CO-ORBITAL MOTION WITH SLOWLY VARYING PARAMETERS

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(Received 20 June 2002; revised: 9 January 2003; accepted 29 January 2003)

Abstract. We consider the dynamics of a test particle co-orbital with a satellite of mass m_s which revolves around a planet of mass $M_0 \gg m_s$ with a mean motion n_s and semi-major axis a_s. We study the long term evolution of the particle motion under slow variations of (1) the mass of the primary, M_0, (2) the mass of the satellite, m_s and (3) the specific angular momentum of the satellite J_s. The particle is not restricted to small harmonic oscillations near L_4 or L_5, and may have any libration amplitude on tadpole or horseshoe orbits. In a first step, no torque is applied to the particle, so that its motion is described by a Hamiltonian with slowly varying parameters. We show that the torque applied to the satellite, as measured by $\epsilon_s = \dot{J}_s/(n_s J_s)$ induces an distortion of the phase space which is entirely described by an asymmetry coefficient $\alpha = \epsilon_s/\mu$, where $\mu = m_s/M$. The adiabatic invariance of action implies furthermore that the long term evolution of the particle co-orbital motion depends only on the variation of $m_s a_s$ with time. Applying a constant torque to the particle, as measured by $\epsilon_p = \dot{J}/(n_s J_s)$ is then merely equivalent to replacing $\alpha = \epsilon_s/\mu$ by $\alpha = (\epsilon_s - \epsilon_p)/\mu$. However, if the torque acting on the particle exhibits a radial gradient, then the action is no more conserved and the evolution of the particle orbit is no more controlled by $m_s a_s$ only. We show that even mild torque gradients can dominate the orbital evolution of the particle, and eventually decide whether the latter will be pulled towards the stable equilibrium points L_4 or L_5, or driven away from them. Finally, we show that when the co-orbital bodies are two satellites with comparable masses m_1 and m_2, we can reduce the problem to that of a test particle co-orbital with a satellite of mass $m_1 + m_2$. This new problem has then parameters varying at rates which are combinations, with appropriate coefficients, of the changes suffered by each satellite.

Key words: co-orbital motion, adiabatic invariant

1. Introduction

Co-orbital bodies are found in several contexts in the solar system. While classical examples are provided by the Trojan asteroids of Jupiter, more recent cases have been exhibited in the Saturnian system after ground-based observations and the *Voyager* encounters: a satellite (Helene) is found near the L_4 point of Dione, while two satellites, Telesto and Calypso, librate near the L_4 and L_5 of Thetys, respectively. In the same vein, the co-orbital satellites Janus and Epimetheus exchange angular momentum and oscillate in horseshoe orbits in a frame rotating with the mean mean motion of the two satellites.

Celestial Mechanics and Dynamical Astronomy **86:** 321–350, 2003.

A question of interest is the origin and evolution of these systems. A possibility is a cogenetic formation, namely accretion of material into various co-orbital bodies right from the beginning. Other scenarios could involve captures, disruptions, re-accretion of co-orbital material in complicated physical processes.

In order to better understand these scenarios, we have first to get a firm understanding of the dynamics of co-orbital bodies. Thus, we will consider here a much simpler, yet rich enough situation, where these co-orbital bodies suffer a slow evolution due to the variation of one of the parameters of the problem. More precisely, we investigate the effect of (1) a variation of the mass of primary, (2) a variation of the mass of one of the co-orbitals and (3) an orbital migration of one of the co-orbitals, either due to a direct external torque, or due to the indirect effect of a mass variation of the primary.

All the dynamics of the system will be described by a unique Hamiltonian, so that the variations of the parameters above mentioned can be studied simultaneously in a coherent frame.

Provided that the variations considered are slow and smooth enough, as quantified herein, we can use adiabatic invariant arguments to derive general results concerning the long term evolution of the co-orbital configuration.

This paper generalized previous results, obtained in particular by Fleming and Hamilton (2000). These authors derive results for small harmonic librations around the stable Lagrange points L_4 and L_5, while our results apply to any libration amplitudes, either on tadpole or horseshoe orbits. We retrieve simple asymptotic results for small tadpole and large horseshoe librations.

Also, we generalize these results in the cases where torques with spatial gradients are applied to the co-orbital bodies. Even though these gradients destroy the Hamiltonian nature of the motion, results can still be derived as to the long term evolution of the particle orbit.

After formulating the problem and obtaining the Hamiltonian for the system (Sections 2–4), we study the distortion of the phase space caused by a torque applied to one of the co-orbital bodies (see Sections 5 and 6).

In Section 7, we show that the adiabatic invariance of action implies that the long term evolution of the co-orbital bodies depends (explicitly) only on the variations of $m_c \times r_c$, where m_c is the combined mass of the co-orbital bodies and r_c is their common orbital radius.

Eventually, we consider in Section 8 the effects of local torque gradients applied to the co-orbitals. These gradients cause in general a much faster evolution of the co-orbital configuration than the variation of parameters mentioned above.

Finally, we treat in the Appendix the case where the co-orbital bodies have comparable masses, and we show it can be reduced to the restricted three-body problem examined in the main body of the paper.

Our aim is to exhibit generic behaviors, using completely non-dimensional and reduced systems. Scaling to the appropriate distances and times allows one to derive numerical results in some specific situations, a task that we will not undertake here.

