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Abstract— **This paper presents two approaches to achieving cluster synchronization in dynamical multi-agent systems. In contrast to the widely studied synchronization behavior, where all the coupled agents converge to the same value asymptotically, in the cluster synchronization problem studied in this paper, we require that all the interconnected agents to evolve into several clusters and each agent only to synchronize within its cluster. The first approach is to add a constant forcing to the dynamics of each agent that are determined by positive diffusive couplings; and the other is to introduce both positive and negative couplings between the agents. Some sufficient and/ or necessary conditions are constructed to guarantee n-cluster synchronization behavior. Simulation results are presented to illustrate the effectiveness of the theoretical analysis.**

I. INTRODUCTION

Recently the study of distributed coordination of multi-agent systems has attracted significant attention from researchers in different disciplines. Simple local coordination rules can sometimes lead to complicated collective behavior, such as synchronization that has been discovered in natural, social and engineering networks and systems [1], [2]. Different types of synchronization phenomena have been investigated, including, for example, complete synchronization [1] and generalized synchronization [2]. In this line of research, various algorithms have been successfully constructed to cause all the agents in a group to converge to the same value asymptotically [3], [4]. However, there is an emerging trend to study how an interconnected group may evolve into different sub-groups called clusters. Here we provide a few motivating examples from diverse backgrounds.

In nature, a group of foraging animals, such as a herd of cows, often need to make collective decisions on where and how to move utilizing social interactions between each other. In [5], Couzin $et\ al.$ study such animal collective decision-making behavior using the models of two types of individual agents. One is called informed agents who have the knowledge about the location of food sources and thus have preferred moving directions, and the other is called naive agents who know nothing about the food sources and have to interact with their neighbors to follow the group. Simulation results have been provided, which illustrate how a group may split into subgroups under certain circumstances. In the study of social networks, different mathematical models have been constructed to describe opinion dynamics in social communities. One such model is the the Krause model [6], [7], in which each agent updates its opinion in the form of a scalar by computing the average of those of its neighbors. In [7], Blondel $et\ al.$ provide some theoretical analysis results on how the agents may evolve into different clusters, where the agents in the same cluster hold the same opinion in the end. In addition, a lower bound on the differences between neighboring clusters has also been provided. The clustering behavior is also potentially useful for the formation control problem for teams of autonomous agents. In [8], one of the main research problems that have been surveyed is to split a formation into sub-formations in order to accomplish covering tasks or avoid obstacles.

Motivated by the above examples, we aim to study in this paper the cluster synchronization problem, in which a coupled multi-agent system is required to split into several clusters, such that the agents synchronize with one another in the same cluster, but differences exist between different clusters. Such problems are beginning to attract attention. For example, in [9] some sufficient conditions have been derived for the coupled oscillators to realize cluster synchronization under pinning control strategies. In this paper, we focus on the n-cluster synchronization problem. We provide two approaches to realizing clustering behavior. One is to add a constant forcing to each agent; and the other is to allow negative coupling between the agents.

The rest of the paper is organized as follows. Problem formulation is give in Section II. The main results are presented in Section III and IV. In Section V, we provide some illustrative examples.

Notations. Throughout this paper, the following notations are used: $\mathbf{1} = (1, \cdots, 1)^T$ with proper dimension and $\mathbf{1}_n = (1, \cdots, 1)^T \in \mathbb{R}^n$; I denotes the identity matrix with proper dimension; for a matrix A, we denote its spectral radius by $\rho(A)$; $A \leq 0$ (resp. $A < 0$) means that A is semi-negative (resp. negative) definite.

II. PROBLEM FORMULATION

The goal of this paper is to design algorithms to realize n-cluster synchronization. First, we define what we mean by n-cluster synchronization. Consider a dynamical system consisting of N agents with dynamics

$$\dot{x}(t) = f(x(t), t), \tag{1}$$

where $x(t) = (x_1(t), \ldots, x_N(t))^T \in \mathbb{R}^N$, $x_i(t)$ is the state of the ith agent, and $f : \mathbb{R}^N \times [0, \infty) \to \mathbb{R}^N$ is a continuous map.

