Cluster synchronization in networks of coupled nonidentical dynamical systems

Wenlian Lu,[1,a)] Bo Liu,[2,b)] and Tianping Chen[2,c)]
[1]*Centre for Computational Systems Biology, Fudan University, Shanghai 200433, People's Republic of China*
[2]*School of Mathematical Sciences, Fudan University, Shanghai 200433, People's Republic of China and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, People's Republic of China*

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In this paper, we study cluster synchronization in networks of coupled nonidentical dynamical systems. The vertices in the same cluster have the same dynamics of uncoupled node system but the uncoupled node systems in different clusters are different. We present conditions guaranteeing cluster synchronization and investigate the relation between cluster synchronization and the unweighted graph topology. We indicate that two conditions play key roles for cluster synchronization: the common intercluster coupling condition and the intracluster communication. From the latter one, we interpret the two cluster synchronization schemes by whether the edges of communication paths lie in inter- or intracluster. By this way, we classify clusters according to whether the communications between pairs of vertices in the same cluster still hold if the set of edges inter- or intracluster edges is removed. Also, we propose adaptive feedback algorithms to adapting the weights of the underlying graph, which can synchronize any bi-directed networks satisfying the conditions of common intercluster coupling and intracluster communication. We also give several numerical examples to illustrate the theoretical results.
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Cluster synchronization is considered to be more momentous than complete synchronization in brain science and engineering control, ecological science and communication engineering, and social science and distributed computation. Most of the existing works only focused on networks with either special topologies such as regular lattices or coupled two/three groups. For the general coupled dynamical systems, theoretical analysis to clarify the relationship between the (unweighted) graph topology and the cluster scheme, including both self-organization and driving, is absent. In this paper, we study this topic and find two essential conditions for an unweighted graph topology to realize cluster synchronization: the common intercluster coupling condition and the intracluster communication. Thus under these conditions, we present two manners of weighting to achieve cluster synchronization. One is adding positive weights on each edge with keeping the invariance of the cluster synchronization manifold and the other is an adaptive feedback weighting algorithm. We prove the availability of each manner. From these results, we give an interpretation of the two clustering synchronization schemes via the communication between pairs of individuals in the same cluster. Thus, we present one way to classify the clusters via whether the set of inter- or intracluster edges is removable if still wanting to keep the communication between vertices in the same cluster.

I. INTRODUCTION

Recent decades witness that chaos synchronization in complex networks has attracted increasing interests from many research and application fields,[1–3] since it was first introduced in Ref. 4. The word "synchronization" comes from Greek, which means "share time" and today, it comes to be considered as "time coherence of different processes." Many new synchronization phenomena appear in a wide range of real systems, such as biology,[5] neural networks,[6] and physiological processes.[7] Among them, the most interesting cases are complete synchronization, cluster synchronization, phase synchronization, imperfect synchronization, lag synchronization, almost synchronization, etc. See Ref. 8 and the references therein.

Complete synchronization is the most special one and characterized by that all oscillators approach to a uniform dynamical behavior. In this situation, powerful mathematical techniques from dynamical systems and graph theory can be utilized. Pecora *et al.*[9] proposed the master stability function for transverse stability analysis[10] of the diagonal synchronization manifold. This method has been widely used to study local completer synchronization in networks of coupled system.[11] References 12–14 proposed a framework of Lyapunov function method to investigate global synchronization in complex networks. One of the most important issues is how the graph topology affects the synchronous

[a)]Electronic mail: wenlian@fudan.edu.cn.
[b)]Electronic mail: 071018024@fudan.edu.cn.
[c)]Electronic mail: tchen@fudan.edu.cn.

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motion.[2] As pointed out in Ref. 15, the connectivity of the graph plays a significant role for chaos synchronization.

Cluster synchronization is considered to be more momentous in brain science[16] and engineering control,[17] ecological science[18] and communication engineering,[19] and social science[20] and distributed computation.[21] This phenomenon is observed when the oscillators in networks are divided into several groups, called clusters, by the way that all individuals in the same cluster reach complete synchronization but the motions in different clusters do not coincide. Cluster synchronization of coupled identical systems is studied in Refs. 22–25. Among them, Jalan *et al.*[25] pointed out two basic formations which realize cluster synchronization. One is *self-organization*, which leads to cluster with dominant intracluster couplings, and the other is *driving*, which leads to cluster with dominant intercluster couplings.

Nowadays, the interest of cluster synchronization is shifting to networks of coupled nonidentical dynamical systems. In this case, cluster synchronization is obtained via two aspects: the oscillators in the same cluster have the same uncoupled node dynamics and the inter- or intracluster interactions realize cluster synchronization via driving or/and self-organizing configurations. Reference 23 proposed cluster synchronization scheme via dominant intracouplings and common intercluster couplings. Reference 26 studied local cluster synchronization for bipartite systems, where no intracluster couplings (driving scheme) exist. Reference 27 investigated global cluster synchronization in networks of two clusters with inter- and intracluster couplings. Belykh *et al.* studied this problem in one-dimensional and two-dimensional lattices of coupled identical dynamical systems in Ref. 22 and nonidentical dynamical systems in Ref. 28, where the oscillators are coupled via inter- or/and intracluster manners. Reference 29 used nonlinear contraction theory[30] to build up a sufficient condition for the stability of certain invariant subspace, which can be utilized to analyze cluster synchronization (concurrent synchronization is called in that literature). However, until now, there are no works revealing the relationship between the (unweighted) graph topology and the cluster scheme, including both self-organization and driving, for general coupled dynamical systems.

The purpose of this paper is to study cluster synchronization in networks of coupled nonidentical dynamical systems with various graph topologies. In Sec. II, we formulate this problem and study the existence of the cluster synchronization manifold. Then, we give one way to set positive weights on each edge and derive a criterion for cluster synchronization. This criterion implies that the communicability between each pair of individuals in the same cluster is essential for cluster synchronization. Thus, we interpret the two communication schemes according to the communication scheme among individuals in the same cluster. By this way, we classify clusters according to the manner by which synchronization in a cluster realizes. In Sec. III, we propose an adaptive feedback algorithm on weights of the graph to achieve a given clustering. In Sec. IV, we discuss the cluster synchronizability of a graph with respect to a given clustering and present the general results for cluster synchronization in networks with general positive weights. We conclude this paper in Sec. V.

II. CLUSTER SYNCHRONIZATION ANALYSIS

In this section, we study cluster synchronization in a network with weighted bidirected graph and a division of clusters. We impose the constraints on graph topology to guarantee the invariance of the corresponding cluster synchronization manifold and derive the conditions for this invariant manifold to be globally asymptotically stable by the Lyapunov function method. Before that, we should formulate the problem.

Throughout the paper, we denote a positive definite matrix Z by $Z>0$ and similarly for $Z<0$, $Z\le 0$, and $Z\ge 0$. We say that a matrix Z is positive definite on a linear subspace V if $u^\top Z u>0$ for all $u\in V$ and $u\neq 0$, denoted by $Z|_V>0$. Similarly, we can define $Z|_V<0$, $Z|_V\ge 0$, and $Z|_V\le 0$. If a matrix Z has all eigenvalues real, then we denote by $\lambda_k(Z)$ the kth largest eigenvalues of Z. $Z^\top$ denotes the transpose of the matrix Z and $Z^s=(Z+Z^\top)/2$ denotes the symmetry part of a square matrix Z. $\#A$ denotes the number of the set A with finite elements.

A. Model description and existence of invariant cluster synchronization manifold

A bidirected unweighted graph $\mathcal{G}$ is denoted by a double set $\{\mathcal{V},\mathcal{E}\}$, where $\mathcal{V}$ is the vertex set numbered by $\{1,\ldots,m\}$, and $\mathcal{E}$ denotes the edge set with $e(i,j)\in\mathcal{E}$ if and only if there is an edge connecting vertices j and i. $\mathcal{N}(i)=\{j\in\mathcal{V}:e(i,j)\in\mathcal{E}\}$ denotes the neighborhood set of vertex i. The graph considered in this paper is always supposed to be simple (without self-loops and multiple edges) and bidirected. A clustering $\mathcal{C}$ is a disjoint division of the vertex set $\mathcal{V}$:$\mathcal{C}=\{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_K\}$ satisfying (*i*). $\cup_{k=1}^{K}\mathcal{C}_k=\mathcal{V}$; (*ii*). $\mathcal{C}_k\cap\mathcal{C}_l=\emptyset$ holds for $k\neq l$.

The network of coupled dynamical system is defined on the graph $\mathcal{G}$. The individual uncoupled system on the vertex i is denoted by an n-dimensional ordinary differential equation $\dot{x}^i=f_k(x^i)$ for all $i\in\mathcal{C}_k$, where $x^i=[x_1^i,\ldots,x_n^i]^\top$ is the state variable vector on vertex i and $f_k(\cdot):\mathbb{R}^n\to R^n$ is a continuous vector-valued function. Each vertex in the same cluster has the same individual node dynamics. The interaction among vertices is denoted by linear diffusion terms. It should be emphasized that f_k for different clusters are distinct, which can guarantee that the trajectories are apparently distinguishing when cluster synchronization is reached.

Consider the following model of networks of linearly coupled dynamical system:[31]

$$\dot{x}^i=f_k(x^i)+\sum_{j\in\mathcal{N}(i)}w_{ij}\Gamma(x^j-x^i),\quad i\in\mathcal{C}_k,\quad k=1,\ldots,K, \quad (1)$$

where w_{ij} is the coupling weight at the edge from vertex j to i and $\Gamma=[\gamma_{uv}]_{u,v=1}^{n}$ denotes the inner connection by the way that $\gamma_{uv}\neq 0$ if the uth component of the vertices can be influenced by the vth component. The graph $\mathcal{G}$ is bidirected and the weights are not requested to be symmetric.