Specific situations where the parameters of the co-orbital system are slowly varied can be found in various papers, see for instance Lissauer et al. (1985) for the case Janus–Epimetheus, or Gomes (1998), Marzari and Scholl (1998a,b) and Fleming and Hamilton (2000) for the case of the Trojan asteroids.

2. Hamiltonian of the Problem

We start with the simplest case, namely the restricted, circular and planar three-body problem. It involves a primary of mass M_0, called the 'planet' for sake of brevity. A body of mass $m_s \ll M_0$ (the 'satellite') orbits the planet with a circular motion, while a test particle of mass $m = 0$ is co-orbital with the satellite, see Figure 1.

The case where the particle and the satellite have comparable masses can be reduced to the previous case, as shown in the Appendix.

In the simplified model considered here, we also neglect in the Hamiltonian small terms arising from the orbital eccentricity of the particle. A more complete approach is used by Yoder et al. (1983, 1989), where second order terms in inclinations and eccentricities (plus the effect of the planet oblateness) are included in the Hamiltonian. These terms introduce small perturbations which average out to zero during the long-term evolution of the orbit in the presence of slowly varying parameters, and therefore, they will not be considered here.

Finally, we assume that the particle and the satellite always remain at several Hill radii from each other at closest approach, and that the orbital eccentricity of the particle is small enough at all time in order to fulfill that condition.

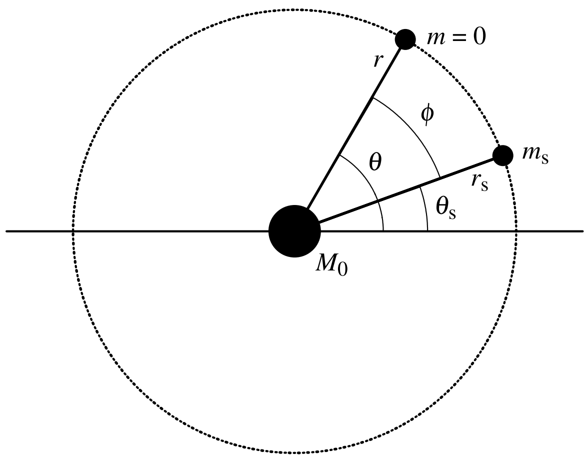

Figure 1. Notations for the restricted circular and planar three-body problem.

The constant of gravitation will be denoted by $\mathcal{G}$, and in the numerical integrations shown in this paper as examples, the units are such that $M_0 = 1$ and $\mathcal{G} = 1$. Thus, a test particle with circular orbit of radius unity will have an orbital period of $T_{\rm orb} = 2\pi$ around the planet.

The angle $\vartheta_{\rm s}$ (resp. ϑ) denotes the true longitude of the satellite (resp. the particle), and

$$\phi = \vartheta - \vartheta_{\rm s}$$

is the angle between the position vector $\mathbf{r}_{\rm s}$ of the satellite and that of the particle, $\mathbf{r}$, while $r_{\rm s} = |\mathbf{r}_{\rm s}|$ and $r = |\mathbf{r}|$ (Figure 1). Because we assume that the satellite orbital eccentricity $e_{\rm s}$ is zero, we have $r_{\rm s} = a_{\rm s}$, where $a_{\rm s}$ is the satellite semi-major axis. We then define:

$$\Delta r = r - a_{\rm s}$$

and we assume that $|\Delta r| \ll a_{\rm s}$, and more precisely that $|\Delta r| \lesssim r_{\rm H}$, where $r_{\rm H}$ is the Hill radius of the satellite, $r_{\rm H} = a_{\rm s}(m_{\rm s}/3M_0)^{1/3}$. This ensures that the satellite and the particle remain co-orbital, and also that they remain at several Hill radii from each other at closest approach, as stated before. Finally, we denote by $\mathbf{\Delta} = \mathbf{r} - \mathbf{r}_{\rm s}$ the position vector of the particle with respect to the satellite.

In the Hamiltonian formulation, the motion of the particle will be described by its polar coordinates (r, ϑ) and by their corresponding conjugate momenta $p_{\rm r} = \dot{r}$ and $J = r^2\dot{\vartheta}$, respectively. Note that $p_{\rm r}$ is the radial component of the *specific* (i.e. per unit mass) momentum of the particle, while J is its specific angular momentum.

In a frame fixed at the center of the planet, and with axes parallel to fixed directions, the motion of the particle is described by the Hamiltonian

$$\mathcal{H}_1(r, p_{\rm r}; \vartheta, J) = \frac{1}{2}\left(p_{\rm r}^2 + \frac{J^2}{r^2}\right) - \frac{\mathcal{G}M_0}{r} - R_{\rm s}, \tag{1}$$

which contains, the kinetic and potential energies of the particle, and the perturbing function $R_{\rm s}$ due to the satellite:

$$R_{\rm s} = \mathcal{G}m_{\rm s}\left[\frac{1}{\Delta} - \frac{\mathbf{r}\cdot\mathbf{r}_{\rm s}}{r_{\rm s}^3}\right].$$