Definition 1: [9] Let $\{C_1, C_2, \ldots, C_n\}$ be a partition of the set $\mathcal{N} = \{1, 2, \ldots, N\}$ into n nonempty subsets, i.e., $C_i \neq \emptyset$, and $\bigcup_{i=1}^n C_i = \{1, \ldots, N\}$. For $i \in \{1, \ldots, N\}$, let $\hat{i}$ denote

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the index of the subset in which the number i lies, i.e., $i \in C_{\hat{i}}$. System (1) is said to realize n-cluster synchronization with the partition $\{C_1, C_2, \ldots, C_n\}$, if $\lim_{t\to\infty} ||x_i(t)-x_j(t)|| = 0$ when $\hat{i} = \hat{j}$, and $\lim_{t\to\infty} ||x_i(t)-x_j(t)|| \neq 0$ when $\hat{i} \neq \hat{j}$.

Remark 1. In [10], a similar concept called the "group consensus" of a multi-agent system is defined, which is weaker than the cluster synchronization defined here because we require in addition that the differences between different clusters do not go to 0. A different type of clustering behavior is considered in [11], [12], where the differences between agents in the same cluster are bounded, while the differences between agents in different clusters grow unbounded as time goes to infinity.

In the sequel, we say that agents i and j are in the same cluster when $\hat{i} = \hat{j}$, and denote the number of agents in the ith cluster by l_i.

III. Cluster synchronization with constant forcing

In this section, we consider the n-cluster synchronization problem for a system consisting of both informed agents and naive agents to capture the features presented in the animal collective decision-making model studied in [5]. Here the informed agents are those under some external constant forcing, whose dynamics are described by

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{N} g_{ij} x_j(t) + a_{\hat{i}}, \tag{2}$$

where $g_{ij} \geq 0$ for $i \neq j$, $\sum_{j=1}^{N} g_{ij} = 0$, and $a_{\hat{i}}$ are constants satisfying $a_{\hat{i}} \neq a_{\hat{j}}$ for $\hat{i} \neq \hat{j}$. The dynamics of the naive agents are

$$\dot{x}_i(t) = \sum_{j=1}^{N} g_{ij} x_j(t). \tag{3}$$

Directed graphs are used to model communication topologies among agents. For an N-dimensional square matrix G, the graph $\mathcal{G}$ associated with G is a directed graph with the node set $\mathcal{V}(\mathcal{G}) = \{v_1, v_2, \ldots, v_N\}$ and the edge set $\mathcal{E}(\mathcal{G}) \subset \{(v_i, v_j) : v_i, v_j \in \mathcal{V}(\mathcal{G})\}$ where (v_i, v_j) is an edge of $\mathcal{G}$ if and only if $g_{ji} \neq 0$ with $i \neq j$. A directed path in $\mathcal{G}$ is a sequence of distinct vertices $v_{i_1}, \ldots, v_{i_k}$ such that $(v_{i_s}, v_{i_{s+1}}) \in \mathcal{V}(\mathcal{G})$ for $s = 1, \ldots, k-1$. A directed graph is strongly connected if there is a directed path from every node to every other node. A graph is balanced if $\sum_{j=1}^{N} g_{ij} = \sum_{j=1}^{N} g_{ji}$ for all i.

Next we give some sufficient and/ or necessary conditions for systems (2) and (3) to converge to n clusters.

A. Systems of informed agents

In this subsection, we consider the simple case when the system only consists of N informed agents described by (2) for $1 \leq i \leq N$. One can write the system into a compact form

$$\dot{x}(t) = -x(t) + Gx(t) + \bar{a} = \bar{G}x(t) + \bar{a}, \tag{4}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_N(t))^T \in \mathbb{R}^N$, $\bar{a} = (\underbrace{a_1, \ldots, a_1}_{l_1}, \cdots, \underbrace{a_n, \ldots, a_n}_{l_n})^T$, $G = (g_{ij})_{N\times N}$, and $\bar{G} = G - I$. Note that the agents in the same cluster have the same constant forcing.

Lemma 1. Let

$$P = (p_{ij})_{N\times N} = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{pmatrix},$$

where $P_{ii} \in \mathbb{R}^{l_i \times l_i}$, $1 \leq i \leq n$, are square matrices and $P_{ij} \in \mathbb{R}^{l_i \times l_j}$ for $i \neq j$. Suppose P is invertible and the inverse of P is

$$Q = (q_{ij})_{N\times N} = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix}. \tag{5}$$

If matrices P_{ij} have constant row sums r_{ij} for $1 \leq i, j \leq n$, then Q_{ij} have constant row sums s_{ij} for $1 \leq i, j \leq n$. In addition, $SR = I_{n\times n}$, where $R = (r_{ij})_{n\times n}$ and $S = (s_{ij})_{n\times n}$.

