# CLUSTERING AND PERCOLATION ON SUPERPOSITIONS OF BERNOULLI RANDOM GRAPHS 

MINDAUGAS BLOZNELIS AND LASSE LESKELÄ


#### Abstract

A simple but powerful network model with $n$ nodes and $m$ partly overlapping layers is generated as an overlay of independent random graphs $G_{1}, \ldots, G_{m}$ with variable sizes and densities. The model is parameterised by a joint distribution $P_{n}$ of layer sizes and densities. When $m$ grows linearly and $P_{n} \rightarrow P$ as $n \rightarrow \infty$, the model generates sparse random graphs with a rich statistical structure, admitting a nonvanishing clustering coefficient together with a limiting degree distribution and clustering spectrum with tunable power-law exponents. Remarkably, the model admits parameter regimes in which bond percolation exhibits two phase transitions: the first related to the emergence of a giant connected component, and the second to the appearance of gigantic single-layer components.


## 1. Introduction

Applications in natural sciences, social sciences, and technology often deal with large networks of nodes linked by pairwise interactions which involve uncertainty due to noisy observations and missing data. Such uncertainties have been investigated using statistical models ranging from classical Bernoulli random graphs and uniform random graphs with given degree distributions to stochastic block models and more complex generative models involving various preferential attachment and rewiring mechanisms [1, 24, 29, 42, 49]. While succeeding to obtain a good fit for degree distributions and tractable percolation analysis, most earlier models fail to capture second-order effects related to clustering and transitivity. Random intersection graphs [5, 11, 17, 33, 44], spatial preferential attachment models [26, 27, $28]$, and hyperbolic random geometric graphs $[13,23,34,35]$ have been introduced to conduct percolation analysis on networks with nonvanishing transitivity and clustering properties.

Despite remarkable methodological advances, most sparse network models still appear somewhat rigid in what comes to modeling finer clustering properties, such as the clustering spectrum (degree-dependent local clustering coefficient) [3, 46, 50], which may significantly impact the percolation properties of the network [4, 18]. A decreasing clustering spectrum manifests the fact that high-degree nodes tend to have sparser local neighbourhoods than lowdegree nodes. Motivated by analysing this phenomenon in a tractable quantitative framework, this article discusses a statistical network model generated as an overlay of mutually independent Bernoulli random graphs $G_{1}, \ldots, G_{m}$ which can be interpreted as layers or communities. The layers have a variable size (number of nodes) and strength (link probability), and they may overlap each other. A key feature of the model is that the layer sizes and layer strengths are assumed to be correlated, which allows to model and analyse a rich class of networks with a tunable frequency of strong small communities and weak large communities.
1.1. Main contributions. This article presents a rigorous mathematical analysis of clustering and percolation of the overlay graph model in the natural sparse limiting regime where the number of nodes $n$ tends to infinity, the number of layers $m$ is linear in the number of nodes, and the joint distribution $P_{n}$ of layer sizes and layer strengths converges to a limiting distribution $P$. We derive exact formulas for the limiting degree distribution, clustering coefficient, clustering spectrum, and the largest component size in terms of cross-factorial

[^0]moments and functional transforms of $P$. We also investigate the model under bond and site percolation, and characterise critical parameter values of the associated phase transitions.

The descriptive power of the model is illustrated by a detailed investigation of an instance where the layer size follows a power law, and the layer strength is a deterministic function of the layer size following another power law. This setting leads to a power-law degree distribution and a power-law clustering spectrum with tunable exponents in ranges $(1, \infty)$ and $[0,2]$, respectively. A special case in which layer strengths are inversely proportional to their sizes corresponds to layers of bounded average degree. In this natural parameter regime we discover a remarkable double phase transition phenomenon with two critical values: the first characterising the emergence of a giant component in the overlay graph, and the second characterising the emergence of gigantic components in layers covering a typical node.

Finally, we highlight that the modelling framework in this article covers both deterministic and random layer types. Our approach of characterising the regularity of layer types using averaged empirical distributions allows both cases to be treated in a uniform manner.
1.2. Related work. The overlay network model discussed in this article is naturally motivated and implicitly described by classical works in social networks [16, 21]. The explanatory power and wide applicability of the model in the context of social, collaboration, and information networks has been demonstrated in [51, 52] by experimental studies of a communityaffiliation graph, which represents an instance of the present model where the node sets of layers are nonrandom or otherwise known to the observer. The superposition of Bernoulli random graphs considered here serves as a null model for sparse community-affiliation graphs.

The mathematical analysis in this article builds on earlier works on component evolution and clustering in inhomogeneous random graphs [14] and random intersection graphs $[8,9]$. The special model instance with unit layer strengths reduces to the so-called passive random intersection graph [25], and as a byproduct, the present article also provides the first rigorous analysis of giant components in general passive random intersection graphs, extending [15, 37]. When layer strengths are constant but not necessarily one, clustering properties and subgraph densities of the model have been analysed in [31, 32, 43], and the recovery of the layers in [20]. Another related work [48] (also part of [47]) on percolation in overlapping community networks assumes that layers are sampled from an arbitrary distribution on the space of finite connected graphs, and the layers are assigned to nodes via a bipartite configuration model. The restriction to connected layers and the use of a configuration model makes the model in [48] and its analysis fundamentally different from the present one, and limits its applicability by ruling out networks composed of weak communities.

Clustering spectra with power-law exponent 1 have been shown for random intersection graph models [7, 9] and spatial preferential attachment models [26, 36], and with a tunable power-law exponent in $[0,1]$ for random intersection graphs $[10,12]$ and recently also for a hyperbolic random geometric graph model [23]. Furthermore, [46] discusses an inhomogeneous Bernoulli graph model where the clustering spectrum vanishes, but its normalised version displays evidence of a power-law behaviour with exponent in range $(0,2)$.

To the best of our knowledge, the present work is the first of its kind where a nonvanishing clustering spectrum with a tunable power-law exponent in the extended range [0,2] is rigorously derived in terms of a simple statistical network model. This model admits a clear explanation of the values of power-law exponents, and introduces a new analytical framework for studying ordinary and double phase transitions in bond and site percolation on sparse networks of overlapping communities of variable size and strength.
1.3. Outline. In the rest of the article, Section 2 presents model details and notations, and Section 3 the main results. Section 4 illustrates the main results in a power-law setting, and confirms the existence of double phase transition. The remaining Sections 5-8 are devoted to proofs, with technical details postponed to Appendix A.

## 2. Model Description

2.1. Multilayer network. A multilayer network model with $n$ nodes and $m$ layers is defined by a list $\left(\left(G_{1}, X_{1}, Y_{1}\right), \ldots,\left(G_{m}, X_{m}, Y_{m}\right)\right)$ of mutually independent random variables with values in $\mathcal{G}_{n} \times\{0, \ldots, n\} \times[0,1]$, where $\mathcal{G}_{n}$ is the set of undirected graphs with node set contained in $\{1, \ldots, n\}$. We assume that conditionally on $\left(X_{k}, Y_{k}\right)$, the probability distribution of $V\left(G_{k}\right)$ is uniform on the subsets of $\{1, \ldots, n\}$ of size $X_{k}$, and conditionally on $\left(V\left(G_{k}\right), X_{k}, Y_{k}\right)$, each node pair of $V\left(G_{k}\right)$ is linked with probability $Y_{k}$, independently of other node pairs. Thus, $G_{k}$ is a Bernoulli random graph on node set $V\left(G_{k}\right)$, with edge set denoted $E\left(G_{k}\right)$. The variables $X_{k}, Y_{k}$, and $\left(X_{k}, Y_{k}\right)$ are called the size, strength, and type of layer $k$, respectively. Aggregation of layers produces an overlay random graph $G$ defined by

$$
\begin{equation*}
V(G)=\{1, \ldots, n\} \quad \text { and } \quad E(G)=\cup_{k=1}^{m} E\left(G_{k}\right) \tag{2.1}
\end{equation*}
$$

This setting includes as special cases: (i) models with deterministic layer types, and (ii) models where the layer types are independent and identically distributed random variables.
2.2. Large networks. A large network is analysed by considering a sequence of network models $\left(\left(G_{1}^{(n)}, X_{1}^{(n)}, Y_{1}^{(n)}\right), \ldots,\left(G_{m}^{(n)}, X_{m}^{(n)}, Y_{m}^{(n)}\right)\right)$ indexed by the number of nodes $n=1,2, \ldots$ so that the number of layers $m=m_{n}$ tends to infinity as $n \rightarrow \infty$. We shall focus on a sparse parameter regime where there exists a probability measure $P$ on $\{0,1, \ldots\} \times[0,1]$ which approximates in sufficiently strong sense the averaged layer type distribution

$$
\begin{equation*}
P_{n}(A)=\frac{1}{m} \sum_{k=1}^{m} \mathbb{P}\left(\left(X_{k}^{(n)}, Y_{k}^{(n)}\right) \in A\right) \tag{2.2}
\end{equation*}
$$

In this fundamental regime, the network features are described by limiting formulas with rich expressive power captured by cross moments and tail characteristics of $P$.
2.3. Notations. We denote $\mathbb{Z}_{+}=\{0,1, \ldots\},(a)_{+}=\max \{0, a\}$, and $(x)_{s}=x(x-1) \cdots(x-$ $s+1)$. The indicator function of a condition $A$ is denoted by $1(A)$ or $1_{A}$, whichever is more convenient. Sets of size $x$ are called $x$-sets. Unordered pairs and triples are abbreviated as $i j=\{i, j\}$ and $i j k=\{i, j, k\}$. We write $\sum_{i, j}^{\prime}$ and $\sum_{i, j, k}^{\prime}$ to indicate sums over ordered pairs and ordered triples with distinct elements. We write $a_{n} \ll b_{n}$ and $a_{n}=o\left(b_{n}\right)$ when $a_{n} / b_{n} \rightarrow 0, a_{n} \lesssim b_{n}$ and $a_{n}=O\left(b_{n}\right)$ when limsup $\left|a_{n} / b_{n}\right|<\infty$, and $a_{n} \sim b_{n}$ when $a_{n} / b_{n} \rightarrow 1$.

A graph is a pair $G=(V, E)$ where $E$ is a set of unordered pairs of elements of $V$. The degree and component of node $i$ in graph $G$ are denoted by $\operatorname{deg}_{G}(i)$ and $C_{G}(i)$, respectively. The transitive closure of graph $G$ is defined as the graph $\bar{G}$ with $V(\bar{G})=V(G)$ and $E(\bar{G})=$ $\left\{i j: i \in C_{G}(j)\right\}$ consisting of unordered node pairs connected by a path in $G$.

The probability distribution of a random variable $X$ is denoted by $\mathcal{L}(X)$. For probability measures, $d_{\mathrm{tv}}(f, g)$ denotes the total variation distance, $f * g$ the convolution, and $f_{n} \xrightarrow{w} f$ refers to weak convergence. On countable spaces, the same letter is used for both a probability measure $f(A)$ and its density $f(t)$ with respect to the counting measure. The Dirac measure at $x$ is denoted by $\delta_{x}$. The densities of the binomial distribution $\operatorname{Bin}(x, y)$ and the Poisson distribution $\operatorname{Poi}(\lambda)$ are denoted by

$$
\operatorname{Bin}(x, y)(t)=\binom{x}{t}(1-y)^{x-t} y^{t}, \quad \operatorname{Poi}(\lambda)(t)=e^{-\lambda} \frac{\lambda^{t}}{t!}
$$

with the convention that the densities are zero for $t$ outside $\{0, \ldots, x\}$ and $\mathbb{Z}_{+}$, respectively. The Bernoulli distribution is denoted $\operatorname{Ber}(y)(t)=\operatorname{Bin}(1, y)(t)$. We also denote by

$$
\begin{equation*}
\operatorname{Bin}^{+}(x, y)(t)=\mathbb{P}\left(\operatorname{deg}_{\bar{H}_{x+1, y}}(1)=t\right) \tag{2.3}
\end{equation*}
$$

the degree distribution of any particular node in the transitive closure $\bar{H}_{x+1, y}$ of a Bernoulli random graph $H_{x+1, y}$ on node set $\{1, \ldots, x+1\}$, where each node pair is linked with probability $y$, independently of other node pairs. Alternatively, $\operatorname{Bin}^{+}(x, y)(t)$ equals the probability
that the connected component of any particular node in $H_{x+1, y}$ has size $t+1$. Both distributions have the same support $\{0, \ldots, n\}$, and $\operatorname{Bin}(x, y) \leq_{\mathrm{st}} \operatorname{Bin}^{+}(x, y)$ in the strong stochastic order. No simple closed form expression is know for $\operatorname{Bin}^{+}(x, y)(t)$, but its values can be efficiently computed with the help of Gontcharoff polynomials $[2,5]$. The compound Poisson distribution with rate parameter $\lambda$ and increment distribution $g$ is denoted $\operatorname{CPoi}(\lambda, g)$; recall that this is the law of a random variable $\sum_{k=1}^{\Lambda} X_{k}$ where $\Lambda, X_{1}, X_{2}, \ldots$ are mutually independent and such that $\mathcal{L}(\Lambda)=\operatorname{Poi}(\lambda)$ and $\mathcal{L}\left(X_{k}\right)=g$.

For any probability measure $P$ on $\mathbb{Z}_{+} \times[0,1]$, any $P$-distributed random variable ( $X, Y$ ), and integers $r, s \geq 0$, we denote

$$
\begin{equation*}
(P)_{r s}=\mathbb{E}(X)_{r} Y^{s}=\int(x)_{r} y^{s} P(d x, d y) \tag{2.4}
\end{equation*}
$$

and when this quantity is finite and nonzero, we define mixed probability distributions $\operatorname{Bin}_{r s}(P)$ and $\operatorname{Bin}_{r s}^{+}(P)$ on $\mathbb{Z}_{+}$with probability mass functions

$$
\begin{align*}
\operatorname{Bin}_{r s}(P)(t) & =\mathbb{E}\left(\operatorname{Bin}(X-r, Y)(t) \frac{(X)_{r} Y^{s}}{(P)_{r s}}\right)  \tag{2.5}\\
\operatorname{Bin}_{r s}^{+}(P)(t) & =\mathbb{E}\left(\operatorname{Bin}^{+}(X-r, Y)(t) \frac{(X)_{r} Y^{s}}{(P)_{r s}}\right) \tag{2.6}
\end{align*}
$$

## 3. Main results

3.1. Degree distribution. The model degree distribution is defined by

$$
\begin{equation*}
f^{(n)}(t)=\frac{1}{n} \sum_{i=1}^{n} \mathbb{P}\left(\operatorname{deg}_{G^{(n)}}(i)=t\right) \tag{3.1}
\end{equation*}
$$

and represents the probability distribution of the number of neighbours of a randomly chosen node. Because $G^{(n)}$ is an exchangeable random graph, we see that $f^{(n)}=\mathcal{L}\left(\operatorname{deg}_{G^{(n)}}(1)\right)$.
Theorem 3.1. Assume that $\frac{m}{n} \rightarrow \mu \in(0, \infty)$ and $P_{n} \rightarrow P$ weakly together with $\left(P_{n}\right)_{10} \rightarrow$ $(P)_{10} \in(0, \infty)$ for some probability measure $P$ on $\mathbb{Z}_{+} \times[0,1]$. Then the model degree distribution $f^{(n)}$ converges weakly to a compound Poisson distribution $f=\operatorname{CPoi}\left(\mu(P)_{10}, \operatorname{Bin}_{10}(P)\right)$.
The limiting degree distribution $f$ in Theorem 3.1 can be represented as the law of $D=\sum_{k=1}^{\Lambda} D_{k}$ where $\Lambda$ is Poisson distributed with mean $\mu(P)_{10}, D_{1}, D_{2}, \ldots$ follow a mixed binomial distribution $\operatorname{Bin}_{10}(P)$, and the random variables in the sum are mutually independent. Here $\Lambda$ represents the number of layers covering a particular node, and $D_{k}$ the number of neighbours in a typical layer covering the node. The mean equals $\mathbb{E}(D)=\mu(P)_{21} \leq \infty$, and the variance equals $\operatorname{Var}(D)=\mu\left((P)_{21}+(P)_{32}\right)$ for $\left(P_{21}\right)<\infty$. Moreover, $\mathbb{E}\left(D^{r}\right)<\infty$ if and only if $(P)_{r+1, r}<\infty$. The generating function is given by $\mathbb{E}\left(z^{D}\right)=e^{\lambda\left(\hat{g}_{10}(z)-1\right)}$, where $\hat{g}_{10}(z)=\int(1-y+y z)^{x-1} \frac{x P(d x, d y)}{(P)_{10}}$. The structure of $P$ determines whether or not the limiting degree distribution is light-tailed or heavy-tailed. Section 4 illustrates both cases and provides examples of power laws with a tunable exponent.
3.2. Clustering. The clustering (a.k.a. transitivity) coefficient of the model is defined by

$$
\tau^{(n)}=\frac{\sum_{i j k}^{\prime} \mathbb{P}\left(G_{i j}^{(n)}, G_{i k}^{(n)}, G_{j k}^{(n)}\right)}{\sum_{i j k}^{\prime} \mathbb{P}\left(G_{i j}^{(n)}, G_{i k}^{(n)}\right)}
$$

where $G_{i j}^{(n)}$ represents the event that node pair $i j$ is linked, and the sums are taken over ordered triples of distinct nodes. We may interpret $\tau^{(n)}$ as the conditional probability that node pair $J K$ is linked given that $I J$ and $I K$ are linked, where $(I, J, K)$ is an ordered triple of distinct nodes selected uniformly at random.

Theorem 3.2. Assume that $\left(P_{n}\right)_{r s} \rightarrow(P)_{r s}<\infty$ for $r s=21,32,33$, and $(P)_{21}>0$. Then the model clustering coefficient is approximated by $\tau^{(n)} \rightarrow \tau$, where

$$
\tau= \begin{cases}\frac{(P)_{33}}{(P)_{32}} & \text { when } m \ll n \text { and }(P)_{32}>0, \\ \frac{(P)_{33}}{(P)_{32}+\mu(P)_{21}^{2}} & \text { when } \frac{m}{n} \rightarrow \mu \in(0, \infty), \\ 0 & \text { when } n \ll m \ll n^{2} .\end{cases}
$$

Remark (Constant layer strengths). When $Y_{k}=q$ is constant for all $k$, we see that $(P)_{r s}=$ $(p)_{r} q^{s}$ where $(p)_{r}$ equals the $r$-th factorial moment of the limiting layer size distribution. In this case the limiting model clustering equals $\frac{q(p)_{3}}{(p)_{3}+\mu(p)_{2}^{2}}$ and agrees with $[9,32]$.

The clustering spectrum of the model is defined by

$$
\sigma^{(n)}(t)=\frac{\sum_{i j k} \mathbb{P}\left(\operatorname{deg}_{G^{(n)}}(i)=t, G_{i j}^{(n)}, G_{i k}^{(n)}, G_{j k}^{(n)}\right)}{\sum_{i j k} \mathbb{P}\left(\operatorname{deg}_{G^{(n)}}(i)=t, G_{i j}^{(n)}, G_{i k}^{(n)}\right)}, \quad t \geq 2,
$$

and can be interpreted as the conditional probability that node pair $J K$ is linked given that $J$ and $K$ are neighbours of a node $I$ with degree $t$, where $(I, J, K)$ is an ordered triple of nodes selected uniformly at random. Section 4 illustrates examples where the limiting clustering spectrum below follows a power law.
Theorem 3.3. Assume that $\frac{m}{n} \rightarrow \mu \in(0, \infty)$, and $P_{n} \rightarrow P$ weakly together with $\left(P_{n}\right)_{r s} \rightarrow$ $(P)_{r s} \in(0, \infty)$ for $r s=10,21,32,33$. Then $\sigma^{(n)} \rightarrow \sigma$ pointwise to the limit

$$
\begin{equation*}
\sigma(t)=\frac{(P)_{33}\left(f * g_{33}\right)(t-2)}{(P)_{32}\left(f * g_{32}\right)(t-2)+\mu(P)_{21}^{2}\left(f * g_{21} * g_{21}\right)(t-2)}, \tag{3.2}
\end{equation*}
$$

where $f=\operatorname{CPoi}\left(\mu(P)_{10}, \operatorname{Bin}_{10}(P)\right)$ is the limiting degree distribution in Theorem 3.1, and the distributions $g_{r s}=\operatorname{Bin}_{r s}(P)$ are defined by (2.5).
3.3. Connected components. We denote by $N_{1}\left(G^{(n)}\right) \geq N_{2}\left(G^{(n)}\right)$ the two largest component sizes in $G^{(n)}$. For a probability distribution $f$ on $\mathbb{Z}_{+}$, we denote by

$$
\rho(f)=1-\min \left\{s \geq 0: \sum_{x \geq 0} s^{x} f(x)=s\right\}
$$

the probability of eternal survival of a Galton-Watson branching process with offspring distribution $f$.
Theorem 3.4. Assume that $\frac{m}{n} \rightarrow \mu \in(0, \infty)$ and $P_{n} \rightarrow P$ weakly together with $\left(P_{n}\right)_{10} \rightarrow$ $(P)_{10} \in(0, \infty)$. Then the largest two component sizes in $G^{(n)}$ are approximated by

$$
\frac{N_{1}\left(G^{(n)}\right)}{n} \xrightarrow{\mathbb{P}} \rho\left(f^{+}\right) \quad \text { and } \quad \frac{N_{2}\left(G^{(n)}\right)}{n} \xrightarrow{\mathbb{P}} 0,
$$

where $f^{+}=\mathrm{CPoi}\left(\mu(P)_{10}, \operatorname{Bin}_{10}^{+}(P)\right)$ is a compound Poisson distribution with rate parameter $\mu(P)_{10}$ and increment distribution $\operatorname{Bin}_{10}^{+}(P)$ defined by (2.6).
3.4. Site percolation. We may analyse how a subset of nodes $S_{n} \subset\{1, \ldots, n\}$ is connected by considering a site-percolated graph defined as the subgraph

$$
\begin{equation*}
\check{G}^{(n)}=G^{(n)}\left[S_{n}\right] \tag{3.3}
\end{equation*}
$$

of $G^{(n)}$ induced by $S_{n}$. The site-percolated graph is an instance of the overlay graph model (2.1) with layers $\left(\check{G}_{1}, \check{X}_{1}, \check{Y}_{1}\right), \ldots,\left(\check{G}_{m}, \check{X}_{m}, \check{Y}_{m}\right)$ such that the conditional distribution of $\check{X}_{k}=\left|V\left(\check{G}_{k}\right)\right|$ given $X_{k}=V\left(G_{k}\right)$ is hypergeometric, and $\check{Y}_{k}=Y_{k}$. An approximation of
the hypergeometric distribution by a binomial distribution $\operatorname{Bin}\left(X_{k}, \theta\right)$ with $\frac{\left|S_{n}\right|}{n} \approx \theta$ suggests replacing the limiting layer type distribution $P$ by

$$
\check{P}(A)=\int\left(\operatorname{Bin}(x, \theta) \times \delta_{y}\right)(A) P(d x, d y)
$$

The following result confirms that this modification is well justified, and summarizes the results of Theorems 3.1-3.4 adjusted to site percolation.
Theorem 3.5. Assume that $\frac{m}{n} \rightarrow \mu \in(0, \infty), P_{n} \rightarrow P$ weakly together with $\left(P_{n}\right)_{10} \rightarrow$ $(P)_{10} \in(0, \infty)$, and $S_{n} \subset\{1, \ldots, n\}$ satisfies $\frac{\left|S_{n}\right|}{n} \rightarrow \theta \in(0,1]$. Then the following approximations are valid for the site-percolated graph $\breve{G}^{(n)}=\check{G}^{(n)}\left[S_{n}\right]$ :
(i) The degree distribution converges weakly to $\check{f}=\operatorname{CPoi}\left(\mu(\check{P})_{10}, \operatorname{Bin}_{10}(\check{P})\right)$.
(ii) The largest two component sizes are approximated by $n^{-1} N_{1} \xrightarrow{\mathbb{P}} \rho\left(\check{f}^{+}\right)$and $n^{-1} N_{2} \xrightarrow{\mathbb{P}} 0$ with $\check{f}^{+}=\operatorname{CPoi}\left(\mu(\check{P})_{10}, \operatorname{Bin}_{10}^{+}(\check{P})\right)$.
If we also assume that $\left(P_{n}\right)_{r s} \rightarrow(P)_{r s} \in(0, \infty)$ for $r s=21,32,33$, then
(iii) The clustering coefficient converges to $\hat{\tau}=\tau$ where $\tau$ is the corresponding limit of the nonpercolated graph $G^{(n)}$.
(iv) The clustering spectrum converges pointwise to $\check{\sigma}$ defined by replacing $f$ and $g_{r s}$ in (3.2) by $\check{f}$ and $\check{g}_{r s}=\operatorname{Bin}_{r s}(\check{P})$.
3.5. Bond percolation. Bond percolation studies how well the nodes of a graph are connected along a subset of links obtained by random sampling. In a multilayer networks, we may either sample (i) a subset of links of the overlay graph, or (ii) independent subsets of links for each layer separately. To analyse these cases for the overlay graph model $G=G^{(n)}$ in (2.1), we define an overlay bond-percolated graph by

$$
\begin{equation*}
\hat{G}=G \cap H \tag{3.4}
\end{equation*}
$$

and a layerwise bond-percolated graph $\tilde{G}$ by

$$
\begin{equation*}
V(\tilde{G})=\{1, \ldots, n\} \quad \text { and } \quad E(\tilde{G})=\cup_{k=1}^{m} E\left(G_{k} \cap H_{k}\right) \tag{3.5}
\end{equation*}
$$

where $H, H_{1}, \ldots, H_{m}$ are mutually independent random graphs on $\{1, \ldots, n\}$ in which each node pair is linked with probability $\theta$, independently of other node pairs, and independently of the layers $\left(G_{k}, X_{k}, Y_{k}\right)$.

In an epidemic modeling context, the standard SIR epidemic model is used to model individuals who infect their neighbours with probability $\theta$, independently of each other [2]. The links of a graph $G$ represent social contacts, and the bond-percolated component of node $i$ corresponds to the set of eventually infected individuals in a population where node $i$ is initially infectious and the other nodes susceptible. Bond percolation on the overlay graph can be used to develop finer models to model contacts of individuals generated by social communities (households, workplaces, schools) of variable size and strength. Layerwise percolation $\hat{G}$ then models the case where infections occur independently inside the communities, and the overlay bond-percolation $\tilde{G}$ models the case where infections occur between individuals regardless of the underlying community structure.

The layerwise bond-percolated graph is an instance of the overlay model (2.1) with layer types $\left(X_{k}, \theta Y_{k}\right)$. This suggests considering a modified limiting layer type distribution

$$
\hat{P}(A)=\int\left(\delta_{x} \times \delta_{\theta y}\right)(A) P(d x, d y)
$$

We expect the overlay bond-percolated model to behave similarly to the layerwise bondpercolated model in sparse regimes where the layers do not overlap much. The following result confirms this, and summarises the results of Theorems 3.1-3.4 adjusted to bond percolation.

Theorem 3.6. Assume that $\frac{m}{n} \rightarrow \mu \in(0, \infty)$, and $P_{n} \rightarrow P$ weakly together with $\left(P_{n}\right)_{10} \rightarrow$ $(P)_{10} \in(0, \infty)$, and $\theta_{n} \rightarrow \theta \in(0,1]$. Then the following approximations are valid for both the overlay bond-percolated graph $\hat{G}^{(n)}$ and the layerwise bond-percolated graph $\tilde{G}^{(n)}$ :
(i) The degree distribution converges weakly to $\hat{f}=\operatorname{CPoi}\left(\mu(\hat{P})_{10}, \operatorname{Bin}_{10}(\hat{P})\right)$.
(ii) The largest two component sizes are approximated by $n^{-1} N_{1} \xrightarrow{\mathbb{P}} \rho\left(\hat{f}^{+}\right)$and $n^{-1} N_{2} \xrightarrow{\mathbb{P}} 0$ with $\hat{f}^{+}=\operatorname{CPoi}\left(\mu(\hat{P})_{10}, \operatorname{Bin}_{10}^{+}(\hat{P})\right)$.
If we also assume that $\left(P_{n}\right)_{r s} \rightarrow(P)_{r s} \in(0, \infty)$ for $r s=21,32,33$, then:
(iii) The clustering coefficient converges to $\hat{\tau}=\theta \tau$ where $\tau$ is the corresponding limit of the nonpercolated graph $G^{(n)}$.
(iv) The clustering spectrum converges pointwise to $\hat{\sigma}$ defined by replacing $P$, $f$, and $g_{r s}$ in (3.2) by $\hat{P}$, $\hat{f}$, and $\hat{g}_{r s}=\operatorname{Bin}_{r s}(\hat{P})$.
3.6. Double phase transition. Theorem 3.6 shows that the largest relative component size in the bond-percolated graph is approximated by the survival probability $\rho\left(\hat{f}^{+}\right)$of a GaltonWatson process with compound Poisson offspring distribution $\hat{f}^{+}=\operatorname{CPoi}\left(\mu(\hat{P})_{10}, \operatorname{Bin}_{10}^{+}(\hat{P})\right)$. The mean of the offspring distribution can be written as ${ }^{1}$

$$
\begin{equation*}
R_{0}(\theta)=\mu \int R(x-1, \theta y) x P(d x, d y) \tag{3.6}
\end{equation*}
$$

where $R(x, y)=\sum_{t \geq 0} t \operatorname{Bin}^{+}(x, y)(t)$ defined using (2.3) represents the expected transitive degree in a homogeneous Bernoulli graph with $x+1$ nodes and link probability $y$. Classical branching process theory tells that $\rho\left(\hat{f}^{+}\right)>0$ if and only if $R_{0}(\theta)>1$. Hence the largest component in the bond-percolated graph is sublinear for $\theta<\theta_{1}$, and linear for $\theta>\theta_{1}$, where the critical threshold is defined by

$$
\theta_{1}=\sup \left\{\theta \in[0,1]: R_{0}(\theta)<1\right\} .
$$

The overlay graph model in studied in this article involves another nontrivial phase transition associated with a critical threshold value

$$
\theta_{2}=\sup \left\{\theta \in[0,1]: R_{0}(\theta)<\infty\right\} .
$$

Section 4 describes an example where $0<\theta_{1}<\theta_{2}<1$.
The first phase transition at $\theta_{1}$ characterises the emergence of a giant component in a bond-percolated overlay graph. To understand the second phase transition, note that $R_{0}(\theta)$ is proportional to the expected number of nodes which can be reached by paths within a typical bond-percolated layer covering a particular node. The second phase transition at $\theta_{2}$ hence amounts to the emergence of gigantic components inside bond-percolated layers covering a typical node.

In the epidemic context discussed in Section 3.5, we note that the critical quantity $R_{0}(\theta)$ does not refer to the number of individuals directly infected by a reference individual in an otherwise susceptible population, unlike in classical SIR models. Rather, $R_{0}(\theta)$ also counts the number of individuals indirectly infected by the reference individual via single-layer infection paths.

## 4. Power-law models

This section illustrates the rich statistical features of the overlay model by discussing the results of Section 3 in a setting where the layer strength is a deterministic function of layer size according to $Y_{k}=q\left(X_{k}\right)$ for some $q: \mathbb{Z}_{+} \rightarrow[0,1]$, and the limiting layer type distribution factorises according to

$$
\begin{equation*}
P(d x, d y)=p(d x) \delta_{q(x)}(d y) \tag{4.1}
\end{equation*}
$$

[^1]where the layer size distribution $p$ is a probability on $\mathbb{Z}_{+}$. For concreteness, we assume that the probability mass function $p(x)$ of the layer size distribution and $q(x)$ follow power laws
\[

$$
\begin{equation*}
p(x)=(a+o(1)) x^{-\alpha} \quad \text { and } \quad q(x)=\left(b+O\left(x^{-1 / 2}\right)\right) x^{-\beta} \tag{4.2}
\end{equation*}
$$

\]

as $x \rightarrow \infty$, with exponents $\alpha>2, \beta \geq 0$ and constants $a, b>0$. In this case

$$
(P)_{r s}=\sum_{x \geq 0}(x)_{r} q(x)^{s} p(x)=\sum_{x \geq 0}\left(a b^{s}+o(1)\right) x^{r-s \beta-\alpha}
$$

shows that $(P)_{r s}$ is finite if and only if $\alpha+s \beta>r+1$.
4.1. Degree distribution and clustering spectrum. Theorems 4.1 and 4.2 below establish power laws for the limiting degree distribution and clustering spectrum. Figures 1 and 2 illustrate how the associated power-law exponents relate to the corresponding exponents of layer sizes and layer strengths. Remarkably, the power law of the clustering spectrum admits a tunable exponent in $[0,2]$. A similar power law with exponent 1 has earlier been established for a random intersection graph [9] and for a spatial preferential attachment random graph [26], and with exponent restricted to $[0,1]$ for inhomogeneous random intersection graphs [7, 10, 12] and a hyperbolic random geometric graph model [23].

Theorem 4.1. Assume (4.2) for some $\alpha>2, \beta \geq 0$, and $a, b>0$.
(i) If $\beta \in(0,1)$, then the limiting degree distribution satisfies

$$
\begin{equation*}
f(t) \sim d t^{-\delta} \tag{4.3}
\end{equation*}
$$

for $\delta=1+\frac{\alpha-2}{1-\beta}$ and $d=\mu(1-\beta)^{-1} a b^{\delta-1}$.
(ii) Relation (4.3) holds also for $\beta=0$, provided that either $b<1$, or $b=1$ and $q(x)=1$ for all but finitely many $x$.
(iii) If $\beta \geq 1$, then the limiting degree distribution is light-tailed with generating function bounded by $\sum_{t \geq 0} z^{t} f(t) \leq e^{\mu(P)_{10}\left(e^{M(z-1)}-1\right)}$ for all $z \geq 0$, where $M=\sup _{x \geq 1}(x-1) q(x)$.


Figure 1. (Color online.) Power-law exponent of degree distribution as a function of layer size exponent $\alpha$ and layer strength exponent $\beta$.

Theorem 4.2. Assume (4.2) for some $\alpha \in(2, \infty)$ and $\beta \in(0,1)$ such that $\alpha+2 \beta>4$, and that $\left(P_{n}\right)_{r s} \rightarrow(P)_{r s} \in(0, \infty)$ for $r s=10,21,32,33$. Then the limiting clustering spectrum defined by (3.2) follows a power law according to

$$
\sigma(t) \sim \begin{cases}c_{1} t^{-\beta /(1-\beta)}, & \beta<2 / 3 \\ c_{2} t^{-2}, & \beta=2 / 3 \\ c_{3} t^{-2}, & \beta>2 / 3\end{cases}
$$

where $c_{1}=b^{1 /(1-\beta)}, c_{3}=\mu(P)_{33}$, and $c_{2}=c_{1}+c_{3}$. Furthermore, if (4.2) holds for $\alpha \in$ $(4, \infty)$ and $\beta=0$, and $q(x)=b \in(0,1]$ for all but finitely many $x$, then $\sigma(t) \sim b$.

Networks with $\sigma(t) \ll t^{-1}$ are sometimes call weakly clustered, and those with $\sigma(t) \gg$ $t^{-1}$ strongly clustered [4]. According to Theorem 4.2, the overlay graph model produces weakly clustered networks for $\beta>\frac{1}{2}$, and strongly clustered networks for $\beta<\frac{1}{2}$. Using techniques in [10], Theorem 4.2 can be generalised to the case where $p(x)$ in (4.2) has a regularly varying tail, and we believe that it can be extended to more general subexponential distributions as well. We do not pursue this line here to avoid unnecessary technicalities.