Because $|\Delta r| \ll a_{\rm s}$, the perturbing function can be written as:

$$R_{\rm s} \approx a_{\rm s}^2 n_{\rm s}^2 \mu f(\phi),$$

where $n_{\rm s}$ is the satellite mean motion and

$$\mu = \frac{m_{\rm s}}{M_0 + m_{\rm s}} \ll 1$$

is the mass of the satellite relative to the total mass

$$M = M_0 + m_{\rm s}$$

of the system. Finally:

$$f(\phi) = -\cos(\phi) + \frac{1}{|2\sin(\phi/2)|} \tag{2}$$

describes the angular dependence of R_s. Because this dependence appears only through $\phi = \vartheta - \vartheta_s$, we can take ϕ and J as conjugate variables, provided that we consider the new Hamiltonian $\mathcal{H}_2 = \mathcal{H}_1 - n_s J$ instead of $\mathcal{H}_1$. Then:

$$\mathcal{H}_2(r, p_r; \phi, J) = \frac{p_r^2}{2} - \frac{\mathcal{G}M_0}{r} + \frac{J^2}{2r^2} - n_s J - a_s^2 n_s^2 \mu f(\phi), \tag{3}$$

which corresponds to the energy of the particle in a frame fixed at the center of the planet and revolving at the angular velocity n_s of the satellite.

The motion derived from $\mathcal{H}_2$ possesses two very different time scales, each associated with one of the two degrees of freedom of the particle, namely the radial and the angular motions. The faster motion is the radial one and it is described by:

$$\begin{aligned} \dot{r} &= \frac{\partial \mathcal{H}_2}{\partial p_r} \\ \dot{p}_r &= -\frac{\partial \mathcal{H}_2}{\partial \mathrm{r}}, \end{aligned}$$

and occurs over a typical time scale T_{orb}, the orbital period of the particle. The slower motion in ϕ, given by:

$$\begin{aligned} \dot{\phi} &= \frac{\partial \mathcal{H}_2}{\partial J} \\ \dot{J} &= -\frac{\partial \mathcal{H}_2}{\partial \phi}, \end{aligned}$$

occurs over much longer time scales and actually describes the co-orbital behavior of the particle. This motion is classically separated between tadpole librations around L_4 or L_5 and inside the homoclinic orbit (the separatrix), and horseshoe librations outside that separatrix, see the upper left panel of Figure 2.

Away from the separatrix (either in tadpole or horseshoe orbits) the typical periods of libration of the particle are of order:

$$T_{lib} \sim \frac{T_{orb}}{\sqrt{\mu}} \gg T_{orb}. \tag{4}$$

Then, we can slowly (i.e. over time scales large with respect to T_{lib}) vary one of the parameters M_0, m_s or a_s, and use adiabatic invariant arguments in order to describe the long term evolution of (1) the orbital eccentricity and (2) the co-orbital motion of the particle.

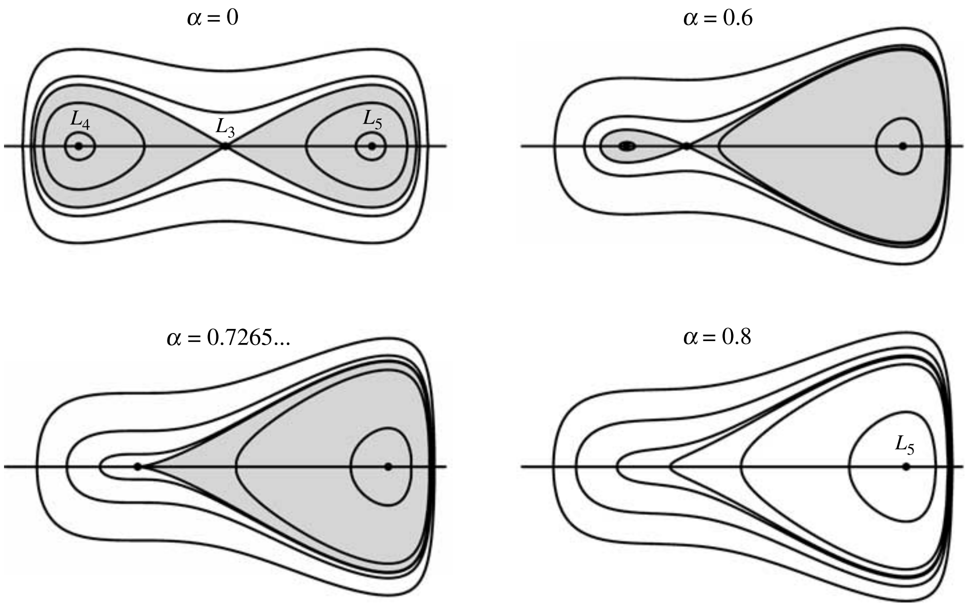

Figure 2. Level curves of the Jacobi constant $C = X^2 + g(\phi)$ in the $[\phi, X]$ space, for various positive values of α, corresponding to an outward migration of the satellite (see text). The horizontal line in each diagram corresponds to $X = 0$ and ranges from $\phi = 0°$ to $\phi = 360°$. The shaded areas fill in the homoclinic orbits. For $\alpha = 0.7265\ldots$ (lower left panel), the two points L_4 and L_3 merge at $\phi = 108°.3510\ldots$ at a cusp singularity. For higher values of α (lower right panel), the homoclinic orbit disappears, and only the singular point L_5 survives. Similar behaviors (but symmetrical with respect to $\phi = \pi$) are observed for negative values of α.

3. Evolution of Eccentricities

Classical calculations allow one to follow up both the orbital eccentricities and semi-major axes of the satellite and the particle, as slow changes are applied to the system, see Jeans (1924), Littlewood (1964) and the review by Boccaletti and Pucacco (1998, Chapter 9). We discuss further in Subsection 4.2 on how these changes have to be applied in order for the adiabatic invariance of actions to be valid. In particular, we will see that the changes of masses of the primary and of the satellites have to occur isotropically.

