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CLUSTER SETS FOR A GENERALIZED LAW OF THE ITERATED LOGARITHM IN BANACH SPACES

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We identify the possible cluster sets for a general law of the iterated logarithm in the Banach space setting, and show that all the possible limit sets arise as cluster sets for some random vector in an arbitrary separable Banach space. This extends previous results obtained in finite dimensional Euclidean spaces.

1. Introduction. Let B denote a real separable Banach space with norm $\|\cdot\|$, and assume $X, X_1, X_2, \ldots$ are i.i.d. B-valued random vectors with $0 < E\|X\| < \infty$ and $E(X) = 0$. As usual, let $S_n = \sum_{j=1}^{n} X_j$ for $n \geq 1$, and write Lt to denote $\log(\max(t, e))$ for $t \geq 0$. The function $L(Lt)$ will be written as $L_2 t$, and B^* denotes the topological dual of B. For any sequence $\{x_n\} \subseteq B$, the set of its limit points is denoted by $C(\{x_n\})$, and is called the cluster set of $\{x_n\}$.

The separability of B and the Hewitt-Savage zero-one law easily imply that for any sequence $\alpha_n \uparrow \infty$, with probability one,

$$C(\{S_n/\alpha_n\}) = A, \tag{1.1}$$

where A is non-random and depends only on $\{\alpha_n\}$ and the distribution of X. See, for example, Lemma 1 in Kuelbs (1981). Of course, if $S_n/\alpha_n \to 0$ a.s., then $A = \{0\}$ and determining the cluster set is trivial. This is no longer the case if one considers sequences $\{\alpha_n\}$ such that with probability one

$$0 < \limsup_{n\to\infty} \|S_n/\alpha_n\| < \infty. \tag{1.2}$$

The classical choice for $\{\alpha_n\}$ in (1.2) is $\{(2nL_2 n)^{1/2}\}$, and the corresponding cluster set problem has been studied extensively. The papers by Alexander, (1989a) and (1989b), provide some final results in this particular case, and also include relevant references. The important fact to be observed here is that the classical norming sequence arises only when the covariance function $E(f(X)g(X))$ is defined on $B^* \times B^*$, and the cluster set A then must be a subset of a canonical set K, which is the unit ball of a reproducing kernel Hilbert space determined by the covariance structure of X. In particular, A is empty or equal to ρK for some $\rho \in [0, 1]$ when X has a covariance function and $\alpha_n = (2nL_2 n)^{1/2}$. Such variety does not appear in every Banach space, but in suitable infinite dimensional spaces there are examples giving all such sets as cluster sets.

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If X fails to have a covariance function, but X is in the domain of attraction of a Gaussian law, then the cluster set is determined by the covariance of the limiting Gaussian law. Hence the cluster set remains canonical even though the normalizing constants $\{\alpha_n\}$ differ from the classical ones, see Kuelbs (1985) and Einmahl (1989). However, much less is known when there is no covariance structure available. Part of the difficulty in this situation is to decide what are the natural candidates for A. When B is a finite dimensional space, and X satisfies the general law of the iterated logarithm in Theorem 3 of Einmahl (1993), then Einmahl (1995) showed every cluster set A must be symmetric about zero, star shaped with respect to the origin, closed, and bounded. Moreover, any cluster set must contain elements with norm 1 in this case. (This is due to the fact that the unit ball is compact in finite-dimensional vector spaces.) Theorem 4 of Einmahl(1995) also shows that any such set in $\mathbb{R}^2$ arises as a cluster set for some random vector $X : \Omega \to \mathbb{R}^2$.

Here we investigate this cluster set problem in the infinite dimensional setting, where we encounter the additional difficulty that we no longer can assume that there are points in $A = C(\{S_n/\alpha_n\})$ which have norm 1. It will turn out that there are cluster sets which are star-like and symmetric about the origin where all elements have norms smaller than or equal to $\rho < 1$, though we still have at the same time $\limsup_{n\to\infty} ||S_n/\alpha_n|| = 1$.

To state things precisely we need some further notation. Following Klass (1976), we associate with any real-valued random variable ξ satisfying $0 < E|\xi| < \infty$ a function $K(\cdot)$ which is defined as the inverse function of the strictly increasing function $G(\cdot)$, given for $y > 0$ by

$$G(y) = y^2 / \int_0^y E(|\xi| I(|\xi| > t))dt.$$

Since $E\|X\| < \infty$ we have $E|f(X)| < \infty$ for all $f \in B^*$, and for any $f \in B^*$ with $E|f(X)| > 0$, let K_f be the K-function corresponding to the real-valued random variable $f(X)$. For $y > 0$, define

$$\tilde{K}(y) = \sup\{K_f(y) : \|f\|_{B^*} \le 1,\ E|f(X)| > 0\}, \tag{1.3}$$

and set for $n \ge 1$,

$$\gamma_n = \sqrt{2}\tilde{K}(n/L_2 n)L_2 n. \tag{1.4}$$

Our theorems characterizing the cluster set follow.

THEOREM 1. *Suppose X is a B-valued mean zero random vector such that $0 < E\|X\| < \infty$ and $\{\gamma_n\}$ is as in* (1.4). *Furthermore, assume*

$$S_n/\gamma_n \xrightarrow{\text{prob}} 0 \tag{1.5}$$

and

$$\sum_{n=1}^{\infty} P(\|X\| > \gamma_n) < \infty. \tag{1.6}$$

Then with probability one,

$$\limsup_{n\to\infty} \|S_n/\gamma_n\| = 1, \tag{1.7}$$

and the cluster set $A = C(\{S_n/\gamma_n\})$ *is such that*

$$A \text{ is closed, non-empty, symmetric about zero} \tag{1.8}$$

and

$$A \text{ is star-like at zero.} \tag{1.9}$$

REMARK. The lim sup in (1.7) was obtained in Theorem 3 of Einmahl (1993) under (1.5) and (1.6). If B is a type-2 Banach space, then Corollary 2 of Einmahl (1993) shows (1.7) is equivalent to (1.6). Of course, (1.7) is an extension of the classical LIL even in finite dimensional spaces, and covers many new situations as well. Einmahl (1993) discusses the various extensions and relationships to previous work, but it is important for us to point out that in all these situations we now know the possible cluster sets must satisfy (1.8) and (1.9).

Our next theorem points out that all such sets arise in every *infinite-dimensional* separable Banach space.

THEOREM 2. *Let $\tilde{A}$ be a closed non-empty subset of the unit ball of B which is symmetric and star-shaped with respect to zero. Then there exists a random vector $X : \Omega \to B$ which satisfies the conditions for the generalized LIL, namely $0 < E\|X\| < \infty$, and both* (1.5) *and* (1.6) *hold, and with probability one,*

$$A = C(\{S_n/\gamma_n\}) = \tilde{A}. \tag{1.10}$$

Moreover, if $\tilde{A} \subseteq V$ where V is an infinite dimensional closed subspace of B we can construct X so that

$$P(X \in V) = 1. \tag{1.11}$$

If $\sup_{x\in\tilde{A}} \|x\| = 1$, *statement* (1.11) *remains true for any* (*possibly finite-dimensional*) *closed subspace $W$of B with $\tilde{A} \subseteq W$.*

REMARK. If B is finite-dimensional, there exists a linear, isometric embedding of B into $(C[0, 1], \|\cdot\|_\infty)$, and using the above theorem one can re-obtain the cluster set result for finite dimensional spaces (i.e., that any set $\tilde{A}$ which is closed, star-like and symmetric about zero and satisfies $\sup_{x\in\tilde{A}} \|x\| = 1$ is a possible cluster set). See also Step 5 of the proof of Theorem 2.

2. Proof of Theorem 1. The proof consists of three parts, and includes several lemmas. The first part consists of a lemma which characterizes points that are in the cluster set $C(\{S_n/\gamma_n\})$, the second part provides probability estimates necessary to apply the lemma, and part three puts things together.

STEP 1. First we point out that (1.7) follows immediately from Theorem 3 in Einmahl (1993), and that (2.3) and (2.4) of Einmahl (1993) implies the sequence $\{\gamma_n\}$ satisfies the following two conditions:

$$\gamma_n/\sqrt{n} \nearrow \tag{2.1}$$

and

$$\gamma_n/n \searrow 0. \tag{2.2}$$

Applying (2.1), and Lemma 1 in Kuelbs (1981), it follows that the cluster set $A = C(\{S_n/\gamma_n\})$ is deterministic with probability one, and Lemma 1 of Einmahl (1995) implies that

$$b \in A \iff \sum_{n=1}^{\infty} \frac{1}{n} P(\|S_n/\gamma_n - b\| < \varepsilon) = \infty \qquad \forall\, \varepsilon > 0. \tag{2.3}$$

The next lemma shows the right hand term in (2.3) can be replaced by a probability involving suitably truncated random variables.

Let $X_{n,j} = X_j I(\|X_j\| \le c_n)$, $\overline{X}_{n,j} = X_{n,j} - E(X_{n,j})$ and $S_{n,n} = \sum_{j=1}^{n} \overline{X}_{n,j}$, where the truncation level c_n will be specified below.

LEMMA 1. *The point b is in the cluster set*

$$A \iff \sum_{n=1}^{\infty} \frac{1}{n} P(\|S_{n,n}/\gamma_n - b\| < \varepsilon) = \infty \qquad \forall\, \varepsilon > 0, \tag{2.4}$$

provided that

$$c_n = \tilde{K}(n/L_2 n)/(Ln)^{\varepsilon_n}, \tag{2.5}$$

with $\{\varepsilon_n\}$ converging to zero.