Figure 2. (Color online.) Power-law exponent of clustering spectrum as a function of layer size exponent $\alpha$ and layer strength exponent $\beta$. The assumptions of Theorem 4.2 do not hold in the grey areas where $(P)_{32}=\infty$.
4.2. Existence of double phase transition. For the power-law model (4.2), the function in (3.6) can be computed as $R_{0}(\theta)=\mu \sum_{x} R(x-1, \theta q(x)) x p(x)$. By applying a classical giant component result for Bernoulli random graphs [29, Theorem 5.4], one may verify that ${ }^{2}$

$$
\begin{align*}
\limsup _{x \rightarrow \infty} \theta x q(x) \leq 1-\varepsilon & \Longrightarrow \quad \limsup _{x \rightarrow \infty} R(x-1, \theta q(x)) \leq 2 \varepsilon^{-2}  \tag{4.4}\\
\liminf _{x \rightarrow \infty} \theta x q(x) \geq 1+\varepsilon & \Longrightarrow \quad \liminf _{x \rightarrow \infty} x^{-1} R(x-1, \theta q(x))>0
\end{align*}
$$

If $\alpha>3$, then the limiting layer size distribution $p$ has a finite second moment and $R(x-$ $1, y) \leq x-1$ implies that $R_{0}(1)<\infty$. Hence $\theta_{2}=1$, and the second phase transition cannot occur. On the other hand, when $\alpha \in(2,3]$, the limiting layer size distribution has infinite second moment. In this case (4.4) yields the following conclusions:
(1) $\beta=1$ with $b>1$. Then $R_{0}(\theta)<\infty$ for $\theta<b^{-1}$, and $R_{0}(\theta)=\infty$ for $\theta>b^{-1}$. Hence $\theta_{2}=b^{-1} \in(0,1)$. Assume in addition that the constant $a$ in (4.2) is large enough so that $\mu \theta(P)_{21} \geq 1$ for $\theta=\frac{1}{2} \theta_{2}$. Then $\hat{f}^{+} \geq_{\text {st }} \hat{f}$ implies that $R_{0}(\theta)=\sum_{t} t \hat{f}^{+}(t) \geq$ $\sum_{t} t \hat{f}(t)=\mu \theta(P)_{21} \geq 1$ for $\theta=\frac{1}{2} \theta_{2}$, and the continuity of $R_{0}(\theta)$ on $\left[0, \theta_{2}\right)$ implies that $\theta_{1} \in\left(0, \frac{1}{2} \theta_{2}\right)$. There are hence two critical values $0<\theta_{1}<\theta_{2}<1$ in which the model displays two distinct phase transitions.
(2) $\beta \in(1, \infty)$, or $\beta=1$ with $b<1$. Then $R_{0}(\theta)<\infty$ for all $\theta \in[0,1]$, so that $\theta_{2}=1$, and the second-type phase transition cannot occur.
(3) $\beta \in[0,1)$. Then one can show that $R_{0}(\theta)=\infty$ for all $\theta \in(0,1]$, and hence $\theta_{1}=\theta_{2}=$ 0 , and there are no phase transitions of either type.
The above observations confirm the existence of a double phase transition in bond percolation, as postulated in [18], for a natural network model admitting tunable power-law exponents for both the degree distribution and the clustering spectrum. Together with Theorems 4.1 and 4.2, this opens up a flexible framework for studying the significance and interrelations of these power laws to bond and site percolation properties in clustered complex

[^2]networks. The investigation of how these phase transitions are reflected in the core-periphery organisation of the network $[4,18]$ remains an important topic for future research.

## 5. Analysis of degree distributions

5.1. Quantitative approximation for deterministic layer types. The following quantitative estimate is valid for every scale.
Proposition 5.1. If the layer types are nonrandom and $\left(P_{n}\right)_{10}>0$, then the model degree distribution $f^{(n)}$ defined by (3.1) is approximated by a compound Poisson distribution $\operatorname{CPoi}\left(\lambda^{(n)}, g_{10}^{(n)}\right)$ with rate parameter $\lambda^{(n)}=\frac{m}{n}\left(P_{n}\right)_{10}$ and increment distribution $g_{10}^{(n)}=$ $\operatorname{Bin}_{10}\left(P_{n}\right)$ defined by (2.5) according to

$$
\begin{equation*}
d_{\mathrm{tv}}\left(f^{(n)}, \operatorname{CPoi}\left(\lambda^{(n)}, g_{10}^{(n)}\right)\right) \leq\left(1+\frac{m}{n}\right)^{2}\|X\|_{\infty}^{4} n^{-1} \tag{5.1}
\end{equation*}
$$

where $\|X\|_{\infty}=\max _{1 \leq k \leq m} X_{k}$.
Proof. We approximate the degree $D_{i}=\operatorname{deg}_{G}(i)$ of node $i$ by a random integer $L_{i}=$ $\sum_{k=1}^{m} \operatorname{deg}_{G_{k}}(i)$. Observe that $L_{i} \neq D_{i}$ if and only if there exists a node $j \neq i$ and some distinct layers $k<\ell$ such that $i j \in E\left(G_{k}\right)$ and $i j \in E\left(G_{\ell}\right)$. Hence by the union bound and the independence of $G_{k}$ and $G_{\ell}$,

$$
\mathbb{P}\left(L_{i} \neq D_{i}\right) \leq \sum_{j \neq i} \sum_{1 \leq k<\ell \leq m} \mathbb{P}\left(i j \in E\left(G_{k}\right)\right) \mathbb{P}\left(i j \in E\left(G_{\ell}\right)\right) .
$$

Hence, noting that $\mathbb{P}\left(\left(i j \in E\left(G_{k}\right)\right)=\frac{\left(X_{k}\right)_{2}}{(n)_{2}} Y_{k} \leq \frac{X_{k}^{2}}{n^{2}} Y_{k} \leq n^{-2}\|X\|_{\infty}^{2}\right.$, it follows that

$$
\begin{equation*}
d_{\mathrm{tv}}\left(D_{i}, L_{i}\right) \leq(n-1)\binom{m}{2}\left(n^{-2}\|X\|_{\infty}^{2}\right)^{2} \leq \frac{m^{2}}{n^{3}}\|X\|_{\infty}^{4} \tag{5.2}
\end{equation*}
$$

Now denote by $W_{x y}=\left\{k:\left(X_{k}, Y_{k}\right)=(x, y)\right\}$ the set of layers with size $x$ and strength $y$, and let $m_{x y}=\left|W_{x y}\right|$. Also denote $S=\left\{(x, y): m_{x y}>0\right\}$. Then we see that $L_{i}=$ $\sum_{(x, y) \in S} \sum_{k \in W_{x y}} \operatorname{deg}_{G_{k}}(i)$. Let us define a random variable

$$
\begin{equation*}
\hat{L}_{i}=\sum_{(x, y) \in S} \sum_{\ell=1}^{M_{x y}} A_{x y}(\ell) \tag{5.3}
\end{equation*}
$$

where $\mathcal{L}\left(M_{x y}\right)=\operatorname{Bin}\left(m_{x y}, \frac{x}{n}\right), \mathcal{L}\left(A_{x y}(\ell)\right)=\operatorname{Bin}(x-1, y)$, and all random variables on the right side are mutually independent. Then for any $(x, y) \in S, \sum_{k \in W_{x y}} \operatorname{deg}_{G_{k}}(i) \stackrel{\text { d }}{=}$ $\sum_{\ell=1}^{M_{x y}} A_{x y}(\ell)$, because the summands on the left are mutually independent, the number of layers $k \in W_{x y}$ containing node $i$ is $\operatorname{Bin}\left(m_{x y}, \frac{x}{n}\right)$-distributed, and because $\mathcal{L}\left(\operatorname{deg}_{G_{k}}(i) \mid V\left(G_{k}\right) \ni\right.$ $i)=\operatorname{Bin}(x-1, y)$ for each $k \in W_{x y}$. As a consequence, it follows that $L_{i} \stackrel{\text { d }}{=} \hat{L}_{i}$.

Now denote $\lambda_{x y}=m_{x y} \frac{x}{n}$ and define a new random variable

$$
\begin{equation*}
L_{i}^{\prime}=\sum_{(x, y) \in S} \underbrace{\sum_{\ell=1}^{M_{x y}^{\prime}} A_{x y}(\ell)}_{L_{x y}^{\prime}}, \tag{5.4}
\end{equation*}
$$

where $M_{x y}^{\prime}$ are $\operatorname{Poi}\left(\lambda_{x y}\right)$-distributed, mutually independent, and independent of the random variables $A_{x y}(\ell)$. Because $\mathcal{L}\left(L_{x y}^{\prime}\right)=\operatorname{CPoi}\left(\lambda_{x y}, \operatorname{Bin}(x-1, y)\right)$, Lemma A. 5 implies that
$\mathcal{L}\left(L_{i}^{\prime}\right)=\operatorname{CPoi}\left(\lambda^{(n)}, g_{10}^{(n)}\right)$ with rate parameter $\lambda^{(n)}=\sum_{(x, y) \in S} \lambda_{x y}=\frac{m}{n}\left(P_{n}\right)_{10}$ and mixed binomial increment distribution

$$
g_{10}^{(n)}=\sum_{(x, y) \in S} \operatorname{Bin}(x-1, y) \frac{\lambda_{x y}}{\lambda^{(n)}}=\sum_{(x, y) \in S} \operatorname{Bin}(x-1, y) \frac{x P_{n}(\{(x, y)\})}{\left(P_{n}\right)_{10}} .
$$

As a consequence of Le Cam's inequality [45] it follows that $d_{\mathrm{tv}}\left(M_{x y}, M_{x y}^{\prime}\right) \leq m_{x y}\left(\frac{x}{n}\right)^{2} \leq$ $n^{-2}\|X\|_{\infty}^{2} m_{x y}$, and hence

$$
d_{\mathrm{tv}}\left(\hat{L}_{i}, L_{i}^{\prime}\right) \leq \sum_{(x, y) \in S} d_{\mathrm{tv}}\left(\sum_{\ell=1}^{M_{x y}} A_{x y}(\ell), \sum_{\ell=1}^{M_{x y}^{\prime}} A_{x y}(\ell)\right) \leq \sum_{(x, y) \in S} d_{\mathrm{tv}}\left(M_{x y}, M_{x y}^{\prime}\right)
$$

implies that $d_{\mathrm{tv}}\left(L_{i}, L_{i}^{\prime}\right)=d_{\mathrm{tv}}\left(\hat{L}_{i}, L_{i}^{\prime}\right) \leq n^{-2} m\|X\|_{\infty}^{2}$. By combining this with (5.2), the claim follows.
5.2. Proof of Theorem 3.1. We prove the claim in three stages: (i) under an extra assumption that the space of layer types is finite, (ii) under an extra assumption that the layer sizes are bounded, (iii) under no extra assumptions. In what follows, $D_{n}=\operatorname{deg}_{G^{(n)}}(1)$ and we consider all models $n=1,2, \ldots$ to be defined on a common probability space (see Section A. 1 for formal details).
(i) Assume that the supports of $P_{n}, n \geq 1$, and $P$ are contained in a finite set $A \subset$ $\mathbb{Z}_{+} \times[0,1]$. Denote by $P_{\theta_{n}}=\frac{1}{m} \sum_{k=1}^{m} \delta_{\left(X_{n, k}, Y_{n, k}\right)}$ the empirical layer type distribution of the $n$-th model, and denote by $\mathcal{L}\left(D_{n} \mid \theta_{n}\right)$ the conditional distribution of $D_{n}$ given layer types $\theta_{n}=\left(\left(X_{n, 1}, Y_{n, 1}\right), \ldots,\left(X_{n, m}, Y_{n, m}\right)\right)$. Let us define $\lambda_{\theta_{n}}=\frac{m}{n}\left(P_{\theta_{n}}\right)_{10}$, and

$$
g_{\theta_{n}}(t)= \begin{cases}\operatorname{Bin}_{10}\left(P_{\theta_{n}}\right), & \left(P_{\theta_{n}}\right)_{10}>0 \\ \delta_{0}(t), & \text { else }\end{cases}
$$

where $\operatorname{Bin}_{10}\left(P_{\theta_{n}}\right)$ is defined by (2.5) and $\delta_{0}$ is the Dirac measure at zero. Then by applying Proposition 5.1 and Lemma A.6,

$$
\begin{aligned}
& d_{\mathrm{tv}}\left(\mathcal{L}\left(D_{n} \mid \theta_{n}\right), \operatorname{CPoi}(\lambda, g)\right) \\
& \leq d_{\mathrm{tv}}\left(\mathcal{L}\left(D_{n} \mid \theta_{n}\right), \operatorname{CPoi}\left(\lambda_{\theta_{n}}, g_{\theta_{n}}\right)\right)+d_{\mathrm{tv}}\left(\operatorname{CPoi}\left(\lambda_{\theta_{n}}, g_{\theta_{n}}\right), \operatorname{CPoi}(\lambda, g)\right) \\
& \leq\left(1+\frac{m}{n}\right)^{2} M^{4} n^{-1}+\left|\lambda_{\theta_{n}}-\lambda\right|+\lambda d_{\mathrm{tv}}\left(g_{\theta_{n}}, g\right),
\end{aligned}
$$

where the inequalities remain valid also on the event that $\left(P_{\theta_{n}}\right)_{10}=0$ because in this case all layers are empty and $\mathcal{L}\left(D_{n} \mid \theta_{n}\right)=\delta_{0}$. On the event that $P_{\theta_{n}} \xrightarrow{w} P$, we see that $\lambda_{\theta_{n}}=$ $\frac{m}{n}\left(P_{\theta_{n}}\right)_{10} \rightarrow \mu(P)_{10}=\lambda$ and $g_{\theta_{n}} \xrightarrow{w} g$ (see Lemma A.10), so that $d_{\mathrm{tv}}\left(\mathcal{L}\left(D_{n} \mid \theta_{n}\right), \operatorname{CPoi}(\lambda, g)\right) \rightarrow$ 0 . Observe next that $d_{\mathrm{tv}}\left(P_{\theta_{n}}, P\right) \xrightarrow{\mathbb{P}} 0$ by Lemma A.2. By applying Lemma A. 1 with $\Phi_{n}\left(\theta_{n}, \xi_{n}\right)=\mathcal{L}\left(D_{n} \mid \theta_{n}\right)$, we conclude that $d_{\mathrm{tv}}\left(\mathcal{L}\left(D_{n} \mid \theta_{n}\right), \operatorname{CPoi}(\lambda, g)\right) \xrightarrow{\mathbb{P}} 0$. Because $d_{\mathrm{tv}}$ is a bounded metric, it follows that $d_{\mathrm{tv}}\left(\mathcal{L}\left(D_{n}\right), \operatorname{CPoi}(\lambda, g)\right) \leq \mathbb{E} d_{\mathrm{tv}}\left(\mathcal{L}\left(D_{n} \mid \theta_{n}\right), \operatorname{CPoi}(\lambda, g)\right) \rightarrow 0$.
(ii) Assume now that the supports of $P_{n}$ and $P$ are all contained in $\{0,1, \ldots, M\} \times[0,1]$. We will discretise the unit interval $[0,1]$ as in Section 7.5. Fix an integer $L \geq 1$, and denote by $G_{n}^{L-}$ (resp. $G_{n}^{L+}$ ) an overlay graph generated by a modified model where the layer strengths $Y_{n, k}$ are replaced by $\left\lfloor Y_{n, k}\right\rfloor_{L}$ (resp. $\left\lceil Y_{n, k}\right\rceil_{L}$ ), defined by (7.26). Denote by $D_{n}^{L-}, D_{n}^{L+}$ the degrees of node 1 in $G_{n}^{L-}, G_{n}^{L+}$, respectively. Under a natural coupling of the Bernoulli variables describing the link indicators of the layers we have $G_{n}^{L-} \subset G_{n} \subset G_{n}^{L+}$ almost surely, and hence

$$
\begin{equation*}
\mathbb{P}\left(D_{n}^{L-} \geq t\right) \leq \mathbb{P}\left(D_{n} \geq t\right) \leq \mathbb{P}\left(D_{n}^{L+} \geq t\right) \tag{5.5}
\end{equation*}
$$

for all integers $t \geq 0$ and $L \geq 1$.

The averaged layer type distribution of $G_{n}^{L \pm}$ is given by $P_{n} \circ \sigma_{L \pm}^{-1}$, where $\sigma_{L-}(x, y)=$ $\left(x,\lfloor y\rfloor_{L}\right)$ and $\sigma_{L+}(x, y)=\left(x,\lceil y\rceil_{L}\right)$. By Lemma 7.12, $P_{n} \circ \sigma_{L \pm}^{-1} \xrightarrow{w} P \circ \sigma_{L \pm}^{-1}$ and $\left(P_{n} \circ\right.$ $\left.\sigma_{L \pm}^{-1}\right)_{10} \rightarrow\left(P \circ \sigma_{L \pm}^{-1}\right)_{10}$. Hence by part (i), it follows that $\mathbb{P}\left(D_{n}^{L \pm} \geq t\right) \rightarrow \mathbb{P}\left(D^{L \pm} \geq t\right)$, where $\mathcal{L}\left(D^{L \pm}\right)=\operatorname{CPoi}\left(\lambda, g_{L \pm}\right)$ with $g_{L \pm}(t)=\operatorname{Bin}_{10}\left(P \circ \sigma_{L \pm}^{-1}\right)$. Hence by (5.5),

$$
\mathbb{P}\left(D^{L-} \geq t\right) \leq \liminf _{n \rightarrow \infty} \mathbb{P}\left(D_{n} \geq t\right) \leq \limsup _{n \rightarrow \infty} \mathbb{P}\left(D_{n} \geq t\right) \leq \mathbb{P}\left(D^{L+} \geq t\right)
$$

Lemma 7.12 also shows that $P \circ \sigma_{L \pm}^{-1} \xrightarrow{w} P$ and $\left(P \circ \sigma_{L \pm}^{-1}\right)_{10} \rightarrow(P)_{10}$, so that (Lemma A.10) $g_{L \pm} \xrightarrow{w} g$ and hence also (Lemma A.6) $\mathcal{L}\left(D^{L \pm}\right) \xrightarrow{w} \mathcal{L}(D)$ as $L \rightarrow \infty$, where $\mathcal{L}(D)=$ $\operatorname{CPoi}(\lambda, g)$. The above inequalities then imply that $\mathbb{P}\left(D_{n} \geq t\right) \rightarrow \mathbb{P}(D \geq t)$ for all $t$. Hence $\mathcal{L}\left(D_{n}\right) \xrightarrow{w} \mathcal{L}(D)$.
(iii) Let us now prove Theorem 3.1 without making any extra assumptions. Let $G_{n}^{M}$ be an overlay graph generated by truncated layers

$$
G_{n, k}^{M}= \begin{cases}G_{n, k} & \text { if }\left|V\left(G_{n, k}\right)\right| \leq M,  \tag{5.6}\\ \text { empty graph } & \text { otherwise }\end{cases}
$$

Denote by $D_{n}^{M}$ the degree of node 1 in $G_{n}^{M}$. Observe that $D_{n} \neq D_{n}^{M}$ implies that there exists a layer $G_{n, k}$ of size larger than $M$ which contains node 1 , and this occurs with probability

$$
\mathbb{P}\left(V\left(G_{n, k}\right) \ni 1,\left|V\left(G_{n, k}\right)\right|>M\right)=\mathbb{E} \frac{X_{n, k}}{n} 1\left(X_{n, k}>M\right) .
$$

Hence by the union bound,

$$
\begin{equation*}
d_{\mathrm{tv}}\left(\mathcal{L}\left(D_{n}\right), \mathcal{L}\left(D_{n}^{M}\right)\right) \leq \sum_{k=1}^{m} \mathbb{E} \frac{X_{n, k}}{n} 1\left(X_{n, k}>M\right) \leq \frac{m}{n} h(M), \tag{5.7}
\end{equation*}
$$

where $h(M)=\sup _{n \geq 1} \int x 1(x>M) P_{n}(d x, d y)$.
Observe next that $G_{n}^{M}$ is an instance of the overlay model with layer types ( $X_{n, k} 1\left(X_{n, k} \leq\right.$ $\left.M), Y_{n, k}\right)$ and averaged layer type distribution $P_{n} \circ \sigma_{M}^{-1}$ where $\sigma_{M}(x, y)=(x 1(x \leq M), y)$. By Lemma 7.12, $P_{n} \circ \sigma_{M}^{-1} \xrightarrow{w} P \circ \sigma_{M}^{-1}$ together with $\left(P_{n} \circ \sigma_{M}^{-1}\right)_{10} \rightarrow\left(P \circ \sigma_{M}^{-1}\right)_{10}$. Hence by part (ii), it follows that

$$
d_{\mathrm{tv}}\left(\mathcal{L}\left(D_{n}^{M}\right), \operatorname{CPoi}\left(\lambda^{M}, g^{M}\right)\right) \rightarrow 0,
$$

where $\lambda^{M}=\mu\left(P \circ \sigma_{M}^{-1}\right)_{10}$ and $g^{M}=\operatorname{Bin}_{10}\left(P \circ \sigma_{M}^{-1}\right)$. Now by (5.7) and Lemma A.6, we find that

$$
\begin{aligned}
d_{\mathrm{tv}}\left(\mathcal{L}\left(D_{n}\right), \operatorname{CPoi}(\lambda, g)\right) \leq & d_{\mathrm{tv}}\left(\mathcal{L}\left(D_{n}^{M}\right), \operatorname{CPoi}\left(\lambda^{M}, g^{M}\right)\right) \\
& +\left|\lambda^{M}-\lambda\right|+\lambda d_{\mathrm{tv}}\left(g^{M}, g\right)+\frac{m}{n} h(M),
\end{aligned}
$$

so that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} d_{\mathrm{tv}}\left(\mathcal{L}\left(D_{n}\right), \operatorname{CPoi}(\lambda, g)\right) \leq\left|\lambda^{M}-\lambda\right|+\lambda d_{\mathrm{tv}}\left(g^{M}, g\right)+\mu h(M) . \tag{5.8}
\end{equation*}
$$

Lemma 7.12 also implies that $h(M) \rightarrow 0$, and that $P \circ \sigma_{M}^{-1} \xrightarrow{w} P$ together with $\left(P \circ \sigma_{M}^{-1}\right)_{10} \rightarrow$ $(P)_{10}$ as $M \rightarrow \infty$. Hence $g^{M} \xrightarrow{w} g$ by Lemma A.10. The claim of Theorem 3.1 now follows because the right side of (5.8) can be made arbitrarily small by choosing a large enough $M$.

## 6. Analysis of clustering

6.1. General subgraph densities. Subgraph frequencies in the overlay graph will be characterised using cross moments

$$
\begin{equation*}
\left(P_{n}\right)_{r s}=\int(x)_{r} y^{s} d P_{n}, \quad\left(P_{n}\right)_{r s, t u}=\int(x)_{r} y^{s}(x)_{t} y^{u} d P_{n} \tag{6.1}
\end{equation*}
$$

of the averaged layer type distribution $P_{n}$ defined by (2.2), and normalised cross moments defined by

$$
\begin{equation*}
\mu_{r s}^{(n)}=\sum_{k=1}^{m} p_{r s}^{(n)}(k), \quad \mu_{r s, t u}^{(n)}=\sum_{k=1}^{m} p_{r s}^{(n)}(k) p_{t u}^{(n)}(k) \tag{6.2}
\end{equation*}
$$

where $p_{r s}^{(n)}(k)=(n)_{r}^{-1} \mathbb{E}\left(X_{k}^{(n)}\right)_{r}\left(Y_{k}^{(n)}\right)^{s}$. These definitions are motivated by the following result, where $G_{k^{*}}=G_{k^{*}}^{(n)}$ represents a randomly chosen layer, and we recall the the mixed binomial distribution $\operatorname{Bin}_{r s}\left(P_{n}\right)$ defined in (2.5).

Lemma 6.1. Let $F_{r s}$ be a graph with node set in $\{1, \ldots, n\}$ such that $\left|V\left(F_{r s}\right)\right|=r$ and $\left|E\left(F_{r s}\right)\right|=s$, and let $i$ be a node in $V\left(F_{r s}\right)$ with $\operatorname{deg}_{F_{r s}}(i)=r-1$. Select $k^{*} \in\{1, \ldots, m\}$ uniformly at random and independently of the layers. Then:
(i) $\mathbb{P}\left(G_{k^{*}} \supset F_{r s}\right)=m^{-1} \mu_{r s}^{(n)}$,
(ii) $\mathbb{P}\left(\operatorname{deg}_{G_{k^{*}}}(i)=t \mid G_{k^{*}} \supset F_{r s}\right)=\operatorname{Bin}_{r s}\left(P_{n}\right)(t-r+1)$ for all $t$.

Proof. (i) Because $\mathbb{P}\left(V\left(G_{k}\right) \supset V\left(F_{r s}\right) \mid X_{k}, Y_{k}\right)=\frac{\left(X_{k}\right)_{r}}{(n)_{r}}$ for any $k$, we see that $\mathbb{P}\left(G_{k} \supset\right.$ $\left.F_{r s}\right)=\mathbb{E} \frac{\left(X_{k}\right)_{r}}{(n)_{r}} Y_{k}^{s}=p_{r s}^{(n)}(k)$. The corresponding probability for a randomly selected $k^{*}$ equals $\mathbb{P}\left(G_{k^{*}} \supset F_{r s}\right)=\frac{1}{m} \sum_{k=1}^{m} p_{r s}^{(n)}(k)=\left(P_{n}\right)_{r s}$.
(ii) Denote $D_{k}=\operatorname{deg}_{G_{k}}(i)$. On the event that $G_{k} \supset F_{r s}$, we see that $D_{k}=d+D_{k}^{\prime}$ where $D_{k}^{\prime}=\left|N_{G_{k}}(i) \backslash V\left(F_{r s}\right)\right|$ and $d=r-1$. Conditionally on $\left(X_{k}, Y_{k}\right)=(x, y)$ and $G_{k} \supset F_{r s}$, the random integer $D_{k}^{\prime}$ is $\operatorname{Bin}(x-r, y)$-distributed. Hence

$$
\mathbb{P}\left(D_{k}=t, G_{k} \supset F_{r s}\right)=\mathbb{E}\left(\operatorname{Bin}\left(X_{k}-r, Y_{k}\right)(t-d) \frac{\left(X_{k}\right)_{r}}{(n)_{r}} Y_{k}^{s}\right)
$$

The corresponding probability for a randomly chosen $k^{*}$ is

$$
\mathbb{P}\left(D_{k^{*}}=t, G_{k^{*}} \supset F_{r s}\right)=\int\left(\operatorname{Bin}(x-r, y)(t-d) \frac{(x)_{r}}{(n)_{r}} y^{s}\right) P_{n}(d x, d y)
$$

so the claim follows by dividing both sides by $\mathbb{P}\left(G_{k^{*}} \supset F_{r s}\right)=(n)_{r}^{-1}\left(P_{n}\right)_{r s}$.
6.2. Triangle densities. The following quantitative bound is valid for every fixed $n$.

Theorem 6.2. Let $K_{3}$ be a triangle with $i \in V\left(K_{3}\right) \subset[n]$. Then:
(i) $\left|\mathbb{P}\left(G \supset K_{3}\right)-\mu_{33}\right| \leq 4 \mu_{21} \mu_{32}+\mu_{21}^{3}$.
(ii) $\mathbb{P}\left(\operatorname{deg}_{G}(i)=t, G \supset K_{3}\right)=\mu_{33} f^{(n)} * g_{33}^{(n)}(t-2)+\varepsilon(t)$, where $f^{(n)}$ is the model degree distribution defined by (3.1), $g_{33}^{(n)}=\operatorname{Bin}_{33}\left(P_{n}\right)$ is defined by (2.5), and the approximation error is bounded by

$$
|\varepsilon(t)| \leq(4+t) \mu_{21} \mu_{32}+\mu_{21}^{3}+2 \mu_{10,33}
$$

Proof. Denote $\mathcal{K}_{3}=\left\{G \supset K_{3}\right\}$. Denote by $\mathcal{A}_{k}=\left\{G_{k} \supset K_{3}\right\}$ the event that all node pairs of the triangle are linked by layer $k$. We also denote $D=\operatorname{deg}_{G}(i), D_{k}=\operatorname{deg}_{G_{k}}(i)$, and $D_{-k}=\operatorname{deg}_{G_{-k}}(i)$ with $G_{-k}=\cup_{k^{\prime} \neq k} G_{k}$.
(i) Denote

$$
\varepsilon_{1}(t)=\mathbb{P}\left(D=t, \mathcal{K}_{3}\right)-\mathbb{P}\left(D=t, \cup_{k} \mathcal{A}_{k}\right)
$$

and observe that $0 \leq \varepsilon_{1}(t) \leq \mathbb{P}\left(D=t, \mathcal{E}_{12}\right)+\mathbb{P}\left(D=t, \mathcal{E}_{111}\right)$, where $\mathcal{E}_{12}$ is the event that there exists one layer covering one link and a different layer covering two links of $K_{3}$, and $\mathcal{E}_{111}$ is the event that three distinct layers cover the links of $K_{3}$. We write $p(a b c)=\mathbb{P}\left(\mathcal{G}_{12}^{a}, \mathcal{G}_{13}^{b}, \mathcal{G}_{23}^{c}\right)$, where $\mathcal{G}_{i j}^{a}$ the event that node pair $i j$ is linked in layer $a$. We note that $p(a b c)=p_{21}(a) p_{21}(b) p_{21}(c), p(a a b)=p_{32}(a) p_{21}(b)$, and $p(a a a)=p_{33}(a)$ for distinct layers $a, b, c$. Hence

$$
\mathbb{P}\left(\mathcal{E}_{12}\right) \leq \sum_{a, b}^{\prime}(p(a a b)+p(a b a)+p(b a a)) \leq 3 \mu_{21} \mu_{32}
$$

and $\mathbb{P}\left(\mathcal{E}_{111}\right) \leq \sum_{a, b, c}^{\prime} p(a b c) \leq \mu_{21}^{3}$. Thus, $\sum_{t \geq 0}\left|\varepsilon_{1}(t)\right| \leq 3 \mu_{21} \mu_{32}+\mu_{21}^{3}$.
Then denote

$$
\varepsilon_{2}(t)=\mathbb{P}\left(D=t, \cup_{k} \mathcal{A}_{k}\right)-\sum_{k} \mathbb{P}\left(D=t, \mathcal{A}_{k}\right) .
$$

Bonferroni's inequalities imply that $0 \leq-\varepsilon_{2}(t) \leq \sum_{k, k^{\prime}}^{\prime} \mathbb{P}\left(D=t, \mathcal{A}_{k}, \mathcal{A}_{k^{\prime}}\right)$, and hence, noting that $\mu_{33} \leq \mu_{32} \leq \mu_{21}$,

$$
\sum_{t \geq 0}\left|\varepsilon_{2}(t)\right| \leq \sum_{k, k^{\prime}}^{\prime} \mathbb{P}\left(\mathcal{A}_{k}, \mathcal{A}_{k^{\prime}}\right)=\sum_{k, k^{\prime}}^{\prime} p_{33}(k) p_{33}\left(k^{\prime}\right) \leq \mu_{33}^{2} \leq \mu_{21} \mu_{32} .
$$

By combining this with the bound for $\varepsilon_{1}(t)$, we conclude that

$$
\mathbb{P}\left(D=t, \mathcal{K}_{3}\right)=\sum_{k} \mathbb{P}\left(D=t, \mathcal{A}_{k}\right)+\varepsilon_{1}(t)+\varepsilon_{2}(t)
$$

where $\sum_{t \geq 0}\left(\left|\varepsilon_{1}(t)\right|+\left|\varepsilon_{2}(t)\right|\right) \leq 4 \mu_{21} \mu_{32}+\mu_{21}^{3}$. Hence claim (i) follows by summing the above equality over $t$, and noting that $\sum_{k} \mathbb{P}\left(\mathcal{A}_{k}\right)=\mu_{33}$.
(ii) We will next approximate

$$
\begin{align*}
\sum_{k} \mathbb{P}\left(D=t, \mathcal{A}_{k}\right) & \approx \sum_{k} \mathbb{P}\left(D_{-k}+D_{k}=t, \mathcal{A}_{k}\right)  \tag{6.3}\\
& =\sum_{k} \sum_{r+s=t} \mathbb{P}\left(D_{-k}=r\right) \mathbb{P}\left(D_{k}=s, \mathcal{A}_{k}\right) \\
& \approx \sum_{k} \sum_{r+s=t} \mathbb{P}(D=r) \mathbb{P}\left(D_{k}=s, \mathcal{A}_{k}\right) . \tag{6.4}
\end{align*}
$$

Lemma 6.1 shows that $\sum_{k} \mathbb{P}\left(D_{k}=s, \mathcal{A}_{k}\right)=\mu_{33} \operatorname{Bin}_{33}\left(P_{n}\right)(s-2)$. Hence the last term above equals $\mu_{33} f^{(n)} * g_{33}^{(n)}(t-2)$, and to prove the claim it suffices to analyse the approximation errors in (6.3)-(6.4).
The approximation error in (6.3) equals $\varepsilon_{3}(t)=\sum_{k} \varepsilon_{3 k}(t)$, where

$$
\varepsilon_{3 k}(t)=\mathbb{P}\left(D=t, \mathcal{A}_{k}\right)-\mathbb{P}\left(D_{-k}+D_{k}=t, \mathcal{A}_{k}\right) .
$$

By applying Lemma A. 14 with $A=\{k\}, B=[n] \backslash\{k\}, \mathcal{E}_{A}=\left\{G_{k} \ni e_{1}, e_{2}, e_{3}\right\}$, and $\mathcal{E}_{B}=\{ \}$ being the sure event, we see that $\left|\varepsilon_{3 k}(t)\right| \leq c_{B} t \mathbb{P}\left(D_{k} \leq t, \mathcal{A}_{k}\right) \leq c_{B} t \mathbb{P}\left(\mathcal{A}_{k}\right)$, where $c_{B}=\mathbb{P}\left(G_{-k} \ni 12\right) \leq \sum_{\ell \neq k} p_{21}(\ell) \leq \mu_{21}$. Hence

$$
\left|\varepsilon_{3}(t)\right| \leq t \mu_{21} \sum_{k} p_{33}(k)=t \mu_{21} \mu_{33} \leq t \mu_{21} \mu_{32} .
$$

The approximation error in (6.4) equals $\varepsilon_{4}(t)=\sum_{k} \varepsilon_{4 k}(t)$ where

$$
\varepsilon_{4 k}(t)=\sum_{r+s=t}\left(\mathbb{P}(D=r)-\mathbb{P}\left(D_{-k}=r\right)\right) \mathbb{P}\left(D_{k}=s, \mathcal{A}_{k}\right) .
$$

By Lemma A.13, $\sum_{t \geq 0}\left|\varepsilon_{4 k}(t)\right| \leq 2 \mathbb{P}\left(D_{k}>0\right) \mathbb{P}\left(\mathcal{A}_{k}\right)$. Because $\mathbb{P}\left(D_{k}>0\right) \leq p_{10}(k)$ and $\mathbb{P}\left(\mathcal{A}_{k}\right)=p_{33}(k)$, it follows that $\sum_{t \geq 0}\left|\varepsilon_{4}(t)\right| \leq 2 \mu_{10,33}$. Claim (ii) follows by combining the above estimates for the total approximation error $\varepsilon(t)=\varepsilon_{1}(t)+\varepsilon_{2}(t)+\varepsilon_{3}(t)+\varepsilon_{4}(t)$.
6.3. Two-star densities. The following quantitative bound is valid for every fixed $n$.