Concerning the satellite, we note that the actions

$$L_s = \sqrt{\mathcal{G} M a_s}$$

(the satellite specific energy divided by its orbital frequency) and

$$J_s = \sqrt{\mathcal{G} M a_s (1 - e_s^2)}$$

(the satellite specific angular momentum) are adiabatically conserved when the total mass $M = M_0 + m_s$ of the system varies isotropically. Consequently, $M a_s$ is adiabatically conserved during such changes (Jeans, 1924), and so is e_s (Littlewood, 1964). Under these conditions, a circular orbit for the satellite will remain so upon changes of M, and we can write

$$J_s = \sqrt{\mathcal{G} M a_s}$$

at any moment.

Concerning the particle, we consider the quantities ϕ and J (and *a fortiori*, M_0, m_s and a_s) as slowly varying parameters for the Hamiltonian $\mathcal{H}_2$ which rules the variations of r and p_r.

Then, provided that the particle motion remains Hamiltonian, we can use the same arguments as for the satellite to follow up the particle semi-major axis and eccentricity, a and e. Namely, $L = \sqrt{\mathcal{G} M a}$ and $J = \sqrt{\mathcal{G} M a(1 - e^2)}$ are adiabatically conserved during the motion, and so is $L - J \approx e^2 \sqrt{\mathcal{G} M a}/2$ for $e \ll 1$.

Now, the particle semi-major axis a can vary due to an external force, or due to a change of M with no external force, or due to a mixture of both kinds of processes. If an external force is applied to the particle, we assume that it does not excite its eccentricity, see Subsection 4.2.

In the case where M remains constant while a only varies, the conservation of $L - J$ implies:

$$e \propto \frac{1}{a^{1/4}}, \tag{5}$$

If the change of a is caused by a change of M *only* (i.e. in the absence of any external force to the particle), then Ma is conserved and:

$$e = \text{constant}. \tag{6}$$

In the case of a mixture of the two processes, only the conservation of $e^2\sqrt{Ma}$ is guaranteed.

These results are derived and discussed in more details by Fleming and Hamilton (2000). These authors note in particular that the mean motion n of the particle must be corrected by a small term of order μ with respect to its Keplerian value (i.e. if there were no co-orbital satellite), which induces corrections of order μ in the two equations above. This effect will not be taken into account herein.

4. The Hamiltonian with Slowly Varying Parameters

4.1. EXPANSION OF THE HAMILTONIAN NEAR a_s

We now eliminate the fast radial motion of the particle by averaging the equations of motion over one orbital period T_{orb}. This yields an orbital radius r = constant and $p_r = 0$. In other words, we consider the evolution of particles on circular orbits only.

Note that because of the adiabatic invariance of $L - J$, a circular orbit ($L - J = 0$) will remain so during the evolution of the system.

Then we have at any moment[1]:

$$J \approx \sqrt{\mathcal{G}Mr} \tag{7}$$

$$J \approx J_{\rm s}\left[1 + \frac{\Delta r}{2a_{\rm s}}\right], \tag{8}$$

Consequently, the Hamiltonian $\mathcal{H}_2$ (Eq. (3)) can be re-written:

$$\mathcal{H}_2 = -\frac{(\mathcal{G}M)^2}{2J^2} - n_{\rm s}J - a_{\rm s}^2 n_{\rm s}^2 \mu f(\phi). \tag{9}$$

We now define

$$J_\phi = J - J_{\rm s} \approx J_{\rm s}\left(\frac{\Delta r}{2a_{\rm s}}\right) \tag{10}$$

as the difference between the specific angular momentum of the particle and that of the satellite. The approximation arises from the fact that $|\Delta r| \ll a_{\rm s}$, so that $|J_\phi/J_{\rm s}| \ll 1$. Thus, $J_\phi/J_{\rm s}$ can be viewed as half of the fractional radial distance of the particle to the satellite orbit.

Note that J_ϕ is *not* the specific angular momentum of the particle in the frame rotating with the satellite, $J_{\rm rot} = r^2\dot{\phi}$, although it is proportional to it: $J_\phi \approx -J_{\rm rot}/3$.

We can expand $\mathcal{H}_2$ to second order in $J_\phi/J_{\rm s}$, near $J_\phi/J_{\rm s} = 0$. Then, to within a constant term that we can drop, we obtain:

$$\mathcal{H}_3(\phi, J_\phi) = -\left(\frac{\mathcal{G}M}{J_{\rm s}}\right)^2\left[\frac{3}{2}\left(\frac{J_\phi}{J_{\rm s}}\right)^2 + \mu f(\phi)\right]. \tag{11}$$

4.2. THE VARYING PARAMETERS

The three parameters of the Hamiltonian $\mathcal{H}_3$ above, namely M, μ and $J_{\rm s}$, can vary slowly, either separately, or simultaneously. We remind that these parameters are related to the orbital radius of the satellite, $a_{\rm s}$, its mass, $m_{\rm s}$, and to the planet mass, M_0 by the system:

$$M = M_0 + m_{\rm s}$$

$$\mu = \frac{m_{\rm s}}{M_0 + m_{\rm s}}$$

$$J_{\rm s} = \sqrt{\mathcal{G}(M_0 + m_{\rm s})a_{\rm s}}.$$

[1]The angular momentum also depends on ϕ, through a term of order μ, which slightly displaces radially the equilibrium point L_3. This effect is irrelevant for the present discussion and will not be considered.