Namely, we do not request $w_{ij}=w_{ji}$ for each pair (i,j) with $e(i,j)\in\mathcal{E}$.

Let $A=[a_{ij}]_{i,j=1}^{m}$ be the adjacent matrix of the graph $\mathcal{G}$. That is, $a_{ij}=1$ if $e(i,j)\in\mathcal{E}$; $a_{ij}=0$ otherwise. Then, model (1) can be rewritten as

$$\dot{x}^i = f_k(x^i) + \sum_{j=1}^{m} a_{ij} w_{ij} \Gamma(x^j - x^i), \quad i \in \mathcal{C}_k, \quad k=1,\ldots,K. \tag{2}$$

In this paper, cluster synchronization is defined as follows.

(1) The differences among trajectories of vertices in the same cluster converge to zero as time goes to infinity, i.e.,

$$\lim_{t\to\infty}[x^i(t) - x^j(t)] = 0, \quad \forall\, i,\ j \in \mathcal{C}_k, \quad k=1,\ldots,K. \tag{3}$$

(2) The differences among the trajectories of vertices in different clusters do not converge to zero, i.e., $\overline{\lim}_{t\to\infty}|x^{i'}(t)-x^{j'}(t)|>0$ holds for each $i'\in\mathcal{C}_k$ and $j'\in\mathcal{C}_l$ with $k\neq l$.

As mentioned above, we suppose that the latter one can be guaranteed by the incoincidence of $f_k(\cdot)$. Under this prerequisite assumption, cluster synchronization is equivalent to the asymptotical stability of the following cluster synchronization manifold with respect to the clustering $\mathcal{C}$:

$$\begin{aligned} \mathcal{S}_{\mathcal{C}}(n) = \{ & [x^{1\top},\ldots,x^{m\top}]^\top : x^i = x^j \in \mathbb{R}^n, \\ & \forall\, i,\ j \in \mathcal{C}_k,\ k=1,\ldots,K. \end{aligned} \tag{4}$$

To investigate cluster synchronization, a prerequisite requirement is that the manifold $\mathcal{S}_{\mathcal{C}}(n)$ should be invariant through Eq. (2). Assume that $x^i(t)=s^k(t)$ for each $i\in\mathcal{C}_k$ is the synchronized solution of the cluster $\mathcal{C}_k$, $k=1,\ldots,K$. By Eq. (2), each s^k must satisfy

$$\dot{s}^k = f_k(s^k) + \sum_{k'=1,k'\neq k}^{K} \alpha_{i,k'} \Gamma(s^{k'} - s^k), \quad \forall\, i \in \mathcal{C}_k, \tag{5}$$

where $\alpha_{i,k'}=\sum_{j\in\mathcal{C}_{k'}} a_{ij}w_{ij}$. This demands $\alpha_{i_1,k'}=\alpha_{i_2,k'}$ for any $i_1\in\mathcal{C}_k$, $i_2\in\mathcal{C}_k$, namely, $\alpha_{i,k'}$ is independent of i. Therefore, we have

$$\alpha_{i,k'} = \alpha(k,k'), \quad i \in \mathcal{C}_k, \quad k \neq k'. \tag{6}$$

This condition is sufficient and necessary for the cluster synchronization manifold $\mathcal{S}_{\mathcal{C}}(n)$ is invariant through the coupled system (2) for general maps $f_k(\cdot)$.

Denote $\mathcal{N}_{k'}(i)=\mathcal{N}(i)\cap\mathcal{C}_{k'}$ and define an index set $\mathcal{L}_k^i=\{k':k'\neq k$ and $N_{k'}(i)\neq\emptyset\}$. The set $\mathcal{L}_k^i$ represents those clusters other than $\mathcal{C}_k$ and have links to the vertex i. To satisfy the condition (6), the following *common intercluster coupling condition* over the unweighted graph topology should be satisfied: for $k=1,\ldots,K$,

$$\mathcal{L}_k^i = \mathcal{L}_k^{i'}, \quad \forall\, i,\ i' \in \mathcal{C}_k. \tag{7}$$

Therefore, we can use $\mathcal{L}_k$ to represent $\mathcal{L}_k^i$ for all $i\in\mathcal{C}_k$ if the common intercluster coupling condition is satisfied.

Throughout this paper, we assume that the vector-valued function $f_k(x)-\alpha\Gamma x:\mathbb{R}^n\to\mathbb{R}^n$ satisfies *decreasing condition* for some $\alpha\in\mathbb{R}$. That is, there exists $\delta>0$ such that

$$(\xi-\zeta)^\top[f_k(\xi)-f_k(\zeta)-\alpha\Gamma(\xi-\zeta)] \leq -\delta(\xi-\zeta)^\top(\xi-\zeta) \tag{8}$$

holds for all $\xi,\zeta\in\mathbb{R}^n$. This condition holds for any globally Lipschitz continuous function $f(\cdot)$ for sufficiently large $\alpha>0$ and $\Gamma=I_n$. However, even though $f(\cdot)$ is only locally Lipschitz, if the solution of the coupled system (1) is essentially bounded, then restricted to such bounded region, the condition (8) also holds for sufficiently large α and $\Gamma=I_n$. In this paper, we suppose that the solution of the coupled system (2) is essentially bounded.

B. Cluster synchronization analysis

In the following, we investigate cluster synchronization of networks of coupled nonidentical dynamical systems with the following weighting scheme:

$$w_{ij} = \begin{cases} \dfrac{c}{d_{i,k}}, & j \in \mathcal{N}_k(i) \quad \text{and} \quad \mathcal{N}_k(i) \in \emptyset \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

where $d_{i,k'}=\#\mathcal{N}_k(i)$ denotes the number of elements in $N_{k'}(i)$ and c denotes the coupling strength. Thus, the coupled system becomes

$$\begin{aligned} \dot{x}^i = {} & f_k(x^i) + c\left[\sum_{\mathcal{N}_{k'}(i)\neq\emptyset} \frac{1}{d_{i,k'}} \sum_{j\in\mathcal{N}_{k'}(i)} \Gamma(x^j - x^i)\right], \\ & i \in \mathcal{C}_k, \quad k=1,\ldots,K. \end{aligned} \tag{10}$$

It can be seen that in Eq. (10), for each $i\in\mathcal{C}_k$, the corresponding $\alpha_{i,k'}=c$ for all $k'\in\mathcal{L}_k$ under the common intercluster coupling condition. The general situation can be handled by the same approach and will be presented in Sec. IV.

We denote the weighted Laplacian of the graph as follows. For each pair (i,j) with $i\neq j$, $l_{ij}=1/d_{i,k}$ if $j\in\mathcal{N}_k(i)$ and $\mathcal{N}_k(i)\neq\emptyset$ for some $k\in\{1,\ldots,K\}$, and $l_{ij}=0$ otherwise; $l_{ii}=-\sum_{j=1}^{m} l_{ij}$. Thus, Eq. (10) can be rewritten as

$$\dot{x}^i = f_k(x^i) + c\sum_{j=1}^{m} l_{ij}\Gamma x^j, \quad i \in \mathcal{C}_k, \quad k=1,\ldots,K. \tag{11}$$

The approach to analyze cluster synchronization is extended from that used in Ref. 14 to study complete synchronization. Let $d=[d_1,\ldots,d_m]^\top$ be a vector with $d_i>0$ for all $i=1,\ldots,m$. We use the vector d to construct a (skew) projection of $x=[x^{1\top},\ldots,x^{m\top}]^\top$ onto the cluster synchronization manifold $\mathcal{S}_{\mathcal{C}}(n)$. Define an average state with respect to d in the cluster $\mathcal{C}_k$ as

$$\bar{x}_d^k = \frac{1}{\sum_{i\in\mathcal{C}_k} d_i} \sum_{i\in\mathcal{C}_k} d_i x^i.$$

Thus, we denote the projection of x on the cluster synchronization manifold $\mathcal{S}_{\mathcal{C}}(n)$ with respect to d as $\bar{x}_d$ $=[\tilde{x}^{1^\top},\ldots,\tilde{x}^{m^\top}]^\top$ is denoted as

$$\bar{x}^i = \bar{x}_d^k \ \text{ if } \ i \in \mathcal{C}_k.$$

Then, the variations $x^i - \bar{x}_d^k$ compose the transverse space

$$\mathcal{T}_{\mathcal{C}}^d(n) = \left\{ u = [u^{1\top}, \ldots, u^{m\top}]^\top \in \mathbb{R}^{mn} : u^i \in \mathbb{R}^n, \ \sum_{i \in \mathcal{C}_k} d_i u^i = 0, \quad \forall\, k = 1, \ldots, K \right\}.$$

In particular, in the case of $n=1$, it denotes

$$\mathcal{T}_{\mathcal{C}}^d(1) = \left\{ u = [u^1, \ldots, u^m]^\top \in \mathbb{R}^m : \sum_{i \in \mathcal{C}_k} d_i u^i = 0, \quad \forall\, k = 1, \ldots, K \right\}.$$

From the definition, we have the following lemma which is repeatedly used below.