PROOF. In view of (2.3) it is obviously enough to show

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\|S_n - S_{n,n}\| > \varepsilon\gamma_n) < \infty \qquad \forall\, \varepsilon > 0. \tag{2.6}$$

To prove (2.6) we first observe that

$$\begin{aligned} &P(\|S_n - S_{n,n}\| > \varepsilon\gamma_n) \\ &\quad \le P\left(\left\|\sum_{j=1}^{n} Z_{n,j} + nE(X_{n,1})\right\| > \varepsilon\gamma_n\right) + nP(\|X\| > \gamma_n), \end{aligned} \tag{2.7}$$

where $Z_{n,j} = X_j I(c_n < \|X_j\| \le \gamma_n)$, $1 \le j \le n$.

Since (1.6) and (2.1) hold, Lemma 7 of Einmahl (1993) implies

$$\lim_{n\to\infty} nE(\|X\|I(\|X\| > \gamma_n))/\gamma_n = 0. \tag{2.8}$$

Using the obvious fact that $E(X_{n,1}) = -(E(Z_{n,1})+E(XI(\|X\| > \gamma_n)))$, we see that (2.6) follows from (1.6), (2.7) and (2.8) provided

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\left\|\sum_{j=1}^{n}(Z_{n,j} - E(Z_{n,j})\right\| > \varepsilon\gamma_n/2\right) < \infty. \tag{2.9}$$

Using standard symmetrization arguments, we have

$$E\left\|\sum_{j=1}^{n}(Z_{n,j} - E(Z_{n,j})\right\| \le 2E\left(\left\|\sum_{j=1}^{n} \varepsilon_j Z_{n,j}\right\|\right),$$

where $\{\varepsilon_j\}$ are independent Rademacher random variables, which are also independent of the sequence $\{X_j\}$. Applying Lemma 6.5, and then Lemma 6.3 of Ledoux-Talagrand (1991), one easily sees that

$$E\left(\left\|\sum_{j=1}^{n} \varepsilon_j Z_{n,j}\right\|\right) \le E\left(\left\|\sum_{j=1}^{n} \varepsilon_j X_j\right\|\right) \le 2E\left(\left\|\sum_{j=1}^{n} X_j\right\|\right).$$

Hence Lemma 9-a of Einmahl (1993) and (1.6) imply that

$$\lim_{n\to\infty} E\left(\left\|\sum_{j=1}^{n}(Z_{n,j} - E(Z_{n,j}))\right\|\right)/\gamma_n = 0. \tag{2.10}$$

Applying (2.10), and Theorem 5 and (3.1) of Einmahl (1993), we have for all n sufficiently large that

$$\begin{aligned} &P\left(\left\|\sum_{j=1}^{n}(Z_{n,j} - E(Z_{n,j}))\right\| > \varepsilon\gamma_n/2\right) \\ &\qquad \le 16\exp\{-\varepsilon^2\gamma_n^2/(2304n\sigma_n^2)\} + c_\varepsilon nE(\|X\|^3 I(\|X\| \le \gamma_n))/\gamma_n^3, \end{aligned} \tag{2.11}$$

where c_ε is a constant depending only on $\varepsilon > 0$, and

$$\sigma_n^2 = \sup_{\|f\|_{B^*}\le 1} E(f^2(Z_{n,1})). \tag{2.12}$$

Lemma 5 of Einmahl (1993), and (1.6), (2.1), imply

$$\sum_{n=1}^{\infty} E(\|X\|^3 I(\|X\| \le \gamma_n))/\gamma_n^3 < \infty, \tag{2.13}$$

so it remains to show

$$\sum_{n=1}^{\infty} \frac{1}{n} \exp\{-\varepsilon^2\gamma_n^2/(2304n\sigma_n^2)\} < \infty. \tag{2.14}$$

To estimate σ_n^2 given by (2.12) we observe that if $f \in B^*$, $\|f\|_{B^*} \le 1$, then

$$\begin{aligned} E(f^2(Z_{n,1})) &\le E(f^2(X)I(|f(X)| \le K_f(n/L_2 n))) \\ &\quad + E(f^2(X)I(|f(X)| > K_f(n/L_2 n),\ \|X\| \le \gamma_n)) \\ &\le K_f^2(n/L_2 n)L_2 n/n + \theta_n(f) \end{aligned}$$

where, by Cauchy–Schwarz,

$$\theta_n(f) = (E(|f(X)|I(|f(X)| > K_f(n/L_2 n))))^{1/2}(E(\|X\|^3 I(\|X\| \le \gamma_n)))^{1/2},$$

and the first term of the second inequality follows from (3.22) in Einmahl (1993). Thus by (1.3) and (1.4) above, and (3.21) in Einmahl (1993),

$$E(f^2(Z_{n,1})) \le \frac{\gamma_n^2}{2n}[(L_2 n)^{-1} + (2\sqrt{2} n E(\|X\|^3 I(\|X\| \le \gamma_n))/\gamma_n^3)^{1/2}]. \tag{2.15}$$

Setting $N_0 = \{n : nE(\|X\|^3 I(\|X\| \le \gamma_n))/\gamma_n^3 \le (L_2 n)^{-2}/(2\sqrt{2})\}$, and using the inequality $\exp\{-x/2\} \le 2/x$, we get

$$\sum_{n \in N_0} n^{-1} \exp\{-\varepsilon^2 \gamma_n^2/(2304 n \sigma_n^2)\} \le 4608 \varepsilon^{-2} \sum_{n=1}^{\infty} (\sigma_n^2/\gamma_n^2)(Ln)^{-\varepsilon^2/4608}. \tag{2.16}$$

Since $c_n = \tilde{K}(n/L_2 n)/(Ln)^{\varepsilon_n}$, the trivial inequality

$$\sigma_n^2 \le E(\|X\|^3 I(\|X\| \le \gamma_n))/c_n$$

implies

$$\begin{aligned} &\sum_{n=1}^{\infty} (\sigma_n^2/\gamma_n^2)(Ln)^{-\varepsilon^2/4608} \\ &\qquad \le \sqrt{2} \sum_{n=1}^{\infty} E(\|X\|^3 I(\|X\| \le \gamma_n))\gamma_n^{-3} L_2 n (Ln)^{\varepsilon_n - \varepsilon^2/4608} < \infty. \end{aligned} \tag{2.17}$$

If $n \notin N_0$, then (2.15) implies

$$\sigma_n^2 \le \frac{\gamma_n^2}{n}(2\sqrt{2} n E(\|X\|^3 I(\|X\| \le \gamma_n))/\gamma_n^3)^{1/2},$$

and it follows from $e^{-x} \le 2/x^2$ that

$$\begin{aligned} &\sum_{n \notin N_0} n^{-1} \exp\left\{-\varepsilon^2 \gamma_n^2/\left(2304 n \sigma_n^2\right)\right\} \\ &\qquad \le \sum_{n \notin N_0} n^{-1} \exp\left\{-\frac{\varepsilon^2}{2304}\left(2\sqrt{2} n E(\|X\|^3 I(\|X\| \le \gamma_n))/\gamma_n^3\right)^{-1/2}\right\} \\ &\qquad \le \sqrt{2}(4608)^2 \varepsilon^{-4} \sum_{n=1}^{\infty} E(\|X\|^3 I(\|X\| \le \gamma_n))/\gamma_n^3 < \infty, \end{aligned} \tag{2.18}$$

where the last inequality follows as in (2.13). Combining (2.16), (2.17) and (2.18) we have (2.14), and hence Lemma 1 is proven. □

STEP 2. Here we set up the notation required to estimate the probabilities in (2.4), and to apply Lemma 1. Recall that $c_n = \tilde{K}(n/L_2 n)/(Ln)^{\varepsilon_n}$ with $\varepsilon_n \geq 0$, $\varepsilon_n \to 0$, to be specified in Lemma 2 below.

Let D be a countable subset of the unit ball of B^* such that $\|x\| = \sup_{f\in D} |f(x)|$, and suppose $b \in A$, $b \neq 0$. Then for any family $D_n \subseteq D$, and $\varepsilon > 0$, we have

$$P(\|S_{n,n}/\gamma_n - b\| < \varepsilon) \leq P\left(\sup_{f\in D_n} |f(S_{n,n}/\gamma_n) - f(b)| < \varepsilon\right). \tag{2.19}$$

In order to determine D_n useful for our purposes, we define on D the semi-metrics

$$\begin{aligned} d_1^n(f, g) &= |f(b) - g(b)|, \\ d_2^n(f, g) &= (nL_2 n)^{1/2} E((f-g)^2(\overline{X}_{n,1}))^{1/2}/\gamma_n, \\ d_3^n(f, g) &= \max(d_1^n(f, g), d_2^n(f, g)). \end{aligned} \tag{2.20}$$

Obviously the d_i^n's satisfy the properties to be metrics, except $d_i^n(f, g) = 0$ need not imply $f = g$ on B. Hence they are semi-metrics.

Setting

$$\alpha_n = E\left(\left\|\sum_{j=1}^n \varepsilon_j \overline{X}_{n,j}\right\| / \gamma_n\right), \tag{2.21}$$

we see from the symmetrization arguments used in the proof of Lemma 1 and the application of (1.6), along with Lemma 9-a of Einmahl (1993), that

$$\lim_{n\to\infty} \alpha_n = 0. \tag{2.22}$$

Let $N(D, d_i^n, \varepsilon)$ denote the minimal number of elements g in D such that for every f in D there exists such a g with $d_i^n(f, g) \leq \varepsilon$. Then $f(b) \in [-\|b\|, \|b\|]$ for all $f \in D$, and hence

$$N(D, d_1^n, \varepsilon) \leq [2\|b\|/\varepsilon] + 2. \tag{2.23}$$

Using the ideas of Lemma 8.3 in Ledoux-Talagrand (1991) we have the following lemma.