The proof will be present in the full-length version of the paper.

Let

$$G = \begin{pmatrix} G_{11} & G_{12} & \cdots & G_{1n} \\ G_{21} & G_{22} & \cdots & G_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n1} & G_{n2} & \cdots & G_{nn} \end{pmatrix}, \tag{6}$$

where $G_{ii} \in \mathbb{R}^{l_i \times l_i}$, $1 \leq i \leq n$, and $G_{ij} \in \mathbb{R}^{l_i \times l_j}$ for $i \neq j$. Since the row sums of $\bar{G}$ are equal to -1 and $\bar{G}$ has positive off-diagonal elements, we know $\bar{G}$ is invertible and the eigenvalues of $\bar{G}$ are all located in the left half plane. The equilibrium of system (4) is $x^* = -\bar{G}^{-1}\bar{a}$. Let $y(t) = x(t) - x^*$, then one has $\dot{y}(t) = \bar{G}y(t)$. It is obvious that $y(t) \to 0$ as $t \to \infty$. Thus x^* is a global stable equilibrium of system (4).

Theorem 1. The system (4) of informed agents achieves n-cluster synchronization for almost all (in the sense of Lebesgue measure) a_i with $1 \leq i \leq n$ and $a_i \neq a_j$ for $i \neq j$, if the block matrices G_{ij}, with $1 \leq i, j \leq n$ and $i \neq j$, have constant row sums.

Proof. Let $Q = (q_{ij})_{N\times N}$ defined in (5) be the inverse of $\bar{G}$. Since $\bar{G}_{ij}$, $i \neq j$, have constant row sums r_{ij} and the row sums of $\bar{G}$ are -1, it follows from Lemma 1 that Q_{ij} have constant row sums s_{ij} for $1 \leq i, j \leq n$, and $S = R^{-1}$, where $R = (r_{ij})_{n\times n}$, and $S = (s_{ij})_{n\times n}$. Then all the agents in the ith cluster have the same asymptotic value $-\sum_{j=1}^{n} s_{ij} a_j$.

Next we show that all the a_i's that do not lead to n-cluster synchronization come from a set which has zero Lebesgue measure. Let $\mathbb{S} = \{x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n : \ x_i = x_j \text{ for some } i \neq j \text{ with } 1 \leq i, j \leq n\}$, and a smooth map $g : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $g(x) = Rx$. Then it is easy to check that $\mathbb{S}$ has zero Lebesgue measure, so does $g(\mathbb{S})$.

Let $U = \{a = (a_1, \dots, a_n)^T \in \mathbb{R}^n : \ a_i \neq a_j$ for $i \neq j$; $(R^{-1}a)_i = (R^{-1}a)_j$ for some $i \neq j$ and $1 \leq i, j \leq n\}$, one has $U \subset g(\mathbb{S})$, which implies that U has zero Lebesgue measure. If $a \notin U$, system (4) realizes n-cluster synchronization, which completes the proof. $\square$

Remark 2. When we look at the interaction topology of the directed graph $\mathcal{G}$ associated with G, we can regard the uth row sum of the matrix $G_{ji},\ i \neq j$, as the weighted information received by the $(l_1 + \dots + l_{j-1} + u)$th node from the ith cluster. The fact that the block matrices $G_{ij},\ i \neq j$, have constant row sums implies that every agent in the same cluster receives the same weighted information from every other cluster.

For the special case when $n = 2$, we have the following result.

Corollary 1. System (4) achieves 2-cluster synchronization if and only if the block matrices $G_{ij}, 1 \leq i, j \leq 2$ and $i \neq j$, have constant row sums.