The variations of M, μ and J_s can be measured by dimensionless small parameters:

$$\epsilon_M = \frac{1}{n_s} \cdot \frac{\dot{M}}{M} \tag{12}$$

$$\epsilon_\mu = \frac{1}{n_s} \cdot \frac{\dot{\mu}}{\mu} \tag{13}$$

$$\epsilon_s = \frac{1}{n_s} \cdot \frac{\dot{J}_s}{J_s}. \tag{14}$$

Some care must be taken when varying M_0, m_s or a_s, first to preserve the Hamiltonian nature of the motion for the particle, and second its adiabaticity. For that, we have to discuss first the physical origin of these variations.

- Variations of the planet mass, M_0. Variations of M_0 can be caused by a mass loss or gain of the central body. This mass variation must occur isotropically in order to preserve the Hamiltonian nature of the particle motion. Anisotropic changes of mass will accelerate the central body and cause indirect accelerations on the particle. In some simple cases, these accelerations could be accounted for by new terms in $\mathcal{H}_3$, but this would complicate our approach, and such effects will not be considered here.

  Then, for slow enough variations of M_0 (and thus also of M), the actions L_s and J_s are adiabatic invariants (see Section 3) and $a_s \propto 1/M$.

  Note that in this case, even though the satellite specific angular momentum does *not* varies (no torque is applied to the satellite: $\dot{J}_s = 0$), the latter can migrate radially ($\dot{a}_s \neq 0$).
- Variations of the satellite mass, m_s. We make the same remark as before concerning the isotropic nature of the mass change for the satellite, in order to keep the Hamiltonian nature of the motion for the particle. Note that a variation of m_s essentially changes the parameter $\mu = m_s/(M_0 + m_s)$ in $\mathcal{H}_4$, while M remains almost constant since $m_s \ll M_0$. Also, an isotropic mass variation of m_s has no effect on J_s.
- Variations of the satellite specific angular momentum, J_s. Changes of J_s demand an external torque (tide, ring, gas, non-isotropic mass change,...). Thus ϵ_s in Equation (14) is actually a non-dimensional measure of the torque exerted on the satellite.

We assume that this change is made so that the orbit of the satellite remains circular at all time, and so that Kepler's third law, $a_s^3 n_s^2 = \mathcal{G}M$, is satisfied at any moment too. (Situations where Kepler's third law is *not* verified anymore arise for instance when the satellite is submitted to a constant radial pressure gradient due to a disk of gas.)

For the moment, we assume that no torque is applied to the particle. External torques on the particle usually destroy the Hamiltonian nature of the motion, preventing the use of the adiabatic invariance of the action. This is discussed further in Section 8.

Note that in the present case the variable J_ϕ defined in Equation (10) is explicitly time dependent through the variation of J_s:

$$\dot{J}_\phi = \dot{J} - \dot{J}_s = \dot{J} - \epsilon_s n_s J_s$$

Then ϕ and J_ϕ are still conjugate provided that a reminder function is added to $\mathcal{H}_3$, which yields the new Hamiltonian[2] $\mathcal{H}_4 = \mathcal{H}_3 + n_s J_s \epsilon_s \phi$.

From $n_s J_s = (\mathcal{G}M/J_s)^2$, we eventually obtain:

$$\mathcal{H}_4(\phi, J_\phi) = -\left(\frac{\mathcal{G}M}{J_s}\right)^2 \left[\frac{3}{2}\left(\frac{J_\phi}{J_s}\right)^2 + \mu g(\phi)\right], \tag{15}$$

where:

$$g(\phi) = f(\phi) - \alpha\phi \tag{16}$$

$$\alpha = \frac{\epsilon_s}{\mu}. \tag{17}$$

The term $-\alpha\phi$ in $g(\phi)$ destroys the symmetry of the Hamiltonian $\mathcal{H}_4$ with respect to $\phi = \pi$, that is, the symmetry of the trajectories $[\phi(t), J_\phi(t)]$ with respect to π. For that reason, α will be called the *asymmetry parameter*, see a discussion and illustrations of this effect in Section 6.

At the point we arrived at, the Hamiltonian $\mathcal{H}_4$ depends on four parameters: M, μ, J_s and α (or equivalently M_0, m_s, a_s and α). Note that because α is the ratio of two small quantities, it is not necessarily small itself, so that the asymmetry of the trajectories with respect to $\phi = \pi$ can be large, even though ϵ_s and μ are small, see an example in Figure 5.

Also, we will see in the next subsection that $\dot{\alpha}/n_s\alpha$ is *a priori* comparable to the coefficients ϵ_M, ϵ_μ and ϵ_s. Thus, the variation of α cannot in general be neglected with respect to the variations of M, μ and J_s.

Note finally that from $J_s = \sqrt{\mathcal{G}Ma_s}$, a migration of the satellite (i.e. a change in a_s) can be caused by a variation of J_s with M constant, or by a variation of M with J_s constant, or by a mixture of the two processes. Thus, when M varies, there is no univocal relationship between J_s and a_s.

4.3. RATE OF CHANGE OF THE PARAMETERS

We examine here the conditions under which the adiabatic invariance of the action $I_\phi = \oint J_\phi d\phi$ is insured. These conditions concerns the rates of change of the parameters and their higher order derivatives.