Lemma 1: For each $k \in 1, \cdots, K$, it holds

$$\sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k) = 0.$$

In fact, note

$$\sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k) = \sum_{i \in \mathcal{C}_k} d_i x^i - \sum_{i \in \mathcal{C}_k} d_i \left(\frac{1}{\sum_{j \in \mathcal{C}_k} d_j} \right) \sum_{i' \in \mathcal{C}_k} d_{i'} x^{i'}$$
$$= \sum_{i \in \mathcal{C}_k} d_i x^i - \sum_{i' \in \mathcal{C}_k} d_{i'} x^{i'} = 0.$$

The lemma immediately follows. As a direct consequence, we have

$$\sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^\top J_k = \left[\sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k) \right]^\top J_k = 0$$

for any J_k with a proper dimension independent of the index i.

Since the dimension of $\mathcal{T}_{\mathcal{C}}^d(n)$ is $n(m-K)$, the dimension of $\mathcal{S}_{\mathcal{C}}$ is nK, and $\mathcal{S}_{\mathcal{C}}(n)$ is disjoint with $\mathcal{T}_{\mathcal{C}}^d(n)$ except the origin $\mathbb{R}^{mn} = \mathcal{S}_{\mathcal{C}}(n) \oplus \mathcal{T}_{\mathcal{C}}^d(n)$, where $\oplus$ denotes the direct sum of linear subspaces. With these notations, the cluster synchronization is equivalent to the *transverse stability* of the cluster synchronization manifold $\mathcal{S}_{\mathcal{C}}(n)$, i.e., the projection of x on the transverse space $\mathcal{T}_{\mathcal{C}}^d(n)$ converges to zero as time goes to infinity.

Theorem 1: *Suppose that the common intercluster coupling condition (7) holds, Γ is symmetry and non-negative definite, and each vector-valued function $f_k(\cdot) - \alpha\Gamma\cdot$ satisfies the decreasing condition (8) for some $\alpha \in \mathbb{R}$. If there exists a positive definite diagonal matrix D such that the restriction of $[D(cL + \alpha I_m)]^s$, restricted to the transverse space $\mathcal{T}_{\mathcal{C}}^d(1)$, is nonpositive definite, i.e.,*

$$[D(cL + \alpha I_m)]^s |_{\mathcal{T}_{\mathcal{C}}^d(1)} \le 0 \tag{12}$$

holds, then the coupled system (11) can cluster synchronize with respect to the clustering $\mathcal{C}$.

Proof: We define an auxiliary function to measure the distance from x to the cluster synchronization manifold as follows:

$$V_k = \frac{1}{2} \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^\top (x_i - \bar{x}_d^k), \quad V(x) = \sum_{k=1}^K V_k.$$

Differentiating V_k along Eq. (11) gives

$$\dot{V}_k = \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^\top \left[f_k(x^i) + c \sum_{j=1}^m l_{ij} \Gamma x^j - \dot{\bar{x}}_d^k \right].$$

Recalling the definitions of l_{ij} and the common intercluster coupling condition (7), we have

$$\sum_{j \in \mathcal{C}_{k'}} l_{ij} = \sum_{j \in \mathcal{C}_{k'}} l_{i'j}, \quad \forall\, i, \ i' \in \mathcal{C}_k, \ k \neq k', \tag{13}$$

which leads

$$\sum_{j \in \mathcal{C}_k} l_{ij} = \sum_{j \in \mathcal{C}_k} l_{i'j}, \quad \forall\, i, \ i' \in \mathcal{C}_k. \tag{14}$$

By Lemma 1, we have

$$\sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^\top \dot{\bar{x}}_d^k = 0, \quad \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^\top f_k(\bar{x}_d^k) = 0,$$

$$\sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^\top \left(\sum_{j \in \mathcal{C}_{k'}} l_{ij} \Gamma \bar{x}_d^{k'} \right) = 0, \quad k' = 1, \ldots, K$$

due to the facts (13) and (14). Therefore, we have

$$\dot{V}_k = \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^\top \Bigg[f_k(x^i) - f_k(\bar{x}_d^k) + f_k(\bar{x}_d^k) + c \sum_{j=1}^m l_{ij} \Gamma (x^j - \bar{x}_d^{k'}) - \dot{\bar{x}}_d^k + c \sum_{k'=1}^K \sum_{j \in \mathcal{C}_{k'}} l_{ij} \Gamma \bar{x}_d^{k'} \Bigg]$$
$$= \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^\top \Bigg[f_k(x^i) - f_k(\bar{x}_d^k) + c \sum_{k'=1}^K \sum_{j \in \mathcal{C}_{k'}} l_{ij} \Gamma (x^j - \bar{x}_d^{k'}) \Bigg].$$

From the decreasing condition (8),

$$(w - v)^\top [f_k(w) - f_k(v) - \alpha \Gamma (w - v)] \le -\delta (w - v)^\top (w - v),$$

we have

$$\dot{V}_k \le -\delta \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^\top (x^i - \bar{x}_d^k) + \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^\top \times \left[c \sum_{k'=1}^K \sum_{j \in \mathcal{C}_{k'}} l_{ij} \Gamma (x^j - \bar{x}_d^{k'}) + \alpha \Gamma (x^i - \bar{x}_d^k) \right].$$

Thus,

$$\dot{V} \le -\delta\sum_{k=1}^{K}\sum_{i\in\mathcal{C}_k} d_i(x^i-\bar{x}_d^k)^\top(x^i-\bar{x}_d^k) + \sum_{k=1}^{K}\sum_{i\in\mathcal{C}_k} d_i(x^i-\bar{x}_d^k)^\top$$
$$\times\left[c\sum_{k'=1}^{K}\sum_{j\in\mathcal{C}_{k'}} l_{ij}\Gamma(x^j-\bar{x}_d^{k'}) + \alpha\Gamma(x^i-\bar{x}_d^k)\right]$$
$$= -\delta\sum_{k=1}^{K}\sum_{i\in\mathcal{C}_k} d_i(x^i-\bar{x}_d^k)^\top(x^i-\bar{x}_d^k) + (x-\bar{x}_d)^\top$$
$$\times\{[D(cL+\alpha I_m)]^s\otimes\Gamma\}(x-\bar{x}_d),$$

where $\otimes$ denotes the Kronecker product and $D = \mathrm{diag}[d_1,\ldots,d_m]$.

It is clear that $[D(cL+\alpha I_m)]^s|_{\mathcal{T}_\mathcal{C}^d(1)}\le 0$ implies $\{[D(cL+\alpha I_m)]^s\otimes I_n\}|_{\mathcal{T}_\mathcal{C}^d(n)}\le 0$. Decompose the positive definite matrix Γ as $\Gamma=C^\top C$ for some matrix C and let $y=[y^{1\top},\ldots,y^{m\top}]^\top$ with $y^i=C(x^i-\bar{x}_d^k)$ for all $i\in\mathcal{C}_k$, i.e., $y=(I_m\otimes C)(x-\bar{x}_d)$. By Lemma 1, it is easy to see that $\Sigma_{i\in\mathcal{C}_k}d_iy^i=\Sigma_{i\in\mathcal{C}_k}d_iC(x^i-\bar{x}_d^k)=0$. This implies that $y\in\mathcal{T}_\mathcal{C}^d(n)$. Therefore,

$$(x-\bar{x}_d)^\top\{[D(cL+\alpha I_m)]^s\otimes\Gamma\}(x-\bar{x}_d)$$
$$=(x-\bar{x}_d)^\top(I_m\otimes C^\top)\{[D(cL+\alpha I_m)]^s\otimes I_n\}$$
$$\times(I_m\otimes C)(x-\bar{x}_d)$$
$$=y^\top\{[D(cL+\alpha I_m)]^s\otimes I_n\}y\le 0. \tag{15}$$

Hence, we have

$$\dot{V}\le-\delta(x-\bar{x}_d)^\top(D\otimes I_n)(x-\bar{x}_d)=-2\delta\times V.$$

This implies that $V(t)\le\exp(-2\delta t)V(0)$. Therefore, $\lim_{t\to\infty}V(t)=0$, namely, $\lim_{t\to\infty}[x(t)-\bar{x}_d(t)]=0$ holds. In other words, $\lim_{t\to\infty}[x^i-\bar{x}_d^k]=0$ for each $i\in\mathcal{C}_k$ and $k=1,\ldots,K$. According to the assumption that $f_k(\cdot)$ are so different that if cluster synchronization is realized, the clusters are also different, we are safe to say that the coupled system (11) can cluster synchronize.

If each uncoupled system $\dot{x}^i=f_k(x^i)$ is unstable, in particular, chaotic, α must be positive in the inequality (8). It is natural to raise the question: Can we find some positive diagonal matrix D such that Eq. (12) satisfies with sufficiently large c and some certain $\alpha>0$? In other words, for the coupled system (10), what kind of unweighted graph topology $\mathcal{G}$ satisfying the common intercluster condition (7) can be a *chaos cluster synchronizer* with respect to the clustering $\mathcal{C}$. It can be seen that if the restriction of $(DL+L^\top D)$ to the transverse subspace $\mathcal{T}_\mathcal{C}^d(1)$ is negative, i.e.,

$$(DL+L^\top D)|_{\mathcal{T}_\mathcal{C}^d(1)}<0 \tag{16}$$

holds, then inequality (12) holds for sufficiently large c.

With these observations, we have

Theorem 2: *Suppose that the common intercluster coupling condition* (7) *holds for the coupled system* (11) *and* $\alpha>0$. There exist a positive diagonal matrix D and a sufficiently large constant c such that inequality (12) *holds if and only if all vertices in the same cluster belong to the same connected component*[38] *in the graph* $\mathcal{G}$.

Proof: We prove the sufficiency for connected graph and unconnected graph separated.