LEMMA 2. *Let* $\{\alpha_n\}$ *be the null sequence in* (2.21), *and suppose* $d_2^n(f, g)$ *is as in* (2.20). *Let* $\varepsilon_n = 2\alpha_n + (L_2 n)^{-1/2}$ *in* (2.5). *Then for all* $\varepsilon > 0$, *and all* n *sufficiently large,*

$$N(D, d_2^n, \varepsilon) \leq \exp\{\alpha_n L_2 n\}. \tag{2.24}$$

PROOF. Suppose this fails. Then there exist infinitely many n and $D_n \subseteq D$ such that for $f, g \in D_n$, $f \neq g$, we have $d_2^n(f, g) \geq \varepsilon$ and

$$\text{card } D_n = [\exp\{\alpha_n L_2 n\}] + 1. \tag{2.25}$$

Setting $h = f - g \neq 0$ for f, g in D_n, then $d_2^n(f, g) \geq \varepsilon$ and we can infer from the Bernstein inequality [see, e.g., Exercise 14 on page 111 of Chow and Teicher(1988)] that for $\varepsilon > 0$ and infinitely many n,

$$\begin{aligned} &P\left(\sum_{j=1}^{n} h^2\left(\overline{X}_{n,j}\right) L_2 n/\gamma_n^2 < \varepsilon^2/2\right) \\ &\quad \leq P\left(\sum_{j=1}^{n}(-h^2(\overline{X}_{n,j}) + E(h^2(\overline{X}_{n,j})))L_2 n/\gamma_n^2 > \varepsilon^2/2\right) \\ &\quad \leq \exp(-x^2/(2s_n^2 + xc)) \leq \exp\{-L_2 n\} \end{aligned} \tag{2.26}$$

where $x = \varepsilon^2\gamma_n^2/(2L_2 n)$,$s_n^2 = n\mathrm{Var}(h^2(\overline{X}_{n,1}))$ and $c = 8c_n^2$. To see the last inequality in (2.26), we note that

$$s_n^2 \leq nE[h^4(\overline{X}_{n,1})] \leq 8n\beta_n^2 c_n^2,$$

where $\beta_n^2 = \sup_{f\in D} E(f^2(\overline{X}_{n,1}))$.

To estimate this quantity we only need slightly modify the proof of relation (2.15) above. Replacing in the inequality following (2.14) the sequence γ_n by c_n, it is easy to see that

$$\beta_n^2 \leq \gamma_n^2/(2nL_2 n) + \sup_{f\in D} \tilde{\theta}_n(f),$$

where

$$\tilde{\theta}_n(f) := c_n E(|f(X)| I\{|f(X)| > K_f(n/L_2 n)\}).$$

Using again relation (3.21) of Einmahl (1993) it follows that as $n \to \infty$,

$$\sup_{f\in D} \tilde{\theta}_n(f) \leq \gamma_n c_n/n = o(\gamma_n^2/(nL_2 n)),$$

which implies in combination with the definition of c_n relation (2.26).

If n is large, we have

$$(\mathrm{card}(D_n))^2 \exp\{-L_2 n\} < 1/4,$$

and hence, for infinitely many n,

$$P\left(\forall\ f, g \in D_n, f \neq g, \sum_{j=1}^{n}(f-g)^2(\overline{X}_{n,j}) > \varepsilon^2\gamma_n^2/(2L_2 n)\right) \geq 3/4. \tag{2.27}$$

Now we turn to the application of Proposition 4.13 in Ledoux-Talagrand (1991) conditionally, which will produce a contradiction to (2.27). That is, (2.21) and (2.22) imply with probability larger than 3/4 that

$$E_\varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j \overline{X}_{n,j} \right\| /\gamma_n = E_\varepsilon\left(\sup_{f\in D}\left|\sum_{j=1}^{n} \varepsilon_j f(\overline{X}_{n,j})\right|/\gamma_n\right) < \varepsilon^2/K \tag{2.28}$$

for every n large enough, where K is the absolute constant of Proposition 4.13 in Ledoux-Talagrand (1991). Considering the set

$$T = T_n = \left\{\left(\left(f(\overline{X}_{n,j})\sqrt{L_2 n}/\gamma_n\right) : 1 \le j \le n\right) : f \in D\right\}$$

in $\mathbb{R}^n$ with standard ℓ^2-distance, we see

$$\sup_{1\le j\le n} \left|f(\overline{X}_{n,j})\right| \sqrt{L_2 n}/\gamma_n \le 2c_n\sqrt{L_2 n}/\gamma_n,$$

and from (2.28) that

$$r(T) = E_\varepsilon\left(\sup_{f\in D}\left|\sum_{j=1}^{n}\varepsilon_j f(\overline{X}_{n,j})\sqrt{L_2 n}/\gamma_n\right|\right) \le \varepsilon^2\sqrt{L_2 n}/K. \tag{2.29}$$

By definition of c_n

$$2c_n(L_2 n)^{1/2}/\gamma_n \le \varepsilon^2(Kr(T))^{-1},$$

and Proposition 4.13 in Ledoux-Talagrand (1991) implies

$$\varepsilon(\log N(T, d_2, \varepsilon))^{1/2} \le Kr(T). \tag{2.30}$$

Here d_2 is standard ℓ^2-distance, and recall the above holds for all sufficiently large n with probability larger than 3/4. Hence with probability at least 1/2, infinitely often in n (2.27) implies

$$\begin{aligned} E_\varepsilon\left\|\sum_{j=1}^{n}\varepsilon_j\overline{X}_{n,j}\right\|/\gamma_n &\ge r(T)/(L_2 n)^{1/2} \\ &\ge \left(\frac{\varepsilon}{K\sqrt{2}}\right)\left(\log N\left(T, d_2, \varepsilon/\sqrt{2}\right)\right)^{1/2}/(L_2 n)^{1/2} \\ &\ge \left(\frac{\varepsilon}{K\sqrt{2}}\right)(\log(\text{card } D_n))^{1/2}/(L_2 n)^{1/2}. \end{aligned} \tag{2.31}$$

Taking expectations on the $\overline{X}_{n,j}$'s, (2.25), (2.31) and the definition of α_n combine to imply that

$$\alpha_n \ge \left(\frac{\varepsilon}{K\sqrt{2}}\right)(\alpha_n L_2 n)^{1/2}/(L_2 n)^{1/2} \tag{2.32}$$

infinitely often in n. This is a contradiction as $\alpha_n \to 0$, so Lemma 2 is proven. □

Given $\varepsilon > 0$, we now define $D_n \subseteq D$ to be an optimal ε-net of D in the d_3^n distance. By merging $\varepsilon/2$-nets for d_1^n and d_2^n and using (2.23) and (2.24) we have for all n large that

$$\text{card}(D_n) \le (4\|b\|/\varepsilon + 2)\exp\{\alpha_n L_2 n\}. \tag{2.33}$$

Hence for any $f \in D$ there exists a function $g_n(f) \in D_n$ such that

$$d_3^n(f, g_n(f)) < \varepsilon. \tag{2.34}$$

Set $D'_n = \{f - g_n(f) : f \in D\}$, and observe that

$$\|S_{n,n}/\gamma_n - b\| \le \sup_{g\in D_n} |g(S_{n,n}/\gamma_n) - g(b)| + \sup_{h\in D'_n} |h(S_{n,n}/\gamma_n) - h(b)|. \tag{2.35}$$

Since $d_3^n = \max(d_1^n, d_2^n)$, (2.34) implies

$$\sup_{h\in D'_n} |h(b)| \le \sup_{f\in D} |(f - g_n(f))(b)| \le \sup_{f\in D} d_1^n(f, g_n(f)) < \varepsilon \tag{2.36}$$

and

$$\Lambda_n = \sup_{h\in D'_n} nE(h^2(\overline{X}_{n,1})) \le \sup_{f\in D} (\gamma_n^2/L_2 n) d_2^n(f, g(f))^2 \le \varepsilon^2\gamma_n^2/L_2 n. \tag{2.37}$$

Applying Theorem 5 of Einmahl (1993) to the semi-norm $\sup_{h\in D'_n} |h(x)|$ on B, and noting that $\lim_n E\|S_{n,n}\|/\gamma_n = 0$ by arguments in Lemma 1 above, we see from (2.37) that for all n sufficiently large,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\sup_{h\in D'_n} |h(S_{n,n})| > 13\varepsilon\gamma_n\right) < \infty, \tag{2.38}$$

where we also use (2.13) again.