Let us denote generically by λ any of the parameters M, μ, J_s or α, normalized to its value at $t = 0$, that is, $\lambda(t) = M(t)/M(0)$, or $\lambda(t) = \mu(t)/\mu(0)$, etc. Thus, λ

[2] Note that this is equivalent to applying a constant torque $-\epsilon_s n_s J_s$ on the particle.

is of order unity and we consider the evolution of the particle after a time T large enough so that λ has also varied by a quantity of order unity.

The parameter $\lambda(t)$ must be a slow and smooth function of time at all orders for the adiabatic invariant theory to be valid (see for instance Henrard, 1993). More precisely, let $n_{\rm lib} = 2\pi/T_{\rm lib}$ be the libration frequency of the co-orbital particle. Since the particle is away from the separatrix, we have $n_{\rm lib} \neq 0$. Then there must be a quantity $\epsilon_0 \ll 1$ such that at any order p we have:

$$\frac{1}{p} \cdot \frac{d^p\lambda}{dt^p} \lesssim \epsilon_0 n_{\rm lib} \frac{d^{p-1}\lambda}{dt^{p-1}}. \tag{18}$$

In particular, for $p = 1$, we get $d\lambda/dt \lesssim \epsilon_0 n_{\rm lib}$. On the other hand, $d\lambda/dt \sim n_{\rm s}\epsilon$, where ϵ is any of the quantities ϵ_M, ϵ_μ or $\epsilon_{\rm s}$. Thus we must have:

$$|\epsilon| \lesssim \epsilon_0 \frac{n_{\rm lib}}{n_{\rm s}} \sim \epsilon_0\sqrt{\mu}, \tag{19}$$

where we have used Equation (4). At second order ($p = 2$), Equation (18) yields $|\dot{\epsilon}| \lesssim n_{\rm lib}\epsilon_0^2$ and using $\alpha = \epsilon_{\rm s}/\mu$, we obtain:

$$\frac{1}{n_{\rm s}} \cdot \frac{\dot{\alpha}}{\alpha} \lesssim \epsilon_0\sqrt{\mu}. \tag{20}$$

Thus, the relative variation of α is *a priori* comparable to the relative variations of M, μ and $J_{\rm s}$, and thus it cannot be neglected in general. This point is discussed in Section 7.

Under the condition specified above (Equation (18)), the adiabatic invariance of the action I_ϕ is guaranteed, except near the separatrix. More precisely, over times $T \sim T_{\rm lib}/\epsilon_0$, the action will suffer relative changes of order ϵ_0 at most.

This is not true near the separatrix, where $T_{\rm lib}$ goes to infinity ($n_{\rm lib}$ goes to zero). There the action can suffer large variations and even discontinuities.

5. Shape of the Trajectories

The Hamiltonian $\mathcal{H}_4$ (Eq. (15)) can be re-written:

$$\mathcal{H}_4(\phi, J_\phi) = -\left(\frac{\mathcal{G}M\sqrt{\mu}}{J_{\rm s}}\right)^2 [X^2 + g(\phi)],$$

where:

$$X = \sqrt{\frac{3}{2\mu}} \cdot \frac{J_\phi}{J_{\rm s}} = \sqrt{\frac{3}{2\mathcal{G}}} \cdot \frac{J_\phi}{\sqrt{m_{\rm s}a_{\rm s}}} \tag{21}$$

The shape of the particle trajectory (i.e. a level curve $\mathcal{H}_4 =$ constant) is entirely determined by the value inside the bracket:

$$C = X^2 + g(\phi), \tag{22}$$

which is a dimensionless and reduced version of the Jacobi constant. Note that C depends on α through $g(\phi) = f(\phi) - \alpha\phi$. Note also that X, C and $g(\phi)$ are of order unity when the particle remains inside the separatrix (tadpole orbit), or follows a horseshoe orbit close to the separatrix. As the amplitude of the horseshoe libration increases, however, C increases up to the point where the particle can get near the Lagrangian point L_1 or L_2, that is, at a distance comparable to the satellite Hill radius $r_{\rm H} = a_{\rm s}(\mu/3)^{1/3}$. In that case C has a value of $\sim \mu^{-1/3}$ and the particle starts to suffer a chaotic motion. This situation will not be considered here, so that we will impose the condition $C \lesssim \mu^{-1/3}$ in all this paper.

When one of the parameters of the problem (M, μ, $J_{\rm s}$ or α) slowly varies, C will slowly change accordingly. For convenience, however, we will still refer to C as the 'Jacobi constant' in the rest of the paper.

For a *given* asymmetry parameter α, the shape of the trajectory (tadpole or horseshoe) at a given moment is entirely determined by the numerical value of C. Examples of such shapes are shown in Figure 2, for various values of α.

The value of C is also a convenient quantity for determining the range of angles ϕ available to the particle, since the relation $C = X^2 + g(\phi)$ implies $g(\phi) \leqslant C$. This range can be graphically derived by diagrams such as of Figure 3.