Proof. (Sufficiency). Let $Q = (q_{ij})_{N\times N} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$ be the inverse of $\bar{G}$. It follows from the fact that $\bar{G}_{ij}$ have constant row sums r_{ij} and Lemma 1 that Q_{ij} have constant row sums s_{ij} and

$$S = \begin{pmatrix} -\frac{r_{21}+1}{r_{12}+r_{21}+1} & -\frac{r_{12}}{r_{12}+r_{21}+1} \\ -\frac{r_{21}}{r_{12}+r_{21}+1} & -\frac{r_{12}+1}{r_{12}+r_{21}+1} \end{pmatrix}.$$

Thus system (4) converges to

$$x^* = -\bar{G}^{-1}\bar{a} = \begin{pmatrix} -a_1 s_{11} - a_2 s_{12} \\ \vdots \\ -a_1 s_{11} - a_2 s_{12} \\ -a_1 s_{21} - a_2 s_{22} \\ \vdots \\ -a_1 s_{21} - a_2 s_{22} \end{pmatrix}.$$

It is easy to check that $-a_1 s_{11} - a_2 s_{12} \neq -a_1 s_{21} - a_2 s_{22}$ since $a_1 \neq a_2$. Thus a two-cluster synchronization has been realized.

(Necessity). Suppose system (4) realizes two cluster synchronization with final values $\bar{x}_1$ and $\bar{x}_2$. Let $\mathcal{K} = \{k \in \mathcal{N}$, the final value of $x_k(t)$ is $\bar{x}_1\}$. We first show that every agent with the same constant forcing is in the same cluster. Suppose on the contrary the ith and jth agents both with constant forcing a_1 have different final values $\bar{x}_1$ and $\bar{x}_2$, then one has

$$\begin{aligned} 0 &= -\bar{x}_1 + a_1 + \sum_{k\in\mathcal{N}/\mathcal{K}, k\neq i} g_{ik}(\bar{x}_2 - \bar{x}_1), \\ 0 &= -\bar{x}_2 + a_1 + \sum_{k\in\mathcal{K}, k\neq j} g_{jk}(\bar{x}_1 - \bar{x}_2). \end{aligned}$$

It follows that $(\bar{x}_2 - \bar{x}_1)(1 + \sum_{k\in\mathcal{N}/\mathcal{K}, k\neq i} g_{ik} + \sum_{k\in\mathcal{K}, k\neq j} g_{jk}) = 0$, which contradicts $\bar{x}_2 - \bar{x}_1 \neq 0$ and $1 + \sum_{k\in\mathcal{N}/\mathcal{K}, k\neq i} g_{ik} + \sum_{k\in\mathcal{K}, k\neq j} g_{jk} > 0$.

From the proof of sufficiency, we find the equilibrium of system (4) is

$$x^* = -\begin{pmatrix} a_1 Q_{11}\mathbf{1}_{l_1} + a_2 Q_{12}\mathbf{1}_{l_2} \\ a_1 Q_{21}\mathbf{1}_{l_1} + a_2 Q_{22}\mathbf{1}_{l_2} \end{pmatrix}.$$

Let the ith row sums of Q_{11} and Q_{12} be t_{i1} and t_{i2} respectively. Then for any $1 \leq i, j \leq l_1$ and $a_1 \neq a_2$, we have $-a_1 t_{i1} - a_2 t_{i2} = -a_1 t_{j1} - a_2 t_{j2}$. It follows that $t_{i1} = t_{j1}$ and $t_{i2} = t_{j2}$ for $1 \leq i, j \leq l_1$. Thus, Q_{11} and Q_{12} have constant row sums. Applying similar arguments to Q_{21} and Q_{22}, one can conclude that G_{12} and G_{21} have constant row sums in view of Lemma 1. $\square$

B. Systems with informed and naive agents

In this section, we consider the system consisting of $n-1$ clusters of informed agents and one cluster of naive agents, which is described by

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{N} g_{ij}x_j(t) + a_{\hat{i}}, \quad 1 \leq i \leq l_1 + \dots + l_{n-1}, \tag{7}$$

and

$$\dot{x}_i(t) = \sum_{j=1}^{N} g_{ij}x_j(t), \qquad l_1 + \dots + l_{n-1} + 1 \leq i \leq N, \tag{8}$$

or in a compact form

$$\dot{x}(t) = \bar{G}x(t) + \bar{a}, \tag{9}$$

where

$$\bar{G} = \begin{pmatrix} G_{11} - I & \cdots & G_{1,n-1} & G_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ G_{n-1,1} & \cdots & G_{n-1,n-1} - I & G_{n-1,n} \\ G_{n1} & \cdots & G_{n,n-1} & G_{nn} \end{pmatrix},$$

$\bar{a} = (\underbrace{a_1, \dots, a_1}_{l_1}, \dots, \underbrace{a_{n-1}, \dots, a_{n-1}}_{l_{n-1}}, \underbrace{0, \dots, 0}_{l_n})^T$, $l_1 + \dots + l_n = N$. In this case, we also call the clusters of informed agents the leader clusters and call an agent in a leader cluster a leader.