STEP 3. We apply the previous steps to show $b \in A \iff tb \in A$ for all $t \in [-1, 1]$. This will complete the proof of Theorem 1. □

LEMMA 3. *If $b \in A$ and $\varepsilon > 0$, then for each $t \in [-1, 1]$,*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\sup_{g\in D_n} |g(S_{n,n}/\gamma_n) - g(tb)| < \varepsilon\right) = \infty. \tag{2.39}$$

PROOF. If $b \in A$, then Lemma 1 and (2.19) implies (2.39) for $t = 1$. Fix $\varepsilon > 0$ and let $d_n = \mathrm{card}(D_n)$, where $D_n \subseteq D$ is an ε-net of D in the d_3^n distance satisfying (2.33). Define $\overline{g} : B \to \mathbb{R}^{d_n}$ to be the vector

$$\overline{g}(x) = (g(x) : g \in D_n),$$

and let G_n be a centered Gaussian $\mathbb{R}^{d_n}$ valued random vector with covariance that of $\overline{g}(\overline{X}_{n,1})$. Also let $U_{d_n}^{\infty}$ and $U_{d_n}^2$ denote the ℓ^∞ and ℓ^2 unit balls in $\mathbb{R}^{d_n}$ and for $-1 \le t \le 1$ set

$$E_n(t) = \overline{g}(tb)\gamma_n + \varepsilon\gamma_n U_{d_n}^{\infty}.$$

Since $U_{d_n}^2 \subseteq U_{d_n}^{\infty}$, it follows from Zaitsev (1987) that there exists an absolute constant c such that for all Borel subsets F of $\mathbb{R}^{d_n}$,

$$P\left(\overline{g}(S_{n,n}) \in F\right) \le P\left(\sqrt{n}G_n \in F + \varepsilon\gamma_n U_{d_n}^{\infty}\right) + cd_n^2 \exp\left(-\varepsilon\gamma_n/(cd_n^2 c_n)\right) \tag{2.40}$$

and

$$P\left(\sqrt{n}G_n \in F\right) \le P(\overline{g}(S_{n,n}) \in F + \varepsilon\gamma_n U_{d_n}^{\infty}) + cd_n^2 \exp(-\varepsilon\gamma_n/(cd_n^2 c_n)). \tag{2.41}$$

Now (2.33) and $\|b\| \leq 1$ implies $d_n \leq 6\varepsilon^{-1}(Ln)^{\alpha_n}$ with $\alpha_n \to 0$ and since $c_n = \tilde{K}(n/L_2 n)/(Ln)^{\varepsilon_n}$ with $\varepsilon_n = 2\alpha_n + (L_2 n)^{-1/2}$ we have

$$\varepsilon\gamma_n/(cd_n^2 c_n) \geq \sqrt{2}\varepsilon^3 \exp((L_2 n)^{1/2}) L_2 n/(36c)$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n}[cd_n^2 \exp\{-\varepsilon\gamma_n/(cd_n^2 c_n)\}] < \infty \tag{2.42}$$

for all $\varepsilon > 0$. In particular, if (2.39) holds for $t = 1$, then, since

$$P\left(\sup_{g\in D_n} |g(S_{n,n}/\gamma_n) - g(tb)| < \varepsilon\right) = P(\overline{g}(S_{n,n}) \in E_n(t)), \tag{2.43}$$

(2.40) [with $F = E_n(1)$] and (2.42) imply

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\sqrt{n}G_n \in E_n(1) + \varepsilon\gamma_n U_{d_n}^{\infty}) = \infty. \tag{2.44}$$

Applying Anderson's inequality [Anderson (1955)], we thus have

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\sqrt{n}G_n \in E_n(t) + \varepsilon\gamma_n U_{d_n}^{\infty}) = \infty \tag{2.45}$$

for each $t \in [-1, 1]$. On the other hand, if (2.45) holds for all $t \in [-1, 1]$, then (2.41) [with $F = E_n(t) + \varepsilon\gamma_n U_{d_n}^{\infty}$], (2.42) and (2.43) imply

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\sup_{g\in D_n} |g(S_{n,n}/\gamma_n) - g(tb)| < 3\varepsilon\right) = \infty \tag{2.46}$$

for all $\varepsilon > 0$ and $t \in [-1, 1]$. Since $\varepsilon > 0$ is arbitrary this gives (2.39), hence Lemma 3. □

LEMMA 4. *If $b \in A$, then $tb \in A$ for all $t \in [-1, 1]$.*

PROOF. Fix $\varepsilon > 0$ and recall (2.36). Then (2.35) implies

$$\begin{aligned}
&P(\|S_{n,n}/\gamma_n - tb\| < 15\varepsilon)\\
&\quad \geq P\left(\sup_{g\in D_n} |g(S_{n,n}/\gamma_n) - g(tb)| < \varepsilon,\right.\\
&\qquad\qquad \left.\sup_{h\in D_n'} |h(S_{n,n}/\gamma_n) - h(tb)| < 14\varepsilon\right)\\
&\quad \geq P\left(\sup_{g\in D_n} |g(S_{n,n}/\gamma_n) - g(tb)| < \varepsilon\right)\\
&\qquad - P\left(\sup_{h\in D_n'} |h(S_{n,n}/\gamma_n)| > 13\varepsilon\right).
\end{aligned} \tag{2.47}$$

Thus (2.38) and (2.39) imply

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\|S_{n,n}/\gamma_n - tb\| < 15\varepsilon) = \infty \tag{2.48}$$

for all $\varepsilon > 0$ and $t \in [-1, 1]$. Since $\varepsilon > 0$ is arbitrary, (2.48) and (2.4) imply $tb \in A$ for all $t \in [-1, 1]$. Thus the lemma is proven.

The properties of A in (1.8) and (1.9) now follow immediately from Lemma 4. Hence Theorem 1 is proven. □

3. Proof of Theorem 2. The proof consists of five steps. We first will define a real-valued random variable $Z : \Omega \to \mathbb{R}$ in the domain of attraction of the normal distribution which satisfies the generalized LIL of (1.7). This is step 1, and in step 2 the desired random vector X is defined as a function of Z so that

$$\|X\| = |Z|. \tag{3.1}$$

Step 3 proves $A \supseteq \tilde{A}$ and step 4 gives $A \subseteq \tilde{A}$. Steps 3–4 are established under the assumption B has a Schauder basis, and step 5 then gives Theorem 2 without the basis assumption.

Hence we first assume $(B, \|\cdot\|)$ is an infinite dimensional (separable) real Banach space which has a Schauder basis $\{e_i : i \geq 1\}$, where we assume without loss of generality that $\|e_i\| = 1$ for $i \geq 1$.

This means that each $x \in B$ determines a unique sequence of real numbers $\alpha_i = \alpha_i(x)$, $i \geq 1$, so that

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{n} \alpha_j(x) e_j - x \right\| = 0. \tag{3.2}$$

It is well known that the mappings α_j on B are linear and continuous, that is, $\alpha_j \in B^*$ on $j \geq 1$. Moreover, if we define operators $\Pi_N : B \to B$ by

$$\Pi_N(x) = \sum_{j=1}^{N} \alpha_j(x) e_j, \qquad x \in B, \tag{3.3}$$

there exists a positive constant C so that uniformly in N

$$\|\Pi_N\| \leq C. \tag{3.4}$$

From (3.4) we readily obtain that the operators $Q_N : B \to B$ which are defined by

$$Q_N(x) = x - \Pi_N(x), \qquad x \in B, \tag{3.5}$$

are continuous and such that

$$\|Q_N\| \leq C + 1. \tag{3.6}$$

Sometimes it will be convenient to identify the finite dimensional space $\Pi_N(B)$ with $\mathbb{R}^N$ by using the isomorphism

$$\sum_{i=1}^{N} \alpha_i e_i \leftrightarrow (\alpha_1, \dots, \alpha_N). \tag{3.7}$$

Since all norms on finite dimensional spaces are equivalent, there exists constants $1 = C_1 \le C_2 \le \cdots \le C_N \le \cdots$ so that

$$C_N^{-1} |(\alpha_1, \dots, \alpha_N)| \le \left\| \sum_{i=1}^{N} \alpha_i e_i \right\| \le C_N |(\alpha_1, \dots, \alpha_N)|, \tag{3.8}$$

where $|\cdot|$ denotes the usual Euclidean norm.

STEP 1. We first observe that $\tilde{A} \neq \{0\}$ can be written as a closure of (at most) countably many open line segments, that is

$$\tilde{A} = \text{cl}\left(\bigcup_{\text{j}=1}^{\infty} \mathscr{L}_{\text{j}} \right), \tag{3.9}$$

where

$$\mathscr{L}_j = \{ t z_j : |t| < \sigma_j \} \tag{3.10}$$

for a suitable unit vector $z_j \in B$ and $0 \le \sigma_j \le \sigma$, with

$$\sigma = \sup_{x \in \tilde{A}} \|x\| \in (0, 1]. \tag{3.11}$$

We will assume that $\sigma_1 = \sigma$ and also $\sigma_j > 0$, $j \ge 2$. (This can always be accomplished by replacing all "trivial" empty line segments $\mathscr{L}_j$ by $\mathscr{L}_1$ if necessary.)

If $\tilde{A} = \{0\}$, we set $\sigma_j = 1$, $j \ge 1$.

Our real random variable Z will be similar to that used in the proof of Theorem 4 of Einmahl (1995).

We set for $k \ge 1$, $m_k = 4^{k^4}$, $m_{k,0} = m_k$, and $m_{k,\ell+1} = m_{k,\ell} + k^3 + 4^{\sigma_{\ell+1}^2 k^4}$ for $0 \le \ell \le k-1$. Furthermore, we define $m_{k,k+1} = m_{k+1}$ and $n_{k,\ell} = m_{k,\ell} + k^3$, $0 \le \ell \le k$.