Case 1: The graph $\mathcal{G}$ is connected. Then, L is irreducible. Perron–Frobenius theorem (see Ref. 32 for more details) tells that the left eigenvector $[\xi_1,\ldots,\xi_m]^T$ of L associated with the eigenvalue 0 has all components $\xi_i>0$, $i=1,\ldots,m$. In this case, we pick $d_i=\xi_i$, $i=1,\ldots,m$, and its symmetric part $[DL]^s=(DL+L^\top D)/2$ has all row sums zero and irreducible with $\lambda_1([DL]^s)=0$ associated with the eigenvector $e=[1,\ldots,1]^\top$ and $\lambda_2([DL]^s)<0$. Therefore, $u^\top(DL)u\le\lambda_2(DL)^su^\top u<0$ for any $u\neq 0$ satisfying $u^\top e=0$.

Now, for any $u=[u_1,\ldots,u_m]^\top\in\mathbb{R}^m$ with $u^\top d=0$, define $\tilde{u}=[\bar{u},\ldots,\bar{u}]^\top$, where $\bar{u}=1/m\Sigma_{i=1}^m u_i$. It is clear that $DL\tilde{u}=0$, $\tilde{u}^\top DL=0$, and $(u-\tilde{u})^\top e=0$. Therefore,

$$u^\top(DL+L^\top D)u=(u-\tilde{u})^\top(DL+L^\top D)(u-\tilde{u})<0,$$

since both hold. This implies that inequality (16) holds.

Case 2: The graph $\mathcal{G}$ is disconnected. In this case, we can divide the bigraph $\mathcal{G}$ into several connected components. If all vertices that belong to the same cluster are in the same connected component, then by the same discussion done in case 1, we conclude that inequality (16) holds for some positive definite diagonal matrix D.

Necessity: We prove the necessity by reduction to absurdity. Considering an arbitrary disconnected graph $\mathcal{G}$, without loss of generality, supposing that L has form

$$L=\begin{bmatrix}L_1 & 0\\ 0 & L_2\end{bmatrix},$$

and letting $\mathcal{V}_1$ and $\mathcal{V}_2$ correspond to the submatrices L_1 and L_2, respectively, we assume that there exists a cluster $\mathcal{C}_1$ satisfying $\mathcal{C}_1\cap\mathcal{V}_i\neq\emptyset$ for all $i=1,2$. That is, there exists at least a pair of vertices in the cluster $\mathcal{C}_1$ which cannot access each other. For each $d=[d_1,\ldots,d_m]^\top$ with $d_i>0$ for all $i=1,\ldots,m$, letting $D=\mathrm{diag}[d_1,\ldots,d_m]$, we can find a nonzero vector $u\in\mathcal{T}_\mathcal{C}^d(1)$ such that $u^\top DLu=0$ (see the Appendix for details). This implies that inequality (16) does not hold. So, inequality (12) cannot hold for any positive α.

In the case that the clustering synchronized trajectories are chaotic with $\alpha>0$, Theorem 2 tells us that chaos cluster synchronization can be achieved (for sufficiently large coupling strength) if and only if all vertices in the same cluster belong to the same connected component in graph $\mathcal{G}$.

In summary, the following two conditions play the key role in cluster synchronization:

(1) common intercluster edges for each vertex in the same cluster and
(2) communicability for each pair of vertices in the same cluster.

The first condition guarantees that the clustering synchronization manifold is invariant through the dynamical system with properly picked weights and the second guarantees that chaos clustering synchronization can be reached with a sufficiently large coupling strength.

TABLE I. Communicability of clusters under edge-removing operations.

	Remove the intracluster edges	Remove the intercluster edges
Cluster type **A**	No	Yes
Cluster type **B**	Yes	No
Cluster type **C**	Yes	Yes
Cluster type **D**	No	No

C. Schemes to cluster synchronization

The theoretical results in Sec. II B indicate that the communication among vertices in the same cluster is important for chaos cluster synchronization. A cluster is said to be *communicable* if every vertex in this cluster can connect any other vertex by paths in the global graph. These paths between vertices are composed of edges, which can be either of intercluster or intracluster. Reference 25 showed that this classification of paths distinguishes the formation of clusters. A self-organized clustering synchronization implies that the intracluster edges are dominant for the communications between vertices in this cluster. Also, a driven cluster synchronization is that the intercluster edges are dominant for the communications between vertices in this cluster. There are various ways to describe "domination." In the following, we consider the unweighted graph topology and investigate the two clustering schemes via the results presented in Secs. II A and II B.

We first describe two schemes for cluster synchronization. The first one represents that the set of intracluster edges is irremovable for the communication between each pair of vertices in the same cluster and the second represents the scheme that the set of intercluster edges is irremovable for the communication between vertices in the same cluster. Thus, we propose the following classification of clusters.

(1) Cluster type **A**: the subgraph of the cluster is connected but when removing the intracluster links of the cluster, there exists at least one pair of vertices such that no paths in the remaining graph can connect them.
(2) Cluster type **B**: the subgraph of the cluster is disconnected, but even when removing all intracluster links of the cluster, each pair of vertices in the cluster can reach each other by paths in the remaining graph.
(3) Cluster type **C**: the subgraph of the cluster is connected and even when removing all intracluster links of the cluster, each pair of vertices in the cluster can reach each other by paths in the remaining graph.
(4) Cluster type **D**: the subgraph of the cluster is disconnected and when removing the intracluster links of the cluster, there exists at least one pair of vertices such that no paths in the remaining graph can link them.

Table I describes the characteristics of each cluster class. Figure 1 shows examples of these four kinds of clusters, which will be used in later numerical illustrations. With this cluster classification, we conclude that any cluster of type **A** or **C** cannot access another of type **A** or **D**. Table II shows all possibilities of accessibility among all kinds of clusters in a connected graph. Moreover, it should be noticed that the cluster in the networks, as illustrated in Fig. 1, may not be connected via the subgraph topologies. For example, the first and third clusters in graph 1, the second and third clusters in graph 3, as well as all clusters in graph 2 are not connected by intercluster subgraph topologies. Certainly, the vertices in the same cluster are connected via inter- and intracluster edges. That is, we can realize cluster synchronization in nonclustered networks.

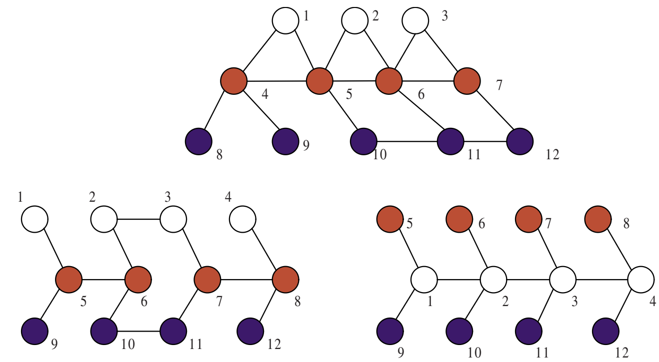

FIG. 1. (Color online) Graphs of examples. In graph 1, the first cluster (vertex set 1–3) is of type **B** since it has no intracluster edges, the second cluster (vertex set 4–7) is of type **C** since each pair of vertices can access each other via only inter- or intraedges, and the third cluster is of type **B** since each pair of vertices can access each other via only the intercluster edges but cannot communicate only via intracluster edges. In graph 2, each cluster of the first and third clusters (vertex sets 1–4 and 9–12) is of type **B** since each pair of vertices can access each other only via intercluster edges but only has a single intracluster edges. However, the second cluster (vertex set 5–8) is recognized as a type-**D** cluster since the sets of inter- or intracluster edges are both necessary for communication between each pair of vertices. In graph 3, the second and third clusters (vertex sets 5–8 and 9–12) are all of type **B** since they do not have intracluster edges and the first cluster (vertex set 1–4) is an example of cluster of type **A** since each pair of vertices can communicate via only the intracluster edges but cannot when removing the intracluster edges.

D. Examples

In this part, we propose several numerical examples to illustrate the theoretical results. In this example, we have $K=3$ clusters. The three graph topologies are shown in Fig. 1. The coupled system is

$$\dot{x}^i = f_k(x^i) + c\left[\sum_{\mathcal{N}_{k'}(i)\neq\emptyset} \frac{1}{d_{i,k'}} \sum_{j\in\mathcal{N}_{k'}(i)} \Gamma(x^j - x^i)\right], \qquad (17)$$

$$i \in \mathcal{C}_k, \quad k = 1,2,3,$$

where $\Gamma = \mathrm{diag}[1,1,0]$ and $f_k(\cdot)$ are nonidentical Chua's circuit

$$f_k(x) = \begin{cases} p_k[-x_1 + x_2 + g(x_1)] \\ x_1 - x_2 + x_3 \\ -q_k x_2, \end{cases} \qquad (18)$$

where $g(x_1) = m_0 x_1 + \frac{1}{2}(m_1 - m_0)(|x_1+1| - |x_1-1|)$. For all $k=1,2,3$, we take $m_0=-0.68$ and $m_1=-1.27$. The parameter pair (p_k, q_k) distinguishes the clusters and is picked as (10.0, 14.87), (9.0, 14.87), (9.0, 12.87) for $k=1,2,3$, respectively. As the Chua's circuits are Lipschitz continuous, any α that is

TABLE II. Possibility of coexistence for two kinds of clusters in connected graph.