Theorem 6.3. Consider a two-star $K_{12}$ with node set $V\left(K_{12}\right) \subset[n]$ and hub node $i$. Then:
(i) $\left|\mathbb{P}\left(G \supset K_{12}\right)-\left(\mu_{32}+\mu_{21}^{2}\right)\right| \leq 6 \mu_{21} \mu_{32}+6 \mu_{21}^{3}+\mu_{21}^{4}+\mu_{21,21}$.
(ii) $\mathbb{P}\left(\operatorname{deg}_{G}(i)=t, G \supset K_{12}\right)=\mu_{32} f^{(n)} * g_{32}^{(n)}(t-2)+\mu_{21}^{2} f^{(n)} * g_{21}^{(n)} * g_{21}^{(n)}(t-2)+\varepsilon(t)$, where $f^{(n)}$ is the degree distribution of $G$, and the approximation error is bounded by

$$
|\varepsilon(t)| \leq(6+2 t)\left(\mu_{21} \mu_{32}+\mu_{21}^{3}\right)+\mu_{21}^{4}+4 \mu_{10,32}+4 \mu_{21} \mu_{10,21}+\mu_{21,21}
$$

Proof. We assume that $K_{12}$ is the two-star with node set $\{1,2,3\}$ and link set $\{12,13\}$, and denote the event under study by $\mathcal{K}_{12}=\left\{G \supset K_{12}\right\}$. We denote by $\mathcal{G}_{i j}^{k}$ the event that $i j \in E\left(G^{k}\right)$ and we set $\mathcal{A}_{k \ell}=\mathcal{G}_{12}^{k} \cap \mathcal{G}_{13}^{\ell}$. We denote $G^{k \ell}=G^{k} \cup G^{\ell}$ and $G^{-k \ell}=\cup_{q \notin\{k, \ell\}} G^{q}$, and we set $D=\operatorname{deg}_{G}(1), D_{k \ell}=\operatorname{deg}_{G^{k \ell}}(1)$ and $D_{-k \ell}=\operatorname{deg}_{G^{-k \ell}}(1)$. We also denote $h_{k \ell}(s)=$ $\mathbb{P}\left(D_{k \ell}=s, \mathcal{A}_{k \ell}\right)$.

First we approximate

$$
\begin{align*}
\mathbb{P}\left(D=t, \mathcal{K}_{12}\right) & \approx \sum_{k, \ell} \mathbb{P}\left(D=t, \mathcal{A}_{k \ell}\right)  \tag{6.5}\\
& \approx \sum_{k, \ell} \mathbb{P}\left(D_{k \ell}+D_{-k \ell}=t, \mathcal{A}_{k \ell}\right)  \tag{6.6}\\
& =\sum_{k, \ell} \sum_{r+s=t} \mathbb{P}\left(D_{-k \ell}=r\right) h_{k \ell}(s) \\
& \approx \sum_{k, \ell} \sum_{r+s=t} \mathbb{P}(D=r) h_{k \ell}(s), \tag{6.7}
\end{align*}
$$

so that

$$
\begin{equation*}
\mathbb{P}\left(D=t, \mathcal{K}_{12}\right) \approx \sum_{r+s=t} f^{(n)}(r) \sum_{k} h_{k k}(s)+\sum_{r+s=t} f^{(n)}(r) \sum_{k, \ell}^{\prime} h_{k \ell}(s) \tag{6.8}
\end{equation*}
$$

Then we note with the help of Lemma 6.1 that $\sum_{k} h_{k k}(s)=\mu_{32} g_{32}^{(n)}(s-2)$. Hence the first term on the right side of (6.8) equals

$$
\begin{equation*}
\sum_{r+s=t} f^{(n)}(r) \sum_{k} h_{k k}(s)=\mu_{32} f^{(n)} * g_{32}^{(n)}(t-2) \tag{6.9}
\end{equation*}
$$

Next we approximate, denoting $h_{k}(s)=\mathbb{P}\left(D_{k}=s, \mathcal{G}_{12}^{k}\right)$,

$$
\begin{align*}
\sum_{k, \ell}^{\prime} h_{k \ell}(s) & =\sum_{k, \ell}^{\prime} \mathbb{P}\left(D_{k \ell}=s, \mathcal{G}_{12}^{k}, \mathcal{G}_{13}^{\ell}\right) \\
& \approx \sum_{k, \ell}^{\prime} \mathbb{P}\left(D_{k}+D_{\ell}=s, \mathcal{G}_{12}^{k}, \mathcal{G}_{13}^{\ell}\right)  \tag{6.10}\\
& =\sum_{k, \ell}^{\prime} \sum_{s_{1}+s_{2}=s} h_{k}\left(s_{1}\right) h_{\ell}\left(s_{2}\right) \\
& \approx \sum_{k, \ell} \sum_{s_{1}+s_{2}=s} h_{k}\left(s_{1}\right) h_{\ell}\left(s_{2}\right) \tag{6.11}
\end{align*}
$$

After noting (see Lemma 6.1) that $\sum_{k} h_{k}(s)=\mu_{21} g_{21}^{(n)}(s-1)$, we conclude that

$$
\sum_{k, \ell} \sum_{s_{1}+s_{2}=s} h_{k}\left(s_{1}\right) h_{\ell}\left(s_{2}\right)=\mu_{21}^{2} g_{21}^{(n)} * g_{21}^{(n)}(s-2)
$$

and hence the second term on the right side of (6.8) is approximately

$$
\begin{equation*}
\sum_{r+s=t} f(r) \sum_{k, \ell}^{\prime} h_{k \ell}(s) \approx \mu_{21}^{2} f * g_{21}^{(n)} * g_{21}^{(n)}(t-2) . \tag{6.12}
\end{equation*}
$$

By combining (6.8), (6.9) and (6.12), we conclude that

$$
\begin{equation*}
\mathbb{P}\left(D=t, \mathcal{K}_{12}\right) \approx \mu_{32} f^{(n)} * g_{32}^{(n)}(t-2)+\mu_{21}^{2} f^{(n)} * g_{21}^{(n)} * g_{21}^{(n)}(t-2) . \tag{6.13}
\end{equation*}
$$

The total approximation error in (6.13) can be written as $\varepsilon(t)=\varepsilon_{1}(t)+\varepsilon_{2}(t)+\varepsilon_{3}(t)+\varepsilon_{4}(t)$, where $\varepsilon_{1}(t), \varepsilon_{2}(t), \varepsilon_{3}(t)$ are the approximation errors in (6.5), (6.6), (6.7), respectively, and the approximation error in (6.12) equals

$$
\varepsilon_{4}(t)=\sum_{r+s=t} f^{(n)}(r)\left(\varepsilon_{41}(s)+\varepsilon_{42}(s)\right)
$$

where $\varepsilon_{41}(s), \varepsilon_{42}(s)$ denote the errors made in (6.10), (6.11), respectively. We will next analyse the individual approximation errors one by one.
(i) The union bound shows that the approximation error $\varepsilon_{1}(t)$ in (6.5) is nonpositive for all $t$, and hence $\sum_{t \geq 0}\left|\varepsilon_{1}(t)\right|=\sum_{k, \ell} \mathbb{P}\left(\mathcal{A}_{k \ell}\right)-\mathbb{P}\left(\bigcup_{k, \ell} \mathcal{A}_{k \ell}\right)$. Bonferroni's inequalities imply that

$$
\sum_{t \geq 0}\left|\varepsilon_{1}(t)\right| \leq \sum_{\left(k_{1}, k_{2}\right),\left(\ell_{1}, \ell_{2}\right)}^{\prime} \mathbb{P}\left(\mathcal{A}_{k_{1} k_{2}}, \mathcal{A}_{\ell_{1} \ell_{2}}\right)=: \Delta .
$$

We split the right side above by $\Delta=\Delta_{2}+\Delta_{3}+\Delta_{4}$, where $\Delta_{i}, i=2,3,4$, is the sum on the right side above over layer pairs $\left(k_{1}, k_{2}\right) \neq\left(\ell_{1}, \ell_{2}\right)$ such that the list ( $k_{1}, k_{2}, \ell_{1}, \ell_{2}$ ) contains precisely $i$ distinct elements. Denote

$$
p\left(k_{1} k_{2} \ell_{1} \ell_{2}\right)=\mathbb{P}\left(G_{k_{1}} \ni e_{1}, G_{k_{2}} \ni e_{2}, G_{\ell_{1}} \ni e_{1}, G_{\ell_{2}} \ni e_{2}\right) .
$$

Then

$$
\begin{aligned}
& \Delta_{2}=\sum_{a, b}^{\prime}(p(a a b b)+p(a b b a)+p(a a a b)+p(a a b a)+p(a b a a)+p(b a a a)) \\
& \Delta_{3}=\sum_{a, b, c}^{\prime}(p(a a b c)+p(a b a c)+p(a b c a)+p(b a a c)+p(b a c a)+p(b c a a))
\end{aligned}
$$

In the sum of $\Delta_{2}$, the terms $p(a a b b)$ and $p(a b b a)$ equal $p_{32}(a) p_{32}(b)$ and the other terms equal $p_{32}(a) p_{21}(b)$. Because $p_{32}(b) \leq p_{21}(b)$, it follows that $\Delta_{2} \leq 6 \sum_{a, b}^{\prime} p_{21}(a) p_{32}(b) \leq 6 \mu_{21} \mu_{32}$. In the sum of $\Delta_{3}$, the terms $p(a b a c)$ and $p(b a c a)$ equal $p_{21}(a) p_{21}(b) p_{21}(c)$ and the other terms equal $p_{32}(a) p_{21}(b) p_{21}(c)$. Because $p_{32}(a) \leq p_{21}(a)$, it follows that $\Delta_{3} \leq 6 \mu_{21}^{3}$. Furthermore, $\Delta_{4}=\sum_{a, b, c, d}^{\prime} p(a b c d) \leq \mu_{21}^{4}$. As a conclusion, it follows that

$$
\sum_{t \geq 0}\left|\varepsilon_{1}(t)\right| \leq 6 \mu_{21} \mu_{32}+6 \mu_{21}^{3}+\mu_{21}^{4} .
$$

Claim (i) now follows by combining the above bound with the equality

$$
\sum_{k, \ell} \mathbb{P}\left(\mathcal{A}_{k \ell}\right)=\sum_{k} p_{32}(k)+\sum_{k, \ell}^{\prime} p_{21}(k) p_{21}(\ell)=\mu_{32}+\mu_{21}^{2}-\mu_{21,21} .
$$

(ii) The approximation error in (6.6) equals $\varepsilon_{2}(t)=\sum_{k, \ell} \varepsilon_{2 k \ell}(t)$ where

$$
\varepsilon_{2 k \ell}(t)=\mathbb{P}\left(D=t, \mathcal{A}_{k \ell}\right)-\mathbb{P}\left(D_{k \ell}+D_{-k \ell}=t, \mathcal{A}_{k \ell}\right) .
$$

By applying Lemma A. 14 with $A=\{k, \ell\}, B=[m] \backslash\{k, \ell\}, \mathcal{E}_{A}=\left\{G_{k} \ni e_{1}, G_{\ell} \ni e_{2}\right\}$, and $\mathcal{E}_{B}=\{ \}$ being the sure event, we see that

$$
\left|\varepsilon_{2 k \ell}(t)\right| \leq t c_{B} \mathbb{P}\left(D_{k \ell} \leq t, \mathcal{A}_{k \ell}\right) \leq t c_{B} \mathbb{P}\left(\mathcal{A}_{k \ell}\right),
$$

where $c_{B} \leq \mathbb{P}\left(G_{-k \ell} \ni 12\right) \leq \mathbb{P}(G \ni 12) \leq \mu_{21}$. Hence

$$
\left|\varepsilon_{2}(t)\right| \leq t \mu_{21} \sum_{k, \ell} \mathbb{P}\left(\mathcal{A}_{k \ell}\right) \leq t\left(\mu_{21} \mu_{32}+\mu_{21}^{3}\right)
$$

(iii) The approximation error in (6.7) equals $\varepsilon_{3}(t)=\sum_{k, \ell} \varepsilon_{3 k \ell}(t)$ where

$$
\varepsilon_{3 k \ell}(t)=\sum_{r+s=t}\left(\mathbb{P}(D=r)-\mathbb{P}\left(D_{-k \ell}=r\right)\right) h_{k \ell}(s)
$$

By applying Lemma A. 13 with $g(s)=\frac{h_{k \ell}(s)}{\mathbb{P}\left(\mathcal{A}_{k \ell}\right)}$, it follows that $\sum_{t \geq 0}\left|\varepsilon_{3 k \ell}(t)\right| \leq 2 \mathbb{P}\left(\mathcal{A}_{k \ell}\right) \mathbb{P}\left(D_{k \ell}>\right.$ $0)$. Observe now that $\mathbb{P}\left(D_{k \ell}>0\right) \leq p_{10}(k)+p_{10}(\ell)$, Hence,

$$
\begin{aligned}
\sum_{t \geq 0}\left|\varepsilon_{3}(t)\right| & \leq 2 \sum_{k, \ell}\left(p_{10}(k)+p_{10}(\ell)\right) \mathbb{P}\left(\mathcal{A}_{k \ell}\right) \\
& =4 \sum_{k} p_{10}(k) p_{32}(k)+4 \sum_{k, \ell}^{\prime} p_{10}(k) p_{21}(k) p_{21}(\ell) \\
& \leq 4 \mu_{10,32}+4 \mu_{21} \mu_{10,21} .
\end{aligned}
$$

(iv) The approximation error in (6.10) equals $\varepsilon_{41}(s)=\sum_{k, \ell}^{\prime} \varepsilon_{4 k \ell}(s)$ where

$$
\varepsilon_{4 k \ell}(s)=\mathbb{P}\left(D_{k \ell}=s, \mathcal{A}_{k \ell}\right)-\mathbb{P}\left(D_{k}+D_{\ell}=s, \mathcal{A}_{k \ell}\right)
$$

By applying Lemma A. 14 with $A=\{k\}$ and $B=\{\ell\}$, together with $\mathcal{E}_{A}=\left\{12 \in G_{k}\right\}$ and $\mathcal{E}_{B}=\left\{13 \in G_{\ell}\right\}$, it follows that $\left|\varepsilon_{4 k \ell}(s)\right| \leq s p_{21}(k) p_{32}(\ell)$. By summing the above inequality with respect to $k, \ell$, it follows that $\left|\varepsilon_{41}(s)\right| \leq s \mu_{21} \mu_{32}$. The approximation error in (6.11) equals

$$
\left|\varepsilon_{42}(s)\right|=\sum_{k} \sum_{s_{1}+s_{2}=s} \mathbb{P}\left(D_{k}=s_{1}, \mathcal{G}_{12}^{k}\right) \mathbb{P}\left(D_{k}=s_{2}, \mathcal{G}_{12}^{k}\right)
$$

Hence $\sum_{s \geq 0}\left|\varepsilon_{42}(s)\right|=\sum_{k} p_{21}(k)^{2}=\mu_{21,21}$. Hence,

$$
\left|\varepsilon_{4}(t)\right| \leq \sum_{r+s=t} f^{(n)}(r)\left(\left|\varepsilon_{41}(s)\right|+\left|\varepsilon_{42}(s)\right|\right) \leq \max _{s \leq t}\left|\varepsilon_{41}(s)\right|+\max _{s \leq t}\left|\varepsilon_{42}(s)\right|
$$

shows that $\left|\varepsilon_{4}(t)\right| \leq t \mu_{21} \mu_{32}+\mu_{21,21}$.
Claim (ii) follows by collecting all the bounds in (i)-(iv) together.

### 6.4. Lemma about cross moments.

Lemma 6.4. Let $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{m}, Y_{m}\right)$ be random variables with values in $\{0, \ldots, n\} \times[0,1]$ and averaged empirical distribution $P_{n}$ defined by (2.2). If $P_{n} \xrightarrow{w} P$ and $\left(P_{n}\right)_{r s} \rightarrow(P)_{r s}<$ $\infty$, then the cross moments defined in (6.1)-(6.2) satisfy $\mu_{10, r s}^{(n)} \ll m(n)_{r}^{-1}$ and $\left(P_{n}\right)_{10, r s} \ll n$. Proof. Denote $A_{k}=X_{k}$ and $B_{k}=\left(X_{k}\right)_{r} Y_{k}^{s}$. Observe that $A_{k} \leq a+A_{k} 1\left(A_{k}>a\right)$ and $B_{k} \leq b+B_{k} 1\left(B_{k}>b\right)$ for any $a, b>0$. Because $A_{k} \leq n$, we find that

$$
\begin{align*}
A_{k} \mathbb{E} B_{k} & \leq\left(a+A_{k} 1\left(A_{k}>a\right)\right) \mathbb{E} B_{k} \\
& \leq a \mathbb{E} B_{k}+b n 1\left(A_{k}>a\right)+n \mathbb{E} B_{k} 1\left(B_{k}>b\right) \tag{6.14}
\end{align*}
$$

By taking expectations and averaging with respect to $k$, we find that

$$
\begin{equation*}
\frac{1}{m} \sum_{k=1}^{m} \mathbb{E} A_{k} \mathbb{E} B_{k} \leq a \mathbb{E} B_{*}+b n \mathbb{P}\left(A_{*}>a\right)+n \mathbb{E} B_{*} 1\left(B_{*}>b\right) \tag{6.15}
\end{equation*}
$$

where $A_{*}=X_{*}, B_{*}=\left(X_{*}\right)_{r} Y_{*}^{s}$, and $\left(X_{*}, Y_{*}\right)$ is a generic $P_{n}$-distributed random variable. Because the left side above equals $m^{-1} n(n)_{r} \mu_{10, r s}^{(n)}$, we conclude

$$
m^{-1}(n)_{r} \mu_{10, r s}^{(n)} \leq \frac{a}{n} c+b \phi(a)+\psi(b)
$$

where $c=\sup _{n}\left(P_{n}\right)_{r s}, \phi(t)=\sup _{n} \int 1(x>t) d P_{n}$, and $\psi(t)=\sup _{n} \int(x)_{r} y^{s} 1\left((x)_{r} y^{s}>t\right) d P_{n}$. Then the tightness of $P_{n}$ implies that $\phi\left(a_{n}\right) \rightarrow 0$ for $a_{n}=n^{1 / 2}$. Hence also $b_{n} \phi\left(a_{n}\right) \rightarrow 0$ where $b_{n}=\phi\left(a_{n}\right)^{-1 / 2} \rightarrow \infty$. The uniform $(x)_{r} y^{s}$-integrability of $P_{n}$ further implies that $\psi\left(b_{n}\right) \rightarrow 0$. Hence the right side above vanishes and first claim follows.

For the second claim, we may repeat the above reasoning to verify that (6.14) holds also with the $\mathbb{E}$-symbol removed. Therefore, (6.15) also holds when the left side is replaced by $\left(P_{n}\right)_{10, r s}=\frac{1}{m} \sum_{k=1}^{m} \mathbb{E} A_{k} B_{k}$. Hence the second claim follows by the same argument.
6.5. Proof of Theorem 3.2. By Theorem 6.2 and Theorem 6.3,

$$
\begin{aligned}
\mathbb{P}\left(\mathcal{K}_{3}\right) & =\mu_{33}+O\left(\mu_{21} \mu_{32}+\mu_{21}^{3}\right) \\
\mathbb{P}\left(\mathcal{K}_{12}\right) & =\mu_{32}+\mu_{21}^{2}+O\left(\mu_{21} \mu_{32}+\mu_{21}^{3}+\mu_{21}^{4}+\mu_{21,21}\right)
\end{aligned}
$$

where $\mu_{r s}=m(n)_{r}^{-1}\left(P_{n}\right)_{r s}$, and the associated cross moments are defined by (6.1)-(6.2). Because $\left(P_{n}\right)_{21} \lesssim 1$ and $\left(P_{n}\right)_{32} \lesssim 1$, it follows that $\mu_{21} \mu_{32} \lesssim m^{2} n^{-5}, \mu_{21}^{3} \lesssim m^{3} n^{-6}$, and $\mu_{21}^{4} \lesssim m^{4} n^{-8}$. Next, we note that $\mu_{21,21} \leq m(n)_{2}^{-2}\left(P_{n}\right)_{21,21}$ by Jensen's inequality. Note also that $\left((x)_{2} y\right)^{2} \leq 2 x(x)_{3} y^{2}$ for $x \geq 3$. Hence $\left((x)_{2} y\right)^{2} \leq 4+2 x(x)_{3} y^{2}$, and $\left(P_{n}\right)_{21,21} \leq$ $4+2\left(P_{n}\right)_{10,32}$. Furthermore, Lemma 6.4 implies that $\left(P_{n}\right)_{10,32} \ll n$. Hence $\mu_{21,21} \ll m n^{-3}$.
(i) Consider the case $\frac{m}{n} \rightarrow \mu \in[0, \infty)$. Then $\mu_{32}=\left((P)_{32}+o(1)\right) m n^{-3}$ and $\mu_{21}^{2}=$ $\left(\mu(P)_{21}^{2}+o(1)\right) m n^{-3}$ imply that

$$
\mathbb{P}\left(\mathcal{K}_{12}\right)=(P)_{32} m n^{-3}+\mu(P)_{21}^{2} m n^{-3}+o\left(m n^{-3}\right) .
$$

Similarly, $\mu_{33}=\left((P)_{33}+o(1)\right) n^{-3} m$ implies

$$
\mathbb{P}\left(\mathcal{K}_{3}\right)=(P)_{33} m n^{-3}+o\left(m n^{-3}\right),
$$

and hence the first two claims of Theorem 3.2 follow.
(ii) Assume now that $n \ll m \ll n^{2}$. Then $m n^{-3}, m^{2} n^{-5}, m^{3} n^{-6} \ll m^{3} n^{-4}$. Hence $\mathbb{P}\left(\mathcal{K}_{3}\right) \ll m^{2} n^{-4}$. Furthermore, $m^{4} n^{-8} \ll m^{2} n^{-4}$, and we conclude that $\mathbb{P}\left(\mathcal{K}_{12}\right)=(P)_{21}^{2} m^{2} n^{-4}+$ $o\left(m^{2} n^{-4}\right)$. Hence $\frac{\mathbb{P}\left(\mathcal{K}_{3}\right)}{\mathbb{P}\left(\mathcal{K}_{12}\right)} \rightarrow 0$ implies the third claim of Theorem 3.2.
6.6. Proof of Theorem 3.3. Let $K_{12}$ be the two-star on $\{1,2,3\}$ with links $\{12,13\}$. Let $K_{3}$ be the triangle on $\{1,2,3\}$. Denote $\mathcal{K}_{3}^{(n)}=\left\{G^{(n)} \supset K_{3}\right\}$ and $\mathcal{K}_{12}^{(n)}=\left\{G^{(n)} \supset K_{12}\right\}$. Let $D^{(n)}=\operatorname{deg}_{G^{(n)}}(1)$. By Theorem 6.2 and Theorem 6.3,

$$
\begin{aligned}
& \mathbb{P}\left(D^{(n)}=t, \mathcal{K}_{3}^{(n)}\right)=\mu_{33} f^{(n)} * g_{33}^{(n)}(t-2)+\varepsilon_{1}^{(n)}(t), \\
& \mathbb{P}\left(D^{(n)}=t, \mathcal{K}_{12}^{(n)}\right)=\mu_{32} f^{(n)} * g_{32}^{(n)}(t-2)+\left(\mu_{21}\right)^{2} f^{(n)} * g_{21}^{(n)} * g_{21}^{(n)}(t-2)+\varepsilon_{2}^{(n)}(t)
\end{aligned}
$$

where the associated cross moments are defined by (6.1)-(6.2), the distributions $g_{r s}^{(n)}=$ $\operatorname{Bin}_{r s}\left(P_{n}\right)$ are defined by (2.5), and

$$
\begin{aligned}
& \left|\varepsilon_{1}^{(n)}(t)\right| \leq(4+t) \mu_{21} \mu_{32}+\left(\mu_{21}\right)^{3}+2 \mu_{10,33} \\
& \left|\varepsilon_{2}^{(n)}(t)\right| \leq(6+2 t)\left(\mu_{21} \mu_{32}+\left(\mu_{21}\right)^{3}\right)+\left(\mu_{21}\right)^{4}+4 \mu_{10,32}+4 \mu_{21} \mu_{10,21}+\mu_{21,21}
\end{aligned}
$$

Now Lemma 6.4 implies that $\mu_{10,21} \ll n^{-1}$ and $\mu_{10,33} \leq \mu_{10,32} \ll n^{-2}$. Also, the argument in the proof of Theorem 3.2 (Section 6.5) implies that $\mu_{21,21} \ll n^{-2}$. Because $\mu_{r s}=$ $m(n)_{r}^{-1}\left(P_{n}\right)_{r s}$ and $\left(P_{n}\right)_{21},\left(P_{n}\right)_{32} \lesssim 1$, it follows that $\mu_{21} \ll n^{-1}$ and $\mu_{21} \mu_{32}+\mu_{21}^{3}+\mu_{21}^{4} \ll$ $n^{-2}$. Hence, $\left|\varepsilon_{1}^{(n)}(t)\right|+\left|\varepsilon_{2}^{(n)}(t)\right| \ll(1+t) n^{-2}$. Note also that $\mu_{32}=(\mu+o(1))(P)_{32} n^{-2}$, $\mu_{33}=(\mu+o(1))(P)_{33} n^{-2}$, together with $\mu_{21}^{2}=(1+o(1)) \mu(P)_{21}^{2} n^{-2}$. Moreover, by Theorem 3.1, $f^{(n)} \xrightarrow{w} f=\mathrm{CPoi}\left(\mu(P)_{10}, g_{10}\right)$. By Lemma A.10, $g_{r s}^{(n)} \xrightarrow{w} g_{r s}$ for $r s=21,32,33$. As
a consequence,

$$
\begin{aligned}
\mu_{33} f^{(n)} * g_{33}^{(n)}(t-2) & =(P)_{33} \mu n^{-2} f * g_{33}(t-2)+o\left(n^{-2}\right), \\
\mu_{32} f^{(n)} * g_{32}^{(n)}(t-2) & =(P)_{32} \mu n^{-2} f * g_{32}(t-2)+o\left(n^{-2}\right), \\
\mu_{21}^{2} f^{(n)} * g_{21}^{(n)} * g_{21}^{(n)}(t-2) & =(P)_{21}^{2} \mu^{2} n^{-2} f * g_{21} * g_{21}(t-2)+o\left(n^{-2}\right),
\end{aligned}
$$

and hence the claim follows.

## 7. Analysis of connectivity

The proof of Theorem 3.4 builds upon the approach developed in [14] and extended to random intersection graphs in [8]. We denote by $C_{G}(i)$ the component of node $i$, by $N_{1}(G) \geq$ $N_{2}(G)$ the largest two component sizes, and by $B_{t}(G)=\left\{i:\left|C_{G}(i)\right|>t\right\}$ be the set of nodes with component larger than $t$ in $G$. Here $\rho_{t}(f)$ denotes the probability that the total progeny of a Galton-Watson process with offspring distribution $f$ is larger than $t$, and $\rho(f)=\lim _{t \rightarrow \infty} \rho_{t}(f)$ is the long-term survival probability (see Appendix A.8). We start by the case with deterministic layer types.
7.1. Quantitative upper bound for deterministic layer types. In this section we prove the following quantitative upper bound which is valid for any model instance with deterministic layer types, without taking limits. The upper bound is characterised by a distribution

$$
\begin{equation*}
f_{\tau, n}=\mathcal{L}\left(\sum_{k=1}^{m} B_{k} T_{k}\right), \tag{7.1}
\end{equation*}
$$

where the random variables on the right are mutually independent and such that $\mathcal{L}\left(B_{k}\right)=$ $\operatorname{Ber}\left(\frac{X_{k}}{n-\tau}\right)$ and $\mathcal{L}\left(T_{k}\right)=\operatorname{Bin}^{+}\left(X_{k}-1, Y_{k}\right)$.
Proposition 7.1. If the layer types are nonrandom with sizes bounded by $M$, then for any $n \geq 3$ and $1 \leq \tau \leq n / 2$, the probability of a node $i$ having a component larger than $\tau$ is bounded by $\mathbb{P}\left(\left|C_{G}(i)\right|>\tau\right) \leq \rho_{\tau}\left(f_{\tau, n}\right)+c \tau^{2} n^{-1} \log n$, where $c=e^{5 M(1+m / n)}$.
7.1.1. Restricted exploration process. The proof of Proposition 7.1 is based on a restricted component exploration process described in Algorithm 1. The algorithm explores each layer at most once, and always discovers a subset of $C_{G}(i)$. This subset may be strict (see Figure 3).

```
Algorithm 1: Restricted exploration.
    Input: Graph layers \(G_{1}, \ldots, G_{m}\), root node \(i\)
    Output: A subset of the \(G\)-connected component of \(i\)
    Initialise: \(\mathcal{Q} \leftarrow\{i\}, \mathcal{M} \leftarrow \emptyset, t \leftarrow 0\)
    while \(\mathcal{Q} \neq \emptyset\) do
        \(t \leftarrow t+1\)
        Node selection: \(v_{t} \leftarrow \min \mathcal{Q}, \mathcal{Q} \leftarrow \mathcal{Q} \backslash\left\{v_{t}\right\}\)
        for \(k=1, \ldots, m\) do
            if \(V\left(G_{k}\right) \ni v_{t}\) and \(k \notin \mathcal{M}\) then
                Layer exploration: \(\mathcal{Z} \leftarrow N_{v_{t}}\left(\bar{G}_{k}\right)\)
                Queue update: \(\mathcal{Q} \leftarrow \mathcal{Q} \cup \mathcal{Z}\)
                Update the set of explored layers: \(\mathcal{M} \leftarrow \mathcal{M} \cup\{k\}\)
    Output node set \(\left\{v_{1}, \ldots, v_{t}\right\}\)
```

7.1.2. Properties of Algorithm 1. We denote by $T_{i}$ the number of steps completed by Algorithm 1 started at root node $i$. For $t=1, \ldots, T_{i}$, we denote by $\mathcal{W}_{t}$ the set of layers which are explored during step $t$. We denote by $\mathcal{M}_{t}^{e}=\cup_{s=1}^{t \wedge T_{i}} \mathcal{W}_{s}$ the set of layers and


Figure 3. The component of node 1 equals $C_{1}=\{1, \ldots, 16\}$, but Algorithm 1 outputs $C_{1} \backslash\{5,9,13\}$. Algorithm 1 discovers nothing while exploring node 10, because layer $G_{1}$ is already explored. A multi-overlap occurs while exploring node 11 when layer $G_{5}$ intersects the already explored layer $G_{1}$.
by $\mathcal{N}_{t}^{e}=\left\{v_{1}, \ldots, v_{t \wedge T_{i}}\right\}$ the set of nodes explored up to time $t$. We denote by $\mathcal{N}_{t}^{d}=$ $\{i\} \cup\left(\cup_{k \in \mathcal{M}_{t}^{e}} V\left(G_{k}\right)\right)$ the set of nodes discovered up to time $t$.

Lemma 7.2. The number of layers explored up to time $t$ is bounded by $\mathbb{P}\left(\left|\mathcal{M}_{t}^{e}\right|>a t\right) \leq$ $t e^{2 M(n-t)^{-1} m-a}$ for all $a \geq 0$.

Proof. Consider an event $\mathcal{E}_{t-1}^{+}=\mathcal{E}_{t-1}^{+}(A, B, C, v)$ that the exploration proceeds to step $t$, in the beginning of which the set of explored nodes equals $\mathcal{N}_{t-1}^{e}=A$, the set of explored layers equals $\mathcal{M}_{t-1}^{e}=B$, the set of discovered nodes equals $\mathcal{N}_{t-1}^{d}=C$, and the currently explored node equals $v_{t}=v$, for some node sets $A \subset C$ with $v \in C \backslash A$ and some layer set $B$ such that the event $\mathcal{E}_{t-1}^{+}$has nonzero probability. The event $\mathcal{E}_{t-1}^{+}$is determined by the random graphs $\left\{G_{k}: k \in B\right\}$ and the indicator variables $\left\{1\left(V\left(G_{k}\right) \ni v\right): v \in A, k \in[m]\right\}$. About the unexplored layers $G_{k}, k \in B^{c}$, the event $\mathcal{E}_{t-1}^{+}$reveals that $V\left(G_{k}\right) \subset A^{c}$, but nothing else. Therefore, given $\mathcal{E}_{t-1}^{+}$, the random graphs $\left\{G_{k}: k \in B^{c}\right\}$ are mutually independent and

$$
\begin{equation*}
\mathcal{L}\left(V\left(G_{k}\right) \mid \mathcal{E}_{t-1}^{+}\right) \text {is uniform among the } X_{k} \text {-sets of } A^{c} . \tag{7.2}
\end{equation*}
$$

Given $\mathcal{E}_{t-1}^{+}$, each unexplored layer $V\left(G_{k}\right)$ hence covers $v$ with probability $\frac{X_{k}}{n-(t-1)} \leq \frac{M}{n-t}$, independently. Therefore, the number of layers explored during step $t$ satisfies $\mathcal{L}\left(\left|\mathcal{W}_{t}\right| \mid \mathcal{E}_{t-1}^{+}\right) \leq_{\text {st }}$ $\operatorname{Bin}\left(m, \frac{M}{n-t}\right)$, and a Chernoff inequality (Lemma A.7) implies that $\mathbb{P}\left(\left|\mathcal{W}_{t}\right|>a \mid \mathcal{E}_{t-1}^{+}\right) \leq$ $e^{2 M(n-t)^{-1} m-a}$. Because the right side of the latter inequality does not depend on the choice of $A, B, C, v$, we conclude that $\mathbb{P}\left(\left|\mathcal{W}_{t}\right|>a \mid T_{i} \geq t\right) \leq e^{2 M(n-t)^{-1} m-a}$. This implies the claim, because the inequality $\left|\mathcal{M}_{t}^{e}\right| \leq t \max _{1 \leq s \leq t \wedge T_{i}}\left|\mathcal{W}_{s}\right|$ implies that

$$
\mathbb{P}\left(\left|\mathcal{M}_{t}^{e}\right|>a t\right) \leq \mathbb{P}\left(\max _{1 \leq s \leq t \wedge T_{i}}\left|\mathcal{W}_{s}\right|>a\right) \leq \sum_{s=1}^{t} \mathbb{P}\left(\left|\mathcal{W}_{s}\right|>a, T_{i} \geq s\right)
$$

During an exploration step $t \leq T_{i}$, a multi-overlap of type 1 occurs if one of the layers covering $v_{t}$ overlaps with previously explored layers in some other node besides $v_{t}$, and a multi-overlap of type 2 occurs if some of the layers covering $v_{t}$ overlap each other in more
than one node. These events can be written as

$$
\begin{aligned}
& \mathcal{O}_{1 t}=\left\{T_{i} \geq t\right\} \cap\left\{V_{k}^{\prime} \cap \mathcal{N}_{t-1}^{d} \neq \emptyset \text { for some } k \in \mathcal{W}_{t}^{+}\right\}, \\
& \mathcal{O}_{2 t}=\left\{T_{i} \geq t\right\} \cap\left\{V_{k}^{\prime} \cap V_{\ell}^{\prime} \neq \emptyset \text { for some distinct } k, \ell \in \mathcal{W}_{t}^{+}\right\},
\end{aligned}
$$

where $V_{k}^{\prime}=V\left(G_{k}\right) \backslash\left\{v_{t}\right\}$ and $\mathcal{W}_{t}^{+}=\left\{k: V\left(G_{k}\right) \ni v_{t}\right\}$. We denote the occurrence of a multi-overlap by $\mathcal{O}_{t}=\mathcal{O}_{1 t} \cup \mathcal{O}_{2 t}$, and we define $\mathcal{O}_{\leq t}=\mathcal{O}_{1} \cup \cdots \cup \mathcal{O}_{t}$.