We assume $H(t) = E(Z^2 I\{|Z| \le t\}), t \ge 0$ satisfies (but Z is still to be defined),

$$H(t) = d_n, \ \exp(n) \le t < \exp(n+1), \qquad n \ge 1,$$

where

$$
\begin{aligned}
d_n &= 0, \qquad 0 \le n < m_4 \quad \text{and for} \quad k \ge 4,\\
d_{m_k} &= \exp(k^3),\\
d_{m_{k,\ell}+j} &= \exp(k^3 + \ell k + j/k^2), \qquad 0 \le j \le k^3,\\
d_m &= \exp(k^3 + (\ell+1)k), n_{k,\ell} \le m \le m_{k,\ell+1}, \qquad 0 \le \ell \le k-1,\\
d_{m_{k,k}+j} &= \exp((2k^2 + 3k + 1) j k^{-3} + k^2 + k^3), \qquad 0 \le j \le k^3,\\
d_m &= \exp((k+1)^3), \qquad n_{k,k} \le m \le m_{k+1}.
\end{aligned}
$$

Arguing as in Lemma 8 of Einmahl (1995) it is easy to see that there exists a symmetric and discrete random variable, which we call Z, and which has an H-function with these properties. Moreover, since $H(t)$ is slowly varying at infinity, it follows that Z is in the domain of attraction of the normal distribution. From Lemma 7 of Einmahl (1995) it also follows that Z satisfies the conditions of the generalized LIL; that is, we have if $Z_1, Z_2, Z_3, \dots$ are independent copies of Z with probability one,

$$
\limsup_{n\to\infty} \left| \sum_{j=1}^{n} Z_j \right| / \overline{\gamma}_n = 1, \tag{3.12}
$$

where $\overline{\gamma}_n = \sqrt{2}\, \overline{K}(n/L_2 n) L_2 n$, $n \ge 1$, and $\overline{K}$ is the K-function corresponding to Z. That is, $\overline{K}$ is the inverse function of $\overline{G}$ which is defined by

$$
\overline{G}(t) = t^2/(H(t) + tM(t))
$$

for $t > 0$ and where $M(t) = E(|Z| I\{|Z| > t\})$. Finally, we note that Z being in the domain of attraction of the normal distribution also implies

$$
\lim_{t\to\infty} tM(t)/H(t) = 0. \tag{3.13}
$$

STEP 2. Given Z, we define X by

$$
X = \sum_{k=1}^{\infty} \sum_{\ell=1}^{k+1} z_{k,\ell} Z I\{\exp(m_{k,\ell-1}) < |Z| \le \exp(m_{k,\ell})\}. \tag{3.14}
$$

where $z_{k,\ell}$, $1 \le \ell \le k+1$, $k \ge 1$ are unit vectors in B.

Since the indicator functions in (3.14) have disjoint supports, for all $f \in B^*$,

$$
f(X) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{k+1} f(z_{k,\ell}) Z I\{\exp(m_{k,\ell-1}) < |Z| \le \exp(m_{k,\ell})\}. \tag{3.15}
$$

Furthermore, since $|f(z_{k,\ell})| \le 1$ if $\|f\|_{B^*} \le 1$, it follows that we have for all $f \in B^*$ with $\|f\|_{B^*} \le 1$ and $E(|f(X)|) > 0$,

$$
K_f(x) \le \overline{K}(x), \qquad x \ge 0 \tag{3.16}
$$

and consequently,

$$\gamma_n \leq \overline{\gamma}_n. \tag{3.17}$$

Letting $\tilde{H}_f(t) = E(f^2(X)I\{|Z| \leq t\})$ and $H_f(t) = E(f^2(X)I\{|f(X)| \leq t\})$ for $t \geq 0$, the following are obvious.

LEMMA 5. (a) $\tilde{H}_f(t) \leq H_f(t)$ *for* $\|f\|_{B^*} \leq 1$, $t > 0$.
(b) $H_f(t) \leq \tilde{H}_f(t) + t^2 P(|Z| \geq t)$ *for* $t > 0$.
(c) $\sup_{\|f\|_{B^*} \leq 1} tM_f(t)/H(t) \to 0$ *as* $t \to \infty$.

LEMMA 6. *We have for* $f \in B^*$:
(a) $\tilde{H}_f(t) \geq f^2(z_{k,\ell})H(t)(1 - e^{-k})$, *and*
(b) $\tilde{H}_f(t) \leq f^2(z_{k,\ell})\{H(t) - H(\exp(m_{k,\ell-1}))\} + \|f\|^2_{B^*} H(\exp(m_{k,\ell-1}))$,
whenever $\exp(n_{k,\ell-1}) \leq t \leq \exp(m_{k,\ell})$, $1 \leq \ell \leq k+1$, $k \geq 1$.

We also need:

LEMMA 7. *We have for* $n \in J_{k,\ell} := \{n : \exp(n_{k,\ell-1}) < \overline{K}(n/L_2 n) \leq \exp(m_{k,\ell})\}$, *and* $1 \leq \ell \leq k+1$, $k \geq 1$

$$\overline{\gamma}_n \leq (1 + \varepsilon_k)(1 - e^{-k})^{-1}\gamma_n \tag{3.18}$$

with $\varepsilon_k \downarrow 0$, *and*

$$\limsup_{n\to\infty} \overline{\gamma}_n/\gamma_n \leq 2. \tag{3.19}$$

PROOF. For (3.18) choose a functional $f = f_{k,\ell}$ so that $f(z_{k,\ell}) = 1$, $\|f\|_{B^*} \leq 1$, and note that for $\exp(n_{k,\ell-1}) < t \leq \exp(m_{k,\ell})$ we have by (3.13) along with Lemmas 5 and 6,

$$\begin{aligned} H_f(t) \geq \tilde{H}_f(t) &\geq H(t)(1 - e^{-k}) \\ &\geq (H(t) + tM(t))(1 + \varepsilon_k)^{-1}(1 - e^{-k}). \end{aligned}$$

Arguing now as in the proof of Lemma 17 (a) in Einmahl (1995) we get (3.18). The proof of (3.19) follows by a similar modification of the proof of Lemma 17(b) in Einmahl (1995). □

Recalling that Z satisfies the conditions for the generalized LIL, we get from (6.5) of Einmahl (1995) that

$$\sum_{n=1}^{\infty} P(|Z| > \delta\overline{\gamma}_n) < \infty \qquad \forall\, \delta > 0. \tag{3.20}$$

Thus (3.17) and (3.19) combine to imply

$$\sum_{n=1}^{\infty} P(|Z| > \gamma_n) < \infty. \tag{3.21}$$

Let us now assume that for some positive constant c,

$$\text{card}\{z_{k,\ell} : 1 \le \ell \le k+1, 1 \le k \le K\} \le cK. \tag{3.22}$$

We next show that then as $n \to \infty$,

$$S_n/\gamma_n \xrightarrow{\text{prob}} 0, \tag{3.23}$$

where $S_n = \sum_{j=1}^n X_j, n \ge 1$, and $X_j, j \ge 1$, are independent copies of X. Therefore, under (3.22) we have from relation (1.7) in Einmahl (1993), that with probability one

$$\limsup_{n\to\infty} \|S_n\|/\gamma_n = 1. \tag{3.24}$$

Hence we must show (3.23) holds.

LEMMA 8. *Under condition* (3.22), *we have* $\lim_{n\to\infty} E(\|S_n\|)/\gamma_n = 0$.

PROOF. First we observe that we can write $X_j = \phi(Z_j)Z_j$ for $j \ge 1$, where $Z_j, j \ge 1$, are independent copies of Z, and ϕ maps $\mathbb{R}$ into the unit ball of B with a countable range. Furthermore, let $\xi_j, j \ge 1$, be a sequence of independent Rademacher variables which is also independent of $\{X_j\}$. Given the symmetry of the random variables $\{X_j\}$, it is obvious that

$$\begin{aligned} &E\left(\left\|\sum_{j=1}^n X_j I\{\|X_j\| \le \overline{K}(n/L_2 n)\}\right\|\right) \\ &\quad = E\left(\left\|\sum_{j=1}^n \xi_j \phi(Z_j) Z_j I\{|Z_j| \le \overline{K}(n/L_2 n)\}\right\|\right). \end{aligned} \tag{3.25}$$

Writing $\{\phi(t) : |t| \le \overline{K}(n/L_2 n)\} = \{v_1, \dots, v_{r_n}\}$, where $v_1, \dots, v_{r_n}$ are unit vectors in B, we get, for $|t_j| \le \overline{K}(n/L_2 n)$, $1 \le j \le n$, that

$$\begin{aligned} E\left(\left\|\sum_{j=1}^n \xi_j \phi(t_j) t_j\right\|\right) &\le \sum_{i=1}^{r_n} E\left(\left\|\sum_{\{j:\phi(t_j)=v_i\}} \xi_j \phi(t_j) t_j\right\|\right) \\ &= \sum_{i=1}^{r_n} E\left|\sum_{\{j:\phi(t_j)=v_i\}} \xi_j t_j\right| \\ &\le r_n E\left|\sum_{j=1}^n \xi_j t_j\right|. \end{aligned}$$

Recalling (3.25), Fubini's theorem implies

$$E\left(\left\|\sum_{j=1}^n X_j I\{\|X_j\| \le \overline{K}(n/L_2 n)\}\right\|\right) \le r_n E\left|\sum_{j=1}^n Z_{n,j}\right|$$

where $Z_{n,j} = Z_j I\{|Z_j| \le \overline{K}(n/L_2 n)\}$, $1 \le j \le n$. Next note that

$$E\left|\sum_{j=1}^{n} Z_{n,j}\right| \le E\left(\left(\sum_{j=1}^{n} Z_{n,j}\right)^2\right)^{1/2} = \sqrt{n} H(\overline{K}(n/L_2 n))^{1/2} \le \overline{\gamma}_n/\sqrt{L_2 n}.$$