	Cluster type **A**	Cluster type **B**	Cluster type **C**	Cluster type **D**
Cluster type **A**	×	√	×	×
Cluster type **B**	√	√	√	√
Cluster type **C**	×	√	×	√
Cluster type **D**	×	√	√	×

greater than the maximum of the Lipschitz constant of f_k can satisfy the decreasing condition. We use the following quantity to measure the variation for vertices in the same cluster:

$$\mathrm{var} = \left\langle \sum_{k=1}^{K} \frac{1}{\#\mathcal{C}_k - 1} \sum_{i \in \mathcal{C}_k} [x^i - \bar{x}_k]^{\top} [x^i - \bar{x}_k] \right\rangle,$$

where $\bar{x}_k = 1/\#\mathcal{C}_k \Sigma_{i \in \mathcal{C}_k} x^i$, $\langle \cdot \rangle$ denotes the time average. The ordinary differential equations (17) are solved by the Runge–Kutta fourth-order formula with a step length of 0.001–0.01 according to the size of the coupling strength. The time interval for computing the average is [50, 100].[34] Figure 2 indicates that for either graph 1, graph 2, or graph 3, the coupled system (17) clustering synchronizes, respectively, if the coupling strength is larger than certain threshold value. The threshold for each graph observed by the plots is clearly larger than the theoretical results, which will be shown in details in Sec. IV A. It is not surprising since the theoretical results only give a sufficient condition that the coupled system can cluster synchronize if the coupling strength c is large enough. It does not exclude the case that the coupled system can still cluster synchronize even if the coupling strength c is small.

The following quantity is used to measure the deviation between clusters:

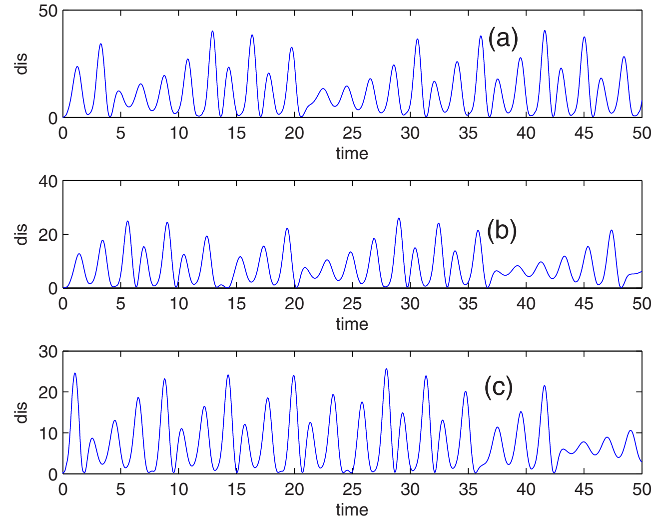

FIG. 3. (Color online) Dynamics of dis(t) through Eq. (10): (a) for graph 1 with c=20; (b) for graph 2 with c=53; (c) for graph 3 with c=53, respectively.

$$\mathrm{dis}(t) = \min_{i \neq j} [\bar{x}_i(t) - \bar{x}_j(t)]^{\top} [\bar{x}_i(t) - \bar{x}_j(t)].$$

Figure 3 shows that the deviation between clusters is apparent, even var≃0, where the coupling strengths are picked in the theoretical region guaranteeing clustering synchronization. It is clear that the difference is caused by the different choice of parameters for different clusters. This illustrates that the cluster synchronization is actually realized.

III. ADAPTIVE FEEDBACK CLUSTER SYNCHRONIZATION ALGORITHM

For a certain network topology, which has weak cluster synchronizability, i.e., the threshold to ensure clustering synchronization is relatively large, which is further studied in Sec. IV A. It is natural to raise the following question: How to achieve cluster synchronization for networks no mat-

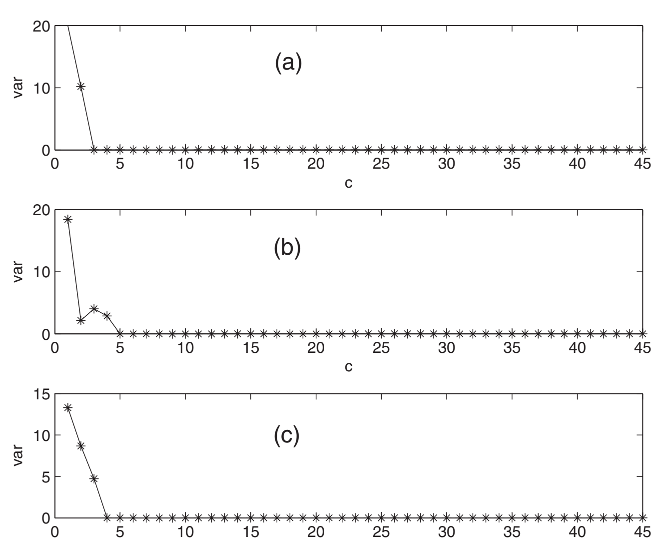

FIG. 2. var with respect to c: (a) for graph 1; (b) for graph 2; (c) for graph 3, respectively.

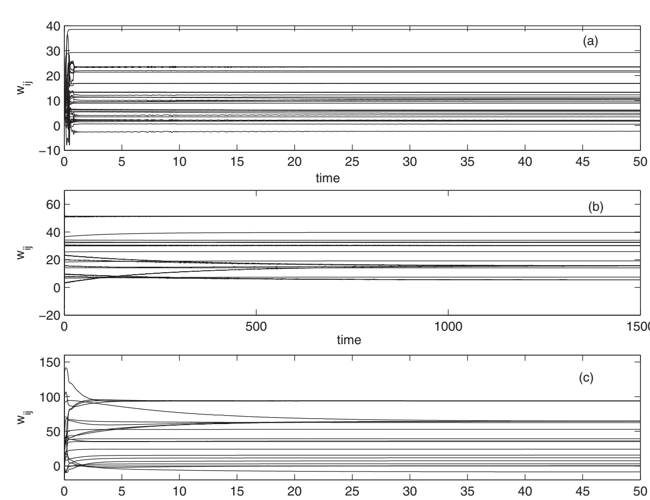

FIG. 4. Convergence dynamics of weights $\{w_{ij},(i,j) \in \mathcal{E}\}$ of edges through equality (10) with the adaptive algorithm (20): (a) for graph 1; (b) for graph 2; (c) for graph 3, respectively.

ter they have "good" topology or not. One approach proposed recently is adding weights to vertices and edges. Reference 35 showed evidences that certain weighting procedures can actually enhance complete synchronization. On the other hand, adaptive algorithm has emerged as an efficient means of weighting to actually enhance complete synchronizability.[36]

In this section, we consider the coupled system

$$\dot{x}^i = f_k(x^i) + \sum_{j=1}^{m} a_{ij} w_{ij} \Gamma(x^j - x^i), \quad i \in \mathcal{C}_k, \quad k = 1, \ldots, K \tag{19}$$

and propose an adaptive feedback algorithm to achieve cluster synchronization for a prescribed graph.

Suppose that the common intercluster and communicability conditions are satisfied. Without loss of generality, we suppose that graph G is undirected and connected. Consider the coupled system (2) with Laplacian L defined as in Eq. (11) and $d^{\top} = [d_1, \ldots, d_m]$ is the left eigenvector of L associated with the eigenvalue 0.

Now, we propose the following adaptive cluster synchronization algorithm

$$\begin{cases} \dot{x}^i(t) = f_k(x^i(t)) + \sum_{j=1}^{m} a_{ij} w_{ij}(t) \Gamma[x^j(t) - x^i(t)], \quad i \in \mathcal{C}_k, \quad k = 1, \ldots, K \\ \dot{w}_{ij}(t) = \rho_{ij} d_i [x^i(t) - \bar{x}_d^k(t)]^{\top} \Gamma [x^i(t) - x^j(t)] \\ \text{for each } e_{ij} \in \mathcal{E} \quad \text{and} \quad i \in \mathcal{C}_k, \quad k = 1, \ldots, K \end{cases} \tag{20}$$

with $\rho_{ij} > 0$ as constants.

Theorem 3: *Suppose that graph $\mathcal{G}$ is connected, all the assumptions of Theorem 1 hold, the system (20) is essentially bounded. Then the system (20) cluster synchronizes for any initial data.*

Proof: First of all, pick l_{ij} as defined in Eq. (11) and a sufficiently large c. Since $\mathcal{G}$ is connected, Theorem 2 tells

$$[D(cL + \alpha I_m)]^s|_{\mathcal{T}_{\mathcal{C}}^d(1)} < 0. \tag{21}$$

Define the following candidate Lyapunov function

$$Q_k(x, W) = \sum_{i \in \mathcal{C}_k} \left[\frac{d_i}{2} (x^i - \bar{x}_d^k)^{\top} (x^i - \bar{x}_d^k) + \frac{1}{2\rho_{ij}} a_{ij} (w_{ij} - c l_{ij})^2 \right],$$

$$Q(x, W) = \sum_{k=1}^{K} Q_k.$$

Differentiating Q_k, we have

$$\dot{Q}_k = \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^{\top} \left\{ f_k(x^i) + \sum_{j=1}^{m} a_{ij} w_{ij} \Gamma(x^j - \bar{x}^j) \right\}$$
$$+ \sum_{i \in \mathcal{C}_k} \sum_{k'=1}^{K} \sum_{j \in \mathcal{N}_{k'}(i)} a_{ij} (w_{ij} - c l_{ij}) d_i (x^i - \bar{x}_d^k)^{\top} \Gamma (x^i - x^j)$$
$$= \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^{\top} \left\{ f_k(x^i) + c \sum_{j=1}^{m} l_{ij} \Gamma(x^j - x^i) - \dot{\bar{x}}_d^k \right\}.$$