Lemma 7.3. For any $n \geq 3$ and $1 \leq \tau \leq n / 2$, the probability that a multi-overlap occurs during the first $\tau$ exploration steps is bounded by $\mathbb{P}\left(\mathcal{O}_{\leq \tau}\right) \leq c \tau^{2} n^{-1} \log n$, where $c=e^{5 M(1+m / n)}$.

Proof. Consider an event $\mathcal{E}_{t-1}^{+}=\mathcal{E}_{t-1}^{+}(A, B, C, v)$ as in the proof of Lemma 7.2. By (7.2), we know that given $\mathcal{E}_{t-1}^{+}$, each layer $k \in B^{c}$ covers $v$ with probability $\frac{X_{k}}{n-(t-1)}$, and given $\mathcal{E}_{t-1}^{+} \cap\left\{V\left(G_{k}\right) \ni v\right\}$, the law of $V_{k}^{\prime}=V\left(G_{k}\right) \backslash\{v\}$ with $k \in B^{c}$ is uniform among the $\left(X_{k}-1\right)$ sets of $\left(A \cup\left\{v_{t}\right\}\right)^{c}$, and hence $V_{k}^{\prime}$ overlaps $C$ with probability at most $\frac{\left(X_{k}-1\right)(|C|-1)}{n-t}$. Because $\left|B^{c}\right| \leq n$ and $|C| \leq M|B|$, the probability of a multi-overlap of type 1 is bounded by

$$
\mathbb{P}\left(\mathcal{O}_{1 t} \mid \mathcal{E}_{t-1}^{+}\right) \leq \sum_{k \in B^{c}} \frac{X_{k}}{n-(t-1)} \frac{\left(X_{k}-1\right)(|C|-1)}{n-t} \leq n \frac{M^{3}|B|}{(n-t)^{2}} .
$$

Similarly, the $\mathcal{E}_{t-1}^{+}$-conditional probability that two distinct layers $k, \ell \in B^{c}$ cover $v$ and overlap each other in some other node is bounded by $\frac{X_{k}}{n-(t-1)} \frac{X_{\ell}}{n-(t-1)} \frac{\left(X_{k}-1\right)\left(X_{\ell}-1\right)}{n-t} \leq \frac{M^{4}}{(n-t)^{3}}$. Hence a multi-overlap of type 2 occurs with probability at most $\mathbb{P}\left(\mathcal{O}_{2 t} \mid \mathcal{E}_{t-1}^{+}\right) \leq\binom{ n}{2} \frac{M^{4}}{(n-t)^{3}}$. Hence for $t \leq n / 2$ and $|B| \leq a t$ with $a, t \geq 1$,

$$
\mathbb{P}\left(\mathcal{O}_{t} \mid \mathcal{E}_{t-1}^{+}\right) \leq 4 M^{3}\left(n^{-1} m+M n^{-2} m^{2}\right) a t n^{-1}
$$

Because the right side above is valid whenever $|B| \leq a t$, the above inequality also holds for $\mathcal{E}_{t-1}^{+}$replaced by the event that $\left|\mathcal{M}_{t-1}^{e}\right| \leq a t$ and $T_{i} \geq t$. Lemma 7.2 now implies that

$$
\begin{aligned}
\mathbb{P}\left(\mathcal{O}_{t}\right) & =\mathbb{P}\left(\mathcal{O}_{t},\left|\mathcal{M}_{t-1}^{e}\right| \leq a t, T_{i} \geq t\right)+\mathbb{P}\left(\mathcal{O}_{t},\left|\mathcal{M}_{t-1}^{e}\right|>a t, T_{i} \geq t\right) \\
& \leq \mathbb{P}\left(\mathcal{O}_{t},\left|\mathcal{M}_{t-1}^{e}\right| \leq a t, T_{i} \geq t\right)+\mathbb{P}\left(\left|\mathcal{M}_{t-1}^{e}\right|>a t\right) \\
& \leq 4\left(M^{3} m / n+M^{4} m^{2} / n^{2}\right) a t n^{-1}+e^{4 M m / n} t e^{-a} .
\end{aligned}
$$

Using $x \leq 1+x \leq e^{x}$ we find that $4\left(M^{3} m / n+M^{4} m^{2} / n^{2}\right)=(2 M)^{2}(M m / n)(1+M m / n) \leq$ $e^{4 M+2 M m / n}$. By plugging in $a=\log n$, it follows that

$$
\mathbb{P}\left(\mathcal{O}_{t}\right) \leq\left(e^{4 M(1+m / n)}+e^{4 M(1+m / n)}\right) t n^{-1} \log n \leq e^{5 M(1+m / n)} t n^{-1} \log n .
$$

Hence the claim follows by the union bound.
Lemma 7.4. The probability that the restricted exploration discovers more than $\tau$ nodes is bounded by $\mathbb{P}\left(Q_{\tau}>0\right) \leq \rho_{\tau}\left(f_{\tau, n}\right)$, where the distribution $f_{\tau, n}$ is defined by (7.1).

Proof. The queue length process satisfies $Q_{0}=1$ and

$$
\begin{equation*}
Q_{t}=\left(Q_{t-1}-1+Z_{t}\right) 1\left(Q_{t-1}>0\right), \quad t=1,2, \ldots, \tag{7.3}
\end{equation*}
$$

where $Z_{t}=\left|\mathcal{Z}_{t}\right|$ is the number of nodes added to the queue in step $t$. Fix $1 \leq t \leq \tau$ and consider an event $\mathcal{E}_{t-1}^{+}=\mathcal{E}_{t-1}^{+}(A, B, C, v)$ as in the proof of Lemma 7.2. On this event,

$$
\begin{equation*}
Z_{t} \leq \sum_{k \in B^{c}} 1\left(V\left(G_{k}\right) \ni v\right) \operatorname{deg}_{\bar{G}_{k}}(v) . \tag{7.4}
\end{equation*}
$$

By recalling (7.2), we know that conditionally on $\mathcal{E}_{t-1}^{+}$, the random variables on the right side of (7.4) are mutually independent, and such that $\mathcal{L}\left(1\left(V\left(G_{k}\right) \ni v\right) \mid \mathcal{E}_{t-1}^{+}\right)=\operatorname{Ber}\left(\frac{X_{k}}{n-(t-1)}\right)$
and $\mathcal{L}\left(\operatorname{deg}_{\bar{G}_{k}}(v) \mid \mathcal{E}_{t-1}^{+}\right)=\operatorname{Bin}^{+}\left(X_{k}-1, Y_{k}\right)$ for all $k \in B^{c}$. We conclude that

$$
\mathcal{L}\left(Z_{t} \mid \mathcal{E}_{t-1}^{+}\right) \leq_{\text {st }} f_{t, n} \leq_{\text {st }} f_{\tau, n} \text { for all } t=1, \ldots, \tau
$$

Because the above inequalities hold for all events $\mathcal{E}_{t-1}^{+}=\mathcal{E}_{t-1}^{+}(A, B, C, v)$ of the above form, it also holds for the event $\left\{Q_{t-1}>0\right\}$ that there is a node to explore at step $t$. Hence it follows that $\left(Q_{0}, \ldots, Q_{\tau}\right) \leq_{\text {st }}\left(Q_{0}^{\prime}, \ldots, Q_{\tau}^{\prime}\right)$ where the right side is defined as in (7.3) but with $Z_{1}, Z_{2}, \ldots$ replaced by independent $f_{\tau, n}$-distributed random integers $Z_{1}^{\prime}, Z_{2}^{\prime}, \ldots$ The claim follows by noting that $\mathbb{P}\left(Q_{\tau}^{\prime}>0\right)=\rho_{\tau}\left(f_{\tau, n}\right)$ (see Appendix A.8).
7.1.3. Proof of Proposition 7.1. Let $Q_{t}=\left|\mathcal{Q}_{t}\right|$ be the exploration queue length in Algorithm 1 started at node $i$. We note that $Q_{\tau}>0$ means that the restricted exploration discovers more than $t$ nodes of the component of $i$. Therefore $Q_{\tau}>0$ implies $\left|C_{G}(i)\right|>\tau$. The converse may not be true (see Figure 3) because the restricted exploration may stop before discovering all nodes in the component of $i$. On the event $\mathcal{O}_{\leq \tau}^{c}$ that multi-overlaps do not occur up to time $\tau$, this cannot happen, and hence $\left\{Q_{\tau}>0\right\} \cap \mathcal{O}_{\leq \tau}^{c}=\left\{\left|C_{G}(i)\right|>\tau\right\} \cap \mathcal{O}_{\leq \tau}^{c}$. Therefore,

$$
\begin{aligned}
\mathbb{P}\left(\left|C_{G}(i)\right|>\tau\right) & =\mathbb{P}\left(Q_{\tau}>0, \mathcal{O}_{\leq \tau}^{c}\right)+\mathbb{P}\left(\left|C_{G}(i)\right|>\tau, \mathcal{O}_{\leq \tau}\right) \\
& \leq \mathbb{P}\left(Q_{\tau}>0\right)+\mathbb{P}\left(\mathcal{O}_{\leq \tau}\right) .
\end{aligned}
$$

The claim follows by combining Lemma 7.3 and Lemma 7.4.
7.2. Double upper bound for deterministic layer types. To obtain an upper bound on the variance of the number of nodes contained in large components, we extend the analysis in Proposition 7.1 to two restricted exploration processes run on the same graph instance.
Proposition 7.5. For any $1 \leq \tau \leq n / 4$, the components sizes of nodes $i \neq j$ are bounded by $\mathbb{P}\left(\left|C_{G}(i)\right|>\tau,\left|C_{G}(j)\right|>\tau\right) \leq \rho_{\tau}\left(f_{2 \tau, n}\right)^{2}+c \tau^{2} n^{-1} \log n$, where $c=e^{9 M(1+m / n)}$.

The proof of Proposition 7.5 requires Lemma 7.6 and 7.7, outlined next.
Lemma 7.6. The probability that Algorithm 1 started at $i$ discovers node $j \neq i$ during $\tau$ steps is bounded by $\mathbb{P}\left(\mathcal{N}_{i, \tau}^{d} \ni j\right) \leq 4 M^{2} n^{-2} m \tau$.
Proof. Fix $1 \leq t \leq \tau$ and consider an event $\mathcal{E}_{t-1}^{+}=\mathcal{E}_{t-1}^{+}(A, B, C, v)$ as in the proof of Lemma 7.2, for some node sets $A \subset C$ with $i \in C \backslash A$ and $C \nexists j$, and some layer set $B$. By recalling (7.2), we know that for any $k \in B^{c}$, the $\mathcal{E}_{t-1}^{+}$-conditional probability that $G_{k}$ covers $v$ and $j$ is bounded by $\frac{X_{k}}{n-(t-1)} \frac{X_{k}-1}{n-t} \leq \frac{M^{2}}{(n-\tau)^{2}}$. Because $\left|B^{c}\right| \leq n$, the union bound implies

$$
\mathbb{P}\left(\mathcal{N}_{i, t}^{d} \ni j \mid \mathcal{E}_{t-1}^{+}\right) \leq m \frac{M^{2}}{(n-t)^{2}} \leq 4 M^{2} n^{-2} m .
$$

Because the above inequality is valid whenever $C \nexists j$, the above inequality also holds with $\mathcal{E}_{t-1}^{+}$replaced by the event $\mathcal{E}_{t-1}^{++}=\left\{\mathcal{N}_{i, t-1}^{d} \nexists j\right\} \cap\left\{T_{i} \geq t\right\}$. Hence the claim follows by noting that $\mathbb{P}\left(\mathcal{N}_{i, \tau}^{d} \ni j\right)=\sum_{t=1}^{\tau} \mathbb{P}\left(\mathcal{N}_{i, t}^{d} \ni j, \mathcal{E}_{t-1}^{++}\right) \leq \sum_{t=1}^{\tau} \mathbb{P}\left(\mathcal{N}_{i, t}^{d} \ni j \mid \mathcal{E}_{t-1}^{++}\right)$.
Lemma 7.7. For any $i \neq j$ and $1 \leq \tau \leq n / 4$, the probability that explorations started at $i$ and $j$ overlap is bounded by $\mathbb{P}\left(\mathcal{N}_{i, \tau}^{d} \cap \mathcal{N}_{j, \tau}^{d} \neq \emptyset\right) \leq c \tau^{2} n^{-1} \log n$, where $c=e^{7 M(1+m / n)}$.
Proof. Consider an event $\mathcal{E}_{i, \tau}=\mathcal{E}_{i, \tau}\left(A_{i}, B_{i}, C_{i}\right)$ that after $\tau$ steps of exploration started from $i$, the set of explored nodes equals $\mathcal{N}_{i, \tau}^{e}=A_{i}$, the set of explored layers equals $\mathcal{M}_{i, \tau}^{e}=B_{i}$, and the set of discovered nodes equals $\mathcal{N}_{i, \tau}^{d}=C_{i}$ for some node sets $A_{i} \subset C_{i}$ such that $C_{i} \nexists j$ and $C_{i} \leq M a \tau$, and some layer set $B_{i}$. Fix $1 \leq t \leq \tau$ and consider an event $\mathcal{E}_{j, t-1}^{+}=\mathcal{E}_{j, t-1}^{+}\left(A_{j}, B_{j}, C_{j}, v\right)$ as in the proof of Lemma 7.2 , for some node sets $A_{j} \subset C_{j}$ with $v \in C_{j} \backslash A_{j}$ and $C_{i} \cap C_{j}=\emptyset$, and some layer set $B_{j}$. Then $\mathcal{N}_{j, t-1}^{d}$ does not overlap $\mathcal{N}_{i, \tau}^{d}$ on the event $\mathcal{F}_{t-1}^{+}=\mathcal{E}_{i, \tau} \cap \mathcal{E}_{j, t-1}^{+}$. We will next analyse the conditional probability that the same is true for $\mathcal{N}_{j, t}^{d}$. We note that on the event $\mathcal{F}_{t-1}^{+}$, the $j$-exploration at step $t$ only explores layers
$k \in\left(B_{i} \cup B_{j}\right)^{c}$ because the layers in $B_{i}$ do not cover $v_{j}$, and the layers in $B_{j}$ have already been explored. We also observe that the event $\mathcal{F}_{t-1}^{+}$is determined by the random graphs $G_{k}, k \in B_{i} \cup B_{j}$, and the indicators $1\left(V\left(G_{k}\right) \ni a\right)$ for $k=1, \ldots, n$ and $a \in A_{i} \cup A_{j}$. Hence given $\mathcal{F}_{t-1}^{+}$, the layers $G_{k}, k \in\left(B_{i} \cup B_{j}\right)^{c}$ are mutually independent, and such that $V\left(G_{k}\right)$ is a uniformly random $X_{k}$-set in $\left(A_{i} \cup A_{j}\right)^{c}$. The probability that a layer $k \in\left(B_{i} \cup B_{j}\right)^{c}$ covers $v$ and overlaps with $C_{i}$, is at most

$$
\frac{X_{k}}{n-\left|A_{i}\right|-\left|A_{j}\right|} \frac{\left(X_{k}-1\right) C_{i}}{n-\left|A_{i}\right|-\left|A_{j}\right|-1} \leq \frac{M^{2} C_{i}}{(n-2 \tau)^{2}} .
$$

Hence, due to $\left|\left(B_{i} \cup B_{j}\right)^{c}\right| \leq n$, and $C_{i} \leq M a \tau$, it follows that

$$
\mathbb{P}\left(\mathcal{N}_{i, \tau}^{d} \cap \mathcal{N}_{j, t}^{d} \neq \emptyset \mid \mathcal{F}_{t-1}^{+}\right) \leq 4 M^{3} a \tau n^{-2} m .
$$

Because the right side above does not depend on $A_{i}, B_{i}, C_{i}, A_{j}, B_{j}, C_{j}, v$, the above inequality remains valid also for $\mathcal{F}_{t-1}^{+}$replaced by $\mathcal{G}_{t-1}^{+}=\left\{\mathcal{N}_{i, \tau}^{d} \cap \mathcal{N}_{j, t-1}^{d}=\emptyset\right\} \cap\left\{\left|\mathcal{N}_{i, \tau}^{d}\right| \leq M a \tau\right\}$. Thus,

$$
\begin{aligned}
\mathbb{P}\left(\mathcal{N}_{i, \tau}^{d} \cap \mathcal{N}_{j, \tau}^{d} \neq \emptyset\left|\mathcal{N}_{i, \tau}^{d} \cap \mathcal{N}_{j, 0}^{d}=\emptyset,\left|\mathcal{N}_{i, \tau}^{d}\right| \leq M a \tau\right)\right. & =\sum_{t=1}^{\tau} \mathbb{P}\left(\mathcal{N}_{i, \tau}^{d} \cap \mathcal{N}_{j, t}^{d} \neq \emptyset \mid \mathcal{G}_{t-1}^{+}\right) \\
& \leq 4 M^{3} a \tau^{2} n^{-2} m .
\end{aligned}
$$

Hence noting that $\mathcal{N}_{i, \tau}^{d} \cap \mathcal{N}_{j, 0}^{d} \neq \emptyset$ if and only if $\mathcal{N}_{i, \tau}^{d} \nexists j$, together with $\left|\mathcal{N}_{i, \tau}^{d}\right| \leq M\left|\mathcal{M}_{i, \tau}^{e}\right|$, applying Lemma 7.2 and Lemma 7.6 , for $a, \tau \geq 1$,

$$
\begin{aligned}
\mathbb{P}\left(\mathcal{N}_{i, \tau}^{d} \cap \mathcal{N}_{j, \tau}^{d} \neq \emptyset\right) & \leq 4 M^{3} a \tau^{2} n^{-2} m+\mathbb{P}\left(\mathcal{N}_{i, \tau}^{d} \ni j\right)+\mathbb{P}\left(\left|\mathcal{N}_{i, \tau}^{d}\right|>M a \tau\right) \\
& \leq 4 M^{3} a \tau^{2} n^{-2} m+4 M^{2} n^{-2} m \tau+\tau e^{4 M m / n-a} .
\end{aligned}
$$

Plugging in $a=\log \frac{n}{\tau}$ and noting that $a \leq \log n$ and $\tau e^{-a}=\tau^{2} n^{-1} \leq \tau^{2} n^{-1} \log n$, implies

$$
\mathbb{P}\left(\mathcal{N}_{i, \tau}^{d} \cap \mathcal{N}_{j, \tau}^{d} \neq \emptyset\right) \leq\left(4 M^{3} \frac{m}{n}+4 M^{2} \frac{m}{n}+e^{4 M m / n}\right) \tau^{2} n^{-1} \log n .
$$

The claim follows after noting that $4 M^{3} \frac{m}{n}+4 M^{2} \frac{m}{n} \leq 8 M^{3} \frac{m}{n}=(2 M)^{3} \frac{m}{n} \leq e^{6 M(1+m / n)}$ implies the term on the right in parentheses is at most $e^{7 M(1+m / n)}$.

Proof of Proposition 7.5. Let $Q_{i t}$ and $Q_{j t}$ be the exploration queue lengths of Algorithm 1 started at distinct nodes $i$ and $j$. We use the notations of Section 7.1.2.

Consider an event $\mathcal{E}_{i \tau}^{+}=\left\{\mathcal{N}_{i \tau}^{e}=A, \mathcal{M}_{i \tau}^{e}=B, Q_{i \tau}>0\right\}$ for some node set $A \not \supset j$ of size $\tau$, and some layer set $B$. Let $Q_{j t}^{\prime}$ be the exploration queue of a modified exploration obtained by running Algorithm 1 started from $j$ with a reduced set of input layers $\left\{G_{k}: k \in B^{c}\right\}$. Then $\left(Q_{j 0}^{\prime}, \ldots, Q_{j \tau}^{\prime}\right)=\left(Q_{0 \tau}, \ldots, Q_{j \tau}\right)$ on the event $\mathcal{M}_{j \tau}^{e} \cap B=\emptyset$. Hence

$$
\begin{align*}
\mathbb{P}\left(Q_{j, \tau}>0, \mathcal{M}_{i \tau}^{e} \cap \mathcal{M}_{j, \tau}^{e}=\emptyset \mid \mathcal{E}_{i \tau}^{+}\right) & =\mathbb{P}\left(Q_{j \tau}^{\prime}>0, \mathcal{M}_{j \tau}^{e} \cap B=\emptyset \mid \mathcal{E}_{i \tau}^{+}\right) \\
& \leq \mathbb{P}\left(Q_{j \tau}^{\prime}>0 \mid \mathcal{E}_{i \tau}^{+}\right) . \tag{7.5}
\end{align*}
$$

The event $\mathcal{E}_{i \tau}^{+}$is determined by the random graphs $\left\{G_{k}: k \in B\right\}$ and the indicators $\left\{1\left(V\left(G_{k}\right) \ni a\right): k \in[m], a \in A\right\}$. Hence the $\mathcal{E}_{i \tau}^{+}$-conditional distribution of $G_{k}, k \in B^{c}$, is such that these layers are mutually independent and $V\left(G_{k}\right)$ is a uniformly random $X_{k}$-set in $A^{c}$. Hence the $\mathcal{E}_{i \tau}^{+}$-conditional law of the $Q_{j t}^{\prime}$-exploration process is the same as the law of the exploration process $Q_{j t}^{\prime \prime}$ obtained by running Algorithm 1 started at $j$ for a model instance with node set $A^{c}$ and layer set $\left\{G_{k}: k \in B^{c}\right\}$. Hence

$$
\begin{equation*}
\mathbb{P}\left(Q_{j \tau}^{\prime}>0 \mid \mathcal{E}_{i \tau}^{+}\right)=\mathbb{P}\left(Q_{j \tau}^{\prime \prime}>0\right) . \tag{7.6}
\end{equation*}
$$

Let $Q_{j t}^{\prime \prime \prime}$ be an exploration queue of Algorithm 1 started at $j$ for a model instance with a full layer set $\left\{G_{k}: k \in[m]\right\}$ and node set $A^{c} \ni j$ of size $n-\tau$. Then $Q_{j t}^{\prime \prime}>0$ implies $Q_{j t}^{\prime \prime \prime}>0$
under a natural coupling, and we conclude with the help of (7.5) and (7.6) that

$$
\mathbb{P}\left(Q_{j, \tau}>0, \mathcal{M}_{i, \tau}^{e} \cap \mathcal{M}_{j \tau}^{e}=\emptyset, \mathcal{E}_{i \tau}^{+}\right) \leq \mathbb{P}\left(\mathcal{E}_{i \tau}^{+}\right) \mathbb{P}\left(Q_{j \tau}^{\prime \prime \prime}>0\right)
$$

Because the probability on the right does not depend on the choice of $A, B$,

$$
\mathbb{P}\left(Q_{i \tau}>0, Q_{j \tau}>0, \mathcal{M}_{i \tau}^{e} \cap \mathcal{M}_{j \tau}^{e}=\emptyset\right) \leq \mathbb{P}\left(Q_{i \tau}>0\right) \mathbb{P}\left(Q_{j \tau}^{\prime \prime \prime}>0\right)
$$

By Lemma 7.4 , we see that $\mathbb{P}\left(Q_{i \tau}>0\right) \leq \rho_{\tau}\left(f_{\tau, n}\right)$. By applying the same lemma again for a model instance with the full layer set $\left\{G_{k}: k \in[m]\right\}$ and a node set of size $n-\tau$, we find that $\mathbb{P}\left(Q_{j \tau}^{\prime \prime \prime}>0\right) \leq \rho_{\tau}\left(f_{\tau, n}^{\prime \prime \prime}\right)$ where $f_{\tau, n}^{\prime \prime \prime}$ is defined as in (7.1) but with $n$ replaced by $n-\tau$. Now we note that $f_{\tau, n}^{\prime \prime \prime}=f_{2 \tau, n}$, and that $f_{\tau, n} \leq_{\text {st }} f_{2 \tau, n}$ implies $\rho_{\tau}\left(f_{\tau, n}\right) \leq \rho_{\tau}\left(f_{2 \tau, n}\right)$. Hence

$$
\mathbb{P}\left(Q_{i \tau}>0, Q_{j \tau}>0\right) \leq \rho_{\tau}\left(f_{2 \tau, n}\right)^{2}+\mathbb{P}\left(\mathcal{M}_{i \tau}^{e} \cap \mathcal{M}_{j \tau}^{e} \neq \emptyset\right)
$$

Because the indicators of $\left\{\left|C_{G}(i)\right|>\tau\right\}$ and $\left\{Q_{i \tau}>0\right\}$ coincide on the event $\mathcal{O}_{i, \leq \tau}^{c}$, and the same is true for $j$, we find that

$$
\begin{aligned}
\mathbb{P}\left(\left|C_{G}(i)\right|>\tau,\left|C_{G}(j)\right|>\tau\right) & \leq \mathbb{P}\left(Q_{i \tau}>0, Q_{j \tau}>0\right)+\mathbb{P}\left(\mathcal{O}_{i, \leq \tau}\right)+\mathbb{P}\left(\mathcal{O}_{j, \leq \tau}\right) \\
& \leq \rho_{\tau}\left(f_{2 \tau, n}\right)^{2}+\mathbb{P}\left(\mathcal{M}_{i \tau}^{e} \cap \mathcal{M}_{j \tau}^{e} \neq \emptyset\right)+2 \mathbb{P}\left(\mathcal{O}_{i, \leq \tau}\right)
\end{aligned}
$$

The claim follows due to $\mathbb{P}\left(\mathcal{M}_{i \tau}^{e} \cap \mathcal{M}_{j \tau}^{e} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{N}_{i \tau}^{d} \cap \mathcal{N}_{j \tau}^{d} \neq \emptyset\right)$ and Lemmas 7.3 and 7.7.
7.3. Quantitative lower bound for deterministic layer types. Proving a lower bound is more complicated than an upper bound, because we need to verify that the types of unexplored layers remain balanced during the exploration. We start by analysing the case with nonrandom layer types in a finite set in Proposition 7.8. The proof is based on analysing a balanced exploration process in Algorithm 2 which uses a randomised selection of disjoint layers in Algorithm 3 as a subroutine.

Proposition 7.8. Fix a finite set $A \subset \mathbb{Z}_{+} \times[0,1]$, integers $1 \leq M, \tau, \nu \leq n$, and a number $\delta \in(0,1)$. Assume that $2 M^{2}|A| \frac{\nu \tau}{n} \leq \delta, \tau \leq \frac{1}{2} n$, and $x \leq M$ for all $(x, y) \in A$. Then

$$
\begin{equation*}
\mathbb{P}\left(\left|C_{G}(i)\right|>\tau\right) \geq \rho_{\tau}\left(f_{\delta, \tau, \nu}\right)-|A| \tau e^{4 M m / n-\nu} \tag{7.7}
\end{equation*}
$$

where

$$
\begin{equation*}
f_{\delta, \tau, \nu}=\mathcal{L}\left(\sum_{(x, y) \in A} \sum_{k=1}^{m_{x y, \tau-1}} B_{x y}(k) T_{x y}(k)\right) \tag{7.8}
\end{equation*}
$$

and the random variables on the right are mutually independent and such that $\mathcal{L}\left(B_{x y}(k)\right)=$ $\operatorname{Ber}\left((1-\delta) \frac{x}{n}\right), \mathcal{L}\left(T_{x y}(k)\right)=\operatorname{Bin}^{+}(x-1, y)$, and $m_{x y, \tau}=\left(m_{x y}-\tau \nu\right)_{+}$where $m_{x y}$ is the number of layers of type $(x, y)$.

```
Algorithm 3: Extracting disjoint sets.
    Input: List of subsets \(\left(V_{1}, \ldots, V_{m}\right)\) of a ground set \(V\), taboo set \(H_{0} \subset V\), parameter
        \(\alpha \in(0,1)\)
    Output: Random index set \(K \subset\{1, \ldots, m\}\)
    Initialise \(K \leftarrow \emptyset\) and \(H \leftarrow H_{0}\)
    for \(k=1, \ldots, m\) do
        \(U_{k} \leftarrow\) uniform random number in \((0,1)\)
        if \(V_{k} \cap H=\emptyset\) and \(U_{k} \leq \alpha\binom{|V|-|H|}{\left|V_{k}\right|}^{-1}\binom{|V|}{\left|V_{k}\right|}\) then
            Add the index \(k\) to \(K\)
            Add the elements of \(V_{k}\) to \(H\)
    Output \(K\)
```

```
Algorithm 2: Balanced exploration.
    Input: Layers \(G_{1}, \ldots, G_{m}\), root node \(i\), parameters \(\nu \in \mathbb{Z}_{+}, \delta \in(0,1)\)
    Output: Subset of \(G\)-component of \(i\).
    State variables:
    \(\mathcal{Q}_{t}=\) Set of nodes in the exploration queue after step \(t\)
    \(\mathcal{N}_{t}=\) Set of discovered nodes after step \(t\)
    \(\mathcal{M}_{t}=\) Set of available layers after step \(t\)
    Initialise: Put node \(i\) into the queue and declare \(i\) discovered; declare all layers available;
        initialise state variables as: \(\mathcal{Q}_{0} \leftarrow\{i\}, \mathcal{N}_{0} \leftarrow\{i\}, \mathcal{M}_{0} \leftarrow\{1, \ldots, m\}\); and set \(t \leftarrow 0\).
    while \(\mathcal{Q}_{t} \neq \emptyset\) do
        Set \(t \leftarrow t+1\) and select node \(v_{t} \leftarrow \min \mathcal{Q}_{t-1}\) for exploration
        Declare the layers in \(\mathcal{W}_{t}^{+} \leftarrow\left\{k \in \mathcal{M}_{t-1}: V\left(G_{k}\right) \ni v_{t}\right\}\) and the nodes in
            \(\mathcal{Z}_{t}^{+} \leftarrow \cup_{k \in \mathcal{W}_{t}^{+}}\left(V\left(G_{k}\right) \backslash\left\{v_{t}\right\}\right)\) as discovered
        Extract a disjoint subcollection of discovered layers \(\mathcal{W}_{t}^{+}\)by computing \(\mathcal{W}_{t} \leftarrow\) Output of
                Algorithm 3 with ground set \(\left\{v_{1}, \ldots, v_{t-1}\right\}^{c}\), input sets \(\left\{V\left(G_{k}\right) \backslash\left\{v_{t}\right\}: k \in \mathcal{W}_{t}^{+}\right\}\), taboo
                set \(\mathcal{N}_{t-1}\), parameter \(\alpha_{t}=(1-\delta)\left(1-\frac{t-1}{n}\right)\)
            Explore the selected layers and determine the node set \(\mathcal{Z}_{t} \leftarrow \cup_{k \in \mathcal{W}_{t}} N_{v_{t}}\left(\bar{G}_{k}\right)\), where \(\bar{G}_{k}\) is
                the transitive closure of \(G_{k}\)
            Update the exploration queue by \(\mathcal{Q}_{t} \leftarrow\left(\mathcal{Q}_{t-1} \backslash\left\{v_{t}\right\}\right) \cup \mathcal{Z}_{t}\) and the set of discovered nodes
                by \(\mathcal{N}_{t} \leftarrow \mathcal{N}_{t-1} \cup \mathcal{Z}_{t}^{+}\)
            Layer balancing: \(\mathcal{M}_{t} \leftarrow \cup_{(x, y) \in A} \mathcal{M}_{x y, t}\) where \(\mathcal{M}_{x y, t}\) is a uniformly random subset of
                \(\mathcal{W}_{x y, t}^{u}=\left\{k \in \mathcal{M}_{t-1} \backslash \mathcal{W}_{t}^{+}: X_{k}=x, Y_{k}=y\right\}\) of size \(\left|\mathcal{W}_{x y, t}^{u}\right| \wedge\left(m_{x y}-\nu t\right)_{+}\)
    Output: \(\left\{v_{1}, \ldots, v_{t}\right\}\)
```

Lemma 7.9. Let $H_{0} \subset V$ be nonrandom sets. Let $V_{1}, \ldots, V_{m}$ be independent uniformly random subsets of $V$ with nonrandom sizes $x_{1}, \ldots, x_{m}$, and assume that $\left|H_{0}\right|+\|x\|_{1} \leq|V|$ and $0 \leq \alpha \leq\left(1-\frac{\left|H_{0}\right|+\|x\|_{1}}{|V|}\right)^{\|x\|_{\infty}}$ where $\|x\|_{1}=\sum_{k=1}^{m} x_{k}$ and $\|x\|_{\infty}=\max _{1 \leq k \leq m} x_{k}$. Then the indicator variables $B_{k}=1(k \in K)$ characterising the output of Algorithm 3 are mutually independent and $\operatorname{Ber}(\alpha)$-distributed, and the sets $\left\{V_{k}: k \in K\right\}$ are mutually disjoint and disjoint from $H_{0}$ almost surely.

Proof. Denote by $H_{k}$ the state of $H$ after finishing round $k$ of Algorithm 3. Then $H_{k}$ equals the union of $H_{0}$ and the sets $V_{j}$ admitted during rounds $j \leq k$, and the if-statement guarantees that a set $V_{k}$ is admitted to $K$ only if it is disjoint from $H_{k-1}$. Hence the family $\left\{V_{k}: k \in K\right\}=\left\{V_{k}: B_{k}=1\right\}$ is surely disjoint and disjoint from $H_{0}$. To investigate the joint distribution of $B_{k}=1_{K}(k), k=1, \ldots, m$, denote by denote by $p(h, x)=\binom{|V|-h}{x}\binom{|V|}{x}^{-1}$ the probability that a random $x$-set in $V$ does not overlap a particular $h$-set of $V$. Let $\mathcal{F}_{k}$ be the sigma-algebra generated by $\left\{\left(U_{j}, V_{j}\right): j \leq k\right\}$. Then $B_{k}, H_{k}$ are $\mathcal{F}_{k}$-measurable, $V_{k}$ is independent of $\mathcal{F}_{k-1}$, and $U_{k}$ is independent of $\left(\mathcal{F}_{k-1}, V_{k}\right)$. Hence,

$$
\begin{align*}
\mathbb{P}\left(V_{k} \cap H_{k-1}=\emptyset \mid \mathcal{F}_{k-1}\right) & =p\left(\left|H_{k-1}\right|, x_{k}\right), \\
\mathbb{P}\left(\left.U_{k} \leq \frac{\alpha}{p\left(\left(H_{k-1}\left|,\left|V_{k}\right|\right)\right.\right.} \right\rvert\, \mathcal{F}_{k-1}\right) & =\frac{\alpha}{p\left(\left|H_{k-1}\right|,\left|V_{k}\right|\right)} \wedge 1 . \tag{7.9}
\end{align*}
$$

A basic computation shows that $p(h, x)=\prod_{r=0}^{x-1}\left(1-\frac{h}{|V|-r}\right) \geq\left(1-\frac{h}{|V|-x}\right)^{x}$, so that $p\left(\left|H_{k-1}\right|, x_{k}\right) \geq p\left(h_{0}+\|x\|_{1}-x_{k}, x_{k}\right)$ with $h_{0}=\left|H_{0}\right|$ implies

$$
p\left(\left|H_{k-1}\right|, x_{k}\right) \geq\left(1-\frac{h_{0}+\|x\|_{1}-x_{k}}{|V|-x_{k}}\right)^{x_{k}} \geq\left(1-\frac{h_{0}+\|x\|_{1}}{|V|}\right)^{\|x\|_{\infty}} \geq \alpha
$$

Hence we may ignore the truncation by one in (7.9), and it follows that $\mathbb{P}\left(B_{k}=1 \mid \mathcal{F}_{k-1}\right)=\alpha$. This implies that $\mathbb{P}\left(B_{k}=1\right)=\alpha$, and that $B_{k}$ is independent of $\mathcal{F}_{k-1}$. Especially, $B_{k}$ is independent of ( $B_{1}, \ldots, B_{k-1}$ ), so we conclude that $B_{1}, \ldots, B_{m}$ are mutually independent.
7.3.1. Proof of Proposition 7.8. It suffices to find a lower bound for the exploration queue length $Q_{t}=\left|\mathcal{Q}_{t}\right|$ in Algorithm 2. This is because Algorithm 2 started at node $i$ discovers a subset of $\left|C_{G}(i)\right|$, and hence $\mathbb{P}\left(\left|C_{G}(i)\right|>\tau\right) \geq \mathbb{P}\left(Q_{\tau}>0\right)$. Denote by $\mathcal{M}_{x y, t}$ the of available $x y$-layers, and recall that $m_{x y, t}=\left(m_{x y}-\nu t\right)_{+}$. Denote by $T_{i}$ the number of steps completed by the algorithm. The queue length obeys the recursion $Q_{t}=\left(Q_{t-1}-1+Z_{t}\right) 1\left(Q_{t-1}>0\right)$ where $Z_{t}=\left|\mathcal{Z}_{t}\right|$. Algorithm 3 guarantees that the node sets $V\left(G_{k}\right) \backslash\left\{v_{t}\right\}$ of the explored layers $k \in \mathcal{W}_{t}$ are mutually disjoint and do not overlap any previously explored layers. Hence

$$
\begin{equation*}
Z_{t}=\sum_{(x, y) \in A} \sum_{k \in \mathcal{M}_{x y, t-1}} B_{1 x y t}(k) B_{2 x y t}(k) T_{x y t}(k), \tag{7.10}
\end{equation*}
$$

where $B_{1 x y t}(k)=1\left(k \in \mathcal{W}_{x y t}^{+}\right), B_{2 x y t}(k)=1\left(k \in \mathcal{W}_{x y t}\right)$, and $T_{x y t}(k)=\left|N_{v_{t}}\left(\bar{G}_{k}\right)\right|$ equals the number of neighbours of node $v_{t}$ in the transitive closure of $G_{k}$.

