CM-VALUES OF HILBERT MODULAR FUNCTIONS

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(October 31, 2006)

This is a joint work with Jan Bruinier, and is a generalization of the well-known work of Gross and Zagier on singular moduli [GZ]. Here is the main result. For detail, please see [BY].

Let $p \equiv 1 \pmod 4$ be a prime number and $F = \mathbb{Q}(\sqrt{p})$. We write $\mathcal{O}_F$ for the ring of integers of F, and $x \mapsto x'$ for the conjugation in F. Let $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$ be the Hilbert modular group associated to F. The corresponding Hilbert modular surface $X = \Gamma\backslash\mathbb{H}^2$ is a normal quasi-projective algebraic variety defined over $\mathbb{Q}$.

Let $K = F(\sqrt{\Delta})$ be a non-biquadratic quartic CM number field (containing F) with discriminant $d_K = p^2 q$ for some prime $q \equiv 1 \mod 4$ (technical condition). Let σ and σ' be the complex embeddings of K given by $\sigma(\sqrt{\Delta}) = \sigma(\sqrt{\Delta'})$, and $\sigma(\sqrt{\Delta}) = -\sqrt{\Delta'}$. Then $\Phi = \{1, \sigma\}$ and $\Phi' = \{1, \sigma'\}$ are two CM types. Let $\mathcal{CM}(K, \Phi)$ be the (formal) sum of CM points in X of CM type (K, Φ) by $\mathcal{O}_K$. Then $\mathcal{CM}(K) = \mathcal{CM}(K, \Phi) + \mathcal{CM}(K, \Phi')$ is an 0-cycle on X defined over $\mathbb{Q}$. If Ψ is a rational modular function on X, then $\Psi(\mathcal{CM}(K))$ is a rational number. An interesting and in general very hard question is to find a factorization formula for this number. We did it successfully when Ψ is a Borcherds product or equivalently has its divisor supported on the Hirzebruch-Zagier divisors, which were constructed in their seminar work in 1970's [HZ].

Let $\tilde{K}$ be the reflex field of (K, Φ) with real quadratic subfield $\tilde{F}$. For a nonzero element $t \in d_{\tilde{K}/\tilde{F}}^{-1}$ (relative discriminant) and a prime ideal $\mathfrak{l}$ of $\tilde{F}$, we define

$$B_t(\mathfrak{l}) = (\mathrm{ord}_{\mathfrak{l}}\, t + 1)\rho(t d_{\tilde{K}/\tilde{F}} \mathfrak{l}^{-1}) \log |\mathfrak{l}| \tag{0.1}$$

and

$$B_t = \sum_{\mathfrak{l}} B_t(\mathfrak{l}). \tag{0.2}$$

Here $|\mathfrak{l}|$ is the norm of $\mathfrak{l}$, and $\rho(\mathfrak{a}) = \rho_{\tilde{K}/\tilde{F}}(\mathfrak{a})$ is defined as

$$\rho(\mathfrak{a}) = \#\{\mathfrak{A} \subset \mathcal{O}_{\tilde{K}} : \ N_{\tilde{K}/\tilde{F}}\mathfrak{A} = \mathfrak{a}\}. \tag{0.3}$$

Date: October 31, 2006.

We mention that there are at most one $B_t(\mathfrak{l}) \neq 0$ if t is not totally positive (negative). Let

$$b_m = \sum_{\substack{t=\frac{n+m\sqrt{q}}{2p} \in d_{\tilde{K}/\tilde{F}}^{-1} \\ |n|<m\sqrt{q}}} B_t. \tag{0.4}$$

Then a special case of our main result is

Theorem 0.1. *Let the notation and assumption be as above. Let Ψ be a 'normalized integral' Hilbert modular function (of weight 0) for the group Γ such that*

$$\operatorname{div}(\Psi) = \sum_{m>0} \tilde{c}(-m) T_m,$$

with integral coefficients $\tilde{c}(-m) \in \mathbb{Z}$. Then

$$\log|\Psi(\mathcal{CM}(K))| = \frac{W_{\tilde{K}}}{4} \sum_{m>0} \tilde{c}(-m) b_m,$$

where $W_{\tilde{K}}$ is the number of roots of unity in $\tilde{K}$.

The case for a similar Hilbert modular form of weight non-zero, we have a similar formula for its Petersson norm (with another term equals to the weight times a transcendental term related to the Faltings height of $\mathcal{CM}(K)$).

Corollary 0.2. *Let the notation and assumption be as in Theorem 0.1. Then*

$$\Psi(\mathcal{CM}(K)) = \pm \prod_{l \text{ rational prime}} l^{e_l} \tag{0.5}$$

with

$$e_l = \frac{W_{\tilde{K}}}{4} \sum_{m>0} \tilde{c}(-m) b_m(l),$$

and

$$b_m(l) \log l = \sum_{\mathfrak{l}|l} \sum_{\substack{t=\frac{n+m\sqrt{q}}{2p} \in d_{\tilde{K}/\tilde{F}}^{-1} \\ |n|<m\sqrt{q}}} B_t(\mathfrak{l}).$$

Moreover, when $K/\mathbb{Q}$ is cyclic, the sign in (0.5) is positive.

Corollary 0.3. *Let the notation and assumption be as in Corollary 0.2. Then $e_l = 0$ unless $4pl | m^2 q - n^2$ for some $m \in M := \{m \in \mathbb{Z}_{>0} : \tilde{c}(-m) \neq 0\}$ and some integer $|n| < m\sqrt{q}$. In particular, every prime factor of $\Psi(\mathcal{CM}(K))$ is less than or equal to $\frac{N^2 q}{4p}$, where $N = \max(M)$.*

We remark that the basic quantity b_m should be the arithmetic intersection number of arithmetic Hirzebruch-Zagier cycle $\mathcal{T}_m$ and arithmetic CM cycle $\mathcal{CM}(K)$ in the arithmetic Hilbert modular surface $\mathcal{X}$ (over $\mathbb{Z}$). I am currently working on this problem and has made some progress.

References

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