Similar to the proof of Theorem 1, we have

$$\sum_{i \in \mathcal{C}_i} \dot{Q}_i = \sum_{i \in \mathcal{C}_i} d_i (x^i - \bar{x}_d^k)^{\top}$$
$$\times \left\{ f_k(x^i) - f(\bar{x}_d^k) + c \sum_{j=1}^{m} l_{ij} \Gamma(x^j - \bar{x}_d^j) \right\}$$

and

$$\dot{Q} = \sum_{k=1}^{K} \dot{Q}_k \le -\delta \sum_{k=1}^{K} \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^{\top} (x_i - \bar{x}_d^k)$$
$$+ \sum_{k=1}^{K} \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^k)^{\top}$$
$$\times \left[\alpha \Gamma (x^i - \bar{x}_d^k) + c \sum_{j=1}^{m} l_{ij} \Gamma (x^j - \bar{x}_d^j) \right]$$
$$= -\delta (x - \bar{x}_d)^{\top} (D \otimes I)(x - \bar{x}_d) + (x - \bar{x}_d)^{\top}$$
$$\times \{ [D(cL + \alpha I_m)]^s \otimes \Gamma \} (x - \bar{x}_d).$$

Inequality (21) implies

$$\dot{Q} \le -\delta (x - \bar{x}_d)^{\top} (D \otimes I)(x - \bar{x}_d) \le 0.$$

This implies

$$\int_0^t \delta (x(s) - \bar{x}_d(s))^{\top} (D \otimes I)(x(s) - \bar{x}_d(s)) ds \le Q(0)$$
$$- Q(t) \le Q(0) < \infty. \tag{22}$$

From the assumption of the boundedness of Eq. (20), we can conclude $\lim_{t \to \infty} [x(t) - \bar{x}_d(t)] = 0$ due to the fact that $x(t)$ is uniform continuous. This completes the proof.

For the disconnected situation, we can split the graph into several connected components and deal with each connected component by the same means as above. The dynamics of the weights $w_{ij}(t)$ is an interesting issue. Even though it is illustrated in Fig. 4 that all weights converge, to our best reasoning, we can only prove that all intraweights converge, i.e., vertices i and j belonging to the same cluster $\mathcal{C}_k$. In fact, by Eq. (22), we have

$$\int_0^\infty [x^i(\tau)-\bar{x}_d^k(\tau)]^\top[x^i(\tau)-\bar{x}_d^k(\tau)]d\tau<+\infty.$$

Thus,

$$\begin{aligned}\int_0^\infty |\dot{w}_{ij}(\tau)|d\tau &= \rho_{ij}d_i\int_0^\infty |[x^i(\tau)-\bar{x}_d^k(\tau)]^\top\Gamma[x^i(\tau)-x^j(\tau)]|d\tau\\ &\le \int_0^\infty \rho_{ij}d_i\|\Gamma\|_2\{|[x^i(\tau)-\bar{x}_d^k(\tau)]^\top[x^i(\tau)-\bar{x}_d^k(\tau)]|+|[x^i(\tau)-\bar{x}_d^k(\tau)]^\top[x^j(\tau)-\bar{x}_d^k(\tau)]|\}d\tau\\ &\le \rho_{ij}d_i\|\Gamma\|_2\left\{\frac{3}{2}\int_0^\infty [x^i(\tau)-\bar{x}_d^k(\tau)]^\top[x^i(\tau)-\bar{x}_d^k(\tau)]d\tau+\frac{1}{2}\int_0^\infty [x^j(\tau)-\bar{x}_d^k(\tau)]^\top[x^j(\tau)-\bar{x}_d^k(\tau)]d\tau\right\}.\end{aligned}$$

Therefore, for any $\epsilon>0$, there exists $T>0$, such that for any $t_1>T$, $t_2>T$, we have

$$|w_{ij}(t_2)-w_{ij}(t_1)|\le\int_{t_1}^{t_2}|\dot{w}_{ij}(\tau)|d\tau<\epsilon.$$

By Cauchy convergence principle, $w_{ij}(t)$ converges to some final weights w_{ij}^* for $i\in\mathcal{C}_k$, $j\in\mathcal{C}_k$ when $t\to\infty$.

On the other hand, to our best reasoning, we cannot prove whether or not the weights $w_{ij}(t)$ converge, if the vertices i and j belong to different clusters. If we assume the convergence of all weights, according to the Lasalle invariant principle,[33] the final weights should guarantee that the cluster synchronization manifold is still invariant. That is to say, if the difference of trajectories $s^{k'}-s^k$ in Eq. (5) are linearly independent, the cluster of the condition (6) still holds for the final weights.

Moreover, we have found out that the final weights in our example sensitively depend on the initial values. Figure 5 gives two sets of weighted topologies of the three graphs, as shown in Fig. 1, after employing the adaptive algorithm with two different sets of initial values of $w_{ij}(0)$ and the same parameters. One can see that the final weight can be quite different for different initial values and even be negative. From this observation, we argue that it may be the adaptive process and not the final weights that counts to reach cluster synchronization. Further investigation of the final weights is out of the scope of the current paper.

A. Examples

To illustrate the adaptive feedback algorithms, we still use graphs 1–3 described in Fig. 1 as the network topology. Also this time we use the Lorenz system as the uncoupled system,

$$f_k(u)=\begin{cases}10(u_2-u_1)\\ \frac{8}{3}u_1-u_2-u_1u_3\\ u_1u_2-b_ku_3,\end{cases} \tag{23}$$

where the parameters $b_1=28$ for the first cluster, $b_2=38$ for the second cluster, and $b_3=58$ for the third cluster are used to distinguish the clusters. As shown in Ref. 37, the boundedness of the trajectories of an array of coupled Lorenz systems can be ensured. Also, this bound is independent of the coupling strength. Therefore, the decreasing condition (8) can be satisfied for a sufficiently large α. In fact, the theoretical estimation of such α is rather large and much larger than the simulating observation (not shown in this paper). However, Theorem 3 indicates that the existence of such α (even very large in theory) is sufficient for the adaptive feedback algorithm (20) succeeding in clustering synchronizing the coupled system.

The ordinary differential equations are solved by the Runge–Kutta fourth-order formula with a step length 0.005. The initial values of the states and the weights are randomly picked in $[-3, 3]$ and $[-5, 5]$, respectively. We use the following quantity to measure the state variance inside each cluster with respect to time:

$$K(t)=\sum_{k=1}^{K}\frac{1}{\#\mathcal{C}_k-1}\sum_{i\in\mathcal{C}_k}[x^i(t)-\bar{x}_k(t)]^\top[x^i(t)-\bar{x}_k(t)].$$

Figure 6 shows that the adaptive algorithm succeeds in clustering synchronizing the network with respect to the pregiven clusters. Figure 7 indicates that the differences between clusters are due to nonidentical parameters b_k. As shown in Fig. 4, the weights converge but the limit values are not always positive. This is not surprising. The right-hand side of the algorithm (20) can be either positive or negative, which causes some weights of edges to be negative. Discussion of the situation with negative weights is out of the scope of this paper.

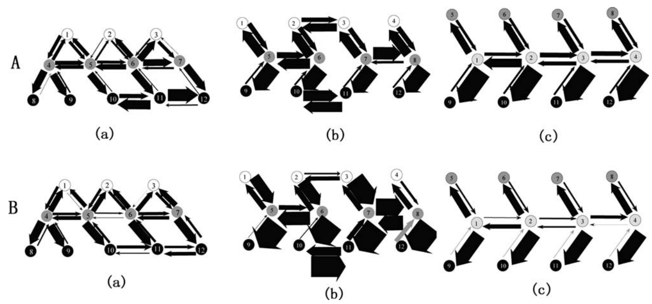

FIG. 5. Two sets of the final weighted topologies of the three graphs in Fig. 1 via employing the adaptive algorithm (20) with two different sets of initial data but the same parameters. Sets A and B correspond to two sets of initial values and (a)–(c) correspond to graphs 1–3 in Fig. 1. The black lines are of positive weights and the gray lines negative widths and the width of the line represents the scale of the weight in modulus.

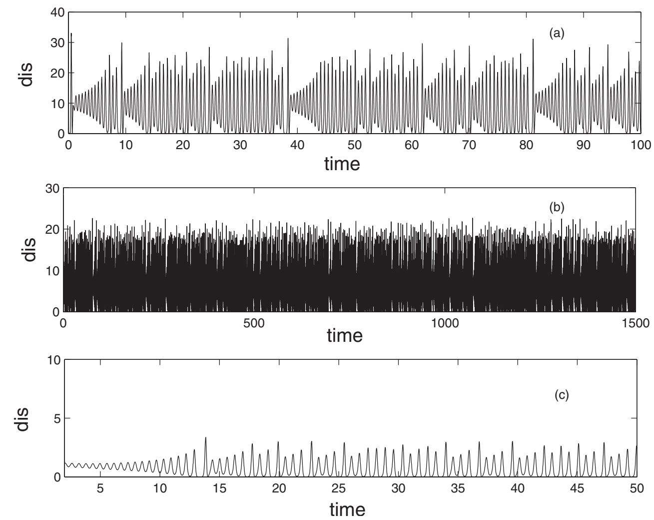

FIG. 7. Dynamics of dis(t) through equality (10) with the adaptive algorithm (20): (a) for graph 1; (b) for graph 2; (c) for graph 3, respectively.

IV. DISCUSSIONS

In this section, we make further discussions for some closely relating issues.