We will compare the queue length process to a random walk defined recursively by $Q_{0}^{\prime}=1$ and $Q_{t}^{\prime}=\left(Q_{t-1}^{\prime}-1+Z_{t}^{\prime}\right) 1\left(Q_{t-1}^{\prime}>0\right)$, where

$$
Z_{t}^{\prime}=\sum_{(x, y) \in A} \sum_{k=1}^{m_{x y, t-1}} B_{1 x y t}^{\prime}(k) B_{2 x y t}^{\prime}(k) T_{x y t}^{\prime}(k),
$$

and where the random variables appearing on the right are mutually independent and such that $\mathcal{L}\left(B_{1 x y t}^{\prime}(k)\right)=\operatorname{Ber}\left(\frac{x}{n-(t-1)}\right), \mathcal{L}\left(B_{2 x y t}^{\prime}(k)\right)=\operatorname{Ber}\left(\alpha_{t}\right)$ with $\alpha_{t}=(1-\delta)\left(1-\frac{t-1}{n}\right)$, and $\mathcal{L}\left(T_{x y t}^{\prime}(k)\right)=\operatorname{Bin}^{+}(x-1, y)$. A key part of the proof is to show that

$$
\begin{equation*}
\mathbb{P}\left(Q_{t}=r, \mathcal{A}_{\leq t}\right)=\mathbb{P}\left(Q_{t}^{\prime}=r, \mathcal{A}_{\leq t}^{\prime}\right) \tag{7.11}
\end{equation*}
$$

for all $r>0$ and $t \leq \tau$, where $\mathcal{A}_{\leq t}=\mathcal{A}_{1} \cap \cdots \cap \mathcal{A}_{t}$ and $\mathcal{A}_{\leq t}^{\prime}=\mathcal{A}_{1}^{\prime} \cap \cdots \cap \mathcal{A}_{t}^{\prime}$ are defined by

$$
\mathcal{A}_{t}=\left\{T_{i} \geq t, \max _{(x, y) \in A}\left|\mathcal{W}_{x y t}^{+}\right| \leq \nu\right\} \quad \text { and } \quad \mathcal{A}_{t}^{\prime}=\left\{\max _{(x, y) \in A} \sum_{k=1}^{m_{x y, t}} B_{1 x y t}^{\prime}(k) \leq \nu\right\}
$$

To verify (7.11), consider an event $\mathcal{E}_{t-1}$ that $Q_{t-1}=q, \mathcal{A}_{\leq t-1}$ is valid, the set of previously explored nodes equals $\hat{\mathcal{N}}_{\leq t-1}^{e}$, the set of previously explored layers equals $\hat{\mathcal{M}}_{\leq t-1}^{e}$, the set of available $x y$-layers after $t-1$ steps is $\hat{\mathcal{M}}_{x y, t-1}$, and node $v_{t}$ is explored on step $t$. This event is determined by the graphs $G_{k}, k \in \hat{\mathcal{M}}_{\leq t-1}^{e}$, the indicator variables $1\left(V\left(G_{k}\right) \ni v\right)$ for $k=[m]$ and $v \in \hat{\mathcal{N}}_{\leq t-1}^{e}$, and the random variables used in the randomised algorithm during steps $s \leq t-1$. On the event $\mathcal{E}_{t-1}$, the number of available $x y$-layers in the beginning of step $t$ equals $m_{x y, t-1}$, and the only thing known about the available layers is that they do not contain any of the explored nodes $\hat{\mathcal{N}}_{\leq t-1}^{e}$. Conditionally on $\mathcal{E}_{t-1}$, the graphs $G_{k}, k \in \hat{\mathcal{M}}_{t-1}$, are hence mutually independent and such that $V\left(G_{k}\right)$ is a uniform $X_{k}$-set in $[n] \backslash \hat{\mathcal{N}}_{\leq t-1}^{e}$.

Conditionally on the event $\mathcal{E}_{t-1}$, each available layer $k \in \hat{\mathcal{M}}_{x y, t-1}$ is discovered with probability $\frac{x}{n-(t-1)}$, independently of other available layers. Hence the indicators $B_{1 x y t}(k)$ in (7.10) are independent and $\operatorname{Ber}\left(\frac{x}{n-(t-1)}\right)$-distributed given $\mathcal{E}_{t-1}$. Let $\mathcal{E}_{t-1}^{+}=\mathcal{E}_{t-1} \cap\left\{\mathcal{W}_{x y t}^{+}=\right.$ $\left.\hat{\mathcal{W}}_{x y t}^{+},(x, y) \in A\right\}$ for some layer sets $\left|\hat{\mathcal{W}}_{x y t}^{+}\right| \leq \nu$ such that $\mathcal{E}_{t-1}^{+}$has nonzero probability. On the event $\mathcal{E}_{t-1}^{+}$, the number of nodes discovered before step $t$ is bounded by $\left|\mathcal{N}_{t-1}\right| \leq$
$M|A| \nu(t-1)$, and $\sum_{k \in \mathcal{W}_{t}^{+}}\left|V\left(G_{k}\right) \backslash\left\{v_{t}\right\}\right| \leq M\left|\mathcal{W}_{t}^{+}\right| \leq M|A| \nu$, and it follows that

$$
\left(1-\frac{\left|\mathcal{N}_{t-1}\right|+\sum_{k \in \mathcal{W}_{t}^{+}}\left|V\left(G_{k}\right) \backslash\left\{v_{t}\right\}\right|}{n-(t-1)}\right)^{M} \geq\left(1-\frac{M|A| \nu t}{n-(t-1)}\right)^{M}
$$

where the right side is at least $\left(1-2 M|A| \frac{\nu t}{n}\right)^{M} \geq 1-2 M^{2}|A| \frac{\nu t}{n} \geq \alpha_{t}$ due to $2 M^{2}|A| \frac{\nu t}{n} \leq \delta$ and $\alpha_{t} \leq 1-\delta$. By Lemma 7.9, we find that the indicators $B_{2 x y t}(k)$ in (7.10) are mutually independent and $\operatorname{Ber}\left(\alpha_{t}\right)$-distributed given $\mathcal{E}_{t-1}^{+}$. Furthermore, also the random integers $T_{x y t}(k)$ in (7.10) are mutually independent, independent of the indicators $B_{2 x y t}(k)$, and such that $\mathcal{L}\left(T_{x y t}(k) \mid \mathcal{E}_{t-1}^{+}\right)=\operatorname{Bin}^{+}(x-1, y)$. These observations allow us to conclude that $\mathcal{L}\left(Z_{t} \mid \mathcal{E}_{t-1}, \mathcal{A}_{t}\right)=\mathcal{L}\left(Z_{t}^{\prime} \mid \mathcal{A}_{t}^{\prime}\right)$ and $\mathbb{P}\left(\mathcal{A}_{t} \mid \mathcal{E}_{t-1}\right)=\mathbb{P}\left(\mathcal{A}_{t}^{\prime}\right)$. Hence for any $r>0$,

$$
\begin{aligned}
\mathbb{P}\left(Q_{t}=r, \mathcal{A}_{t} \mid \mathcal{E}_{t-1}\right) & =\mathbb{P}\left(\mathcal{A}_{t} \mid \mathcal{E}_{t-1}\right) \mathbb{P}\left(q-1+Z_{t}=r \mid \mathcal{E}_{t-1}, \mathcal{A}_{t}\right) \\
& =\mathbb{P}\left(\mathcal{A}_{t}^{\prime}\right) \mathbb{P}\left(q-1+Z_{t}^{\prime}=r \mid \mathcal{A}_{t}^{\prime}\right) \\
& =\mathbb{P}\left(q-1+Z_{t}^{\prime}=r, \mathcal{A}_{t}^{\prime}\right) \\
& =\mathbb{P}\left(Q_{t}^{\prime}=r, \mathcal{A}_{t}^{\prime} \mid Q_{t-1}^{\prime}=q, \mathcal{A}_{\leq t-1}^{\prime}\right)
\end{aligned}
$$

By multiplying both sides above by $\mathbb{P}\left(\mathcal{E}_{t-1}\right)$ and summing over all $\mathcal{E}_{t-1}$ which are subsets of the event $\left\{Q_{t-1}=q\right\} \cap \mathcal{A}_{\leq t-1}$, it follows that

$$
\mathbb{P}\left(Q_{t}=r, \mathcal{A}_{t} \mid Q_{t-1}=q, \mathcal{A}_{\leq t-1}\right)=\mathbb{P}\left(Q_{t}^{\prime}=r, \mathcal{A}_{t}^{\prime} \mid Q_{t-1}^{\prime}=q, \mathcal{A}_{\leq t-1}^{\prime}\right)
$$

Because the above equality holds for all $q, r>0$, a simple induction argument, based on

$$
\mathbb{P}\left(Q_{t}=r, \mathcal{A}_{\leq t}\right)=\sum_{q>0} \mathbb{P}\left(Q_{t-1}=q, \mathcal{A}_{\leq t-1}\right) \mathbb{P}\left(Q_{t}=r, \mathcal{A}_{t} \mid Q_{t-1}=q, \mathcal{A}_{\leq t-1}\right)
$$

confirms (7.11). With the help of (7.11), we now find that

$$
\mathbb{P}\left(Q_{\tau}>0\right) \geq \mathbb{P}\left(Q_{\tau}>0, \mathcal{A}_{\leq \tau}\right)=\mathbb{P}\left(Q_{\tau}^{\prime}>0, \mathcal{A}_{\leq \tau}^{\prime}\right) \geq \mathbb{P}\left(Q_{\tau}^{\prime}>0\right)-\mathbb{P}\left(\left(\mathcal{A}_{\leq \tau}^{\prime}\right)^{c}\right)
$$

Denote $M_{1 x y t}^{\prime}=\sum_{k=1}^{m_{x y, t-1}} B_{1 x y t}^{\prime}(k)$ and observe that $m_{x y, t-1} \leq m$ and $\mathcal{L}\left(B_{1 x y t}^{\prime}(k)\right) \leq_{s t}$ $\operatorname{Ber}\left(2 \frac{M}{n}\right)$ imply that $\mathcal{L}\left(M_{1 x y t}^{\prime}\right) \leq_{\text {st }} \operatorname{Bin}\left(m, \frac{2 M}{n}\right)$ for $t \leq n / 2$. The moment generating function of the latter distribution, evaluated at one, is bounded by $\left(1+\frac{2 M}{n}(e-1)\right)^{n} \leq e^{2 M(e-1) m / n} \leq$ $e^{4 M m / n}$. Therefore, Markov's inequality for $e^{M_{1 x y t}^{\prime}}$ implies $\mathbb{P}\left(M_{1 x y t}^{\prime}>\nu\right) \leq e^{4 M m / n-\nu}$, and

$$
\mathbb{P}\left(\left(\mathcal{A}_{\leq \tau}^{\prime}\right)^{c}\right) \leq \sum_{t=1}^{\tau} \mathbb{P}\left(\left(\mathcal{A}_{t}^{\prime}\right)^{c}\right) \leq|A| \tau e^{4 M m / n-\nu}
$$

Finally, observe that the distribution of $Z_{t}^{\prime}$ coincides with $f_{\delta, t, \nu}$ defined by (7.8). Moreover, $f_{\delta, t, \nu} \geq_{\text {st }} f_{\delta, \tau, \nu}$ for all $t=1, \ldots, \tau$. Therefore, $\mathbb{P}\left(Q_{\tau}^{\prime}>0\right) \geq \mathbb{P}\left(Q_{\tau}^{\prime \prime}>0\right)$ where $\left(Q_{0}^{\prime \prime}, \ldots, Q_{\tau}^{\prime \prime}\right)$ is defined as before, but with $Z_{1}^{\prime}, \ldots, Z_{\tau}^{\prime}$ replaced by mutually independent $f_{\delta, \tau, \nu}$-distributed random integers $Z_{1}^{\prime \prime}, \ldots, Z_{\tau}^{\prime \prime}$. The claim follows by noting that $\mathbb{P}\left(Q_{\tau}^{\prime \prime}>0\right)=\rho_{\tau}\left(f_{\delta, \tau, \nu}\right)$.

### 7.4. Component analysis for a finite layer type space.

Lemma 7.10. Under the assumptions and notations of Theorem 3.4, together with the extra assumption that the supports of $P$ and $\left(P_{n}\right)_{n \geq 1}$ are all contained in a finite set $A \subset$ $\{0, \ldots, M\} \cap[0,1]$, the component size of any particular node $i$ satisfies

$$
\begin{array}{ll}
\mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\tau\right) \rightarrow \rho_{\tau}\left(f^{+}\right) & \text {for any constant } \tau \geq 1 \\
\mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\omega\right) \rightarrow \rho\left(f^{+}\right) & \text {for } 1 \ll \omega \ll n \log ^{-1} n \tag{7.13}
\end{array}
$$

the relative frequencies of nodes with large components satisfy

$$
\begin{array}{ll}
n^{-1}\left|B_{\tau}\left(G^{(n)}\right)\right| & \xrightarrow{\mathbb{P}} \rho_{\tau}\left(f^{+}\right) \\
n^{-1}\left|B_{\omega}\left(G^{(n)}\right)\right| & \text { for any constant } \tau \geq 1,  \tag{7.15}\\
\mathbb{P} & \left(f^{+}\right)
\end{array} \quad \text { for } 1 \ll \omega \ll n \log ^{-1} n, ~ l
$$

and the largest component size in $G^{(n)}$ satisfies

$$
\begin{equation*}
n^{-1} N_{1}\left(G^{(n)}\right) \xrightarrow{\mathbb{P}} \rho\left(f^{+}\right) . \tag{7.16}
\end{equation*}
$$

Proof. We start by making an additional assumption that all layer types are nonrandom. The extension to random layer types is treated in the end.
(i) Upper bound for (7.12). Fix $1 \leq \tau \leq n / 2$. Then by Proposition 7.1,

$$
\begin{equation*}
\mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\tau\right) \leq \rho_{\tau}\left(f_{\tau, n}\right)+c \tau^{2} n^{-1} \log n \tag{7.17}
\end{equation*}
$$

where $c=e^{5 M(1+m / n)}$, and $f_{\tau, n}$ is the distribution defined by (7.1). A natural coupling implies that $\left|\rho_{\tau}\left(f_{\tau, n}\right)-\rho_{\tau}\left(f^{+}\right)\right| \leq \tau d_{\mathrm{tv}}\left(f_{\tau, n}, f^{+}\right)$. Hence by (7.17) it follows that

$$
\begin{equation*}
\mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\tau\right) \leq \rho_{\tau}\left(f^{+}\right)+c \tau^{2} n^{-1} \log n+\tau d_{\mathrm{tv}}\left(f_{\tau, n}, f^{+}\right) \tag{7.18}
\end{equation*}
$$

Define $\tilde{f}_{\tau, n}$ using the same formula (7.1), but with the $\operatorname{Ber}\left(\frac{X_{k}}{n-\tau}\right)$-distributed random variables $B_{k}$ replaced by $\operatorname{Poi}\left(\frac{X_{k}}{n-\tau}\right)$-distributed random variables $\tilde{B}_{k}$. $\operatorname{Because} d_{\mathrm{tv}}(\operatorname{Ber}(p), \operatorname{Poi}(p))=$ $p\left(1-e^{-p}\right) \leq p^{2}$ for all $0 \leq p \leq 1$, a natural coupling implies that

$$
d_{\mathrm{tv}}\left(f_{\tau, n}, \tilde{f}_{\tau, n}\right) \leq \sum_{k=1}^{m}\left(\frac{X_{k}}{n-\tau}\right)^{2} \leq 4 \frac{M^{2}}{n^{2}} m
$$

Then we see by Lemma A. 5 that $\tilde{f}_{\tau, n}=\operatorname{CPoi}\left(\frac{m}{n-\tau}\left(P_{n}\right)_{10}, g_{n}\right)$ where $g_{n}=\operatorname{Bin}_{10}^{+}\left(P_{n}\right)$. Lemma A. 6 implies that $d_{\mathrm{tv}}\left(\tilde{f}_{\tau, n}, f^{+}\right) \leq\left|\frac{m}{n-\tau}\left(P_{n}\right)_{10}-\mu(P)_{10}\right|+\mu(P)_{10} d_{\mathrm{tv}}\left(g_{n}, g\right)$, and we conclude that

$$
d_{\mathrm{tv}}\left(f_{\tau, n}, f^{+}\right) \leq\left|\frac{m}{n-\tau}\left(P_{n}\right)_{10}-\mu(P)_{10}\right|+\mu(P)_{10} d_{\mathrm{tv}}\left(g_{n}, g\right)+4 \frac{M^{2}}{n^{2}} m .
$$

Because $g_{n} \xrightarrow{w} g$ by Lemma A.10, it follows that $d_{\mathrm{tv}}\left(f_{\tau, n}, f^{+}\right) \rightarrow 0$ as $n \rightarrow \infty$. Hence by (7.18) it follows that $\lim \sup _{n \rightarrow \infty} \mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\tau\right) \leq \rho_{\tau}\left(f^{+}\right)$.
(ii) Upper bound for (7.13). Fix $\varepsilon>0$ and select a large enough $t$ such that $\rho_{t}\left(f^{+}\right) \leq$ $\rho\left(f^{+}\right)+\varepsilon$. Define $\tau_{n}=\left\lfloor\omega_{n} \wedge n^{1 / 3}\right\rfloor$. Then $\tau_{n} \geq t$ for large values of $n$, and by (7.17),

$$
\mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\tau_{n}\right) \leq \rho_{\tau_{n}}\left(f_{\tau_{n}, n}\right)+c \tau_{n}^{2} n^{-1} \log n \leq \rho_{t}\left(f_{\tau_{n}, n}\right)+c n^{-1 / 3} \log n
$$

A natural coupling implies that $\left|\rho_{t}\left(f_{\tau_{n}, n}\right)-\rho_{t}\left(f^{+}\right)\right| \leq t d_{\mathrm{tv}}\left(f_{\tau_{n}, n}, f^{+}\right)$. Hence it follows that

$$
\mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\omega_{n}\right) \leq \mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\tau_{n}\right) \leq \rho\left(f^{+}\right)+c n^{-1 / 3} \log n+t d_{\mathrm{tv}}\left(f_{\tau_{n}, n}, f^{+}\right)+\varepsilon
$$

The upper bound analysis of (7.12) shows that $d_{\mathrm{tv}}\left(f_{\tau, n}, f^{+}\right) \rightarrow 0$ also for $\tau=\tau_{n} \gg 1$. Hence we conclude that $\limsup \operatorname{sum}_{n \rightarrow \infty} \mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\omega_{n}\right) \leq \rho\left(f^{+}\right)$.
(iii) Lower bound for (7.12). Fix $\varepsilon>0$. To avoid trivialities we assume that $(P)_{10}>0$, in which case $\left(P_{n}\right)_{10}>0$ for all large values of $n$. Define $f_{\delta}=\operatorname{CPoi}((1-\delta) \lambda, g)$ with $\lambda=\mu(P)_{10}$. Lemma A. 6 then implies that $f_{\delta} \xrightarrow{w} f^{+}$as $\delta \rightarrow 0$. Hence by Lemma A. 15 we may choose a small $\delta \in(0,1)$ such that $\rho_{\tau}\left(f_{\delta}\right) \geq \rho_{\tau}\left(f^{+}\right)-\varepsilon$. Define $\nu_{n}=\lceil 2 \log n\rceil$. Then $2 M^{2}|A| \frac{\tau \nu_{n}}{n} \leq \delta$ for large values of $n$, and Lemma 7.8 implies, recalling that $\left|\rho_{\tau}\left(f_{\delta, \tau, \nu_{n}}^{(n)}\right)-\rho_{\tau}\left(f_{\delta}\right)\right| \leq \tau d_{\mathrm{tv}}\left(f_{\delta, \tau, \nu_{n}}^{(n)}, f_{\delta}\right)$,

$$
\begin{align*}
\mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\tau\right) & \geq \rho_{\tau}\left(f_{\delta, \tau, \nu_{n}}^{(n)}\right)-|A| e^{4 M m / n} \tau n^{-2} \\
& \geq \rho_{\tau}\left(f_{\delta}\right)-\tau d_{\mathrm{tv}}\left(f_{\delta, \tau, \nu_{n}}^{(n)}, f_{\delta}\right)-|A| e^{4 M m / n} \tau n^{-2}  \tag{7.19}\\
& \geq \rho_{\tau}\left(f^{+}\right)-\varepsilon-\tau d_{\mathrm{tv}}\left(f_{\delta, \tau, \nu_{n}}^{(n)}, f_{\delta}\right)-|A| e^{4 M m / n} \tau n^{-2}
\end{align*}
$$

where $f_{\delta, \tau, \nu_{n}}^{(n)}$ is the distribution defined in (7.8). Hence it suffices to verify that $d_{\mathrm{tv}}\left(f_{\delta, \tau, \nu_{n}}^{(n)}, f_{\delta}\right) \rightarrow$ 0 . To do this, define modifications of $f_{\delta, \tau, \nu_{n}}^{(n)}$ by

$$
f_{\delta}^{(n)}=\mathcal{L}\left(\sum_{(x, y) \in A} \sum_{k=1}^{m_{x y}} B_{x y}(k) T_{x y}(k)\right), \quad \tilde{f}_{\delta}^{(n)}=\mathcal{L}\left(\sum_{(x, y) \in A} \sum_{k=1}^{m_{x y}} \tilde{B}_{x y}(k) T_{x y}(k)\right)
$$

where the random variables are mutually independent and such that $\mathcal{L}\left(B_{x y}(k)\right)=\operatorname{Ber}((1-$ $\left.\delta) \frac{x}{n}\right), \mathcal{L}\left(\tilde{B}_{x y}(k)\right)=\operatorname{Poi}\left((1-\delta) \frac{x}{n}\right)$, and $\mathcal{L}\left(T_{x y}(k)\right)=\operatorname{Bin}^{+}(x-1, y)$. A natural coupling implies

$$
\begin{equation*}
d_{\mathrm{tv}}\left(f_{\delta, \tau, \nu_{n}}^{(n)}, f_{\delta}^{(n)}\right) \leq \sum_{(x, y) \in A}\left(m_{x y}-m_{x y, \tau-1}\right)(1-\delta) \frac{x}{n} \leq M|A| \frac{\tau \nu_{n}}{n} \tag{7.20}
\end{equation*}
$$

Because $d_{\mathrm{tv}}(\operatorname{Ber}(p), \operatorname{Poi}(p))=p\left(1-e^{-p}\right) \leq p^{2}$ for all $0 \leq p \leq 1$, and $\sum_{(x, y) \in A} \sum_{k=1}^{m_{x y}} \leq m$, a natural coupling implies that

$$
\begin{equation*}
d_{\mathrm{tv}}\left(f_{\delta}^{(n)}, \tilde{f}_{\delta}^{(n)}\right) \leq \sum_{(x, y) \in A} \sum_{k=1}^{m_{x y}}\left((1-\delta) \frac{x}{n}\right)^{2} \leq \frac{M^{2}}{n^{2}} m \tag{7.21}
\end{equation*}
$$

Now let us observe that $\mathcal{L}\left(\sum_{k=1}^{m_{x y}} \tilde{B}_{x y}(k) T_{x y}(k)\right)=\operatorname{CPoi}\left((1-\delta) m_{x y} \frac{x}{n}, \operatorname{Bin}^{+}(x-1, y)\right)$, so by Lemma A. 5 we see that $\tilde{f}_{\delta}^{(n)}=\operatorname{CPoi}\left((1-\delta) \lambda_{n}, g_{n}\right)$, where $\lambda_{n}=\frac{m}{n}\left(P_{n}\right)_{10}$, and $g_{n}=\operatorname{Bin}_{10}^{+}\left(P_{n}\right)$. Then Lemma A. 6 implies $d_{\mathrm{tv}}\left(\tilde{f}_{\delta}^{(n)}, f_{\delta}\right) \leq\left|\lambda_{n}-\lambda\right|+\lambda d_{\mathrm{tv}}\left(g_{n}, g\right)$, and combining this with (7.20)-(7.21) shows that

$$
d_{\mathrm{tv}}\left(f_{\delta, \tau, \nu}^{(n)}, f_{\delta}\right) \leq M|A| \frac{\tau \nu_{n}}{n}+\frac{M^{2}}{n^{2}} m+\left|\lambda_{n}-\lambda\right|+\lambda d_{\mathrm{tv}}\left(g_{n}, g\right)
$$

Because $\lambda_{n} \rightarrow \lambda$ and $g_{n} \xrightarrow{w} g$ (Lemma A.10), we see that $d_{\mathrm{tv}}\left(f_{\delta, \tau, \nu}^{(n)}, f_{\delta}\right) \rightarrow 0$, and in light of (7.19), it follows that $\liminf _{n \rightarrow \infty} \mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\tau\right) \geq \rho\left(f^{+}\right)$.
(iv) Lower bound for (7.13). Fix $\varepsilon>0$, define $\nu=\lceil 2 \log n\rceil$, and let $\tau_{n}=\omega_{n}$. Again let us choose a small $\delta \in(0,1)$ such that $\rho\left(f_{\delta}\right) \geq \rho(f)-\varepsilon$. Recall that Lemma 7.8 implies

$$
\mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\tau\right) \geq \rho_{\tau}\left(f_{\delta, \tau, \nu_{n}}^{(n)}\right)-|A| e^{4 M m / n} \tau n^{-2} \geq \rho\left(f_{\delta, \tau, \nu_{n}}^{(n)}\right)-|A| e^{4 M m / n} \tau n^{-2}
$$

Inspection of the previous part of the proof shows that $d_{\mathrm{tv}}\left(f_{\delta, \tau, \nu}^{(n)}, f_{\delta}\right) \rightarrow 0$ also for $\tau=\tau_{n}$ with $1 \ll \tau_{n} \ll n \log ^{-1} n$. Hence also $\rho\left(f_{\delta, \tau, \nu}^{(n)}\right) \rightarrow \rho\left(f_{\delta}\right)$ and $\liminf _{n \rightarrow \infty} \mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\omega\right) \geq \rho\left(f^{+}\right)$.
(iv) Proof of (7.14). Denote $p_{i}=\mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\tau\right)$ and $p_{i j}=\mathbb{P}\left(\left|C_{G^{(n)}}(i)\right|>\tau, C_{j}\left(G^{(n)}\right)>\right.$ $\tau)$. Symmetry implies that $\mathbb{E}\left|B_{\tau}\left(G^{(n)}\right)\right|=n p_{1}$ and $\operatorname{Var}\left|B_{\tau}\left(G^{(n)}\right)\right|=n p_{1}\left(1-p_{1}\right)+(n)_{2}\left(p_{12}-\right.$ $p_{1}^{2}$ ). Then (7.12) implies that $n^{-1} \mathbb{E}\left|B_{\tau}\left(G^{(n)}\right)\right| \rightarrow \rho_{\tau}$. If $\rho_{\tau}=0$, the claim follows by Markov's inequality. Assume next that $\rho_{\tau}>0$. Proposition 7.5 shows that $p_{12} \leq \rho_{\tau}\left(f_{2 \tau, n}\right)^{2}+$ $c \tau^{2} n^{-1} \log n$ where $c=e^{9 M(1+m / n)}$ and $f_{2 \tau, n}$ is defined by (7.1). The analysis of the upper bound for (7.12) shows that $\rho_{\tau}\left(f_{2 \tau, n}\right) \rightarrow \rho_{\tau}\left(f^{+}\right)$. Hence for any $\varepsilon>0$, we see that $p_{12} \leq \rho_{\tau}(f)^{2}+\varepsilon$ for all sufficiently large $n$. Because $p_{1} \rightarrow \rho_{\tau}\left(f^{+}\right)$by (7.12), we conclude that $p_{12}-p_{1}^{2} \leq 2 \varepsilon$ for large $n$. Hence $\operatorname{Var}\left|B_{\omega}\left(G^{(n)}\right)\right| \leq n p_{1}+2 n^{2} \varepsilon$ for large $n$, and

$$
\frac{\operatorname{Var}\left|B_{\tau}\left(G^{(n)}\right)\right|}{\left(\mathbb{E}\left|B_{\tau}\left(G^{(n)}\right)\right|\right)^{2}} \leq \frac{n p_{1}}{\left(n p_{1}\right)^{2}}+\frac{2 n^{2} \varepsilon}{\left(n p_{1}\right)^{2}}
$$

Because $p_{1} \asymp 1$, the ratio on the left vanishes and (7.14) follows by Chebyshev's inequality.
(v) Proof of (7.15) for $\omega \asymp \log n$. Now (7.13) implies that $p_{1}=n^{-1} \mathbb{E}\left|B_{\omega}\left(G^{(n)}\right)\right| \rightarrow \rho$. If $\rho=0$, the claim follows by Markov's inequality. For $\rho>0$, Proposition 7.5 shows that $p_{12} \leq$ $\rho_{\omega}\left(f_{2 \omega, n}\right)^{2}+c \omega^{2} n^{-1} \log n$. By a similar argument as in (ii), we conclude $\frac{\operatorname{Var}\left|B_{\omega}\left(G^{(n)}\right)\right|}{\left(\mathbb{E} B_{\omega}\left(G^{(n)}\right)\right)^{2}} \ll 1$, so that Chebyshev's inequality now yields (7.15) for $\omega \asymp \log n$.
(iv) Proof of (7.15) for $1 \ll \omega \ll n \log ^{-1} n$. Let $\omega^{\prime} \asymp \log n$. Then $\left|1\left(C_{i}>\omega\right)-1\left(C_{i}>\omega^{\prime}\right)\right|=$ $1\left(C_{i}>\omega \wedge \omega^{\prime}\right)-1\left(C_{i}>\omega \vee \omega^{\prime}\right)$ together with the triangle inequality shows that $\left|\left|B_{\omega}\right|-\left|B_{\omega^{\prime}}\right|\right| \leq$ $\left|B_{\omega \wedge \omega^{\prime}}\right|-\left|B_{\omega \vee \omega^{\prime}}\right|$. Taking expectations and Markov's inequality imply that $\mathbb{P}\left(\left|\left|B_{\omega}\right|-\left|B_{\omega^{\prime}}\right|\right|>\right.$ $\varepsilon n) \leq \varepsilon^{-1} n^{-1} \mathbb{E}\left|B_{\omega \wedge \omega^{\prime}}\right|=\varepsilon^{-1} \mathbb{P}\left(C_{i}>\omega \wedge \omega^{\prime}\right) \ll 1$ by (7.13). Hence $\left|B_{\omega}\right|-\left|B_{\omega^{\prime}}\right|=o_{\mathbb{P}}(n)$. In the previous step we saw that $\left|B_{\omega^{\prime}}\right|=\rho n+o_{\mathbb{P}}(m)$. Hence $\left|B_{\omega}\right|=\rho n+o_{\mathbb{P}}(n)$, and (7.15) holds also for general $1 \ll \omega \ll n \log ^{-1} n$.
(v) Proof of an upper bound for (7.16). Fix $\varepsilon>0$, and let $\omega \asymp \log m$. Then $(\rho+\varepsilon) n \geq \omega$ for large values of $n$. If $N_{1}\left(G^{(n)}\right)>(\rho+\varepsilon) n$, then every node in a largest component has its component bigger than $\omega$, and hence $\left|B_{\omega}\right| \geq N_{1}\left(G^{(n)}\right) \geq(\rho+\varepsilon) n$. Hence by (7.13),

$$
\mathbb{P}\left(n^{-1} N_{1}\left(G^{(n)}\right)>\rho+\varepsilon\right) \leq \mathbb{P}\left(n^{-1}\left|B_{\omega}\right|>\rho+\varepsilon\right) \rightarrow 0 .
$$

(vi) Proof of a lower bound for (7.16). We assume that $(P)_{21}>0$ because otherwise $g$ and $f^{+}=\operatorname{CPoi}(\lambda, g)$ both degenerate to the Dirac measure at zero, and the lower bound is trivial. Fix $\varepsilon>0$. Fix $\delta \in(0,1)$ so small that $\rho_{\delta}(\operatorname{CPoi}((1-\delta) \lambda, g))$ satisfies $\rho_{\delta} \geq \rho-\varepsilon / 2$. Denote by $m_{x y}=m P_{n}(x, y)$ the number of $x y$-layers. Let us partition the set of layers into two categories called red and blue, so that the number of red $x y$-layers equals $m_{x y}^{(r)}=\left\lfloor(1-\delta) m_{x y}\right\rfloor$ for each layer type $(x, y) \in A$, and denote by $G^{(n, r)}$ the overlay graph on $[n]$ generated by the red layers. Then $\frac{m_{x y}^{(r)}}{m} \rightarrow(1-\delta) P(x, y)$ implies that the total number of red layers equals $m^{(r)} \sim(1-\delta) m$ and the empirical layer type distribution of the red layers satisfies $P_{n}^{(r)} \xrightarrow{w} P$ with $\left(P_{n}^{(r)}\right)_{10} \rightarrow(P)_{10}$. By applying (7.15) to the overlay graph $G^{(n, r)}$, it follows that the relative proportion of nodes with a large red component is approximated by

$$
\begin{equation*}
n^{-1}\left|B_{\omega}\left(G^{(n, r)}\right)\right| \xrightarrow{\mathbb{P}} \rho_{\delta} \tag{7.22}
\end{equation*}
$$

for any $1 \ll \omega \ll n \log ^{-1} n$. Furthermore, denoting $\mathcal{E}_{n}=\left\{B_{\omega}\left(G^{(n, r)}\right)\right.$ is $G^{(n)}$-connected $\}$,

$$
\begin{aligned}
\mathbb{P}\left(n^{-1} N_{1}\left(G^{(n)}\right)<\rho-\varepsilon\right) & \leq \mathbb{P}\left(n^{-1} N_{1}\left(G^{(n)}\right)<\rho-\varepsilon, \mathcal{E}_{n}\right)+\mathbb{P}\left(\mathcal{E}_{n}^{c}\right) \\
& \leq \mathbb{P}\left(n^{-1}\left|B_{\omega}\left(G^{(n, r)}\right)\right|<\rho-\varepsilon\right)+\mathbb{P}\left(\mathcal{E}_{n}^{c}\right) \\
& \leq \mathbb{P}\left(n^{-1}\left|B_{\omega}\left(G^{(n, r)}\right)\right|<\rho_{\delta}-\varepsilon / 2\right)+\mathbb{P}\left(\mathcal{E}_{n}^{c}\right) .
\end{aligned}
$$