Lemma 2. $\bar{G}$ is invertible if and only if for any naive agent, there is a directed path from some leader.

The proof is omitted here due to the length limit. In the following discussion, we assume that for any naive agent there is a directed path from some leader. Since $\bar{G}$ is invertible, the equilibrium x^* of system (9) is $x^* = -\bar{G}^{-1}\bar{a}$. Let $y(t) = x(t) - x^*$, then one has $\dot{y}(t) = \bar{G}y(t)$. It is obvious that $y(t) \to 0$ as $t \to \infty$. Thus x^* is a global stable equilibrium of system (9).

In order to ensure that agents in the same cluster have the same final values, we make the following requirements. Suppose $\bar{G}_{ij}$ have constant row sums r_{ij} for $i = 1, \dots, n-1,\ j = 1, \dots, n$, and the ith row sums of $G_{n1}, \dots, G_{n,n-1}$ are $m_i h_1, \dots, m_i h_{n-1}$ for $1 \leq i \leq l_n$, which can be regarded as the weighted information received from leader clusters, where m_i are positive constants. We require that there is at least one $h_i \neq 0$ with $1 \leq i \leq n-1$. Without loss of generality, suppose $h_1, \dots, h_k \neq 0,\ 1 < k \leq n-1$, and $h_{k+1} = \dots = h_{n-1} = 0$, it is easy to see that the row sums of G_{nn} are $-m_i \sum_{j=1}^{n-1} h_j$. Expanding the equation $Q\bar{G} = I$, one has

$$\begin{cases} r_{11} \sum_{j=1}^{l_1} q_{ij} + \cdots + q_{i,N-l_n+1} m_1 h_1 \\ \qquad\qquad + \cdots + q_{iN} m_{l_n} h_1 = 1, \\ r_{12} \sum_{j=1}^{l_1} q_{ij} + \cdots + q_{i,N-l_n+1} m_1 h_2 \\ \qquad\qquad + \cdots + q_{iN} m_{l_n} h_2 = 0, \\ \vdots \\ r_{1n} \sum_{j=1}^{l_1} q_{ij} + \cdots - q_{i,N-l_n+1} m_1 \sum_{j=1}^{n-1} h_j \\ \qquad\qquad - \cdots - q_{iN} m_{l_n} \sum_{j=1}^{n-1} h_j = 0, \end{cases}$$

where $1 \le i \le l_1$. Let

$$M = \begin{pmatrix} h_2 r_{11} - h_1 r_{12} & \cdots & h_2 r_{n-1,1} - h_1 r_{n-1,2} \\ \vdots & \cdots & \vdots \\ h_k r_{11} - h_1 r_{1k} & \ddots & h_k r_{n-1,1} - h_1 r_{n-1,k} \\ \vdots & \ddots & \vdots \\ r_{1,n-1} & \cdots & r_{n-1,n-1} \\ -1 & \cdots & -1 \end{pmatrix},$$

one has

$$M \begin{pmatrix} \sum_{j=1}^{l_1} q_{ij} \\ \sum_{j=l_1+1}^{l_1+l_2} q_{ij} \\ \vdots \\ \sum_{j=N-l_n+1}^{N} q_{ij} \end{pmatrix} = \begin{pmatrix} h_2 \\ \vdots \\ h_k \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

M is invertible because of the invertibility of $\bar{G}$. By some simple calculations, it is easy to derive that $Q_{ij},\ 1 \le i \le n,\ 1 \le j \le n-1$, have constant row sums s_{ij}, and

$$\begin{pmatrix} s_{11} & \cdots & s_{1,n-1} \\ \vdots & \ddots & \vdots \\ s_{n1} & \cdots & s_{n,n-1} \end{pmatrix} = \begin{pmatrix} h_2 \ \cdots \ h_k & 0 \ \cdots \ 0 & 1 \\ -h_1 I & 0 & \mathbf{1} \\ 0 & I & \mathbf{1} \end{pmatrix} M^{-T}.$$