Observing that (3.22) implies $r_n = O((L_2 n)^{1/4})$ as $n \to \infty$, we readily obtain that as $n \to \infty$,

$$E\left(\left\|\sum_{j=1}^{n} X_j I\{\|X_j\| \le \overline{K}(n/L_2 n)\}\right\|\right) = O(\gamma_n/(L_2 n)^{1/4}).$$

Finally, note that

$$E\|S_n\| \le E\left\|\sum_{j=1}^{n} X_j I\{\|X_j\| \le \overline{K}(n/L_2 n)\}\right\| + nE(\|X\| I\{\|X\| > \overline{K}(n/L_2 n)\})$$

where, in view of (3.13), as $n \to \infty$

$$E(\|X\| I\{\|X\| > \overline{K}(n/L_2 n)\}) = o(H(\overline{K}(n/L_2 n))/\overline{K}(n/L_2 n)) = o(\overline{\gamma}_n/n).$$

Hence (3.19) implies the assertion of Lemma 8. □

Having established general global properties of the random variables given as in (3.14), we now specify the vectors $z_{k,\ell}$, $1 \le \ell \le k+1$, $k \ge 1$ so that we obtain the desired cluster sets. There are two cases to consider

CASE 1. $0 \le \sigma < 1$. Let $\{z_j\}$ be as in (3.10) and assume V is an arbitrary infinite dimensional vector space which contains $\tilde{A}$. Using Riesz's lemma [see, e.g., page 2 of Diestel (1983)], we can find a sequence of unit vectors v_i, $i \ge 1$, in V and such that for $n \ge 1$,

$$\text{dist}(v_{n+1}, \text{ span } \{v_1, \dots, v_n\}) \ge 1/2. \tag{3.26}$$

If $\sigma > 0$, we set $z_{k,\ell} = z_\ell$, $1 \le \ell \le k$, $z_{k,k+1} = v_k$.

If $\sigma = 0$, we set $z_{k,\ell} = v_k$, $1 \le \ell \le k+1$.

CASE 2. $\sigma = 1$. Set $z_{k,\ell} = z_\ell$, $1 \le \ell \le k$, $z_{k,k+1} = z_1$.

Obviously, in case 1 we have $P(X \in V) = 1$, whereas in case 2 we even have $P(X \in W) = 1$ for any closed subspace W containing $\tilde{A}$. It is also easy to see that in both cases condition (3.22) is satisfied so that the conclusions of Lemma 8 hold.

STEP 2 ($A \supseteq \tilde{A}$). We assume that $\sigma > 0$ since $\sigma = 0$ implies $\tilde{A} = \{0\}$ and then the inclusion $A \supseteq \tilde{A}$ is trivial since $S_n/\gamma_n \xrightarrow{\text{prob}} 0$.

Since A as a cluster set is closed, we only need to show

$$\mathscr{L}_j \subseteq A, \ j \ge 1. \tag{3.27}$$

In light of Lemma 1, (3.1), (3.17), and (3.19) it suffices to prove for any chosen $j \geq 1, 0 < |t| < \sigma_j$ and $\varepsilon > 0$,

$$\sum_{n \geq 1} n^{-1} P(\|S_{n,n}/\gamma_n - tz_j\| < \varepsilon) = \infty, \tag{3.28}$$

where

$$S_{n,n} = \sum_{j=1}^{n} X_j I\{|Z_j| \leq \overline{K}(n/L_2 n)\}.$$

In order to prove (3.28), we will show that as k tends to infinity,

$$\sum_{n \in J_{k,j}} n^{-1} P(\|S_{n,n}/\gamma_n - tz_j\| < \varepsilon) \to \infty, \tag{3.29}$$

for all $|t| < \sigma_j$, $\varepsilon > 0$. To that end we first note that

$$\begin{aligned} &P(\|S_{n,n}/\gamma_n - tz_j\| < \varepsilon) \\ &\quad \geq P(\|\Pi_M(S_{n,n}/\gamma_n - tz_j)\| < \varepsilon/2, \|Q_M(S_{n,n}/\gamma_n - tz_j)\| < \varepsilon/2) \\ &\quad \geq P(\|\Pi_M(S_{n,n}/\gamma_n - tz_j)\| < \varepsilon/2) - P(\|Q_M(S_{n,n}/\gamma_n)\| \geq 35\varepsilon/72) \end{aligned}$$

provided we have chosen M so large that

$$\|Q_M(z_j)\| < \varepsilon/72. \tag{3.30}$$

We next show that for $n \in J_{k,j}$ and large k,

$$P(\|Q_M(S_{n,n})\| > 35\varepsilon\gamma_n/72) \leq A((Ln)^{-4/3} + \varepsilon^{-3} n\delta_n), \tag{3.31}$$

where A is an absolute constant and $\delta_n = E(|Z|^3 I\{|Z| \leq \gamma_n\})/\gamma_n^3$. To verify (3.31) we apply the Fuk–Nagaev inequality in Banach space as given in Einmahl (1993), where we set $p = 3$. Note that by (3.6) and standard symmetry considerations

$$E\|Q_M(S_{n,n})\| \leq (C+1)E\|S_{n,n}\| \leq (C+1)E\|S_n\|$$

which is of order $o(\gamma_n)$ by Lemma 8. Thus for large k,

$$\begin{aligned} &P(\|Q_M(S_{n,n})\| \geq 35\varepsilon\gamma_n/72) \\ &\quad \leq P(\|Q_M(S_{n,n})\| > (\varepsilon/3)\gamma_n + 352E\|Q_M(S_{n,n})\|) \\ &\quad \leq 22\exp(-\varepsilon^2\gamma_n^2/(1296\Lambda_n)) + A'\varepsilon^{-3} n\delta_n, \end{aligned}$$

where A' is an absolute constant and

$$\Lambda_n = n \sup_{\|f\|_{B^*} \leq 1} E(f^2(Q_M(X)) I\{|Z| \leq \overline{K}(n/L_2 n)\}).$$

Clearly, if k is large enough and $n \in J_{k,j}$, then

$$\begin{aligned}\Lambda_n &\le nE(\|Q_M(X)\|^2 I\{|Z| \le \overline{K}(n/L_2 n)\}) \\ &\le n\{\|Q_M(z_j)\|^2 H(\overline{K}(n/L_2 n)) + (C+1)^2 H(\exp(m_{k,j-1}))\} \\ &\le n\{(\varepsilon/72)^2 + e^{-k}(C+1)^2\}\overline{\gamma}_n^2/(2nL_2 n) \\ &\le 3(\varepsilon/72)^2\gamma_n^2/L_2 n,\end{aligned}$$

where we have again used (3.19), and (3.30). Hence it is easy to see (3.31) holds as indicated.

We now need a good lower bound for the finite-dimensional probability

$$P(\|\Pi_M(S_{n,n}/\gamma_n) - t\Pi_M(z_j)\| < \varepsilon/2).$$

Identifying the finite dimensional space $\Pi_M(B)$ with the Euclidean space $\mathbb{R}^M$ and applying (3.8) we have

$$P(\|\Pi_M(S_{n,n}/\gamma_n) - t\Pi_M(z_j)\| < \varepsilon/2) \ge P(|T_{n,n}/\gamma_n - tz_j^{(M)}| < \varepsilon/(2C_M)),$$

where $T_{n,n} = \sum_{r=1}^n Y_{n,r}$, $Y_{n,r} = (Y_{n,r}^{(1)}, \ldots, Y_{n,r}^{(M)})$, with $Y_{n,r}^{(i)} = \alpha_i(X_r)I\{|Z_r| \le \overline{K}(n/L_2 n)\}$, $1 \le i \le M$, $1 \le r \le n$, $n \ge 1$, and $z_j^{(M)} = (\alpha_1(z_j), \ldots, \alpha_M(z_j))$. Furthermore, set $v_j^{(M)} = |z_j^{(M)}|^{-1} z_j^{(M)}$, and define τ_j to be the projection onto the orthogonal complement of the line $\{tv_j^{(M)} : t \in \mathbb{R}\}$; that is,

$$\tau_j(y) = y - \left\langle y, v_j^{(M)}\right\rangle v_j^{(M)}, \qquad y \in \mathbb{R}^M. \tag{3.32}$$

Then, of course, we have

$$\begin{aligned}&P\left(\left|T_{n,n}/\gamma_n - tz_j^{(M)}\right| < \varepsilon/(2C_M)\right) \\ &\quad\ge P\left(\left|\left\langle T_{n,n}/\gamma_n, v_j^{(M)}\right\rangle - t\left|z_j^{(M)}\right|\right| < \varepsilon/(4C_M)\right) \\ &\qquad - P(|\tau_j(T_{n,n})| > \varepsilon\gamma_n/(4C_M)).\end{aligned}$$

Next note that by Lemma 6(a), for $n \in J_{k,j}$ and large k,

$$\begin{aligned}\lambda_n^2(j) &= \operatorname{Var}\left(\left\langle Y_{n,1}, v_j^{(M)}\right\rangle\right) \\ &\ge \left|z_j^{(M)}\right|^2 H(\overline{K}(n/L_2 n))(1 - e^{-k}) \\ &\ge \left|z_j^{(M)}\right|^2 (H(\overline{K}(n/L_2 n)) + \overline{K}(n/L_2 n)M(\overline{K}(n/L_2 n))(1 - \delta_{k,1}) \\ &= \left|z_j^{(M)}\right|^2 (1 - \delta_{k,1})\overline{\gamma}_n^2/(2nL_2 n),\end{aligned}$$

where $\delta_{k,1} \downarrow 0$. Recalling $\overline{\gamma}_n \ge \gamma_n$, we can conclude that for $n \in J_{k,j}$ and large k

$$\lambda_n^2(j) \ge (1 - \delta_{k,1})|z_j^{(M)}|^2 \gamma_n^2/(2nL_2 n). \tag{3.33}$$

By a similar argument, where we have to use Lemma 6(b) and (3.18), we can also prove that for $n \in J_{k,j}$ and large k,

$$\lambda_n^2(j) \le (1+\delta_{k,2})|z_j^{(M)}|^2\gamma_n^2/(2nL_2n), \tag{3.34}$$

where $\delta_{k,2} \downarrow 0$.