In light of (7.22), it suffices to show that $\mathcal{E}_{n}$ occurs with high probability.
On the complement of $\mathcal{E}_{n}$, there exists a pair of distinct $G^{(n, r)}$-components $C^{\prime}, C^{\prime \prime} \subset$ $B_{\omega}\left(G^{(n, r)}\right)$ such that there are no $G^{(n)}$-links between $C^{\prime}, C^{\prime \prime}$. Especially, there are no links between $C^{\prime}, C^{\prime \prime}$ generated by the blue layers. Denote by $p_{x y}$ the conditional probability that a particular blue layer of type $(x, y)$ connects $C^{\prime}$ and $C^{\prime \prime}$ by a link, given the red layers and the event that $C^{\prime}, C^{\prime \prime}$ are distinct $G^{(n, r)}$-components both larger than $\omega$. Then by applying Lemma 7.11 and noting that $(x)_{2} \leq M^{2} 1(x \geq 2)$, it follows that

$$
\begin{equation*}
p_{x y} \geq \frac{2\left|C^{\prime}\right|\left|C^{\prime \prime}\right|}{(n)_{2}} 1(x \geq 2) y \geq M^{-2}\left(\frac{\omega}{n}\right)^{2}(x)_{2} y . \tag{7.23}
\end{equation*}
$$

Denote by $M_{b}$ the number of blue layers generating at least one link between $C^{\prime}$ and $C^{\prime \prime}$. Then using $1-x \leq e^{-x}$,

$$
\mathbb{E} e^{-s M_{b}}=\prod_{(x, y) \in A}\left(\left(1-p_{x y}+p_{x y} e^{-s}\right)^{m_{x y}^{(b)}} \leq e^{-\left(1-e^{-s}\right) \sum_{(x, y) \in A} m_{x y}^{(b)} p_{x y}}\right.
$$

By noting that $m_{x y}^{(b)} \geq \delta m P_{n}(x, y)$ and applying (7.23), we see that for $\omega=n^{2 / 3}, \sum_{(x, y) \in A} m_{x y}^{(b)} p_{x y} \geq$ $\delta M^{-2} m n^{-2 / 3}\left(P_{n}\right)_{21} \geq c_{1} n^{1 / 3}$ for large value of $n$, where $c_{1}=\frac{1}{2} \delta \mu M^{-2}(P)_{21}$. Markov's inequality implies that for any $a, s>0$,

$$
\begin{equation*}
\mathbb{P}\left(M_{b}<a\right) \leq e^{s a} \mathbb{E} e^{-s M_{b}} \leq e^{s a-\left(1-e^{-s}\right) c_{1} n^{1 / 3}} \tag{7.24}
\end{equation*}
$$

By noting that $1-p_{x y} \leq e^{-p_{x y}}$ it follows that the conditional probability that there are no blue links between $C^{\prime}$ and $C^{\prime \prime}$ is bounded by

$$
\prod_{(x, y) \in A}\left(1-p_{x y}\right)^{m_{x y}^{(b)}} \leq e^{-\sum_{(x, y) \in A} M^{-2}\left(\frac{\omega}{n}\right)^{2}(x)_{2} y m_{x y}^{(b)}}=e^{-M^{-2}\left(\frac{\omega}{n}\right)^{2} m^{(b)}\left(P_{n}^{(b)}\right)_{21}}
$$

Note that there are at most $\frac{n}{\omega}=n^{1 / 3}$ distinct $G^{(n, r)}$-components inside $B_{\omega}\left(G^{(n, r)}\right)$. The number of such component pairs is hence at most $\frac{1}{2} n^{2 / 3}$, and the union bound together with (7.24) with $a=s=1$ then confirms that

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{E}_{n}^{c}\right) \leq \frac{1}{2} n^{2 / 3} e^{1-\left(1-e^{-1}\right) c_{1} n^{1 / 3}} \rightarrow 0 \tag{7.25}
\end{equation*}
$$

This fact together with (7.22) implies that $n^{-1} N_{1}\left(G^{(n)}\right) \geq \rho-\varepsilon$ with high probability.
(vii) Finally, let us extend the proofs to random layer types. Denote by $\mathbb{P}_{\theta_{n}}$ the regular conditional distribution of the $n$-th model given layer types $\theta_{n}=\left(\left(X_{n, 1}, Y_{n, 1}\right), \ldots,\left(X_{n, m}, Y_{n, m}\right)\right)$, see Section A. 1 for formal details. In this case the earlier analysis of (7.16) confirms that $\mathbb{P}_{\theta_{n}}\left(\left|n^{-1} N_{1}\left(G^{(n)}\right)-\rho\left(f^{+}\right)\right|>\varepsilon\right) \rightarrow 0$ for any realisation of $\left(\theta_{1}, \theta_{2}, \ldots\right)$ for which the empirical layer type distributions converge according to $d_{\mathrm{tv}}\left(P_{\theta_{n}}, P\right) \rightarrow 0$. Because $P_{n} \xrightarrow{w} P$ it follows by Lemma A. 2 that $d_{\mathrm{tv}}\left(P_{\theta_{n}}, P\right) \xrightarrow{\mathbb{P}} 0$. Now by applying Lemma A. 1 with $\Phi_{n}\left(\theta_{n}, \xi_{n}\right)=n^{-1} N_{1}\left(G_{n}\right)$ and $G_{n}=G_{n}\left(\xi_{n}\right)$, we find that $\Phi_{n} \xrightarrow{\mathbb{P}} \rho\left(f^{+}\right)$, and hence (7.16) also holds for random layer types. The same argument also confirms (7.12)-(7.15) for random layer types.

Lemma 7.11. Let $C_{1}, C_{2}$ be disjoint subsets of $[n]$ of sizes $c_{1}, c_{2}$. Let $V$ be a uniformly random $x$-set in $[n]$ with $x \geq 2$. Then the probability that $V$ intersects both $C_{1}$ and $C_{2}$ is at least $\frac{2 c_{1} c_{2}}{n(n-1)}$.

Proof. Denote $p_{x}=\mathbb{P}\left(V \cap C_{1} \neq \emptyset, V \cap C_{2} \neq \emptyset\right)$ for $V$ being a uniformly random $x$-set in [ $n$ ]. Define a random set $V^{\prime}$ so that the conditional distribution of $V^{\prime}$ given $V$ is uniformly random among the 2-subsets of $V$. Then $V^{\prime} \subset V$ with probability one, and the unconditional distribution of $V^{\prime}$ is uniform among the 2-subsets of $[n]$. Hence it follows that

$$
p_{x} \geq \mathbb{P}\left(V^{\prime} \cap C_{1} \neq \emptyset, V^{\prime} \cap C_{2} \neq \emptyset\right)=p_{2}=\frac{c_{1} c_{2}}{\binom{n}{2}}
$$

7.5. Discretising layer types. Layer sizes are compactified using the function $\sigma_{M}:(x, y) \mapsto$ $(x 1(x \leq M), y)$ which simply sets the layer size to zero. In the proofs we also need to discretise layer strengths. Some care is needed to avoid possible atoms of the limiting layer type distribution. Given a probability measure $P$ on $\mathbb{Z}_{+} \times[0,1]$, for every integer $L \geq 1$ we define functions $\sigma_{L-}, \sigma_{L+}: \mathbb{Z}_{+} \times[0,1] \rightarrow \mathbb{Z}_{+} \times[0,1]$ as follows. First, let $B_{P}$ be the set of points $y \in(0,1)$ for which $P\left(\mathbb{Z}_{+} \times\{y\}\right)>0$. Because $B_{P}$ is countable, for every integer $L \geq 1$ we may select a set of points $0=s_{0}<s_{1}<\cdots<s_{L}=1$ such that $\left\{s_{1}, \ldots, s_{L-1}\right\} \cap B_{P}=\emptyset$ and $\left|s_{i}-s_{i-1}\right| \leq 2 L^{-1}$ for all $i=1, \ldots, L$. Then we define

$$
\begin{align*}
\lfloor y\rfloor_{L} & =\sum_{i=1}^{L} s_{i-1} 1\left(s_{i-1} \leq y<s_{i}\right)+s_{L-1} 1(y=L) \\
\lceil y\rceil_{L} & =s_{1} 1(y=0)+\sum_{i=1}^{L} s_{i} 1\left(s_{i-1}<y \leq s_{i}\right) \tag{7.26}
\end{align*}
$$

and set $\sigma_{L-}(x, y)=\left(x,\lfloor y\rfloor_{L}\right)$ and $\sigma_{L+}(x, y)=\left(x,\lceil y\rceil_{L}\right)$.

Lemma 7.12. Consider probability measures on $\mathbb{Z}_{+} \times[0,1]$ such that $P_{n} \xrightarrow{w} P$ and $\left(P_{n}\right)_{10} \rightarrow$ $(P)_{10} \in[0, \infty)$. Then (i) $P \circ \sigma_{M}^{-1} \xrightarrow{w} P$ and $\left(P \circ \sigma_{M}^{-1}\right)_{10} \rightarrow(P)_{10}$ as $M \rightarrow \infty$; (ii) $P_{n} \circ \sigma_{M}^{-1} \xrightarrow{w}$ $P \circ \sigma_{M}^{-1}$ and $\left(P_{n} \circ \sigma_{M}^{-1}\right)_{10} \rightarrow\left(P \circ \sigma_{M}^{-1}\right)_{10}$ as $n \rightarrow \infty$; (iii) $P \circ \sigma_{L \pm}^{-1} \xrightarrow{w} P$ and $\left(P \circ \sigma_{L \pm}^{-1}\right)_{10} \rightarrow(P)_{10}$ as $L \rightarrow \infty$; (iv) $P_{n} \circ \sigma_{L \pm}^{-1} \xrightarrow{w} P \circ \sigma_{L \pm}^{-1}$ and $\left(P_{n} \circ \sigma_{L \pm}^{-1}\right)_{10} \rightarrow\left(P \circ \sigma_{L \pm}^{-1}\right)_{10}$ as $n \rightarrow \infty$; and (v) $h(M)=\sup _{n \geq 1} \int x 1(x>M) P_{n}(d x, d y) \rightarrow 0$.

Proof. (i) Let $f$ be bounded and continuous. Then $f \circ \sigma_{M} \rightarrow f$ pointwise as $M \rightarrow \infty$ and $\left|f \circ \sigma_{M}\right| \leq|f|$ pointwise, so that by Lebesgue's dominated convergence, $P \circ \sigma_{M}^{-1}(f)=$ $P\left(f \circ \sigma_{M}\right) \rightarrow P(f)$. The same argument applied to $f(x, y)=x$ shows that $\left(P \circ \sigma_{M}^{-1}\right)_{10}=$ $P\left(f \circ \sigma_{M}\right) \rightarrow P(f)=(P)_{10}$.
(ii) Because $f \circ \sigma_{M}$ is bounded and continuous whenever $f$ is, it follows that $P_{n} \circ \sigma_{M}^{-1}(f)=$ $P_{n}\left(f \circ \sigma_{M}\right) \rightarrow P\left(f \circ \sigma_{M}\right)=P \circ \sigma_{M}^{-1}(f)$. For $f(x, y)=x$, we find that $\left(P_{n} \circ \sigma_{M}^{-1}\right)_{10}=$ $P_{n}\left(f \circ \sigma_{M}\right) \rightarrow P\left(f \circ \sigma_{M}\right)=\left(P \circ \sigma_{M}^{-1}\right)_{10}$.
(iii) The construction in (7.26) guarantees that $\lfloor y\rfloor_{L} \rightarrow y$ and $\lceil y\rceil_{L} \rightarrow y$ as $L \rightarrow \infty$ for every $y \in[0,1]$. Therefore the functions $\sigma_{L-}, \sigma_{L+}$ converge pointwise to the identity map on $\mathbb{Z}_{+} \times[0,1]$ as $L \rightarrow \infty$. Hence by Lebesgue's dominated convergence, $P \circ \sigma_{L \pm}^{-1}(f)=$ $P\left(f \circ \sigma_{L \pm}\right) \rightarrow P(f)$ for any bounded continuous $f$. The same argument applied to $f(x, y)=x$ implies that $\left(P \circ \sigma_{L \pm}^{-1}\right)_{10}=P\left(f \circ \sigma_{L \pm}\right) \rightarrow P(f)=(P)_{10}$.
(iv) By Skorohod's coupling [30, Proposition 4.30], there exist random variables ( $X_{n}, Y_{n}$ ) and $(X, Y)$ such that $\mathcal{L}\left(X_{n}, Y_{n}\right)=P_{n}, \mathcal{L}(X, Y)=P$, and $\left(X_{n}, Y_{n}\right) \rightarrow(X, Y)$ almost surely. Hence $\sigma_{L-}\left(X_{n}, Y_{n}\right)=\left(X_{n},\left\lfloor Y_{n}\right\rfloor_{L}\right) \rightarrow\left(X,\lfloor Y\rfloor_{L}\right)$ whenever $Y \in[0,1] \backslash\left\{s_{1}, \ldots, s_{L-1}\right\}$. Now $\mathbb{P}\left(Y=s_{i}\right)=P\left(\mathbb{Z}_{+} \times\left\{s_{i}\right\}\right)=0$ by construction, so we conclude that $\sigma_{L-}\left(X_{n}, Y_{n}\right) \rightarrow$ $\sigma_{L-}(X, Y)$ almost surely. Hence $P_{n} \circ \sigma_{L-}^{-1} \xrightarrow{w} P \circ \sigma_{L-}^{-1}$. The same argument also works for $P_{n} \circ \sigma_{L+}^{-1}$, and $\left(P_{n} \circ \sigma_{L \pm}^{-1}\right)_{10} \rightarrow\left(P \circ \sigma_{L \pm}^{-1}\right)_{10}$ then follows by dominated convergence.
(v) Let ( $X_{n}, Y_{n}$ ) and ( $X, Y$ ) be random variables distributed according to $P_{n}$ and $P$, respectively. Then $X_{n} \rightarrow X$ in distribution and $\mathbb{E} X_{n}=\left(P_{n}\right)_{10} \rightarrow(P)_{10}=\mathbb{E} X<\infty$. Hence $\left(X_{n}\right)_{n \geq 1}$ is uniformly integrable, and $h(M)=\sup _{n \geq 1} \mathbb{E} X_{n} 1\left(X_{n}>M\right) \rightarrow 0$.
7.6. Proof of Theorem 3.4. Denote $P^{M}=P \circ \sigma_{M}^{-1}$ where $\sigma_{M}:(x, y) \mapsto(x 1(x \leq M), y)$. Let $f^{+}=\operatorname{CPoi}\left(\mu(P)_{10}, g\right)$ and $f^{M}=\operatorname{CPoi}\left(\mu\left(P^{M}\right)_{10}, g^{M}\right)$, where $g=\operatorname{Bin}_{10}^{+}(P)$ and $g^{M}=$ $\mathrm{Bin}_{10}^{+}\left(P^{M}\right)$ are defined by (2.6). Then by Lemma 7.12, $P^{M} \xrightarrow{w} P$ together with $\left(P^{M}\right)_{10} \rightarrow$ $(P)_{10}$. Lemma A. 10 implies that $g^{M} \xrightarrow{w} g$. Hence $f^{M} \xrightarrow{w} f^{+}$(Lemma A.6), implying that $\rho_{t}\left(f^{M}\right) \rightarrow \rho_{t}\left(f^{+}\right)$for all $t$ and $\rho\left(f^{M}\right) \rightarrow \rho\left(f^{+}\right)$(Lemma A.15).
(i) Lower bound. Fix $\varepsilon>0$. Fix a large enough $M$ such that $\rho\left(f^{M}\right) \geq \rho\left(f^{+}\right)-\varepsilon$. Then apply the layer strength discretisation procedure (7.26) to $P^{M}$, and define $\sigma_{L-}$ accordingly. Define $P^{M L-}=P^{M} \circ \sigma_{L-}^{-1}$. Lemma 7.12 then implies that $P^{M L-} \xrightarrow{w} P^{M}$ and $\left(P^{M L-}\right)_{10} \rightarrow$ $\left(P^{M}\right)_{10}$ as $L \rightarrow \infty$. The same argument as above then implies that $f^{M L-} \xrightarrow{w} f^{M}$ and $\rho\left(f^{M L-}\right) \rightarrow \rho\left(f^{M}\right)$ as $L \rightarrow \infty$, where $f^{M L-}=\operatorname{CPoi}\left(\mu\left(P^{M L-}\right)_{10}, g^{M L-}\right)$ and $g^{M L-}=$ $\operatorname{Bin}_{10}^{+}\left(P^{M L-}\right)$. Hence we may fix a large $L$ so that $\rho\left(f^{M L-}\right) \geq \rho\left(f^{M}\right)-\varepsilon$.

Now for each $n$, consider a modification $G^{(n M L-)}$ of $G^{(n)}$ where each layer of type $(x, y)$ is replaced by a layer of type $\left(x 1(x \leq M),\lfloor y\rfloor_{L}\right)$. Under a natural coupling, $N_{1}\left(G^{n M L-}\right) \leq$ $N_{1}\left(G^{n}\right)$ almost surely for every $n$, and

$$
\frac{N_{1}\left(G^{(n)}\right)}{n} \geq \frac{N_{1}\left(G^{n M L-}\right)}{n} \geq \rho\left(f^{+}\right)-2 \varepsilon+\left(\frac{N_{1}\left(G^{n M L-}\right)}{n}-\rho\left(f^{M L-}\right)\right)
$$

The averaged layer type distribution of $G^{(n M L-)}$ equals $P_{n}^{M L-}=P_{n} \circ \sigma_{M}^{-1} \circ \sigma_{L-}^{-1}$. In light of Lemma 7.12, we see that $P_{n}^{M L-} \xrightarrow{w} P^{M L-}$ and $\left(P_{n}^{M L-}\right)_{10} \rightarrow\left(P^{M L-}\right)_{10}$ as $n \rightarrow \infty$. A
suitable lower bound follows from the above inequality, because $\frac{N_{1}\left(G^{n M L-}\right)}{m} \xrightarrow{\mathbb{P}} \rho\left(f^{M L-}\right)$ due to Lemma 7.10.
(ii) Upper bound. Given $\delta, \varepsilon>0$, choose a large enough $t$ so that $\rho_{t}\left(f^{+}\right) \leq \rho\left(f^{+}\right)+\varepsilon / 5$. Then choose a large enough $M$ so that $\rho_{t}\left(f^{M}\right) \leq \rho_{t}\left(f^{+}\right)+\varepsilon / 5$ and $h(M) \leq \frac{\delta \varepsilon}{40 \mu t}$ where $h(M)=\sup _{n} \int x 1(x>M) d P_{n}$ (see Lemma 7.12). By similar arguments as in the proof of the lower bound, we find that $P^{M L+} \xrightarrow{w} P^{M}, g^{M L+} \xrightarrow{w} g^{M}$, and $f^{M L+} \xrightarrow{w} f^{M}$ as $L \rightarrow \infty$, where $f^{M L+}=\operatorname{CPoi}\left(\mu\left(P^{M L+}\right)_{10}, g^{M L+}\right)$ with $P^{M L+}=P^{M} \circ \sigma_{L+}^{-1}$ and $g^{M L+}=\operatorname{Bin}_{10}^{+}\left(P^{M L+}\right)$. Hence we may choose (Lemma A.15) a large enough $L$ so that $\rho_{t}\left(f^{M L+}\right) \leq \rho_{t}\left(f^{M}\right)+\varepsilon / 5$. Hence $\rho_{t}\left(f^{M L+}\right) \leq \rho\left(f^{+}\right)+\frac{3}{5} \varepsilon$.

Let $G^{(n, M)}$ and $G^{(n, M L+)}$ be modified overlay graphs in which each layer of type $(x, y)$ is replaced by a layer of type $(x 1(x \leq M), y)$ and $\left(x 1(x \leq M),\lceil y\rceil_{L}\right)$, respectively. We fix a natural coupling under which $G^{(n, M)} \subset G^{(n, M L+)}$ almost surely. Then by Lemma A.12,

$$
\frac{N_{1}\left(G^{(n)}\right)}{n} \leq \frac{\left|B_{t}\left(G^{(n, M L+)}\right)\right|}{n}+\frac{t}{n}\left(Z_{n, M}+1\right)
$$

where $Z_{n, M}$ is the number of nodes covered by layers larger than $M$ in the nontruncated model $G^{(n)}$. By Lemma 7.10, we may choose an integer $n_{0}$ such that $\frac{1}{m} \leq h(M), \frac{m}{n} \leq 2 \mu$, and

$$
\begin{equation*}
\mathbb{P}\left(\frac{\left|B_{t}\left(G^{(n, M L+)}\right)\right|}{n}>\rho_{t}\left(f^{(M L+)}\right)+\frac{\varepsilon}{5}\right) \leq \frac{\delta}{2} \tag{7.27}
\end{equation*}
$$

for all $n \geq n_{0}$. Then we note that

$$
\mathbb{E} Z_{n, M} \leq \mathbb{E} \sum_{k=1}^{m} X_{k}^{(n)} 1\left(X_{k}^{(n)}>M\right) \leq m h(M)
$$

so that $\mathbb{E} \frac{t}{n}\left(Z_{n, M}+1\right) \leq \frac{t}{n}(m h(M)+1) \leq 4 \mu t h(M) \leq \frac{\delta \varepsilon}{10}$, and by Markov's inequality, $\mathbb{P}\left(\frac{t}{n}\left(Z_{n, M}+1\right)>\frac{\varepsilon}{5}\right) \leq \frac{\delta}{2}$. Hence for all $n \geq n_{0}$,

$$
\frac{N_{1}\left(G^{(n)}\right)}{n} \leq \rho_{t}\left(f^{(M L+)}\right)+\frac{2}{5} \varepsilon \leq \rho\left(f^{+}\right)+\varepsilon
$$

with probability at least $1-\delta$.
(iii) Upper bound on the second largest component. Fix $\delta, \varepsilon, t, M, L$ as in part (ii) and define $G^{(n, M)}$ and $G^{(n, M L+)}$ in the same way. By Lemma A. 11 and Lemma A.12,

$$
N_{1}\left(G^{(n)}\right)+N_{2}\left(G^{(n)}\right) \leq\left|B_{t}\left(G^{(n)}\right)\right|+2 t \leq\left|B_{t}\left(G^{(n, M)}\right)\right|+2 t+t Z_{n, M}
$$

Under a natural coupling, $\left|B_{t}\left(G^{(n, M)}\right)\right| \leq\left|B_{t}\left(G^{(n, M L+)}\right)\right|$, so that

$$
\frac{N_{2}\left(G^{(n)}\right)}{n} \leq \frac{\left|B_{t}\left(G^{(n, M L+)}\right)\right|}{n}-\frac{N_{1}\left(G^{(n)}\right)}{n}+\frac{t}{n}\left(Z_{n, M}+2\right)
$$

By part (i), $\frac{N_{1}\left(G^{(n)}\right)}{n} \geq \rho\left(f^{+}\right)-\varepsilon / 5$ with probability at least $1-\delta / 2$, whereas part (ii) implies that $\mathbb{P}\left(\frac{t}{n}\left(Z_{n, M}+1\right)>\frac{\varepsilon}{5}\right) \leq \frac{\delta}{2}$. Together with (7.27) it follows that $\frac{N_{2}\left(G^{(n)}\right)}{n} \leq \frac{6}{5} \varepsilon$ with probability at least $1-\frac{3}{2} \delta$, whenever $n$ is large enough. Hence $\frac{N_{2}\left(G^{(n)}\right)}{n} \xrightarrow{\mathbb{P}} 0$.

### 7.7. Proofs for percolation models.

7.7.1. Site percolation. Proof of Theorem 3.5. The site-percolated graph $\check{G}^{(n)}$ is an instance of the overlay graph model (2.1) with $\check{n}=\left|S_{n}\right|$ nodes and $m$ layers $\check{G}_{1}, \ldots, \check{G}_{m}$ where $\check{G}_{k}$ is the subgraph of $G_{k}$ induced by $S_{n}$, and $G_{1}, \ldots, G_{m}$ are the original layers generating the graph $G$. The layer types ( $X_{k}, Y_{k}$ ) in the site-percolated model are mutually independent, and $\mathcal{L}\left(\check{X}_{k} \mid X_{k}=x_{k}\right)$ is hypergeometric with probability mass function

$$
\operatorname{Hyp}\left(n, \check{n}, x_{k}\right)(t)=\frac{\binom{\check{n}}{t}\binom{n-\check{n}}{x_{k}-t}}{\binom{n}{x_{k}}} .
$$

The site-percolated graph is hence an instance of the overlay model with $\check{n}$ nodes, $m$ layers, and averaged layer type distribution

$$
\check{P}_{n}(A)=\int\left(\operatorname{Hyp}(n, \check{n}, x) \times \delta_{y}\right)(A) P_{n}(d x, d y)
$$

Define probability kernels $K_{n}, K$ on $\mathbb{Z}_{+} \times[0,1]$ by formulas $K_{n}((x, y), A)=(\operatorname{Hyp}(n, \check{n}, x) \times$ $\left.\delta_{y}\right)(A)$ and $K((x, y), A)=\left(\operatorname{Bin}(x, \theta) \times \delta_{y}\right)(A)$. By [19, Theorem 4], $d_{\mathrm{tv}}\left(\operatorname{Hyp}(n, \check{n}, x), \operatorname{Bin}\left(x, \frac{\check{n}}{n}\right)\right) \leq$ $2 \frac{x}{n}$. A basic coupling of coin flips implies that $d_{\mathrm{tv}}\left(\operatorname{Bin}\left(x, \frac{\check{n}}{n}\right), \operatorname{Bin}(x, \theta)\right) \leq\left|\frac{\check{n}}{n}-\theta\right| x$. Then for any bounded continuous function $\phi$ on $\mathbb{Z}_{+} \times[0,1]$,

$$
\left|K_{n} \phi(x, y)-K \phi(x, y)\right| \leq 2\|\phi\|_{\infty} d_{\mathrm{tv}}(\operatorname{Hyp}(n, \check{n}, x), \operatorname{Bin}(x, \theta)) \leq 2\|\phi\|_{\infty}\left(\frac{2}{n}+\left|\frac{\check{n}}{n}-\theta\right|\right) x
$$

for all $x, y$. Hence $K_{n} \phi \rightarrow K \phi$ uniformly on the compact subsets of $\mathbb{Z}_{+} \times[0,1]$. Because $\left(P_{n}\right)_{n \geq 1}$ is tight, it follows that $\check{P}_{n}(\phi)=P_{n}\left(K_{n} \phi\right) \rightarrow P(K \phi)=\check{P}(\phi)$ for any bounded continuous $\phi$. Hence $\check{P}_{n} \xrightarrow{w} \check{P}$. Direct computations show that $\left(\check{P}_{n}\right)_{10}=\frac{\check{n}}{n}\left(P_{n}\right)_{10} \rightarrow \theta(P)_{10}=$ $(\check{P})_{10}$, and $\frac{m}{\check{n}} \rightarrow \check{\mu}=\theta^{-1} \mu$. Theorem 3.5:(i)-(ii) now follow by applying Theorems 3.1 and 3.4 to $\check{G}^{(n)}$ and noting that $\check{\mu}(\check{P})_{10}=\mu(P)_{10}$.

Assume next that $\left(P_{n}\right)_{r s} \rightarrow(P)_{r s} \in(0, \infty)$ for $r s=21,32,33$. A direct computation using the binomial distribution shows that $(\check{P})_{r s}=\theta^{r}(P)_{r s}$. Theorem 3.5:(iii) now follows by applying Theorem 3.2 to conclude that the clustering coefficient of $\check{G}^{(n)}$ converges to $\check{\tau}=\frac{(\check{P})_{33}}{(\check{P})_{32}+\check{\mu}(\tilde{P})_{21}^{2}}=\frac{(P)_{33}}{(P)_{32}+\mu(P)_{21}^{2}}=\tau$. Theorem 3.5:(iv) follows similarly from Theorem 3.3.
7.7.2. Layerwise bond percolation. Proof of Theorem 3.6 for the layerwise bond-percolated graph $\tilde{G}^{(n)}$. The graph $\tilde{G}_{n}$ is an instance of the overlay model with $n$ nodes and $m$ layers $\tilde{G}_{1}, \ldots, \tilde{G}_{m}$ where $\tilde{G}_{k}$ has size $X_{k}$ and strength $\theta Y_{k}$. The layers ( $\tilde{G}_{k}, X_{k}, \theta Y_{k}$ ) are mutually independent, with averaged layer type distribution

$$
\tilde{P}_{n}(A)=\int\left(\delta_{x} \times \delta_{\theta y}\right)(A) P_{n}(d x, d y)
$$

converging according to $\tilde{P}_{n} \xrightarrow{w} \hat{P}$ and $(\tilde{P})_{10} \rightarrow(\hat{P})_{10}$. Furthermore, a direct computation shows that $(\hat{P})_{r s}=\theta^{s}(P)_{r s}$. Statements (i)-(ii) of Theorem 3.6 now follow by Theorems 3.1 and 3.4, and noting that $(\hat{P})_{10}=(P)_{10}$. Statements (iii)-(iv) follow analogously by Theorems 3.2 and 3.3, and noting that $\hat{\tau}=\frac{(\hat{P})_{33}}{(\hat{P})_{32}+(\hat{P})_{21}^{2}}=\theta \frac{(P)_{33}}{(P)_{32}+\mu(P)_{21}^{2}}=\theta \tau$.
7.7.3. Bond percolation coupling. We will utilise the fact that the overlay bond-percolated graph does not differ much from the layerwise bond-percolated graph $\tilde{G}^{(n)}$, for which the theorem has already been proved. The conditional distribution of $\hat{G}^{(n)}$ given the layers ( $G_{k}, X_{k}, Y_{k}$ ) is an inhomogeneous Bernoulli graph on $\{1, \ldots, n\}$ where each node pair $i j$ is linked with probability $\hat{p}_{i j}=\theta\left(M_{i j} \wedge 1\right)$ where $M_{i j}=\sum_{k} 1\left(E\left(G_{k}\right) \ni i j\right)$ is the number of layers linking a node pair $i j$. The corresponding conditional distribution of $\tilde{G}^{(n)}$ is a similar inhomogeneous Bernoulli graph with link probabilities $\tilde{p}_{i j}=1-(1-\theta)^{M_{i j}}$. Because $\hat{p}_{i j} \leq \tilde{p}_{i j}$, this suggest the following coupling construction:
(i) Sample the layers $\left(G_{k}, X_{k}, Y_{k}\right), k=1, \ldots, m$.
(ii) Sample independent inhomogeneous Bernoulli graphs $\tilde{H}$ and $H^{*}$ with link probabilities $\tilde{p}_{i j}$ and $p_{i j}^{*}=\frac{\hat{p}_{i j}}{\hat{p}_{i j}}$ with the convention $\frac{0}{0}=0$.
(iii) Define $\hat{G}=G \cap \hat{H}$ and $\tilde{G}=G \cap \tilde{H}$ with $G$ defined by (2.1) and $\hat{H}=\tilde{H} \cap H^{*}$.