For $1 \le i \le n-1$, $\sum_{j=1}^{n} r_{ij} = -1$, and $\sum_{j=1}^{n} r_{nj} = 0$, it is easy to show that $\sum_{j=1}^{n-1} s_{ij} = -1$, for $1 \le i \le n$. Moreover, for $1 \le i \le n-1,\ 1 \le k \le l_n$, one can derive from $\bar{G}Q = I$ that

$$m_k h_1 s_{1i} + \cdots + m_k h_{n-1} s_{n-1,i} - m_k \sum_{j=1}^{n-1} h_j s_{ni} = 0.$$

It follows that

$$s_{ni} = \frac{\sum_{k=1}^{n-1} h_k s_{ki}}{\sum_{j=1}^{n-1} h_j}.$$

Suppose $\bar{x}_1, \ldots, \bar{x}_n$ are the final values for the n clusters, then each cluster converges to $\bar{x}_i = -\sum_{j=1}^{n-1} s_{ij} a_j$, and in addition

$$\begin{aligned} \bar{x}_n &= -\sum_{t=1}^{n-1} s_{nt} a_t = -\sum_{t=1}^{n-1} \sum_{k=1}^{n-1} \frac{h_k s_{kt}}{\sum_{j=1}^{n-1} h_j} a_t \\ &= \sum_{k=1}^{n-1} \frac{h_k}{\sum_{j=1}^{n-1} h_j} \left(-\sum_{t=1}^{n-1} s_{kt} a_t\right) = \sum_{k=1}^{n-1} \frac{h_k \bar{x}_k}{\sum_{j=1}^{n-1} h_j} \end{aligned}$$

which implies that the final values of the naive agents are a linear combination of the final values of the leader clusters. The coefficients $\frac{h_k}{\sum_{j=1}^{n-1} h_j}$ are determined by the row sums of $G_{n1}, \ldots, G_{n,n-1}$. If $\bar{x}_i,\ 1 \le i \le n$, are not equal to each other, then the n-cluster synchronization of system (10) is realized.

Note that these final values only depend on the row sums of the sub-matrices of $\bar{G}$, i.e., the weighted information received by each leader cluster or by each naive agent. It does not depend on the number of agents and the proportion of informed agents in the system.

In the next section, we present a different approach to achieving cluster synchronization.

IV. Cluster Synchronization with Negative Weights

Consider the linear time-invariant multi-agent system

$$\dot{x}(t) = Gx(t), \tag{10}$$

where $G \in \mathbb{R}^{N \times N}$ is in the form of (6). It is well-known that if the interaction topology associated with G contains a directed spanning tree, then the system achieves consensus [4]. Here we discuss the n-cluster synchronization problem for system (10). Let $\eta_1 = (\underbrace{1, \ldots, 1}_{l_1}, 0, \ldots, 0)^T$, $\eta_2 = (0, \ldots, 0, \underbrace{1, \ldots, 1}_{l_2}, 0, \ldots, 0)^T$, $\ldots$, $\eta_n = (0, \ldots, 0, \underbrace{1, \ldots, 1}_{l_n})^T$ be n independent right eigenvectors associated with 0, and $\alpha_1, \ldots, \alpha_n$ be the corresponding n left eigenvectors satisfying $\eta_i^T \alpha_j = 1$, if $i = j$ and $\eta_i^T \alpha_j = 0$, if $i \ne j$. Since the solution of (10) is $x(t) = e^{Gt} x(0)$, it is obvious that if the following matrix equation

$$\lim_{t \to \infty} e^{Gt} = \sum_{i=1}^{n} \eta_i \alpha_i^T \tag{11}$$

holds, then n-cluster synchronization is achieved provided that we have some constraints on the initial conditions. We obtain the following necessary and sufficient condition under which (11) holds. The proof will be present in the full-length version of the paper.

Lemma 3. (11) holds if and only if

$$G\eta_i = 0,\ \alpha_i^T G = 0, \tag{12}$$

where $i = 1, \ldots, n$, and G has exactly n zero eigenvalues and all the other eigenvalues have negative real parts.

Remark 3. From Lemma 3, one can see that in order to realize n-cluster synchronization, it is required that G_{ij} have zero row sums, which means that the coupling between clusters need to be canceled out after n-cluster synchronization is realized.

In the following discussion, suppose G satisfies the condition that the row sums of G_{ij}, $1 \le i, j \le n$, are 0, then G has a zero eigenvalue with the geometric multiplicity at least n.