A. Clustering synchronizability

Synchronizability is used to measure the capability of synchronization for the graph. It can be described by the threshold of the coupling strength to guarantee that the coupled system can synchronize. For complete synchronization, it was formulated as a function of the eigenvalues of symmetric Laplacian[11] or certain Rayleigh quotient of asymmetric Laplacian.[15] How the topology of the underlying graph affects synchronizability is an important issue for the study of complex networks.[2] Here, similarly, we are also interested in how to formulate and analyze the cluster synchronizability of a graph $\mathcal{G}$ and a clustering $\mathcal{C}$.

Consider the model (11) of coupled system. Theorem 1 tells us that under the common intercluster condition, the cluster synchronization condition (12) can be rewritten as

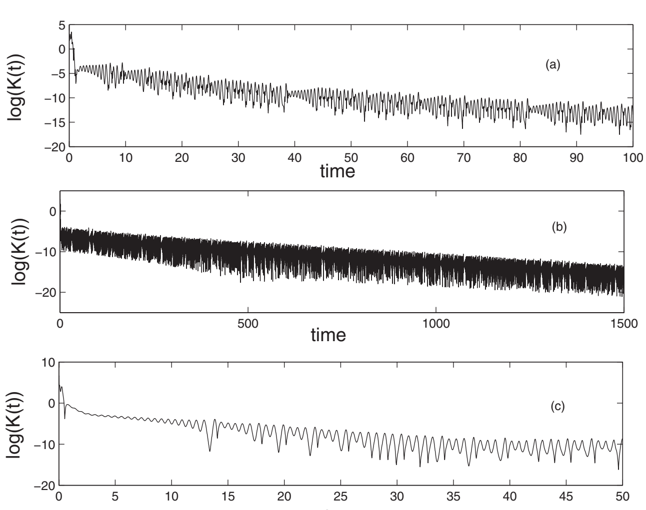

FIG. 6. Dynamics of the logarithm of $K(t)$ through equality (10) with the adaptive algorithm (20): (a) for graph 1; (b) for graph 2; (c) for graph 3, respectively.

$$c > \frac{\alpha}{\min_{u\in\mathcal{T}_{\mathcal{C}}^{d}(1),u\neq 0}\dfrac{-u^{\top}(DL)^{s}u}{u^{\top}Du}} \tag{24}$$

for some positive definite diagonal D. Therefore, we take the Rayleigh–Hitz quotient

$$\mathrm{CS}_{\mathcal{G},\mathcal{C}} = \max_{D\in\mathcal{D}} \min_{u\in\mathcal{T}_{\mathcal{C}}^{d}(1),u\neq 0} \frac{-u^{\top}(DL)^{s}u}{u^{\top}Du}$$

to measure the cluster synchronizability for graph $\mathcal{G}$ and clustering $\mathcal{C}$, where $\mathcal{D}$ denotes the set of positive definite diagonal matrices of dimension m. It can be seen that the larger the $\mathrm{CS}_{\mathcal{G},\mathcal{C}}$ is, the smaller the coupling strength c can be, such that the coupled system (11) clusteringly synchronizes. In particular, if L is symmetric, then $\mathrm{CS}_{\mathcal{G},\mathcal{C}}$ is just the maximum eigenvalue of $-L$ in the transverse space $\mathcal{T}_{\mathcal{C}}^{e}(1)$, where $e=[1,1,\ldots,1]^{\top}$. It is an interesting topic about how the two schemes discussed above affect the cluster synchronizability for a given graph topology. It will be a possible topic in our future research.

Reconsidering the examples in Sec. II D, we can use MATLAB LMI and Control Toolbox to obtain the numerical values of $\mathrm{CS}_{\mathcal{G},\mathcal{C}}$ for three graphs shown in Fig. 1. Thus, we can derive the values of $\mathrm{CS}_{\mathcal{G},\mathcal{C}}$: 0.472, 0.178, and 0.172, respectively. So, we can obtain the minimal estimation of the coupling strength c as

$$c^{*} = \frac{\alpha}{\mathrm{CS}_{\mathcal{G},\mathcal{C}}}.$$

The globally Lipschitz continuity of Chua's circuit allows us to obtain $\alpha < 9.062$. Thus, we obtain estimations of the infimum of c: 19.20 for graph 1, 50.91 for graph 2, and 52.69 for graph 3. The details of algebras are omitted here. One can see that they are all located in the region of cluster synchronization, as numerically illustrated in Fig. 2, but less accurate since the estimation of α is very loose. However, the theoretical value of $\mathrm{CS}_{\mathcal{G},\mathcal{C}}$ provides information on the relative synchronizability of coupling structure, independent of the node dynamics set on the network.

B. Generalized weighted topologies

Previous discussions can also be available toward the coupled system (2) with general weights,

$$\dot{x}^i = f_k(x^i) + \sum_{j=1}^{m} a_{ij} w_{ij} \Gamma(x^j - x^i), \;\; i \in \mathcal{C}_k, \;\; k = 1, \ldots, K. \tag{25}$$

Here, the graph may be directed, i.e., $a_{ij}=1$, if there is an edge from vertex j to vertex i, otherwise, $a_{ij}=0$. Weights are even not required positive. For the existence of invariant cluster synchronization manifold, we assume

$$\sum_{j \in \mathcal{N}_{k'}(i)} w_{ij} = \sum_{j' \in \mathcal{N}_{k'}(i')} w_{i'j'} \tag{26}$$

holds for all $i, i' \in \mathcal{C}_k$ and $k \neq k'$. Define its Laplacian $G=[g_{ij}]_{i,j=1}^m$ as follows:

$$g_{ij} = \begin{cases} w_{ij}, & a_{ij} = 1 \\ 0, & i \neq j \;\; \text{and} \;\; a_{ij} = 0 \\ -\sum_{k=1, k\neq i}^{m} g_{ik}, & i = j. \end{cases}$$

Thus, Eq. (25) becomes

$$\dot{x}^i = f_k(x^i) + \sum_{j=1}^{m} g_{ij} \Gamma x^j, \;\; i \in \mathcal{C}_k, \;\; k = 1, \ldots, K. \tag{27}$$

Replacing cl_{ij} by g_{ij} and following the routine of the proof of Theorem 1, we can obtain following results.

Theorem 4: *Suppose that the common intercluster coupling condition (26) is satisfied, each $f_k(\cdot) - \alpha\Gamma\cdot$ satisfies the decreasing condition for some $\alpha \in \mathbb{R}$*, and Γ *is non-negative definite. If there exists a positive definite diagonal matrix D* such that

$$[D(G + \alpha I_m)]^s_{\mathcal{T}^d_{\mathcal{C}}(1)} \leq 0 \tag{28}$$

holds, then the coupled system (27) can cluster synchronize with respect to the clustering $\mathcal{C}$.

Also, we use the same discussions as in Theorem 2 to obtain the following general result.

Theorem 5: *Suppose that the common intercluster coupling condition (7) is satisfied. For a bidirected unweighted graph $\mathcal{G}$*, there exist positive weights to the graph $\mathcal{G}$ *such that inequality (28) holds if and only if all vertices in the same cluster belong to the same connected component in graph $\mathcal{G}$.*

In fact, the proofs of Theorems 4 and 5 simply repeat those of Theorems 1 and 2, respectively.

Here, we compare the results in a closely relating work[26] with this paper. First, investigate the local cluster synchronization of interconnected clusters by extending the master stability function method. Instead, in this paper, we are concerned with the global cluster synchronization. Second, the models of the two papers are different. The topologies discussed in Ref. 26 exclude intracluster couplings. In this paper, we consider more general graph topology. Third, Ref. 26 studied the situation of nonlinear coupling function and we consider the linear case. Despite that Ref. 26 considered different coupling strengths for clusters and we consider a common one in Sec. II, Theorem 4 can apply to discussion of the models proposed in Ref. 26, too.

V. CONCLUSIONS

The idea for studying synchronization in networks of coupled dynamical systems sheds light on cluster synchronization analysis. In this paper, we study cluster synchronization in networks of coupled nonidentical dynamical systems. Cluster synchronization manifold is defined as that the dynamics of the vertices in the same cluster are identical. The criterion for cluster synchronization is derived via linear matrix inequality. The differences between clustered dynamics are guaranteed by the nonidentical dynamical behaviors of different clusters. The algebraic graph theory tells that the communicability between each pair of vertices in the same cluster is a doorsill for chaos cluster synchronization. This leads to a description of two schemes to realize cluster synchronization: the set of intracluster edges is irremovable for the communication between each pair of vertices in the same cluster; the set of intercluster edges is irremovable for the communication between vertices in the same cluster. One can see that the latter scheme implies that cluster synchronization can be realized in a network without community structure, for example, graph 2 in Fig. 1. Adaptive feedback algorithm is used to enhance cluster synchronization.

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APPENDIX: PROOF OF NECESSITY IN THEOREM 2

In this appendix, for each positive d, we give the details to find a $u \in \mathcal{T}^d_{\mathcal{C}}(1)$ with $u \neq 0$, such that $u^\top DLu=0$ in the case that there exists a cluster $\mathcal{C}_1$ that does not belong to the same connected component. Without loss of generality, suppose L has the following form:

$$L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}.$$

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ correspond to submatrices L_1 and L_2, respectively, and $\mathcal{C}_1 \cap \mathcal{V}_i \neq \emptyset$ for all $i=1,2$. There are two cases.