Then $(\hat{G}, \tilde{G}, G)$ constitutes a coupling of the overlay bond-percolated, layerwise bond-percolated, and nonpercolated graphs such that $\hat{G} \subset \tilde{G} \subset G$ almost surely.
7.7.4. Proof of Theorem 3.6:(i) for overlay bond percolation. Let us denote by $\hat{D}_{n}=\operatorname{deg}_{\hat{G}^{(n)}}(i)$ and $\tilde{D}_{n}=\operatorname{deg}_{\tilde{G}^{(n)}}(i)$ the degrees of node $i$ in the overlay bond-percolated and layerwise bondpercolated graph, respectively. Using the coupling of Section 7.7 .3 , we observe that $\hat{D}_{n}=\tilde{D}_{n}$ on the event $M_{i} \leq 1$, where $M_{i}=\max _{j \neq i} M_{i j}$. Hence $d_{\mathrm{tv}}\left(\mathcal{L}\left(\hat{D}_{n}\right), \mathcal{L}\left(\tilde{D}_{n}\right)\right) \leq \mathbb{P}\left(M_{i}>1\right)$. The union bound implies that

$$
\mathbb{P}\left(M_{i j}>1\right) \leq \sum_{k, \ell}^{\prime} \mathbb{P}\left(E\left(G_{k}\right) \ni i j\right) \mathbb{P}\left(E\left(G_{\ell}\right) \ni i j\right) \leq\left(\sum_{k} \mathbb{P}\left(E\left(G_{k}\right) \ni i j\right)\right)^{2}
$$

By noting that $\mathbb{P}\left(E\left(G_{k}\right) \ni i j\right)=\mathbb{E} \frac{\left(X_{k}\right)_{2}}{(n)_{2}} Y_{k}$, we conclude that

$$
\begin{equation*}
\mathbb{P}\left(M_{i j}>1\right) \leq\left(m(n)_{2}^{-1}\left(P_{n}\right)_{21}\right)^{2} \tag{7.28}
\end{equation*}
$$

Another union bound shows that $\mathbb{P}\left(M_{i}>1\right) \leq \sum_{j \neq i} \mathbb{P}\left(M_{i j}>1\right)$ and hence

$$
\begin{equation*}
d_{\mathrm{tv}}\left(\mathcal{L}\left(\hat{D}_{n}\right), \mathcal{L}\left(\tilde{D}_{n}\right)\right) \leq(m / n)^{2}\left(P_{n}\right)_{21}^{2}(n-1)^{-1} \tag{7.29}
\end{equation*}
$$

Because $\mathcal{L}\left(\tilde{D}_{n}\right) \xrightarrow{w} \operatorname{CPoi}\left(\mu(P)_{10}, \operatorname{Bin}_{10}(\hat{P})\right)$, the same result for the bond-percolation graph follows from (7.29) in case of bounded layer sizes. In the general case, we truncate layers as in (5.6), and denote by $\hat{D}_{n}^{M}$ (resp. $\tilde{D}_{n}^{M}$ ) the degree of node $i$ in $\hat{G}_{n}^{M}$ (resp. $\tilde{G}_{n}^{M}$ ). Then (7.29) implies that $d_{\mathrm{tv}}\left(\mathcal{L}\left(\hat{D}_{n}^{M}\right), \mathcal{L}\left(\tilde{D}_{n}^{M}\right)\right) \leq c M^{4} n^{-1}$ for all large values of $n$, with $c=2 \mu^{2}$. The reasoning in (5.7) works also for bond-percolated models, and hence $d_{\mathrm{tv}}\left(\mathcal{L}\left(\hat{D}_{n}\right), \mathcal{L}\left(\hat{D}_{n}^{M}\right)\right) \leq$ $h(M)$ and $d_{\mathrm{tv}}\left(\mathcal{L}\left(\tilde{D}_{n}\right), \mathcal{L}\left(\tilde{D}_{n}^{M}\right)\right) \leq h(M)$ where $h(M)=\sup _{n \geq 1} \int x 1(x>M) P_{n}(d x, d y)$. We conclude that

$$
d_{\mathrm{tv}}\left(\mathcal{L}\left(\hat{D}_{n}\right), \mathcal{L}\left(\tilde{D}_{n}\right)\right) \leq c M^{4} n^{-1}+2 h(M)
$$

for all $M$. By choosing $M \asymp n^{1 / 5}$, we see that $d_{\mathrm{tv}}\left(\mathcal{L}\left(\hat{D}_{n}\right), \mathcal{L}\left(\tilde{D}_{n}\right)\right) \rightarrow 0$, and Theorem 3.6:(i) follows for $\hat{G}^{(n)}$.
7.7.5. Proof of Theorem 3.6:(iii) for overlay bond percolation. For any distinct nodes $i, j, k$, we see that $\mathbb{P}\left(\hat{G}_{i j}^{(n)}, \hat{G}_{i k}^{(n)}, \hat{G}_{j k}^{(n)}\right)=\theta^{3} \mathbb{P}\left(G_{i j}^{(n)}, G_{i k}^{(n)}, G_{j k}^{(n)}\right)$ and $\mathbb{P}\left(\hat{G}_{i j}^{(n)}, \hat{G}_{i k}^{(n)}\right)=\theta^{2} \mathbb{P}\left(G_{i j}^{(n)}, G_{i k}^{(n)}\right)$. Hence $\hat{\tau}^{(n)}=\theta \tau^{(n)}$ for every $n$, and the claim follows by applying Theorem 3.2 to the nonpercolated model.
7.7.6. Proof of Theorem 3.6:(iv) for overlay bond percolation. Fix any distinct nodes $i, j, k$, and note that the clustering spectrum of $\hat{G}^{(n)}$ can be written as $\hat{\sigma}^{(n)}(t)=\mathbb{P}\left(\hat{\mathcal{A}}_{n, t}\right) / \mathbb{P}\left(\hat{\mathcal{B}}_{n, t}\right)$ where

$$
\begin{aligned}
\hat{\mathcal{A}}_{n, t} & =\left\{\operatorname{deg}_{\hat{G}^{(n)}}(i)=t, \hat{G}_{i j}^{(n)}, \hat{G}_{i k}^{(n)}, \hat{G}_{j k}^{(n)}\right\}, \\
\hat{\mathcal{B}}_{n, t} & =\left\{\operatorname{deg}_{\hat{G}^{(n)}}(i)=t, \hat{G}_{i j}^{(n)}, \hat{G}_{i k}^{(n)}\right\} .
\end{aligned}
$$

A similar formula also holds for the clustering spectrum $\tilde{\sigma}^{(n)}(t)$ of the layerwise bondpercolated graph, with $\tilde{\mathcal{A}}_{n, t}, \tilde{\mathcal{B}}_{n, t}$ defined analogously. Observe that $1\left(\hat{\mathcal{A}}_{n, t}\right)=1\left(\tilde{\mathcal{A}}_{n, t}\right)$ and $1\left(\hat{\mathcal{B}}_{n, t}\right)=1\left(\tilde{\mathcal{B}}_{n, t}\right)$ on the event that $M_{i}=\max _{j \neq i} M_{i j} \leq 1$ and $M_{j k} \leq 1$, where $M_{i j}$ refers to the number of layers linking node pair $i j$ in the coupling construction of Section 7.7.3. By exchangeability, the union bound, estimate (7.28), and $\left(P_{n}\right)_{21} \lesssim 1$, it follows that

$$
\mathbb{P}\left(M_{i}>1 \text { or } M_{j k}>1\right) \leq n \mathbb{P}\left(M_{i j}>1\right) \leq n\left(m(n)_{2}^{-1}\left(P_{n}\right)_{21}\right)^{2} \lesssim n^{-1} .
$$

Hence $\mathbb{P}\left(\hat{\mathcal{A}}_{n, t}\right)=\mathbb{P}\left(\tilde{\mathcal{A}}_{n, t}\right)+O\left(n^{-1}\right)$ and $\mathbb{P}\left(\hat{\mathcal{B}}_{n, t}\right)=\mathbb{P}\left(\tilde{\mathcal{B}}_{n, t}\right)+O\left(n^{-1}\right)$. Hence $\hat{\tau}^{(n)}(t)=(1+$ $o(1)) \tilde{\tau}^{(n)}(t)$, and the claim follows from the corresponding result for the layerwise bondpercolated model.
7.7.7. Proof of Theorem 3.6:(ii) for overlay bond percolation. The coupling construction in Section 7.7 .3 shows that all components in $\hat{G}^{(n)}$ are stochastically smaller than their counterparts in $\tilde{G}^{(n)}$. Hence the upper bounds concerning component sizes in $\hat{G}^{(n)}$ follow directly from the result of Theorem 3.6:(ii) for $\tilde{G}^{(n)}$. Therefore, we only need to prove that with high probability $\hat{G}^{(n)}$ contains a component of size $\left(1+o_{\mathbb{P}}(1)\right) \rho\left(\hat{f}^{+}\right) n$.

Let us investigate how Lemma 7.10 behaves when $G^{(n)}$ is replaced by $\hat{G}^{(n)}=G^{(n)} \cap H$ where $H$ is a homogeneous Bernoulli graph on $\{1, \ldots, n\}$ with link probability $\theta$. Define a modification of Algorithm 2 where the layer exploration step is replaced by $\mathcal{Z}_{t} \leftarrow \cup_{k \in \mathcal{W}_{t}} N_{v_{t}}\left(\hat{G}_{k}^{\prime}\right)$ where $\hat{G}_{k}^{\prime}$ is the transitive closure of $\hat{G}_{k}=G_{k} \cap H$. By construction, the modified version of Algorithm 2 discovers a subset of the $\hat{G}^{(n)}$-component of the root. Furthermore, the algorithm avoids multi-overlaps, and therefore the output of Algorithm 2 is the same as if it were run for the layerwise bond-percolated model $\tilde{G}^{(n)}$ with mutually independent layers $\tilde{G}_{k}=G_{k} \cap H_{k}$ as in (3.5). Hence Lemma 7.8 is valid for the overlay bond-percolated model, with the same lower bound as for the layerwise bond-percolated model. Hence the statements in (7.12)-(7.15) of Lemma 7.10 are valid just the same as for the layer-percolated model.

To finish extending Lemma 7.10 to the overlay bond-percolated graph, we still need to verify the sprinkling argument in the proof of the lower bound for (7.16). To do this, we modify the earlier argument slightly using a modified coupling. As in the earlier proof for the nonpercolated model, fix a small $\delta \in(0,1)$, partition the set of layers into red layers and blue layers, and denote by $G^{(r)}$ and $G^{(b)}$ the overlay graphs generated by the red and blue layers. Let $\theta^{(b)}=\delta$ and define $\theta^{(r)}=1-\frac{1-\theta}{1-\delta}$. Let $H^{(r)}, H^{(b)}$ be mutually independent homogeneous Bernoulli graphs on $[n]$ with link probabilities $\theta^{(b)}$ and $\theta^{(r)}$, respectively, sampled independently of the layers. Then $\tilde{G}=G \cap H$ with $G=G^{(r)} \cup G^{(b)}$ and $H=H^{(r)} \cup H^{(b)}$ is an instance of the bond-percolated overlay graph. For a lower bound, we note that $\tilde{G} \supset \tilde{G}^{(r)} \cup \tilde{G}^{(b)}$ where $\tilde{G}^{(r)}=G^{(r)} \cap H^{(r)}$ and $\tilde{G}^{(b)}=G^{(b)} \cap H^{(b)}$. Note that $\theta-\delta / 2 \leq \theta^{(r)} \leq \theta$ for $0<\delta \leq \frac{1}{2}$.

Let $B=B_{\omega}\left(\tilde{G}^{(r)}\right)$ be the set of nodes having $\tilde{G}^{(r)}$-component larger than $\omega=n^{2 / 3}$. Then by (7.12)-(7.15) of Lemma 7.10, it follows that $B \geq\left(\rho\left(\hat{f}^{+}\right)-\varepsilon\right) n$ with high probability, where $\varepsilon>0$ becomes arbitrarily small after choosing a small enough $\delta>0$. We claim that $B$ is $\tilde{G}$-connected with high probability for $\omega=n^{2 / 3}$. If $B$ is not $\tilde{G}$-connected, then there exist disjoint $\tilde{G}^{(r)}$-components $C^{\prime}, C^{\prime \prime}$ both of size at least $\omega$, between which there are no $\tilde{G}$-links and hence no $\tilde{G}^{(b)}$-links. Let us condition on the red layers and $H^{(r)}$. Given these, the blue layers and $H^{(b)}$ behave independently. Denote by $M_{b}$ the number of blue layers containing at least one link between $C^{\prime}$ and $C^{\prime \prime}$. Denote by $L_{b}=\left|E\left(G^{(b)}, C^{\prime}, C^{\prime \prime}\right)\right|$ (resp. $\left.\tilde{L}_{b}=\left|E\left(\tilde{G}^{(b)}, C^{\prime}, C^{\prime \prime}\right)\right|\right)$ the number of $G^{(b)}$-links (resp. $\tilde{G}^{(b)}$-links) between $C^{\prime}$ and $C^{\prime \prime}$. Let $s=\delta^{-1} \log n$ and $t=3 \delta^{-1} \log n$, and observe that

$$
\mathbb{P}\left(\tilde{L}_{b}=0 \mid L_{b} \geq s\right) \leq(1-\delta)^{s} \leq e^{-\delta s}=n^{-1} .
$$

Given $M_{b} \geq t$, we know that $L_{b} \geq_{\text {st }} N_{t}$ where $N_{t}$ is the number of distinct coupon types obtained after collecting $t$ random coupons from a collection of $n_{0}=\left|C^{\prime} \times C^{\prime \prime}\right|$ coupon types. By Lemma A.16, for large enough $n$ such that $1+s \leq \frac{1}{2} t$ and $t \leq n_{0}^{1 / 4}$,

$$
\mathbb{P}\left(L_{b}<s \mid M_{b} \geq t\right) \leq \mathbb{P}\left(N_{t}<s\right) \leq n_{0}^{-1} \leq \omega^{-2} .
$$

By applying (7.24) and noting that $t \leq\left(\frac{1}{2}-e^{-1}\right) c_{1} n^{1 / 3}$ for large $n$, we see that $\mathbb{P}\left(M_{b}<t\right) \leq$ $e^{t-\left(1-e^{-1}\right) c_{1} n^{1 / 3}} \leq e^{-c_{2} n^{1 / 3}}$ with $c_{1}=\frac{1}{2} \delta \mu M^{-2}(P)_{21}$ and $c_{2}=\frac{1}{4} \delta \mu M^{-2}(P)_{21}$. Now

$$
\begin{aligned}
\mathbb{P}\left(\tilde{L}_{b}=0\right) & \leq \mathbb{P}\left(\tilde{L}_{b}=0 \mid L_{b} \geq s\right)+\mathbb{P}\left(L_{b}<s\right) \\
& \leq \mathbb{P}\left(\tilde{L}_{b}=0 \mid L_{b} \geq s\right)+\mathbb{P}\left(L_{b}<s \mid M_{b} \geq t\right)+\mathbb{P}\left(M_{b}<t\right) \\
& \leq n^{-1}+\omega^{-2}+e^{-c_{2} n^{1 / 3}} \\
& \leq 3 n^{-1} .
\end{aligned}
$$

Now there are at most $\frac{n}{\omega}=n^{1 / 3}$ such components $C^{\prime}, C^{\prime \prime}$, and hence at most $\frac{1}{2} n^{2 / 3}$ such component pairs. Hence the probability that there exists a component pair $C^{\prime}, C^{\prime \prime}$ with no $\tilde{G}^{(b)}$-links in between, is at most $\frac{3}{2} n^{-1 / 3}$. We conclude that $B$ is $\tilde{G}$-connected with high probability. This confirms that the lower bound for (7.16) in Lemma 7.10 extends to the overlay bond-percolated setting.

All the rest in the proof of Theorem 3.4 extends to the overlay bond-percolated setting in a straightforward manner. This concludes the proof of Theorem 3.6:(ii).

## 8. Analysis of power-law models

8.1. Mixed binomial power laws. When the limiting layer type distribution factorises according to (4.1)-(4.2) and $\alpha+s \beta>r+1$, we find that the mixed binomial distribution in (2.5) can be written as

$$
\operatorname{Bin}_{r s}(P)(t)=\sum_{x=1}^{\infty} \operatorname{Bin}(x-r, q(x))(t) \tilde{p}_{r s}(x),
$$

where $\tilde{p}_{r s}(x)=\frac{(x)_{r} q(x)^{s} p(x)}{(P)_{r s}}$ is a biased layer size distribution. Assumptions (4.2) imply that the biased layer size distribution follows a power law $\tilde{p}_{r s}(x) \sim \frac{a b^{s}}{(P)_{r s}} x^{-(\alpha+s \beta-r)}$. If $\beta>0$ or $b<1$, then Lemma A. 4 shows that also the mixed binomial distribution follows a power law

$$
\begin{equation*}
\operatorname{Bin}_{r s}(P)(t) \sim d_{r s} t^{-\delta_{r s}} \tag{8.1}
\end{equation*}
$$

with parameters

$$
\begin{equation*}
\delta_{r s}=1+\frac{\alpha+s \beta-r-1}{1-\beta} \quad \text { and } \quad d_{r s}=\frac{a b^{s}}{(P)_{r s}} \frac{b^{\delta_{r s}-1}}{1-\beta} \tag{8.2}
\end{equation*}
$$

8.2. Proof of Theorem 4.1. The limiting degree distribution given by Theorem 3.1 equals $f=\operatorname{CPoi}\left(\mu(P)_{10}, g_{10}\right)$ with $g_{10}=\operatorname{Bin}_{10}(P)$.
(i) Assume first that $0 \leq \beta<1$ and that either $\beta>0$ or $b<1$. By (8.1), we find that $g_{10}(t) \sim d_{10} t^{-\delta_{10}}$. The above formula implies that $g_{10}$ is subexponential [22, Theorem 4.14] and it follows that [22, Theorem 4.30] $f(t) \sim \mu(P)_{10} g_{10}(t) \sim \mu(P)_{10} d_{10} t^{-\delta_{10}}$.
(ii) Consider the case with $\beta=0$ and $b=1$, and assume that $q(x)=1$ for all but finitely many $x$. Then $\operatorname{Bin}(x-1, q(x))=\delta_{x-1}$ for large values of $x$, and it follows that $g_{10}(t)=\tilde{p}_{10}(t+1)$ for all large $t$. Hence $g_{10}(t) \sim \tilde{p}_{10}(t)$, and the claim follows as in (i).
(iii) If $\beta \geq 1$, then $M=\sup _{x \geq 1}(x-1) q(x)<\infty$. The generating function of the limiting degree distribution equals $\sum_{t \geq 0} z^{t} f(t)=e^{\lambda\left(\hat{g}_{10}(z)-1\right)}$, where

$$
\hat{g}_{10}(z)=\sum_{x \geq 1}(1-q(x)+q(x) z)^{x-1} \tilde{p}_{10}(x) .
$$

Because $1-y+y z \leq e^{y(z-1)}$ for all real numbers $z$, it follows that $\hat{g}_{10}(z) \leq e^{M(z-1)}$ and hence $\sum_{t \geq 0} z^{t} f(t)$ is finite for all $z>0$.
8.3. Proof of Theorem 4.2. The limiting clustering spectrum $\sigma(t)$ in Theorem 3.3 is represented using convolutions of the limiting degree distribution $f=\operatorname{CPoi}\left(\mu(P)_{10}, g_{10}\right)$ and distributions $g_{r s}=\operatorname{Bin}_{r s}(P)$ defined by (2.5). Theory of discrete subexponential densities [22, Lemmas 4.9 and 4.14] implies that $\left(f_{1} * f_{2}\right)(t) \sim f_{1}(t)+f_{2}(t)$ for all probability densities on the positive integers such that $f_{i}(t) \sim a_{i} t^{-\alpha_{i}}$ with $a_{i}>0$ and $\alpha_{i}>1$. By Theorem 4.1, we know that $f(t) \sim \mu(P)_{10} d_{10} t^{-\delta_{10}}$, and by (8.1), we find that $g_{r s}(t) \sim d_{r s} t^{-\delta_{r s}}$ with parameters given by (8.2). Because $\delta_{32}<\delta_{21}<\delta_{10}$, it follows that

$$
\left(f * g_{32}\right)(t) \sim f(t)+g_{32}(t) \sim g_{32}(t)
$$

and

$$
\left(f * g_{21} * g_{21}\right)(t) \sim f(t)+g_{21}(t)+g_{21}(t) \ll g_{32}(t) .
$$

Hence by formula (3.2),

$$
\sigma(t) \sim \frac{(P)_{33}}{(P)_{32}} \frac{f(t)+g_{33}(t)}{g_{32}(t)} \sim \frac{(P)_{33}}{(P)_{32}} \frac{\mu(P)_{10} d_{10} t^{-\delta_{10}}+d_{33} t^{-\delta_{33}}}{d_{32} t^{\delta_{32}}} .
$$

Because $\delta_{33}-\delta_{10}=\frac{3 \beta-2}{1-\beta}$, we see that $\sigma(t)$ follows a power law with density exponent $\delta_{33}-\delta_{32}=\frac{\beta}{1-\beta}$ for $\beta \leq \frac{2}{3}$, and density exponent $\delta_{10}-\delta_{32}=2$ for $\beta \geq \frac{2}{3}$. The constant term of the power law is determined by (8.2).

## Appendix A. Supplementary results

A.1. Formal model definition. Fix integers $n, m \geq 1$. Let $p_{n, 1}, \ldots, p_{n, m}$ be probability measures on $\mathbb{Z}_{+} \times[0,1]$, and let $q_{n}$ be a probability kernel from $\mathbb{Z}_{+} \times[0,1]$ into $\mathcal{G}_{n}$ defined by $\left.\left.q_{n}((x, y), g)=\binom{n}{x}^{-1}(1-y)\right)^{n} \begin{array}{c}n \\ 2\end{array}\right)-|E(g)| ~ y^{|E(g)|}$. The space of possible layer type configurations $\theta_{n}=\left(\left(x_{1}, y_{1}\right), \ldots,\left(x_{m}, y_{m}\right)\right)$ is denoted by $\Omega_{1, n}=\left(\mathbb{Z}_{+} \times[0,1]\right)^{m}$, and the space of possible layer configurations $\xi_{n}=\left(g_{1}, \ldots, g_{m}\right)$ by $\Omega_{2, n}=\mathcal{G}_{n}^{m}$. Define a probability measure $\bar{p}_{n}$ on $\Omega_{1, n}$ and a probability kernel $\bar{q}_{n}$ from $\Omega_{1, n}$ to $\Omega_{2, n}$ by

$$
\bar{p}_{n}\left(d \theta_{n}\right)=\prod_{k=1}^{m} p_{n, k}\left(d x_{k}, d y_{k}\right), \quad \bar{q}_{n}\left(\theta_{n}, \xi_{n}\right)=\prod_{k=1}^{m} q_{n}\left(\left(x_{k}, y_{k}\right), g_{k}\right) .
$$

The joint probability distribution of layers and their types is a probability measure $\mathbb{P}_{n}=$ $\bar{p}_{n} \otimes \bar{q}_{n}$ on $\Omega_{n}=\Omega_{1, n} \times \Omega_{2, n}$. We denote by $\mathbb{P}_{\theta_{n}}(A)=\bar{q}_{n}\left(\theta_{n}, A\right)$ the regular conditional distribution of the layers given layer types $\theta_{n}$. The empirical layer type distribution is defined by $P_{\theta_{n}}=\frac{1}{m} \sum_{k=1}^{m} \delta_{x_{k}, y_{k}}$. The averaged layer type distribution is denoted by $P_{n}=$ $\frac{1}{m} \sum_{k=1}^{m} p_{n, k}$.

By defining $\mathbb{P}$ as the product measure on $\Omega=\Omega_{1} \times \Omega_{2} \times \cdots$ we may consider all models on all scales simultaneously on a common probability space. Then $\theta_{n}, \xi_{n}$, and the graphs $G_{n}=G_{n}\left(\theta_{n}, \xi_{n}\right)$ can be viewed as random variables on $\Omega$ defined using canonical coordinate projections $(\theta, \xi) \rightarrow \theta_{n},(\theta, \xi) \rightarrow \xi_{n}$, and the deterministic map $\xi_{n} \mapsto G_{n}\left(\xi_{n}\right)=$ $\left(\{1, \ldots, n\}, \cup_{k=1}^{m} E\left(g_{k}\right)\right)$.
Lemma A.1. Let $\Phi_{n}: \Omega_{n} \rightarrow \mathbb{R}$ measurable functions such that $\mathbb{P}_{\theta_{n}}\left(\left\{\xi_{n}:\left|\Phi_{n}\left(\theta_{n}, \xi_{n}\right)-c\right|>\right.\right.$ $\varepsilon\}) \rightarrow 0$ for all $\varepsilon>0$ and for all $\left(\theta_{1}, \theta_{2}, \ldots\right)$ such that $d_{\mathrm{tv}}\left(P_{\theta_{n}}, P\right) \rightarrow 0$. Assume that $d_{\mathrm{tv}}\left(P_{\theta_{n}}, P\right) \xrightarrow{\mathbb{P}} 0$. Then $\Phi_{n} \xrightarrow{\mathbb{P}} c$.
Proof. We will apply the result [30, Lemma 4.2] that $X_{n} \xrightarrow{\mathbb{P}} X$ if and only if for any subsequence of $\mathbb{N}$ there exists a further subsequence along which the convergence takes place $\mathbb{P}$-almost surely. Fix a subsequence $\mathbb{N}^{\prime} \subset \mathbb{N}$. Because $d_{\text {tv }}\left(P_{\theta_{n}}, P\right) \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$ along $\mathbb{N}^{\prime}$, there exists a further subsequence $\mathbb{N}^{\prime \prime} \subset \mathbb{N}$ such that $d_{\mathrm{tv}}\left(P_{\theta_{n}}, P\right) \rightarrow 0 \mathbb{P}$-almost surely along $\mathbb{N}^{\prime \prime}$. Then for any $\varepsilon>0$, the random variables $Z_{\varepsilon, n}=\mathbb{P}_{\theta_{n}}\left(\left\{\xi_{n}:\left|\Phi_{n}\left(\theta_{n}, \xi_{n}\right)-c\right|>\varepsilon\right\}\right)$ satisfy $Z_{\varepsilon, n} \rightarrow 0 \mathbb{P}$-almost surely along $\mathbb{N}^{\prime \prime}$. Dominated convergence then implies that
$\mathbb{P}\left(\left|\Phi_{n}-c\right|>\varepsilon\right)=\mathbb{E} Z_{\varepsilon, n} \rightarrow 0$ along $\mathbb{N}^{\prime \prime}$. Then there exists a further subsequence $\mathbb{N}^{\prime \prime \prime}$ such that $\Phi_{n} \rightarrow c \mathbb{P}$-almost surely along $\mathbb{N}^{\prime \prime \prime}$.

Lemma A.2. Assume that $P_{n}, n \geq 1$, and $P$ are supported on a finite set $A \subset \mathbb{Z}_{+} \times[0,1]$, and that $P_{n} \xrightarrow{w} P$. Then $d_{\mathrm{tv}}\left(P_{\theta_{n}}, P\right) \xrightarrow{\mathbb{P}} 0$.
Proof. Now $\mathbb{E} P_{\theta_{n}}(x, y)=P_{n}(x, y) \rightarrow P(x, y)$ for all $(x, y) \in A$. Because the layer types are independent, $\operatorname{Var} P_{\theta_{n}}(x, y)=\frac{1}{m^{2}} \sum_{k=1}^{m} \operatorname{Var} 1\left(X_{n, k}=x, Y_{n, k}=y\right) \leq m^{-1}$. Hence by Chebyshev's inequality, $P_{\theta_{n}}(x, y) \xrightarrow{\mathbb{P}} P(x, y)$ for all $(x, y) \in A$, and the claim follows.

## A.2. Elementary analysis.

Lemma A.3. Fix integers $a<b$ and let $f:[a, b] \rightarrow[0, \infty)$ be unimodular in the sense that there exists $s^{*} \in[a, b]$ such that $f$ is nondecreasing on $\left[a, s^{*}\right]$ and nonincreasing on $\left[s^{*}, b\right]$. Then $\left|\sum_{k=a}^{b} f(k)-\int_{a}^{b} f(s) d s\right| \leq\|f\|_{\infty}$.

Proof. Let us abbreviate $\int_{a}^{b} f=\int_{a}^{b} f(s) d s$ and $\sum_{a}^{b} f=\sum_{a \leq k \leq b} f(k)$. Denote $r_{1}=\left\lfloor s^{*}\right\rfloor$ and $r_{2}=\left\lceil s^{*}\right\rceil$. Then by writing

$$
\sum_{k=a}^{r_{1}-1} f(k)=\int_{a}^{r_{1}} f(\lfloor s\rfloor) d s \quad \text { and } \quad \sum_{k=r_{2}+1}^{b} f(k)=\int_{r_{2}}^{b} f(\lceil s\rceil) d s
$$

we find that $\sum_{a}^{r_{1}-1} f \leq \int_{a}^{r_{1}} f$ and $\sum_{r_{2}+1}^{b} f \leq \int_{r_{2}}^{b} f$. If $r_{1}=r_{2}=s^{*}$, then $f\left(r_{1}\right)=f\left(r_{2}\right)=$ $\|f\|_{\infty}$, and we see that $\sum_{a}^{b} f \leq \int_{a}^{b} f+\|f\|_{\infty}$. If $r_{1}=r_{2}-1$, then $f\left(r_{1}\right) \wedge f\left(r_{2}\right) \leq f(s)$ for $s \in\left[r_{1}, r_{2}\right]$ implies that

$$
f\left(r_{1}\right)+f\left(r_{2}\right)=f\left(r_{1}\right) \wedge f\left(r_{2}\right)+f\left(r_{1}\right) \vee f\left(r_{2}\right) \leq \int_{r_{1}}^{r_{2}} f+\|f\|_{\infty}
$$

and hence $\sum_{a}^{b} f \leq \int_{a}^{b} f+\|f\|_{\infty}$ also in this case.
To obtain a lower bound, a similar reasoning shows that $\int_{a}^{r_{1}} f \leq \sum_{a+1}^{r_{1}} f$ and $\int_{r_{2}}^{b} f \leq$ $\sum_{r_{2}}^{b-1} f$. Together with the fact that $\int_{r_{1}}^{r_{2}} f \leq\|f\|_{\infty} 1\left(r_{1}<r_{2}\right)$, it follows that $\int_{a}^{b} f \leq$ $\sum_{a+1}^{r_{1}} f+\sum_{r_{2}}^{b-1} f+\|f\|_{\infty} 1\left(r_{1}<r_{2}\right)$. Now, because $\sum_{a+1}^{r_{1}} f+\sum_{r_{2}}^{b-1} f=\sum_{a+1}^{b-1} f+\|f\|_{\infty} 1\left(r_{1}=\right.$ $\left.r_{2}\right) \leq \sum_{a}^{b} f+\|f\|_{\infty} 1\left(r_{1}=r_{2}\right)$, it follows that $\int_{a}^{b} f \leq \sum_{a}^{b} f+\|f\|_{\infty}$.
A.3. Power laws. The following result characterises conditions under which a mixed binomial distribution follows a power law.

Lemma A.4. Consider a mixed binomial distribution $g(r)=\sum_{k \geq 1} p_{k} f_{k}(r)$ where $f_{k}=$ $\operatorname{Bin}\left(x_{k}, y_{k}\right)$ and $\left(p_{k}\right)$ is a probability distribution on $\{1,2, \ldots\}$. Assume that

$$
x_{k}=\left(a+O\left(k^{-\alpha / 2}\right)\right) k^{\alpha}, \quad y_{k}=\left(b+O\left(k^{-\alpha / 2}\right)\right) k^{-\beta}, \quad p_{k}=(c+o(1)) k^{-\gamma},
$$

for some $0 \leq \beta<\alpha<\beta+2$ and $\gamma>1$, and some $a, b, c>0$ such that $\beta>0$ or $b<1$. Then

$$
g(r)=(d+o(1)) r^{-\delta}
$$

where $\delta=1+\frac{\gamma-1}{\alpha-\beta}$ and $d=(a b)^{\delta-1} c /(\alpha-\beta)$.
Proof. Denote the mean and variance of $f_{k}$ by $\mu_{k}=x_{k} y_{k}$ and $\sigma_{k}^{2}=x_{k} y_{k}\left(1-y_{k}\right)$. Denote $x_{k}=\left(1+\varepsilon_{1, k}\right) a k^{\alpha}, y_{k}=\left(1+\varepsilon_{2, k}\right) b k^{-\beta}$, and define $\varepsilon_{k}$ by the formula $1+\varepsilon_{k}=\left(1+\varepsilon_{1, k}\right)\left(1+\varepsilon_{2, k}\right)$. Then $\varepsilon_{k}=O\left(k^{-\alpha / 2}\right)$, and we may fix constants $k_{0}, M>0$ such that $\left|\varepsilon_{k}\right| \leq M k^{-\alpha / 2} \leq \frac{1}{4}$ for all $k \geq k_{0}$. Then

$$
\mu_{k}=\left(1+\varepsilon_{k}\right) a b k^{\rho}
$$

where $\rho=\alpha-\beta$. Define

$$
A_{r}=\left\{k \in \mathbb{N}:\left|a b k^{\rho}-r\right| \leq \Delta_{r}\right\}
$$

where $\Delta_{r}=r^{1 / 2} \log r$. Let us choose $r_{0}$ large enough so that $\max _{k<k_{0}} x_{k}<r_{0}$ and $4 M\left(\frac{5 a}{4}\right)^{1 / 2} r^{1 / 2} \leq$ $\Delta_{r} \leq \frac{1}{2} r$ for all $r \geq r_{0}$.
(i) We will first verify that for all $r \geq r_{0}$,

$$
\begin{equation*}
\sum_{k \notin A_{r}} f_{k}(r) p_{k}=\sum_{k: k \geq k_{0}: x_{k} \geq r, k \notin A_{r}} f_{k}(r) p_{k} \leq e^{-\frac{\Delta_{r}^{2}}{10 r}} . \tag{A.1}
\end{equation*}
$$

Because $f_{k}(r)=0$ for $x_{k}<r$, we observe that only indices $k$ with $k \geq k_{0}$ and $x_{k} \geq r$ appear in the sum $g(r)=\sum_{k: x_{k} \geq r} p_{k} f_{k}(r)$ when $r \geq r_{0}$. This confirms the equality in (A.1). For such $k, r \leq x_{k}$ and $x_{k} \leq\left(1+\frac{1}{4}\right) a k^{\alpha}$ imply $k \geq\left(\frac{4}{5 a}\right)^{1 / \alpha} r^{1 / \alpha}$, and this further shows that $\left|\varepsilon_{k}\right| \leq M k^{-\alpha / 2} \leq M\left(\frac{5 a}{4}\right)^{1 / 2} r^{-1 / 2}$, so that $\left|\varepsilon_{k}\right| r \leq \frac{1}{4} \Delta_{r}$. Then by writing

$$
\mu_{k}-r=\left(1+\varepsilon_{k}\right)\left(a b k^{\rho}-r\right)+\varepsilon_{k} r,
$$

we find that when $r \geq r_{0},\left|\mu_{k}-r\right| \geq\left(1-\left|\varepsilon_{k}\right|\right) \Delta_{r}-\left|\varepsilon_{k}\right| r \geq \frac{1}{2} \Delta_{r}$ for all $k$ such that $x_{k} \geq r$ and $k \notin A_{r}$. For such values of $k$, Chernoff inequalities for the binomial distribution (Lemma A.7) imply (using $\Delta_{r} \leq \frac{1}{2} r$ ) that

$$
f_{k}(r) \leq e^{-\frac{\Delta_{r}^{2}}{8\left(r+\frac{1}{2} \Delta_{r}\right)}} \leq e^{-\frac{\Delta_{r}^{2}}{1 r}} .
$$

(ii) For $r \geq r_{0}$ and for values $k \in A_{r}$, we have $\frac{1}{2} r \leq a b k^{\rho} \leq 2 r$ due to $\Delta_{r} \leq \frac{1}{2} r$, and hence $c_{0} r^{1 / \rho} \leq k \leq c_{0}^{\prime} r^{1 / \rho}$, where $c_{0}=(2 a b)^{-1 / \rho}$ and $c_{0}^{\prime}=(a b / 2)^{-1 / \rho}$. Then let

$$
\varepsilon_{r}^{\prime}=\max _{k \geq c_{0} r^{1 / \rho}}\left|\varepsilon_{k}\right| .
$$

Then $\varepsilon_{r}^{\prime}$ is decreasing and nonnegative. Now $\left|\varepsilon_{k}\right| \leq M k^{-\alpha / 2} \leq c_{0}^{-\alpha / 2} M r^{-\alpha /(2 \rho)}$ for $k \geq k_{0}$ and $k \geq c_{0} r^{1 / \rho}$. Hence $\varepsilon_{r}^{\prime}=O\left(r^{-\alpha /(2 \rho)}\right)$. Now it follows that the mean of $f_{k}$ is approximated by

$$
\mu_{k}=\left(1+O\left(r^{-1} \Delta_{r}\right)+O\left(\varepsilon_{r}^{\prime}\right)\right) r
$$

uniformly for $k \in A_{r}$. Next, we note that $y_{k}=\Theta\left(r^{-\beta / \rho}\right)$ for $\beta>0$, and $y_{k}=b+O\left(k^{-\alpha / 2}\right)=$ $b+O\left(r^{-\alpha /(2 \rho)}\right)$ for $\beta=0$, uniformly for $k \in A_{r}$. It follows that, denoting $\beta^{\prime}=\beta$ for $\beta>0$ and $\beta^{\prime}=\alpha / 2$ for $\beta=0$,

$$
\sigma_{k}^{2}=\left(1+O\left(r^{-1} \Delta_{r}\right)+O\left(r^{-\beta^{\prime} / \rho}\right)+O\left(\varepsilon_{r}^{\prime}\right)\right) \sigma_{0}^{2} r
$$

where $\sigma_{0}^{2}=1-b$ for $\beta=0$ and $\sigma_{0}^{2}=1$ for $\beta>0$. Also,

$$
k^{-\gamma}=(a b)^{\gamma / \rho}\left(a b k^{\rho}\right)^{-\gamma / \rho}=\left(1+O\left(r^{-1} \Delta_{r}\right)\right)(a b)^{\gamma / \rho} r^{-\gamma / \rho} .
$$