Theorem 2. Suppose the initial values of system (10) satisfy that $\alpha_i^T x(0)$ with $1 \le i \le n$ are not equal to each other, then n-cluster synchronization can be achieved if and only if G has exactly n zero eigenvalues and all the other eigenvalues have negative real parts.

Remark 4. Similar result has been given in [10], where a weaker notion of cluster synchronization is considered. Compared to the conditions in [10], the additional requirement in Theorem 2 is that $\alpha_i^T x(0)$, $1 \le i \le n$, are not equal to each other.

The condition given in Theorem 2 for achieving n-cluster synchronization is an algebraic condition, which is difficult to check in application. Our aim then is to develop algorithms to construct proper coupling topologies that satisfy the algebraic condition.

Lemma 4. [13] Let A and B be $N \times N$ Hermitian matrices and let the eigenvalues $\lambda_i(A)$, $\lambda_i(B)$, and $\lambda_i(A+B)$ be arranged in decreasing order as

$$\lambda_N(\cdot) \le \lambda_{N-1}(\cdot) \le \cdots \le \lambda_1(\cdot).$$

For each $k = 1, 2 \cdots, N$ we have

$$\lambda_k(A) + \lambda_N(B) \le \lambda_k(A+B) \le \lambda_k(A) + \lambda_1(B).$$

Intuitively, if the inner couplings within the clusters are strong enough, system (10) can achieve cluster synchronization. In fact, we have the following result.

Proposition 1. Let

$$\begin{aligned} G &= diag\{c_1 G_{11}, \dots, c_n G_{nn}\} \\ &\quad + \begin{pmatrix} 0 & G_{12} & \cdots & G_{1n} \\ G_{21} & 0 & \cdots & G_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n1} & G_{n2} & \cdots & 0 \end{pmatrix} \end{aligned}$$

be a symmetric matrix, $G_1 = diag\{c_1 G_{11}, \dots, c_n G_{nn}\}$, $G_2 = G - G_1$. Suppose G_{ij} have zero row sums, matrices G_{ii} are irreducible and the off-diagonal elements of G_{ii} are nonnegative. If $c_i > \frac{\rho(G_2)}{-\max_{1 \le i \le n} \lambda_2(G_{ii})}$, then G has exactly n zero eigenvalues and all the other eigenvalues are negative.

Proof. Since G_{ij} have zero row sums, G has at least n zero eigenvalues. Using Lemma 4, one has

$$\lambda_N(G_2) \le \lambda_i(G) - \lambda_i(G_1) \le \lambda_1(G_2),$$

which leads to $|\lambda_i(G) - \lambda_i(G_1)| \le \rho(G_2)$. It follows from $c_i > \frac{\rho(G_2)}{-\max_{1 \le i \le n} \lambda_2(G_{ii})}$ that $\max_{1 \le i \le n} c_i \lambda_2(G_{ii}) + \rho(G_2) < 0$. From the assumptions, one has $-G_{ii}$ are irreducible Laplacian matrices. It follows that $\lambda_1(G_1) = \cdots = \lambda_n(G_1) = 0$, and $\lambda_{n+1}(G_1) = \max_{1 \le i \le n} c_i \lambda_2(G_{ii})$. Thus one concludes $\lambda_{n+1}(G) \le \max_{1 \le i \le n} c_i \lambda_2(G_{ii}) + \rho(G_2) < 0$. $\square$

Proposition 2. Suppose the graphs $\mathcal{G}_1, \dots, \mathcal{G}_n$ associated with $G_1, \dots, G_n$ are balanced and strongly connected, then for any positive definite matrix S with proper dimension, zero is an eigenvalue of $Sdiag\{G_1, \dots, G_n\}$ of algebraic and geometric multiplicity n, and all the other eigenvalues of $Sdiag\{G_1, \dots, G_n\}$ have negative real parts.

The Proposition can be proved using a similar argument as the proof of Theorem 4.5 in [14].

Proposition 2 provides a way to construct a graph satisfying the condition in Theorem 2. Let $\mathcal{G}'$ be a graph with n disconnected components, which are strongly connected and balanced. Let the matrix associated with $\mathcal{G}'$ be G', then multiplying from the left a positive definite matrix S gives us a matrix $G = SG'$ satisfying the condition in Theorem 2.