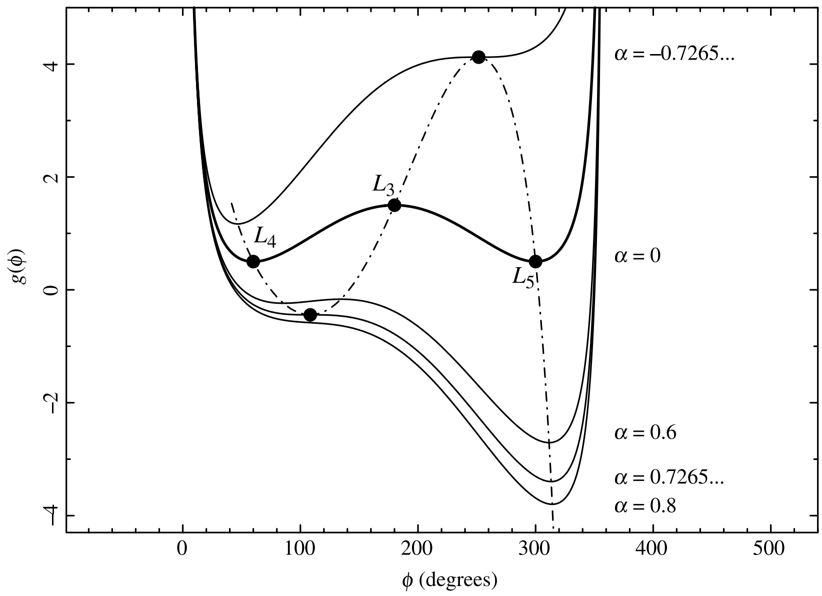

Figure 3. Solid curves: the function $g(\phi)$ for various values of the parameter α, as indicated by the labels on the right. The four positive values of α are used in Figure 2 to generate the four diagrams of level curves. The thicker curve ($\alpha = 0$) gives the classical positions of the fixed points L_4, L_3 and L_5 at $\phi = 60°$, 180° and 300°. Dash-dotted line: the locations of the fixed points when the parameter α varies. For $\alpha = 0.7265\ldots$ the two points L_4 and L_3 merge and then disappear for larger values of α, see also Figure 2. For $\alpha = -0.7265\ldots$, L_3 and L_5 merge and then disappear.

In the case $\alpha = 0$, $C = 1/2$ corresponds to the stable points L_4 and L_5, $1/2 \leqslant C < 3/2$ corresponds to the tadpole orbits, $C = 3/2$ corresponds the separatrix between tadpole and horseshoe orbits, while $C > 3/2$ corresponds to the horseshoe orbits, see the thicker curve in Figure 3.

If α changes with time, the situation is more complicated, since the graph of $g(\phi)$ is itself time-dependent. This problem is examined in Section 7.

6. Distortion of the Phase Space

In this section, we consider an example where only J_s varies, due to a torque ϵ_s applied to the satellite, while M_0 and m_s are fixed, so that $\epsilon_M = \epsilon_\mu = 0$. Then, $\epsilon_s \geqslant 0$ (resp. $\leqslant 0$) corresponds to an outward (resp. inward) orbital migration for the satellite. Also, we assume for clarity that the asymmetry parameter α is constant, that is, that ϵ_s is constant.

Note again that if the satellite migrates because of a variation of M_0 (or m_s), then J_s remains constant and *no* asymmetries of the trajectories with respect to $\phi = \pi$ appears. Physically, this can be understood by the fact that a variation of M_0 induces the same radial migration for the particle and for the satellite.

Note also that the two small dimensionless parameters ϵ_s and μ have no hierarchical relation *a priori* (except for the relation $|\epsilon| \lesssim \epsilon_0\sqrt{\mu}$, see Eq. (19)), so that nothing can be said about the value of α in general. In any case, the distortion of the phase space, measured for instance by α or by the shifts suffered by L_4, L_3 or L_5 (Equations (23) and (24)) may be several orders of magnitude larger than ϵ_s since $\mu \ll 1$. In particular, in the numerical integrations shown in Figure 4, $\alpha = 0.3$ is of order unity, even though both ϵ_s and μ are of order of 10^{-4}.

As mentioned in the previous section, the shape of the trajectory $[\phi(t), J_s(t)]$ (tadpole or horseshoe) is entirely determined by the value of $C = X^2 + g(\phi)$ and α. It is thus convenient to plot the trajectories using the variables $[\phi(t), X(t)]$ instead of $[\phi(t), J_\phi(t)]$. In this case, however, ϕ and X are *not* conjugate variables for $\mathcal{H}_4$, so that the adiabatic conservation of the area enclosed in the trajectories is *not* verified in the space (ϕ, X), see the next subsection.

Various level curves for various values of α are shown in Figure 2. These diagrams illustrate how the level curves are distorted by a non-zero value of α. In particular, the fixed points are given by $X = 0$ and $g'(\phi) = 0$, that is, $X = 0$ and $f'(\phi) = \alpha$, where the prime denotes the first derivative with respect to ϕ. Figure 3 shows how the two points L_4 and L_3 get closer as α increases, and merge together at the inflexion point $\phi_0 = 108°.4\ldots$, where the second derivatives verify $g''(\phi_0) = f''(\phi_0) = 0$ for the value $\alpha = f'(\phi_0) = 0.7265\ldots$. For higher values of α, L_4 and L_3 disappear, leaving only the point L_5.

Symmetrically, the points L_3 and L_5 get closer as α decreases, then they merge together for $\alpha = -0.7265\ldots$ at $\phi = 360° - \phi_0 = 251°.6\ldots$, and disappear for smaller values of α, leaving only the point L_4.