First, in the case that $\mathcal{C}_1$ is isolated from other clusters. In this case, there are no edges connecting $\mathcal{C}_1$ to other clusters. Define

$$u_i = \begin{cases} \alpha, & i \in \mathcal{C}_1 \cap \mathcal{V}_1 \\ \beta, & i \in \mathcal{C}_1 \cap \mathcal{V}_2 \\ 0, & \text{otherwise}, \end{cases}$$

$a=\Sigma_{j\in\mathcal{C}_1\cap\mathcal{V}_1} d_j$, and $b=\Sigma_{j\in\mathcal{C}_1\cap\mathcal{V}_2} d_j$. Then, by picking α and β satisfying $a\alpha+b\beta=0$ with $\alpha,\beta\neq 0$, we have $u\in\mathcal{T}^d_{\mathcal{C}}(1)$. In addition, $u^\top D L u=0$ due to $Lu=0$.

In the second case, $\mathcal{C}_1$ is not isolated. Suppose the network has K clusters and L_1 and L_2 are connected (otherwise, we only consider the connection components of L_1 and L_2 that contain vertices from $\mathcal{C}_1$). Due to the common intercluster coupling condition and the absence of isolated cluster, we have $\mathcal{C}_i\cap\mathcal{V}_j\neq\emptyset$ for all $i=1,\ldots,K$ and $j=1,2$. Pick a vector $u=[u_1,\ldots,u_m]^\top$ with

$$u_i = \begin{cases} \alpha_k, & i \in \mathcal{C}_k \cap \mathcal{V}_1 \\ \beta_k, & i \in \mathcal{C}_k \cap \mathcal{V}_2. \end{cases}$$

Denote $d_k^1=\Sigma_{i\in\mathcal{C}_k\cap\mathcal{V}_1} d_i$, $d_k^2=\Sigma_{i\in\mathcal{C}_k\cap\mathcal{V}_2} d_i$ and $\bar{u}_1=[\alpha_1,\ldots,\alpha_K]^\top$, $\bar{u}_2=[\beta_1,\ldots,\beta_K]^\top$, $\bar{u}=[\bar{u}_1^\top,\bar{u}_2^\top]^\top$, $\bar{D}_1=\text{diag}[d_1^1,\ldots,d_K^1]$, $\bar{D}_2=\text{diag}[d_1^2,\ldots,d_K^2]$, and $\bar{D}=\text{diag}\times[\bar{D}_1,\bar{D}_2]$. Define a $K\times K$ matrix W^1 from L_1 in such a way that for $i\neq j$, $W^1_{ij}=1$ if there is interaction between cluster i and j, and $W^1_{ij}=0$ otherwise. $W^1_{ii}=-\Sigma_{j=1,j\neq i}^K W^1_{ij}$. Define W^2 in the same way according to L_2 due to the *common intercluster condition*, it is easy to see that $W^1=W^2$. Denote $W=\text{diag}[W^1,W^2]$.

By some algebras, we can conclude that for any given positive definite diagonal matrix $D=\text{diag}[d_1,\ldots,d_m]$, $u^\top DLu=\bar{u}^\top\bar{D}W\bar{u}$ holds. For $u\in\mathcal{T}^d_{\mathcal{C}}$, $\bar{u}_2=-\bar{D}_1\bar{D}_2^{-1}\bar{u}_1$. Letting $v=\bar{D}_1\bar{u}_1$, we have $\bar{u}^\top\bar{D}W\bar{u}=[v^\top v^\top]W\bar{D}^{-1}[v^\top v^\top]^\top=v^\top W^1(\bar{D}_1^{-1}+\bar{D}_2^{-1})v$. This implies that if we can find v satisfying $v^\top W^1(\bar{D}_1^{-1}+\bar{D}_2^{-1})v=0$, then there exists $u\in\mathcal{T}^d_{\mathcal{C}}(1)$ such that $u^\top DLu=0$. Since $W^1(\bar{D}_1^{-1}+\bar{D}_2^{-1})$ has rank at most $K-1$, we can pick v as the eigenvector corresponding to the zero eigenvalue of $W^1(\bar{D}_1^{-1}+\bar{D}_2^{-1})$, and this completes the proof.

In summary, in each case, we can find a nonzero vector u belonging to the transverse space $\mathcal{T}^d_{\mathcal{C}}(1)$ such that $u^\top DLu=0$.

[1]A. Pikovsky, M. Roseblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, England, 2001).
[2]S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).
[3]X. F. Wang and G. Chen, IEEE Circuits Syst. Mag. **3**, 6 (2003).
[4]H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983); **72**, 885 (1984); V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **29**, 795 (1986).
[5]S. H. Strogatz and I. Stewart, Sci. Am. **269**, 102 (1993).
[6]C. M. Gray, J. Comput. Neurosci. **1**, 11 (1994).
[7]L. Glass, Nature (London) **410**, 277 (2001).
[8]S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, Phys. Rep. **366**, 1 (2002).
[9]L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990); J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. E **50**, 1874 (1994).
[10]P. Ashwin, J. Buescu, and I. Stewart, Nonlinearity **9**, 703 (1996).
[11]J. Jost and M. P. Joy, Phys. Rev. E **65**, 016201 (2001); X. F. Wang and G. Chen, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **49**, 54 (2002); G. Rangarajan and M. Ding, Phys. Lett. A **296**, 204 (2002); Y. H. Chen, G. Rangarajan, and M. Ding, Phys. Rev. E **67**, 026209 (2003).
[12]C. W. Wu and L. O. Chua, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **42**, 430 (1995).
[13]I. V. Belykh, V. N. Belykh, and M. Hasler, Physica D **195**, 159 (2004); **195**, 188 (2004); J. Cao, P. Li, and W. Wang, Phys. Lett. A **353**, 318 (2006).
[14]W. Lu and T. Chen, Physica D **213**, 214 (2006).
[15]C. W. Wu, Nonlinearity **18**, 1057 (2005).
[16]A. Schnitzler and J. Gross, Nat. Rev. Neurosci. **6**, 285 (2005).
[17]P. R. Chandler, M. Patcher, and S. Rasmussen, Proceedings of the American Control Society, 2001, p. 20 (unpublished); K. M. Passino, IEEE Control Syst. Mag. **22**, 52 (2002); J. Finke, K. Passino, and A. G. Sparks, *ibid.* **14**, 789 (2006).
[18]B. Blasius, A. Huppert, and L. Stone, Nature (London) **399**, 354 (1999); E. Montbrió, J. Kurths, and B. Blasius, Phys. Rev. E **70**, 056125 (2004).
[19]N. F. Rulkov, Chaos **6**, 262 (1996).
[20]L. Stone, R. Olinky, B. Blasius, A. Huppert, and B. Cazelles, AIP Conf. Proc. **662**, 476 (2002).
[21]E. Jones, B. Browning, M. B. Dias, B. Argall, M. Veloso, and A. Stentz, Proceedings of the IEEE International Conference on Robotics and Automation, Orlando, 2006, p. 570 (unpublished); K.-S. Hwang, S.-W. Tan, and C.-C. Chen, IEEE Trans. Fuzzy Syst. **12**, 569 (2004).
[22]V. N. Belykh, I. V. Belykh, and M. Hasler, Phys. Rev. E **62**, 6332 (2000); V. N. Belykh, I. V. Belykh, and E. Mosekilde, *ibid.* **63**, 036216 (2001).
[23]Z. Ma, Z. Liu, and G. Zhang, Chaos **16**, 023103 (2006).
[24]W. Wu and T. Chen, Physica D **238**, 355 (2009); W. Wu, W. Zhou, and T. Chen, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **54**, 829 (2009).
[25]S. Jalan and R. E. Amritkar, Phys. Rev. Lett. **90**, 014101 (2003); S. Jalan, R. E. Amritkar, and C.-K. Hu, Phys. Rev. E **72**, 016211 (2005); **72**, 016212 (2005).
[26]F. Sorrentino and E. Ott, Phys. Rev. E **76**, 056114 (2007).
[27]L. Chen and J. Lu, J. Syst. Sci. Complexity **20**, 21 (2008).
[28]I. V. Belykh, V. N. Belykh, and M. Hasler, Chaos **13**, 165 (2003).
[29]Q.-C. Pham and J.-J. Slotine, Neural Networks **20**, 62 (2007).
[30]W. Lohmiller and J.-J. Slotine, Automatica **34**, 683 (1998).
[31]L. O. Chua, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **42**, 559 (1995).
[32]P. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, England, 1985).
[33]J. P. Lasalle, IRE Trans. Circuit Theory **7**, 520 (1960).
[34]The reason why we choose initial time for average from 50 not 0 is that even if the coupled system synchronizes clustering, the variance as calculated by var can be very large ($>10^3$). This implies that it will take a very long time for average to make the variance near zero. To save calculating amount, we pick the initial time apart from zero.
[35]M. Chavez, D.-U. Hwang, H. G. E. Hentschel, and S. Boccaletti, Phys. Rev. Lett. **94**, 218701 (2005); C. Zhou and J. Kurths, *ibid.* **96**, 164102 (2006); S. Boccaletti, D.-U. Hwang, M. Chavez, A. Amann, J. Kurths, and L. M. Pecora, Phys. Rev. E **74**, 016102 (2006); C. S. Zhou, A. E. Motter, and J. Kurths, Phys. Rev. Lett. **96**, 034101 (2006).
[36]J. Lu and J. Cao, Chaos **15**, 043901 (2005); D. Huang, Phys. Rev. E **71**, 037203 (2005); J. Zhou, J.-A. Lu, and J. Lü, IEEE Trans. Autom. Control **51**, 652 (2006); W. Lu, Chaos **17**, 023122 (2007).
[37]X. Liu and T. Chen, Physica D **237**, 630 (2008).
[38]Connected component in a direct graph $\mathcal{G}$ is a maximal vertex set of which each vertex can access all others.