Using a non-uniform version of the Berry-Esseen inequality [see, e.g., page 125 of Petrov (1975)], we get, for small enough ε, with G being a standard normal random variable,

$$\begin{aligned} &P\left(\left|\left\langle T_{n,n}/\gamma_n, v_j^{(M)}\right\rangle - t\left|z_j^{(M)}\right|\right| < \varepsilon/(4C_M)\right) \\ &\quad \ge P\left(\left|\sqrt{n}\lambda_n(j)G - t\gamma_n\left|z_j^{(M)}\right|\right| < \varepsilon\gamma_n/(4C_M)\right) \\ &\quad\quad - A|t|^{-3}\left|z_j^{(M)}\right|^{-3} n\delta_n, \end{aligned} \tag{3.35}$$

where δ_n is as in (3.31), and A is an absolute constant. Recalling (3.33) and (3.34) and using similar arguments as in the proof of Proposition 1–(ii, iii) of Einmahl (1995), we readily obtain for $n \in J_{k,j}$, large k, and small $\varepsilon > 0$,

$$P\left(\left|\sqrt{n}\lambda_n(j)G - t\gamma_n\left|z_j^{(M)}\right|\right| < \varepsilon\gamma_n/(4C_M)\right) \ge C(Ln)^{-t^2}, \tag{3.36}$$

where $C > 0$ is a constant depending on ε and t.

We still need an upper bound for $P(|\tau_j(T_{n,n})| > \varepsilon\gamma_n/(4C_M))$. We again use the Fuk–Nagaev inequality in Banach space. Now clearly

$$\begin{aligned} E(|\tau_j(T_{n,n})|) &\le E|T_{n,n}| \le C_M E\|\Pi_M(S_{n,n})\| \le C_M CE\|S_{n,n}\| \le C_M CE\|S_n\| \\ &= o(\gamma_n) \end{aligned}$$

by Lemma 8. Therefore, for large n,

$$\begin{aligned} &P(|\tau_j(T_{n,n})| \ge \varepsilon\gamma_n/(4C_M)) \\ &\quad \le 22\exp\left(-A_1\varepsilon^2\gamma_n^2/\left(nC_M^2\tilde{\lambda}_n^2(j)\right)\right) + A_2C_M^3 n\delta_n/\varepsilon^3, \end{aligned} \tag{3.37}$$

where A_1, A_2 are absolute constants and

$$\begin{aligned} \tilde{\lambda}_n^2(j) &= \sup\{E(\langle w, \tau_j(Y_{n,1})\rangle^2) : |w| \le 1, w \in \mathbb{R}^M\} \\ &= \sup\{E(\langle \tau_j(w), Y_{n,1}\rangle^2) : |w| \le 1, w \in \mathbb{R}^M\} \end{aligned}$$

since τ_j is self-adjoint on $\mathbb{R}^M$. Observing that for $w \in \mathbb{R}^M$,

$$E(\langle \tau_j(w), Y_{n,1}\rangle^2) = E(f^2(X)I\{|Z| \le \overline{K}(n/L_2n)\}),$$

where $f \in B^*$, $f(\cdot) = \sum_{i=1}^M \alpha_i(\cdot)\tau_j^{(i)}(w)$, and $f(z_j) = 0$, we readily obtain from Lemma 6(b) that for $n \in J_{k,j}$ and large k,

$$\tilde{\lambda}_n^2(j) \le e^{-k}H(\overline{K}(n/L_2n)) \le 3e^{-k}\gamma_n^2/(nL_2n).$$

Replacing $\tilde{\lambda}_n^2(j)$ in (3.37) by this upper bound, we finally see that for $n \in J_{k,j}$ and large k,

$$P(|\tau_j(T_{n,n})| \geq \varepsilon\gamma_n/C_M) \leq (Ln)^{-2} + A_2(C_M/\varepsilon)^3 n\delta_n. \tag{3.38}$$

Combining (3.31), (3.35), (3.36) and (3.38), we get, for large k,

$$\sum_{n\in J_{k,j}} n^{-1}P(\|S_{n,n}/\gamma_n - tz_j\| < \varepsilon) \geq C \sum_{n\in J_{k,j}} n^{-1}(Ln)^{-t^2} - \sum_{n\in J_{k,j}} \tilde{\delta}_n \tag{3.39}$$

where $\sum_n \tilde{\delta}_n < \infty$, which of course implies

$$\lim_{k\to\infty} \sum_{n\in J_{k,j}} \tilde{\delta}_n = 0. \tag{3.40}$$

Arguing as in the proof of (4.53) of Einmahl (1995), we get, for large k,

$$\sum_{n\in J_{k,j}} n^{-1} \geq (m_{k,j} - n_{k,j-1})/2 = 4^{k^4\sigma_j^2}/2. \tag{3.41}$$

Using the fact that $Ln \leq 4^{(k+1)^4}$ when $n \in J_{k,j}$, we readily obtain for $|t| < \sigma_j$,

$$\lim_{k\to\infty} \sum_{n\in J_{k,j}} n^{-1}(Ln)^{-t^2} = \infty, \tag{3.42}$$

which in combination with (3.39) and (3.40) implies (3.29), and hence (3.27). This ends step 3.

STEP 4 ($A \subseteq \tilde{A}$). Applying Lemma 2 again, it is enough to show that for any $x \notin \tilde{A}$ there exists an $\varepsilon > 0$ so that

$$\sum_{n=1}^{\infty} n^{-1}P(\|S_{n,n}/\gamma_n - x\| < \varepsilon) < \infty, \tag{3.43}$$

where $S_{n,n}$ is defined as in (3.28).

Let $J'_{k,\ell} = \{n : \exp(m_{k,\ell-1}) < \overline{K}(n/L_2 n) \leq \exp(n_{k,\ell-1})\}$, for $1 \leq \ell \leq k+1$, and let $J_{k,\ell}$ be defined as in Lemma 7. Then it is obviously enough to show for a suitable $\varepsilon > 0$,

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{k}\sum_{n\in J_{k,\ell}} n^{-1}P(\|S_{n,n}/\gamma_n - x\| < \varepsilon) < \infty, \tag{3.44}$$

$$\sum_{k=1}^{\infty}\sum_{n\in J_{k,k+1}} n^{-1}P(\|S_{n,n}/\gamma_n - x\| < \varepsilon) < \infty \tag{3.45}$$

and

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{k+1}\sum_{n\in J'_{k,\ell}} n^{-1}P(\|S_{n,n}/\gamma_n - x\| < \varepsilon) < \infty. \tag{3.46}$$

We first establish (3.44), where the following lemma will come in handy.

LEMMA 9. *Given $x \in B$, $w \in W$, where W is a closed subspace of B with* $\operatorname{dist}(x, W) = \rho > 0$, *there exists $f \in B^*$ so that $f(x) = \|x\|$, $f(w) = 0$, and* $\|f\|_{B^*} \leq \|x\|/\rho$.

PROOF. Define $f(\lambda x + \mu w) = \lambda\|x\|$ for $\lambda, \mu \in \mathbb{R}$, and note that if $\lambda \neq 0$,

$$\begin{aligned}|f(\lambda x + \mu w)| &= \|\lambda x + \mu w\|\|x\|\|x + (\mu/\lambda)w\|^{-1}\\ &\leq (\|x\|/\rho)\|\lambda x + \mu w\|.\end{aligned}$$

By the Hahn–Banach theorem f can be extended to B without increasing its norm. □

We now assume that $\sigma > 0$. For the proof of (3.44) when $\sigma = 0$, see the remarks at the end of the proof of (3.45).

Set $\beta = \operatorname{dist}(x, \tilde{A}) > 0$.

Let W_j be the line $\{tz_j : t \in \mathbb{R}\}$ and set $\beta_j = \operatorname{dist}(x, W_j)$ for $j \geq 1$. If $\beta_j > \beta/4$, choose via Lemma 9 a functional $f_j \in B^*$ satisfying $f_j(z_j) = 0$, $f_j(x) = \|x\|$, and $\|f_j\|_{B^*} \leq 4\|x\|/\beta$. Set $\varepsilon = \beta/8$. Then we have, for $n \in J_{k,j}$,

$$\begin{aligned}P(\|S_{n,n}/\gamma_n - x\| < \varepsilon) &\leq P(|f_j(S_{n,n}/\gamma_n) - \|x\|\,| < \|x\|/2)\\ &\leq P(|f_j(S_{n,n})| \geq \|x\|\gamma_n/2).\end{aligned}$$

Using the non-uniform version of the Berry–Esseen inequality, we can estimate the last probability from above by

$$\exp\left(-\|x\|^2\gamma_n^2/(8n\sigma_{n,j}^2)\right) + An\beta^{-3}\delta_n,$$

where A is an absolute constant and $\sigma_{n,j}^2 = E(f_j^2(X_{n,1}))$. (Notice that $\|x\| \geq \beta$ since $0 \in \tilde{A}$ and dist(x, Ã) $= \beta$.) Recalling Lemma 6(b), it is easy to see that

$$\sigma_{n,j}^2 \leq \frac{8\|x\|^2}{\beta^2} e^{-k}\overline{\gamma}_n^2/(nL_2 n)$$

and we can conclude that for $n \in J_{k,j}$, $1 \leq j \leq k$, and large k:

$$P(\|S_{n,n}/\gamma_n - x\| < \varepsilon) \leq (Ln)^{-2} + An\beta^{-3}\delta_n \tag{3.47}$$

provided $\beta_j > \beta/4$.

