dynkin quiver Q. Then $\alpha \in D(\beta)$ if and only if $\langle \alpha, \beta\rangle = 0$.*

Proof. If $\alpha \in D(\beta)$, then $\langle \alpha, \beta\rangle = 0$ by definition. Conversely, if $\langle \alpha, \beta\rangle = 0$, then $\hom_Q(\alpha, \beta) = \operatorname{ext}_Q(\alpha, \beta) = 0$ since they cannot both be non-zero. This implies that $\hom_Q(\alpha, \beta') = 0$ for any $\beta' \hookrightarrow \beta$. So $\langle \alpha, \beta' \rangle \le 0$ and $\alpha \in D(\beta)$. □

Theorem 8.1.5. *Let β be a positive root of Q. Then $D(\beta) \subset \mathbb{R}^n$ is the set of all non-negative real linear combinations of positive roots α such that $\langle \alpha, \beta\rangle = 0$ and negative projective roots $-p(v_i)$ such that $\langle p(v_i), \beta\rangle = \beta_{v_i} = 0$.*

Proof. Since $D(\beta)$ is given by homogeneous linear equations and inequalities with integer coefficients, it suffices to prove the theorem in $\mathbb{Z}^n$ instead of $\mathbb{R}^n$. So let $\alpha \in \mathbb{Z}^n \cap D(\beta)$. By Theorem 6.5.11 and its corollary, this set is the same as the support of C_V where V is the unique indecomposable representation with dimension β. Proposition 5.1.4 implies that there is a module M and a projective module P over $\Bbbk Q$ with $\underline{\dim}\, M - \underline{\dim}\, P = \alpha$ and such that $\operatorname{Hom}_Q(M, V) = \operatorname{Ext}^1_Q(M, V) = 0$ and $\operatorname{Hom}_Q(P, V) = 0$. But then the same holds for all indecomposable direct summands M_i of M and all indecomposable direct summands $-P_j$ of $-P$. Thus the corresponding roots α_i lie in $D(\beta)$. So $\alpha = \sum \underline{\dim}\, M_i - \sum \underline{\dim}\, P_j$ is a positive linear combination of the required roots. □

Corollary 8.1.6. *For any $x \in D(\beta)$ there is a virtual semi-tilting set $\{\alpha_j\}$, all of whose elements lie in $D(\beta)$, such that x is a non-negative linear combination of the α_j.*

Proof. Since the set of all x satisfying this condition is closed, it suffices to show that it holds for a dense subset of $D(\beta)$. So, we may assume that $x \in \mathbb{Q}^n$. Upon multiplying by the denominator, we may assume that $x \in \mathbb{Z}^n$. Now, repeat the last step of the proof of Theorem 8.1.5. □

Our main theorem identifies the codimension-one skeleton of $\mathcal{T}(Q)$ with the supports of semi-invariants.

Theorem 8.1.7. *The image in S^{n-1} of the $n-2$ skeleton of $\mathcal{T}(Q)$ under the homeomorphism λ is the union of supports of semi-invariants:*

$$\lambda(|\mathcal{T}(Q)^{n-2}|) = \bigcup_{\beta\in\Phi_+} D(\beta)\cap S^{n-1}.$$

Proof. The statement is equivalent to saying that $\bigcup D(\beta)$ is equal to the union of rays emanating from 0 and passing through the $n-2$ skeleton of $\mathcal{T}(Q)$. Corollary 8.1.6 implies that each $D(\beta)$ is contained in this union of rays. To prove the converse, it suffices to show that any partial tilting set $\{\alpha_1, \ldots, \alpha_{n-1}\}$ is contained in $D(\beta)$ for some positive root β; by Lemma 8.1.4, this is equivalent to saying that $\langle\alpha_i, \beta\rangle = 0$ for each α_i. This is trivially true when $n = 1$. So let us assume that $n \geq 2$ and that the statement holds for all Dynkin quivers with fewer vertices.

Suppose, for the moment, that one of the α_i is a shifted projective $p(v)[1]$. If v is minimal, then all other roots α_j have support disjoint from v. So we can delete the vertex v and delete the root $p(v)[1]$ to obtain a partial tilting set on Q', the subquiver of Q, by deleting the vertex v and all arrows to and from v. Using a counting argument, we see that this consists of a partial tilting set on one of the components of Q' and a complete tilting set on the other components. By induction on n, there is a positive root β on the first component so that $\langle|\alpha_i|, \beta\rangle = 0$ for all the i. This gives the desired root for Q and proves the theorem in this particular case.

If none of the α_i is a shifted projective, we use the inverse translation τ^{-1}. Let $m > 0$ be the minimal value such that at least one $\tau^{-m}\alpha_i$ is a shifted projective. By the previous case, there is a positive root β such that $\langle|\tau^{-m}\alpha_i|, \beta\rangle = 0$ for all i. Then, $\langle\alpha_i, |\tau^m\beta|\rangle = 0$ for all i and so $\{\alpha_i\} \subset D(|\tau^m\beta|)$ as claimed. □

Acknowledgement

The authors thank the referee, whose comments have significantly improved the exposition in this paper.

References

ASS06 I. Assem, D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras. Volume 1: techniques of representation theory*, London Mathematical Society Student Texts, vol. 65 (Cambridge University Press, Cambridge, 2006).

BMRRT06 A. B. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), 572–618.

CFZ02 F. Chapoton, S. Fomin and A. Zelevinsky, *Polytopal realizations of generalizad associahedra*, Canad. Math. Bull. **45** (2002), 537–566.

DW00 H. Derksen and J. Weyman, *Semi-invariants of quivers and saturation for Littlewood–Richardson coefficients*, J. Amer. Math. Soc. **13** (2000), 467–479.

DW02 H. Derksen and J. Weyman, *On canonical decomposition for quiver representations*, Compositio. Math. **133** (2002), 245–265.

HU89 D. Happel and L. Unger, *Almost complete tilting modules*, Proc. Amer. Math. Soc. **107** (1989), 603–610.

IO01 K. Igusa and K. Orr, *Links, pictures and the homology of nilpotent groups*, Topology **40** (2001), 1125–1166.

Kac82 V. G. Kac, *Infinite root systems, representations of graphs and invariant theory. II*, J. Algebra **78** (1982), 141–162.

Sch91 A. Schofield, *Semi-invariants of quivers*, J. London Math. Soc. **43** (1991), 383–395.

Sch92 A. Schofield, *General Representations of quivers*, Proc. London Math. Soc. (3) **65** (1992), 46–64.

Ste64 R. Steinberg, *Lectures on Chevalley groups*, Yale University Lecture Notes (1964).

Ung96 L. Unger, *The simplicial complex of tilting modules over quiver algebras*, Proc. London Math. Soc (3) **73** (1996), 27–46.

Kiyoshi Igusa igusa@brandeis.edu
Department of Mathematics, Brandeis University, 415 South Street, Waltham, MA 02454-9110, USA

Kent Orr korr@indiana.edu
Department of Mathematics, Indiana University, 831 E. 3rd Street, Bloomington, IN 47405-7106, USA

Gordana Todorov todorov@neu.edu
Department of Mathematics, Northeastern University, 360 Huntington Avenue, Boston, MA 02115, USA

Jerzy Weyman j.weyman@neu.edu
Department of Mathematics, Northeastern University, 360 Huntington Avenue, Boston, MA 02115, USA

https://doi.org/10.1112/S0010437X09004151 Published online by Cambridge University Press