V. Illustrative Examples

In this section, several examples are given to illustrate the theoretical analysis results. First, consider the system consisting of two clusters of informed agents and one cluster of naive agents with $l_1 = l_2 = l_3 = 2$ and $a_1 = 1,\ a_2 = 7$. The coupling matrix is given by

$$G = \begin{pmatrix} -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 & -3 & 0 \\ 0 & 2 & 4 & 0 & 0 & -6 \end{pmatrix},$$

Since the final value for the first and second clusters are 4 and 5.5, respectively, the values of the naive agents converge to $4 \times \frac{1}{3} + 5.5 \times \frac{2}{3} = 5$. Fig. 1 shows the evolution of the three clusters.

Let

$$G' = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Obviously the associated graph $\mathcal{G}'$ contains two disconnected components, which are balanced and strongly connected. Multiplying from the left a positive definite matrix S leads to the matrix $G = SG'$

$$\begin{pmatrix} -1.3992 & -0.0542 & 1.4534 & 0.6371 & -0.6371 \\ 2.2438 & -1.6874 & -0.5563 & 0.2567 & -0.2567 \\ 0.6106 & 0.1734 & -0.7839 & -0.0629 & 0.0629 \\ 0.7467 & -0.2281 & -0.5186 & -1.5581 & 1.5581 \\ -0.1471 & -0.0343 & 0.1814 & 0.3770 & -0.3770 \end{pmatrix}.$$

G has exactly two zero eigenvalues and the rest three eigenvalues have negative real parts. Fig. 2 shows the evolution of the system states, from which we find that 2-cluster synchronization is achieved.

An interesting graph that realizes 2-cluster synchronization has the topology shown in Fig. 3. The associated matrix G has two zero eigenvalues and the rest eigenvalues have negative real parts. Let groups 1, 2, 3 be $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, respectively. It is easy to find from Figs. 3 and 4 that,

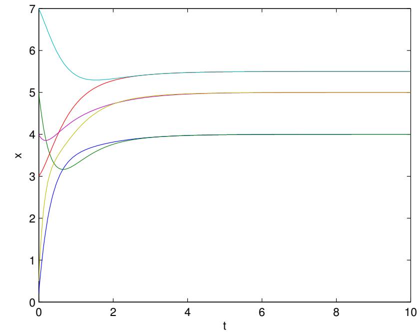

Fig. 1. The evolution of the system states.

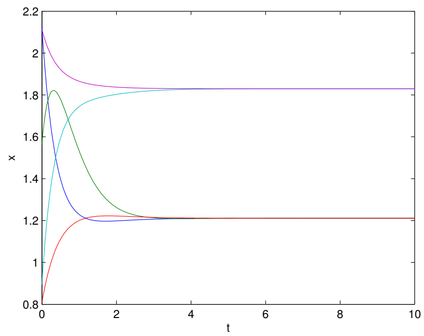

Fig. 2. The evolution of the system states.

although there is no direct connection between groups 1 and 2, the states of the agents in these two groups finally achieve the same value via the interconnection with agents in group 3, which have a different final state.

VI. Concluding remarks

This paper has investigated the approaches for realizing n-cluster synchronization in multi-agent systems. First, some sufficient conditions for the system containing informed and naive agents to achieve n-clusters are given. Second, we provide a systematic way to construct the coupling matrix with negative weights. Numerical examples are given to verify the effectiveness of our methods. The two approaches presented in this paper are just examples for constructing cluster synchronization algorithms. It is envisioned that more

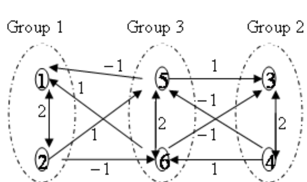

Fig. 3. The communication topology of a network.

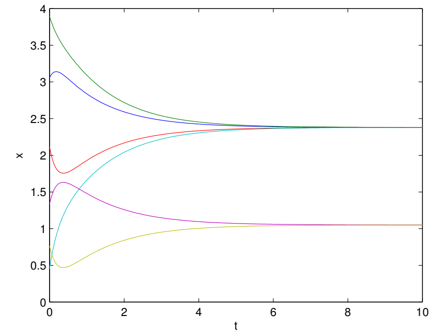

Fig. 4. State trajectories. (Agents 1,2,3,4 are in the same cluster)

such algorithms may appear and then their advantages and disadvantages can be compared. The constructed algorithms might lead to insight into the clustering behavior in natural and man-made systems, and in the end help to design efficient coordination algorithms for dynamic multi-agent systems.

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