If $\beta_j \leq \beta/4$, choose $y_j = \lambda_j z_j$ so that $\|x - y_j\| \leq \beta/4$. Clearly, $|\lambda_j - t| = \|y_j - tz_j\| \geq \|x - tz_j\| - \beta/4 \geq 3\beta/4$, provided $|t| < \sigma_j$, and consequently $|\lambda_j| \geq \sigma_j + 3\beta/4$. Now choose a functional $\tilde{f}_j \in B^*$ with $\|\tilde{f}_j\| = 1 = \tilde{f}_j(z_j)$. Then,

$$\begin{aligned}P(\|S_{n,n}/\gamma_n - x\| < \beta/8) &\leq P(\|S_{n,n}/\gamma_n - y_j\| < 3\beta/8)\\ &\leq P(|\tilde{f}_j(S_{n,n}/\gamma_n) - \lambda_j| < 3\beta/8)\\ &\leq P(|\tilde{f}_j(S_{n,n})| > (\sigma_j + 3\beta/8)\gamma_n).\end{aligned}$$

Using once more the non-uniform version of the Berry-Esseen inequality, along with the trivial bound

$$E(\tilde{f}_j^2(X_{n,1})) \le H(\overline{K}(n/L_2n)) \le \overline{\gamma}_n^2/(2nL_2n),$$

and (3.18), we can conclude that for $n \in J_{k,j}$, and large k,

$$P(\|S_{n,n}/\gamma_n - x\| < \beta/8) \le (Ln)^{-\sigma_j^2 - \beta^2/10} + A'n\beta^{-3}\delta_n \tag{3.48}$$

provided that $\beta_j \le \beta/4$.

Combining (3.47) and (3.48), a bit of calculation then obtains (3.44).

We next turn to the proof of (3.45), where we assume $\sigma < 1$. If $\sigma = 1$, (3.45) follows directly from (3.47) and (3.48), that is, use W_1 in the previous argument.

Let V_0 be the smallest closed subspace which contains the sequence $\{v_i\}$; that is, $V_0 = \mathrm{cl}(\bigcup_{n=1}^{\infty} \mathrm{V_n})$, where $V_n =$ span $\{v_1, \dots, v_n\}$. Set $\rho =$ dist (x, V_0).

If $\rho > 0$, then using Lemma 9, we can find for any k a functional $f_k \in B^*$ so that $f_k(x) = \|x\|$, $f_k(v_k) = 0$, and $\|f_k\|_{B^*} \le \|x\|/\rho$, and we get, for $n \in J_{k,k+1}$, $\varepsilon \le \rho/2$,

$$P(\|S_{n,n}/\gamma_n - x\| < \varepsilon) \le P(|f_k(S_{n,n}/\gamma_n)| > \|x\|/2),$$

which, as in (3.47), can be estimated from above by

$$(Ln)^{-2} + An\beta^{-3}\delta_n,$$

and we readily obtain (3.45) in this case.

If $\rho = 0$, we need a further lemma.

LEMMA 10. *Let W be a closed subspace of B and let v be a unit vector so that* dist$(v, W) \ge 1/2$. *Given $y \in W$, we can find a functional $f \in B^*$ so that* $f(y) = \|y\|$, $f(v) = 0$, *and* $\|f\|_{B^*} \le 3$.

PROOF. Let f be a linear functional in the subspace generated by $\{v, y\}$ satisfying $f(y) = \|y\|$ and $f(v) = 0$. Then f is such that for $\mu, \lambda \ne 0$,

$$\begin{aligned} |f(\mu y + \lambda v)| &= \|\mu y\|\ |\lambda|^{-1}\|v + (\mu/\lambda)y\|^{-1}\|\mu y + \lambda v\| \\ &\le 2\|\mu y\|\ |\lambda|^{-1}\|\mu y + \lambda v\| \\ &\le 3\|\mu y + \lambda v\| \end{aligned}$$

whenever $\|\mu y\| \le 3|\lambda|/2$. If $\|\mu y\| > 3|\lambda|/2$, then trivially,

$$\|\mu y + \lambda v\| \ge \|\mu y\| - |\lambda| \ge \|\mu y\|/3,$$

and therefore

$$|f(\mu y + \lambda v)| = \|\mu y\| \le 3\|\mu y + \lambda v\|$$

in both cases. Now extend f to all of B by the Hahn–Banach theorem. □

If $\rho = 0$, we can find $k_0 \geq 1, \lambda_1, \ldots, \lambda_{k_0} \in \mathbb{R}$ so that

$$\left\| x - \sum_{j=1}^{k_0} \lambda_j v_j \right\| < \varepsilon.$$

Setting $y = \sum_{j=1}^{k_0} \lambda_j v_j$, and choosing for $k \geq k_0+1$ linear functionals $\tilde{f}_k \in B^*$ with $\tilde{f}_k(v_k) = 0$, $\tilde{f}_k(y) = \|y\|$, $\|\tilde{f}_k\| \leq 3$, we can conclude for $\varepsilon < \|x\|/14$ that

$$\begin{aligned} P(\|S_{n,n}/\gamma_n - x\| < \varepsilon) &\leq P(\|S_{n,n}/\gamma_n - y\| < 2\varepsilon) \\ &\leq P(|\tilde{f}_k(S_{n,n}/\gamma_n) - \|y\|\,| < 6\varepsilon) \\ &\leq P(|\tilde{f}_k(S_{n,n}/\gamma_n)| \geq \|x\| - 7\varepsilon) \\ &\leq P(|\tilde{f}_k(S_{n,n}/\gamma_n)| \geq \|x\|/2). \end{aligned}$$

As in (3.47), the above implies we now have, for $n \in J_{k,k+1}$, $\varepsilon < \|x\|/14$ and large k, that

$$P(\|S_{n,n}/\gamma_n - x\| < \varepsilon) \leq (Ln)^{-2} + An\beta^{-3}\delta_n,$$

which yields (3.45).

If $\sigma = 0$, the two previous bounds also apply if $n \in J_{k,\ell}$, $1 \leq \ell \leq k$, and we obtain (3.44) as well when $\sigma = 0$.

It remains to verify (3.46) which is quite easy since by definition of $J'_{k,\ell}$, $1 \leq \ell \leq k+1$, we have for every $\delta > 0$ that

$$\sum_{k=1}^{\infty} \sum_{\ell=1}^{k+1} \sum_{n \in J'_{k,\ell}} n^{-1}(Ln)^{-\delta} < \infty. \tag{3.49}$$

Applying once more the Fuk–Nagaev inequality in Banach space, we get, for large n and $\varepsilon < \|x\|/2$ that

$$\begin{aligned} P(\|S_{n,n}/\gamma_n - x\| < \varepsilon) &\leq P(\|S_{n,n}/\gamma_n\| > \|x\|/2) \\ &\leq A((Ln)^{-\delta} + \beta^{-3} n\delta_n), \end{aligned}$$

where A is an absolute constant and $\delta = \delta(x) > 0$. Recalling $\sum_{n=1}^{\infty} \delta_n < \infty$ we get (3.46) from (3.49). Thus Theorem 2 is proven provided B has a Schauder basis.

STEP 5. Let now $(B, \|\cdot\|)$ be an arbitrary real separable Banach space, and let $\tilde{A} \subseteq B$ be a closed star shaped, symmetric subset of the unit ball of B.

Let $\Lambda : (B, \|\cdot\|) \to (C[0,1], \|\cdot\|_\infty)$ be a linear isometric embedding. Then $\Lambda(\tilde{A}) \subseteq C[0,1]$ is a closed, star shaped, and symmetric subset of the unit ball of $(C[0,1], \|\cdot\|_\infty)$. In view of steps 1-4 we can find a random variable $Y : \Omega \to C[0,1]$ such that Y satisfies the bounded LIL with respect to $\{\gamma_n\}$ and $C(\{\sum_{j=1}^n Y_j/\gamma_n\}) = \Lambda(\tilde{A})$, where $\{\gamma_n\}$ is defined via the random vector Y. Moreover, if either $\dim(B) = \dim(\Lambda(B)) = \infty$, or $\dim(B) = \dim(\Lambda(B)) < \infty$ and $\sup_{x\in\tilde{A}} \|x\| = 1$, we can assume that $P(Y \in \Lambda(B)) = 1$. (See the end of step 2.) Setting $X = \Lambda^{-1}(Y)$, then $X : \Omega \to B$ and noticing that the $\tilde{K}$-functions for X and Y are identical, we see X satisfies the bounded LIL with respect to $\{\gamma_n\}$ and with probability one $\{C(\frac{S_n}{\gamma_n})\} = \Lambda^{-1}(\{C(\sum_{j=1}^{n} Y_j/\gamma_n)\}) = \tilde{A}$. Thus Theorem 2 is proven. □

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