Hence,

$$
\begin{equation*}
p_{k}=(1+o(1)) c_{1} r^{-\gamma / \rho} . \tag{A.2}
\end{equation*}
$$

for $c_{1}=(a b)^{\gamma / \rho} c$, uniformly for $k \in A_{r}$.
(iii) We will next approximate the binomial density $f_{k}$ by a normal density with the same mean and variance. By a local limit theorem [53, Lemma 5] (see also [10, 40]),

$$
\left|f_{k}(r)-\frac{1}{\sigma_{k}} \phi\left(\frac{r-\mu_{k}}{\sigma_{k}}\right)\right| \leq 0.516 \sigma_{k}^{-2}
$$

for all $0 \leq r \leq k-1$ and all $k \geq 2$, where $\phi(s)=(2 \pi)^{-1 / 2} e^{-s^{2} / 2}$ is the standard normal density. Hence

$$
\begin{equation*}
f_{k}(r)=\frac{1}{\sigma_{k}} \phi\left(\frac{r-\mu_{k}}{\sigma_{k}}\right)+O\left(r^{-1}\right) \tag{A.3}
\end{equation*}
$$

uniformly for $k \in A_{r}$.
(iv) We will approximate the parameters of the normal density in (A.3) by $\mu_{k} \approx a b k^{\rho}$ and $\sigma_{k} \approx \sigma_{0} r^{1 / 2}$. To see that these approximations hold uniformly, denote $s_{k, r}=\frac{\mu_{k}-r}{\sigma_{k}}$ and $t_{k, r}=\frac{a b k^{\rho}-r}{\sigma_{0} r^{1 / 2}}$. Note that

$$
s_{k, r}=\sigma_{k}^{-1}\left(1+O\left(\varepsilon_{r}^{\prime}\right)\right)\left(a b k^{\rho}-r\right)
$$

and

$$
\begin{equation*}
\sigma_{k}^{-1}=\left(1+O\left(r^{-1} \Delta_{r}\right)+O\left(r^{-\beta^{\prime} / \rho}\right)+O\left(\varepsilon_{r}^{\prime}\right)\right) \sigma_{0}^{-1} r^{-1 / 2} \tag{A.4}
\end{equation*}
$$

Hence

$$
s_{k, r}=\left(1+O\left(r^{-1} \Delta_{r}\right)+O\left(r^{-\beta^{\prime} / \rho}\right)+O\left(\varepsilon_{r}^{\prime}\right)\right) t_{k, r}
$$

Note that $s^{2}-t^{2}=(2+u) u t^{2}$ for $s=(1+u) t$. By applying this formula with $u$ being the above approximation error, using $\left|t_{k, r}\right|=O\left(r^{-1 / 2} \Delta_{r}\right)$, we find that

$$
\begin{aligned}
s_{k, r}^{2}-t_{k, r}^{2} & =\left(O\left(r^{-1} \Delta_{r}\right)+O\left(r^{-\beta^{\prime} / \rho}\right)+O\left(\varepsilon_{r}^{\prime}\right)\right) O\left(t_{k, r}^{2}\right) \\
& =O\left(r^{-2} \Delta_{r}^{3}\right)+O\left(r^{-1-\beta^{\prime} / \rho} \Delta_{r}^{2}\right)+O\left(\varepsilon_{r}^{\prime} r^{-1} \Delta_{r}^{2}\right)
\end{aligned}
$$

uniformly for $k \in A_{r}$. Our choice of $\Delta_{r}=r^{1 / 2} \log r$ implies that $s_{k, r}^{2}-t_{k, r}^{2}=o(1)$ uniformly with respect to $k \in A_{r}$. Then $\left|e^{t}-1\right| \leq e|t|$ for $|t| \leq 1$ implies

$$
\frac{\phi\left(s_{k, r}\right)}{\phi\left(t_{k, r}\right)}=e^{\frac{1}{2}\left(t_{k, r}^{2}-s_{k, r}^{2}\right)}=1+O\left(\left|t_{k, r}^{2}-s_{k, r}^{2}\right|\right)=1+o(1)
$$

and

$$
\phi\left(\frac{\mu_{k}-r}{\sigma_{k}}\right)=(1+o(1)) \phi\left(\frac{a b k^{\rho}-r}{\sigma_{0} r^{1 / 2}}\right)
$$

uniformly for $k \in A_{r}$. Together with (A.3) and (A.4), it follows that

$$
\begin{equation*}
f_{k}(r)=(1+o(1)) \frac{1}{\sigma_{0} r^{1 / 2}} \phi\left(\frac{a b k^{\rho}-r}{\sigma_{0} r^{1 / 2}}\right)+O\left(r^{-1}\right) \tag{A.5}
\end{equation*}
$$

uniformly for $k \in A_{r}$.
(v) By Lemma A.3, it follows that

$$
\sum_{k \in A_{r}} \frac{1}{\sigma_{0} r^{1 / 2}} \phi\left(\frac{a b k^{\rho}-r}{\sigma_{0} r^{1 / 2}}\right)=\int_{A_{r}} \frac{1}{\sigma_{0} r^{1 / 2}} \phi\left(\frac{a b s^{\rho}-r}{\sigma_{0} r^{1 / 2}}\right) d s+O\left(r^{-1 / 2}\right)
$$

By a change of variables $s=\nu(t)$ with $\nu(t)=(t / a b)^{1 / \rho}$, we find that

$$
\begin{aligned}
\int_{A_{r}} \frac{1}{\sigma_{0} r^{1 / 2}} \phi\left(\frac{a b s^{\rho}-r}{\sigma_{0} r^{1 / 2}}\right) d s & =\int_{r-\Delta_{r}}^{r+\Delta_{r}} \frac{1}{\sigma_{0} r^{1 / 2}} \phi\left(\frac{t-r}{\sigma_{0} r^{1 / 2}}\right) \nu^{\prime}(t) d t \\
& =\mathbb{E}\left(\nu^{\prime}\left(r+\sigma_{0} r^{1 / 2} Z\right) 1\left(\sigma_{0} r^{1 / 2}|Z| \leq \Delta_{r}\right)\right)
\end{aligned}
$$

where $\mathcal{L}(Z)$ is standard normal. Because $\nu^{\prime}(r)=c_{2} r^{1 / \rho-1}$ with $c_{2}=\rho^{-1}(a b)^{-1 / \rho}$, we see that $\nu^{\prime}\left(r+\sigma_{0} r^{1 / 2} z\right)=(1+o(1)) \nu^{\prime}(r)$ uniformly for $|z| \leq \sigma_{0}^{-1} r^{-1 / 2} \Delta_{r}$. Hence it follows by Lebesgue's dominated convergence that

$$
\int_{A_{r}} \frac{1}{\sigma_{0} r^{1 / 2}} \phi\left(\frac{a b s^{\rho}-r}{\sigma_{0} r^{1 / 2}}\right) d s=(1+o(1)) \nu^{\prime}(r)=\left(c_{2}+o(1)\right) r^{1 / \rho-1}
$$

Because $r^{-1 / 2} \ll r^{1 / \rho-1}$ due to $\rho<2$, it follows that

$$
\begin{equation*}
\sum_{k \in A_{r}} \frac{1}{\sigma_{0} r^{1 / 2}} \phi\left(\frac{a b k^{\rho}-r}{\sigma_{0} r^{1 / 2}}\right) \sim c_{2} r^{1 / \rho-1} \tag{A.6}
\end{equation*}
$$

A similar computation also shows that

$$
\begin{equation*}
\left|A_{r}\right|=\int_{r-\Delta_{r}}^{r+\Delta_{r}} \nu^{\prime}(t) d t \sim 2 \Delta_{r} r^{1 / \rho-1} \tag{A.7}
\end{equation*}
$$

(vi) By combining (A.2), (A.5), (A.6), and (A.7) we now conclude that

$$
\begin{aligned}
\sum_{k \in A_{r}} f_{k}(r) p_{k} & \sim c_{1} r^{-\gamma / \rho} \sum_{k \in A_{r}} f_{k}(r) \\
& \sim c_{1} r^{-\gamma / \rho} \sum_{k \in A_{r}}\left(\frac{1}{\sigma_{0} r^{1 / 2}} \phi\left(\frac{a b k^{\gamma}-r}{\sigma_{0} r^{1 / 2}}\right)+O\left(r^{-1}\right)\right) \\
& \sim c_{1} r^{-\gamma / \rho} c_{2} r^{1 / \rho-1} .
\end{aligned}
$$

Together with (A.1), this now implies the claim, because $e^{-\frac{\Delta_{r}^{2}}{10 r}} \ll r^{-\delta}$ for $\delta=1+\frac{\gamma-1}{\alpha-\beta}$.
A.4. Compound Poisson and binomial distributions. Recall that $\operatorname{CPoi}(\lambda, f)$ denotes the compound Poisson distribution with rate parameter $\lambda$ and increment distribution $f$. The following three elementary results are included for ease of reference, although they are rather immediately available in the literature (e.g. [6, 29]).

Lemma A.5. Let $X=\sum_{i} X_{i}$ be a sum of independent random variables such that $\mathcal{L}\left(X_{i}\right)=$ $\operatorname{CPoi}\left(\lambda_{i}, g_{i}\right)$ with $0<\sum_{i} \lambda_{i}<\infty$. Then $\mathcal{L}(X)=\operatorname{CPoi}(\lambda, g)$ with $\lambda=\sum_{i} \lambda_{i}$ and $g=\sum_{i} \frac{\lambda_{i}}{\lambda} g_{i}$.

Proof. The probability generating function of a compound Poisson distribution $\operatorname{CPoi}\left(\lambda_{i}, g_{i}\right)$ equals $\exp \left(\lambda_{i}\left(G_{g_{i}}(z)-1\right)\right.$. Hence the probability generating function of $\sum_{i} X_{i}$ equals

$$
G_{X}(z)=\prod_{i} G_{X_{i}}(z)=\exp \left(\sum_{i} \lambda_{i}\left(G_{g_{i}}(z)-1\right)\right)=\exp \left(\lambda\left(G_{g}(z)-1\right)\right),
$$

where $G_{g}(z)$ is the probability generating function of $g=\sum_{i} \frac{\lambda_{i}}{\lambda} g_{i}$.
Lemma A.6. For any $\lambda, \lambda^{\prime} \geq 0$ and any probability measures $f, f^{\prime}$ on $\mathbb{R}$,

$$
d_{\mathrm{tv}}\left(\operatorname{CPoi}(\lambda, f), \operatorname{CPoi}\left(\lambda^{\prime}, f^{\prime}\right)\right) \leq \min \left\{\lambda, \lambda^{\prime}\right\} d_{\mathrm{tv}}\left(f, f^{\prime}\right)+\left|\lambda-\lambda^{\prime}\right| .
$$

Proof. By symmetry, we may assume that $\lambda \leq \lambda^{\prime}$. Denote $g=\operatorname{CPoi}(\lambda, f), g^{\prime}=\operatorname{CPoi}\left(\lambda, f^{\prime}\right)$, and $g^{\prime \prime}=\operatorname{CPoi}\left(\lambda^{\prime}, f^{\prime}\right)$. By triangle inequality, it suffices to verify that $d_{\mathrm{tv}}\left(g, g^{\prime}\right) \leq \lambda d_{\mathrm{tv}}\left(f, f^{\prime}\right)$ and $d_{\mathrm{tv}}\left(g^{\prime}, g^{\prime \prime}\right) \leq \lambda^{\prime}-\lambda$.
(i) Let $\left(X, X^{\prime}\right)$ a coupling of $f$ and $f^{\prime}$ which is optimal in the sense that $\mathbb{P}\left(X \neq X^{\prime}\right)=$ $d_{\mathrm{tv}}\left(f, f^{\prime}\right)$. Define a coupling of $g$ and $g^{\prime}$ by

$$
Y=\sum_{j=1}^{\Lambda} X_{j} \quad \text { and } \quad Y^{\prime}=\sum_{j=1}^{\Lambda} X_{j}^{\prime},
$$

where $\Lambda,\left(X_{1}, X_{1}^{\prime}\right),\left(X_{2}, X_{2}^{\prime}\right), \ldots$ are mutually independent random variables such that $\mathcal{L}(\Lambda)=$ $\operatorname{Poi}(\lambda)$ and $\mathcal{L}\left(X_{j}, X_{j}^{\prime}\right)=\mathcal{L}\left(X, X^{\prime}\right)$ for all $j$. Then by the union bound, we see that

$$
\mathbb{P}\left(Y \neq Y^{\prime} \mid \Lambda=\ell\right)=\mathbb{P}\left(\sum_{j=1}^{\ell} X_{j} \neq \sum_{j=1}^{\ell} X_{j}^{\prime}\right) \leq \ell \mathbb{P}\left(X \neq X^{\prime}\right) .
$$

By summing both sides weighted by $\mathbb{P}(\Lambda=\ell)$, it follows that $\mathbb{P}\left(Y \neq Y^{\prime}\right) \leq \mathbb{E}(\Lambda) \mathbb{P}(X \neq$ $X^{\prime}$ ) and hence $d_{\mathrm{tv}}\left(g, g^{\prime}\right) \leq \lambda d_{\mathrm{tv}}\left(f, f^{\prime}\right)$.
(ii) Let $Y^{\prime}$ and $\Delta$ be independent random numbers such that $\mathcal{L}\left(Y^{\prime}\right)=\operatorname{CPoi}\left(\lambda, f^{\prime}\right)$ and $\mathcal{L}(\Delta)=\operatorname{CPoi}\left(\delta, f^{\prime}\right)$ with $\delta=\lambda^{\prime}-\lambda$. Define $Y^{\prime \prime}=Y^{\prime}+\Delta$ and note by Lemma A. 5 that $\mathcal{L}\left(Y^{\prime \prime}\right)=\operatorname{CPoi}\left(\lambda^{\prime}, f^{\prime}\right)$. Hence

$$
d_{\mathrm{tv}}\left(g^{\prime}, g^{\prime \prime}\right) \leq \mathbb{P}\left(Y^{\prime} \neq Y^{\prime \prime}\right)=\mathbb{P}(\Delta \neq 0) \leq 1-e^{-\delta} \leq \delta=\lambda^{\prime}-\lambda .
$$

Lemma A.7. If $X$ is $\operatorname{Bin}(n, p)$-distributed with mean $\mu=n p$, then (i) $\mathbb{P}(X>a) \leq e^{2 \mu-a}$ for all $a \geq 0$, (ii) $\mathbb{P}(X \leq a) \leq e^{-\mu / 8}$ for any $a \leq \frac{1}{2} \mu$, and (iii) $\mathbb{P}(X=r) \leq e^{-\frac{s^{2}}{2(r+s)}}$ for any $s>0$ and for all integers $r$ such that $|r-\mu| \geq s$.

Proof. (i) Because $\mathbb{E} e^{X}=(1+p(e-1))^{n} \leq e^{(e-1) \mu} \leq e^{2 \mu}$, Markov's inequality implies that $\mathbb{P}(X>a)=\mathbb{P}\left(e^{X}>e^{a}\right) \leq e^{-a} \mathbb{E} e^{X} \leq e^{2 \mu-a}$.
(ii) Because $(\mu-a)^{2} \geq \frac{1}{4} \mu^{2}$, it follows by [29, Theorem 2.1] that $\mathbb{P}(X \leq a) \leq e^{-(\mu-a)^{2} /(2 \mu)} \leq$ $e^{-\mu / 8}$.
(iii) The approximation $\mathbb{P}(X=r) \leq \min \{\mathbb{P}(X \leq r), \mathbb{P}(X \geq r)\}$ combined with suitable Chernoff bounds [29, Theorem 2.1] will do the job, as shown below. Fix an integer $r \geq 0$ and consider the following two cases:
(a) If $r \leq \mu-s$. Then the bound $\mathbb{P}(X \leq \mu-t) \leq e^{-\frac{t^{2}}{2 \mu}}$ for $t=\mu-r$, together with the fact that $t \mapsto \frac{(t-r)^{2}}{2 t}$ is increasing on $(r, \infty)$, implies that

$$
\mathbb{P}(X \leq r)=\mathbb{P}(X \leq \mu-(\mu-r)) \leq \exp \left(-\frac{(\mu-r)^{2}}{2 \mu}\right) \leq \exp \left(-\frac{s^{2}}{2(r+s)}\right)
$$

(b) If $r \geq \mu+s$. Then the bound $\mathbb{P}(X \geq \mu+t) \leq e^{-\frac{t^{2}}{2(\mu+t / 3)}}$ for $t=s$, and the fact that $\mu+s / 3 \leq r \leq r+s$ imply that

$$
\mathbb{P}(X \geq r)=\mathbb{P}(X \geq \mu+s) \leq \exp \left(-\frac{s^{2}}{2(\mu+s / 3)}\right) \leq \exp \left(-\frac{s^{2}}{2(r+s)}\right)
$$

A.5. Biased and truncated probability measures. Below $P(\psi)=\int \psi(x) P(d x)$ is used as a shorthand for integrals. When $P(\psi) \in(0, \infty)$, we denote by $P^{\psi}=\frac{\psi(x) P(d x)}{P(\psi)}=\frac{\psi d P}{P(\psi)}$ the $\psi$-biased probability measure $P^{\psi}(A)=\frac{\int_{A} \psi(x) P(d x)}{P(\psi)}$. For a probability measure $P$ and a probability kernel $K$ we denote by $P K$ the probability measure $P K(A)=\int K(x, A) P(d x)$. For a function $\phi$, we define a function $K \phi$ by $K \phi(x)=\int \phi(y) K(x, d y)$. The following three results are proved for ease of reference, although they are rather immediate consequences of standard Wasserstein-type estimates of probability kernels (e.g. [38, 41]).
Lemma A.8. Let $P_{n}, P$ be probability measures on a separable metric space such that $P_{n} \xrightarrow{w}$ $P$ and $P_{n}(\psi) \rightarrow P(\psi) \in(0, \infty)$ for some continuous function $\psi \geq 0$. Then $P_{n}^{\psi} \xrightarrow{w} P^{\psi}$.

Proof. By Skorohod coupling [30, Proposition 4.30] there exist random variables $X_{n}, X$ such that $\mathcal{L}\left(X_{n}\right)=P_{n}, \mathcal{L}(X)=P$, and $X_{n} \rightarrow X$ almost surely. Let $\phi$ be a bounded and continuous. Then $Y_{n}=\phi\left(X_{n}\right) \psi\left(X_{n}\right)$ converges almost surely to $Y=\phi(X) \psi(X)$, and $\left|Y_{n}\right| \leq$ $\|\phi\|_{\infty} \psi\left(X_{n}\right)$ almost surely for all $n$. Because $\mathbb{E} \phi\left(X_{n}\right) \rightarrow \mathbb{E} \phi(X)<\infty$, Lebesgue's dominated convergence theorem (as stated in [30, Theorem 1.21]) implies that $P_{n}(\phi \psi)=\mathbb{E} Y_{n} \rightarrow \mathbb{E} Y=$ $P(\phi \psi)$. Hence $P_{n}^{\psi}(\phi)=\frac{P_{n}(\phi \psi)}{P_{n}(\psi)} \rightarrow \frac{P(\phi \psi)}{P(\psi)}=P^{\psi}(\phi)$.

Lemma A.9. Let $P_{n}, P$ be probability measures on $\mathbb{Z}_{+} \times[0,1]$, and let $K$ be a probability kernel from $\mathbb{Z}_{+} \times[0,1]$ into $\mathbb{Z}_{+}$such that $y \mapsto K((x, y), t)$ is continuous for every $x, t \in \mathbb{Z}_{+}$. If $P_{n} \xrightarrow{w} P$, then $P_{n} K \xrightarrow{w} P K$.

Proof. Let $\phi: \mathbb{Z}_{+} \rightarrow \mathbb{R}$ be bounded. Assume that $\left(x_{n}, y_{n}\right) \rightarrow(x, y)$. Then the probability measures on $\mathbb{Z}_{+}$defined by $Q_{n}(A)=K\left(\left(x_{n}, y_{n}\right), A\right)$ and $Q(A)=K((x, y), A)$ converge according to $Q_{n}(\{t\}) \rightarrow Q(\{t\})$ for all $t \in \mathbb{Z}_{+}$, and hence also weakly. Hence the function $K \phi$ defined by $K \phi(x, y)=\sum_{t} K((x, y), t) \phi(t)$ is bounded and continuous. Now $P_{n} \xrightarrow{w} P$ implies that $P_{n} K(\phi)=P_{n}(K \phi) \rightarrow P(K \phi)=P K(\phi)$. Hence $P_{n} K \xrightarrow{w} P K$.

Lemma A.10. If $P_{n} \xrightarrow{w} P$ and $\left(P_{n}\right)_{r s} \rightarrow(P)_{r s} \in(0, \infty)$, then the laws in (2.5)-(2.6) satisfy $\operatorname{Bin}_{r s}\left(P_{n}\right) \xrightarrow{w} \operatorname{Bin}_{r s}(P)$ and $\operatorname{Bin}_{r s}^{+}\left(P_{n}\right) \xrightarrow{w} \operatorname{Bin}_{r s}^{+}(P)$.

Proof. Define $\psi$-biased probability measures $P_{n}^{\psi}, P^{\psi}$ using $\psi(x, y)=(x)_{r} y^{s}$. Then $P_{n}^{*} \xrightarrow{w}$ $P^{*}$ by Lemma A.8. Observe next that the kernels $K((x, y), t)=\operatorname{Bin}(x-r, y)(t)$ and $K^{+}((x, y), t)=\operatorname{Bin}^{+}(x-r, y)(t)$ are continuous in $y$ (being polynomials of finite order). The claims now follow by Lemma A. 9 because $\operatorname{Bin}_{r s}\left(P_{n}\right)=P_{n}^{\psi} K$ and $\operatorname{Bin}_{r s}^{+}\left(P_{n}\right)=P_{n}^{\psi} K^{+}$.
A.6. Graph components. Denote by $N_{1}(G) \geq N_{2}(G)$ the largest two component sizes in $G$ (with $N_{2}(G)=0$ if $G$ is connected.) Let $B_{t}(G)=\left\{i \in V(G):\left|C_{i}(G)\right|>t\right\}$ be the set of nodes with component larger than $t$.

Lemma A.11. For all $t \geq 0$ : (i) $N_{1}(G) \leq \max \left\{\left|B_{t}(G)\right|, t\right\}$ and (ii) $N_{1}(G)+N_{2}(G) \leq$ $\left|B_{t}(G)\right|+2 t$.

Proof. (i) Let $C_{1}$ be a component of $G$ of size $\left|C_{1}\right|=N_{1}(G)$. If $\left|C_{1}\right| \leq t$, there is nothing to prove. If $\left|C_{1}\right|>t$, then every node in $C_{1}$ has component larger than $t$, and hence $C_{1} \subset B_{t}(G)$ implies $\left|N_{1}(G)\right| \leq\left|B_{t}(G)\right|$.
(ii) If $N_{2}(G) \leq N_{1}(G) \leq t$, the claim is clear. If $N_{2}(G) \leq t<N_{1}(G)$, the claim follows from (i). Assume now that $t<N_{2}(G) \leq N_{1}(G)$, and let $C_{1}, C_{2}$ be components of $G$ with sizes $\left|C_{1}\right|=N_{1}(G)$ and $\left|C_{2}\right|=N_{2}(G)$. Then every node in $C_{1} \cup C_{2}$ has component larger than $t$, and the claim follows from $N_{1}(G)+N_{2}(G)=\left|C_{1} \cup C_{2}\right| \leq\left|B_{t}(G)\right|$.
A.7. Graph superpositions. Let $G_{1}, \ldots, G_{m}$ be graphs such that $V\left(G_{k}\right) \subset V$ for all $k$. For $A \subset[m]$ we denote by $G_{A}$ the overlay graph with $V\left(G_{A}\right)=V$ and $E\left(G_{A}\right)=\cup_{a \in A} E\left(G_{a}\right)$.

Lemma A.12. For any $A, B \subset[m]$ and $t \geq 0$,

$$
\begin{aligned}
\left|B_{t}\left(G_{A \cup B}\right)\right| & \leq\left|B_{t}\left(G_{A}\right)\right|+t\left|U_{B}\right|, \\
N_{1}\left(G_{A \cup B}\right) & \leq \max \left\{\left|B_{t}\left(G_{A}\right)\right|+t\left|U_{B}\right|, t\right\},
\end{aligned}
$$

where $U_{B}=\cup_{k \in B} V\left(G_{k}\right)$.
Proof. Assume that $i \in B_{t}\left(G_{A \cup B}\right) \backslash B_{t}\left(G_{A}\right)$. Then $\left|C_{i}\left(G_{A \cup B}\right)\right|>t$ but $\left|C_{i}\left(G_{A}\right)\right| \leq t$, and we see that $C_{i}\left(G_{A}\right)$ must contain some node $j \in U_{B}$. Then $i \in C_{j}\left(G_{A}\right)$ and $\left|C_{j}\left(G_{A}\right)\right| \leq t$. We conclude that

$$
B_{t}\left(G_{A \cup B}\right) \backslash B_{t}\left(G_{A}\right) \subset \bigcup_{j \in U_{B}:\left|C_{j}\left(G_{A}\right)\right| \leq t} C_{j}\left(G_{A}\right)
$$

Hence

$$
\left|B_{t}\left(G_{A \cup B}\right)\right| \leq\left|B_{t}\left(G_{A}\right)\right|+\left|B_{t}\left(G_{A \cup B}\right) \backslash B_{t}\left(G_{A}\right)\right| \leq\left|B_{t}\left(G_{A}\right)\right|+t\left|U_{B}\right|
$$

The second inequality follows because $N_{1}\left(G_{A \cup B}\right) \leq \max \left\{\left|B_{t}\left(G_{A \cup B}\right)\right|, t\right\}$ by Lemma A.11.

In the following two results, we denote by $N_{A}$ the set of neighbours of node $i$ in $G_{A}$, and we set $D_{A}=\left|N_{A}\right|$ to denote the degree of $i$ in $G_{A}$.

Lemma A.13. Let $g$ be an arbitrary probability density on $\mathbb{Z}_{+}$. Let $\varepsilon(t)=\sum_{r+s=t}\left(\mathbb{P}\left(D_{A \cup B}=\right.\right.$ $\left.r)-\mathbb{P}\left(D_{A}=r\right)\right) g(s)$. Then $\sum_{t \geq 0}|\varepsilon(t)| \leq 2 \mathbb{P}\left(D_{B}>0\right)$.

Proof. Denote the densities of the degrees by $f_{A \cup B}=\mathcal{L}\left(D_{A \cup B}\right)$ and $f_{A}=\mathcal{L}\left(D_{A}\right)$. Then $\sum_{t>0}|\varepsilon(t)|=\left\|f_{A \cup B} * g-f_{A} * g\right\|_{1}=2 d_{\mathrm{tv}}\left(f_{A \cup B} * g, f_{A * g}\right) \leq 2 d_{\mathrm{tv}}\left(f_{A \cup B}, f_{A}\right)$. Further, $d_{\mathrm{tv}}\left(f_{A \cup B}, f_{A}\right) \leq$ $\mathbb{P}\left(\bar{D}_{A \cup B} \neq D_{A}\right) \leq \mathbb{P}\left(D_{B}>0\right)$.

Lemma A.14. Assume that $G_{1}, \ldots, G_{m}$ are mutually independent, let $A, B \subset[m]$ be disjoint, and let $\mathcal{E}_{A}, \mathcal{E}_{B}$ be events determined by $\left(G_{a}\right)_{a \in A}$ and $\left(G_{b}\right)_{b \in B}$, respectively. Then

$$
\mathbb{P}\left(D_{A \cup B}=t, \mathcal{E}_{A}, \mathcal{E}_{B}\right)=\mathbb{P}\left(D_{A}+D_{B}=t, \mathcal{E}_{A}, \mathcal{E}_{B}\right)+\varepsilon(t)
$$

where the error term is bounded by $|\varepsilon(t)| \leq c_{B} t \mathbb{P}\left(D_{A} \leq t, \mathcal{E}_{A}\right)$, and where $c_{B}=\max _{j \neq i} \mathbb{P}(i j \in$ $\left.E\left(G_{B}\right), \mathcal{E}_{B}\right)$.

Proof. Because $D_{A \cup B}=D_{A}+D_{B}$ outside the event $\mathcal{F}=\left\{\left|N_{A} \cap N_{B}\right|>0\right\}$, we see that

$$
\varepsilon(t)=\mathbb{P}\left(D_{A \cup B}=t, \mathcal{E}_{A}, \mathcal{E}_{B}, \mathcal{F}\right)-\mathbb{P}\left(D_{A}+D_{B}=t, \mathcal{E}_{A}, \mathcal{E}_{B}, \mathcal{F}\right)
$$

Hence it follows that $|\varepsilon(t)| \leq \mathbb{P}\left(D_{A} \leq t, \mathcal{E}_{A}, \mathcal{E}_{B}, \mathcal{F}\right)$, where the upper bound can be expressed as

$$
\mathbb{P}\left(D_{A} \leq t, \mathcal{E}_{A}, \mathcal{E}_{B}, \mathcal{F}\right)=\sum_{U:|U| \leq t, i \notin U} \mathbb{P}\left(N_{A}=U, \mathcal{E}_{A}\right) \mathbb{P}\left(\left|U \cap N_{B}\right|>0, \mathcal{E}_{B}\right)
$$

Because $\mathbb{P}\left(\left|U \cap N_{B}\right|>0, \mathcal{E}_{B}\right) \leq \sum_{j \in U} \mathbb{P}\left(i j \in E\left(G_{B}\right), \mathcal{E}_{B}\right) \leq c_{B} t$ whenever $|U| \leq t$, the claim follows.
A.8. Galton-Watson processes. Let $f$ be a probability measure on $\mathbb{Z}_{+}$and consider a Galton-Watson branching process with offspring distribution $f$. The exploration queue length of the corresponding tree [49, Section 3.3] satisfies the recursion $Q_{0}=1$ and $Q_{t}=$ $1\left(Q_{t-1}>0\right)\left(Q_{t-1}-1+Z_{t}\right)$ where $Z_{1}, Z_{2}, \ldots$ are independent $f$-distributed random integers. The total progeny equals $T=\inf \left\{t \geq 1: Q_{t}=0\right\} \in[0, \infty]$. We denote $\rho_{t}(f)=\mathbb{P}(T>t)$ and $\rho(f)=\mathbb{P}(T=\infty)$. We also note that $\mathbb{P}(T>t)=\mathbb{P}\left(Q_{t}>0\right)$.

Lemma A.15. If $f_{n} \xrightarrow{w} f$, then (i) $\rho_{t}\left(f_{n}\right) \rightarrow \rho_{t}(f)$ for all $t \geq 0$. If $f_{n} \xrightarrow{w} f$ and $f(0)>0$, then (ii) $\rho\left(f_{n}\right) \rightarrow \rho(f)$, and (iii) $\rho_{\omega_{n}}\left(f_{n}\right) \rightarrow \rho(f)$ for all sequences $\omega_{n} \rightarrow \infty$.

Proof. A natural coupling of exploration processes implies that $\left|\rho_{t}\left(f_{n}\right)-\rho_{t}(f)\right| \leq t d_{\mathrm{tv}}\left(f_{n}, f\right)$ for all $t$. Hence (i) follows by noting that weak convergence and total variation convergence are equivalent for probability measures $f_{n}, f$ on the countable space $\mathbb{Z}_{+}$. Claim (ii) follows by [39, Lemma 2.6]. For (iii), we first note that $\rho_{t}(f) \rightarrow \rho(f)$ as $t \rightarrow \infty$. Hence given any $\varepsilon>0$, we may choose $t$ so that $\rho(f) \leq \rho_{t}(f) \leq \rho(f)+\varepsilon$. Then, we see that

$$
\rho\left(f_{n}\right) \leq \rho_{\omega_{n}}\left(f_{n}\right) \leq \rho_{t}\left(f_{n}\right)+\varepsilon
$$

for all sufficient large values of $n$ such that $\omega_{n} \geq t$. Now (iii) follows by noting that $\rho\left(f_{n}\right) \rightarrow$ $\rho(f)$ by (i), and $\rho_{t}\left(f_{n}\right) \rightarrow \rho_{t}(f)$ by (ii).
A.9. Coupon collection. The classical coupon collector's problem involves a collector who at each round receives a coupon with type selected uniformly at random among a set of $n$ types, independently of previous rounds. We denote by $N_{t}$ the number of distinct coupon types obtained after collecting $t$ coupons.
Lemma A.16. Fix integers $k, t, n \geq 1$ such that $\frac{1}{k} \geq \frac{1}{t}+\frac{1}{n}$. Then the probability that the number of distinct coupon types obtained after collecting $t$ coupons is less than $k$ is at most

$$
\begin{equation*}
\mathbb{P}\left(N_{t}<k\right) \leq\left(\frac{t}{k}\right)^{k}\left(\frac{n}{k}-\frac{n}{t}\right)^{-(t-k)} \tag{A.8}
\end{equation*}
$$

Especially, $\mathbb{P}\left(N_{t}<k\right) \leq n^{-\alpha}$ whenever $\alpha+k \leq(1-\beta) t$ and $t \leq n^{\beta / 2}$ for some $\alpha>0$ and $\beta \in(0,1)$
Proof. Fix $s=\log \left(\frac{n}{k}-\frac{n}{t}\right)$. Then $s \geq 0$ and $\frac{k}{n} e^{s}=1-\frac{k}{t}<1$. Denote by $T_{k}$ the number of coupons needed to obtain $k$ distinct coupon types. Then $T_{j+1}-T_{j}$ is geometrically distributed with moment generating function $\mathbb{E} e^{s\left(T_{j+1}-T_{j}\right)}=\frac{(1-j / n) e^{s}}{1-(j / n) e^{s}}$. Hence

$$
\mathbb{E} e^{s T_{k}}=\prod_{j=0}^{k-1} \frac{(1-j / n) e^{s}}{1-(j / n) e^{s}} \leq\left(\frac{e^{s}}{1-\frac{k}{n} e^{s}}\right)^{k}
$$

Markov's inequality applied to $e^{s T_{k+1}}$ hence shows that

$$
\mathbb{P}\left(N_{t}<k\right)=\mathbb{P}\left(T_{k}>t\right) \leq e^{-s t} \mathbb{E} e^{s T_{k+1}}=\frac{e^{s(k-t)}}{\left(1-\frac{k}{n} e^{s}\right)^{k}}=\frac{\left(\frac{n}{k}-\frac{n}{t}\right)^{k-t}}{\left(\frac{k}{t}\right)^{k}}
$$

Observe next that $\frac{1}{k}-\frac{1}{t} \geq t^{-2}$ implies that the right side of (A.8) is at most $\left(\frac{t}{k}\right)^{k}\left(\frac{t^{2}}{n}\right)^{t-k} \leq$ $t^{2 t} n^{-(t-k)} \leq n^{\beta t-(t-k)}$ for $t \leq n^{\beta / 2}$. Hence $\mathbb{P}\left(N_{t}<k\right) \leq n^{-\alpha}$ when we also assume that $\alpha+k \leq(1-\beta) t$.

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[^0]:    Date: November 4, 2020.

[^1]:    ${ }^{1} R_{0}(\theta)$ can be interpreted as the basic reproduction number " $R$ naught" in the epidemiological context.

[^2]:    ${ }^{2}$ The first implication in (4.4) follows by noting that if $x y<1$, then the proof of [29, Theorem 5.4] shows that $\mathbb{E}\left|C_{H_{x y}}(1)\right|=\sum_{t \geq 1} \mathbb{P}\left(\left|C_{H_{x y}}(1)\right| \geq t\right) \leq \sum_{t \geq 1} e^{-\frac{1}{2}(1-x y)^{2} t} \leq \int_{0}^{\infty} e^{-\frac{1}{2}(1-x y)^{2} t} \leq 2(1-x y)^{-2}$, so that $R(x-1, y)=\mathbb{E}\left|C_{H_{x y}}(1)\right|-1 \leq 2(1-x y)^{-2}$.