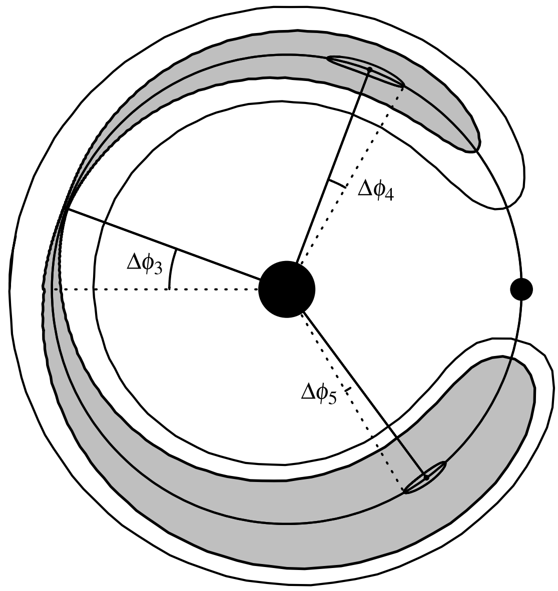

Figure 4. Shift of the Lagrange equilibrium points and distortion of the orbit in the presence of a torque applied to the satellite. A part of the particle trajectory of Figure 5 (lower panel) is plotted when the particle is close to the separatrix. The trajectory is shown in a reference frame rotating with the satellite, and expanding as a_s increases, so that the satellite (dot at the right) appears motionless. The radial displacement of the particle with respect to the satellite orbit is furthermore multiplied by a factor 10 for clarity. The dotted lines point toward the classical Lagrangian points ($\phi = \pi/3$, π and $5\pi/3$), while the solid lines point towards the actual equilibrium points.

At the moment of disappearance, the singular point is neither elliptic nor hyperbolic, but of the cusp kind, see Figure 2.

In the case where $\alpha \ll 1$, the distortion of the phase space remains small, and the equilibrium points L_4, L_3 and L_5 remain close to their classical values of $\phi_i = \pi/3$, π and $5\pi/3$ radians, for $i = 4$, 3 and 5, respectively.

Near these equilibrium points, $g'(\phi) \approx f''(\phi_i)\Delta\phi_i - \alpha$, where $\Delta\phi_i = \phi - \phi_i$. The new equilibrium points are given by $g'(\phi) = 0$, so that each equilibrium point is shifted by $\Delta\phi_i = \alpha/f''(\phi_i)$. We have $f''(\pi/3) = f''(5\pi/3) = 9/4$ and $f''(\pi) = -7/8$. Consequently, the two points L_4 and L_5 are both shifted angularly by the same amount (to first order in α):

$$\Delta\phi_4 \approx \Delta\phi_5 \approx \frac{\alpha}{f''(\pi/3)} = +\frac{4}{9}\alpha \text{ rad}, \tag{23}$$

while the point L_3 is shifted in the opposite direction by an amount:

$$\Delta\phi_3 \approx \frac{\alpha}{f''(\pi)} = -\frac{8}{7}\alpha \text{ rad}. \tag{24}$$

Examples of such shifts are shown in Figure 4 for $\alpha = 0.3$. Because α is not so small in this case, we can detect a little difference between $\Delta\phi_4$ and $\Delta\phi_5$ upon close examination of the figure.

Note finally that the asymmetry of the phase space described in this section might have applications to the observed asymmetry in the populations of the Jupiter Trojan asteroids, the L_4 region being more populated than the L_5 zone, see Marzari et al. (2003). Thus, an *inward* migration of Jupiter during the formation of the Trojans increases the size of the L_4 region with respect to the L_5 region, and could in principle explain this asymmetry. However, preliminary integrations that we made showed that the transfer of asteroids from the L_5 to the L_4 region is rather complex, and that the transferred bodies tend to be captured back to their original region when the migration stops.

Moreover, the very existence of a real asymmetry between the two populations of Trojans is still debated, and might only be due to biases in the surveys conducted up to now (Marzari et al., 2003). For these reasons, it is still premature to directly relate our work to this asymmetry issue. A detailed analysis of this effect is nevertheless interesting, but stays beyond the scope of this paper.

7. Long Term Evolution of the Orbits

During one libration period T_{lib}, the particle Jacobi constant C varies by a small quantity of order ϵ_0. On the long term, however ($T \sim T_{\text{lib}}/\epsilon_0$), C can vary by a quantity of order unity, due to the slow change of one or several of the parameters M, μ, J_{s} and α.

If we can keep track of the variations of C over this long time, we can know how the shape of the particle orbit evolves. In particular, we can know when it crosses the separatrix between tadpole and horseshoe orbits, how the libration amplitude around L_4 or L_5 evolves, etc.

7.1. TRAJECTORY OF THE PARTICLE IN VARIOUS SPACES

The trajectory of the particle can be plotted using various spaces, each illustrating an aspect of the problem.

As an example, we show in Figure 5 the trajectory of the particle in the presence of a satellite migration with parameters $\epsilon_{\text{s}} = +3 \times 10^{-5}$ and $\mu = 10^{-4}$ (with $\epsilon_M = \epsilon_\mu = 0$), first using the conjugate variables (ϕ, J_ϕ), and then in the (ϕ, X) space.

The trajectory shown in this figure is the result of a *direct* numerical integration of the motion of a test particle under the action of the planet and the satellite, the latter undergoing a slow migration due to a torque which changes J_{s}. In particular, this integration uses none of the approximations involved in the derivation of Equation (15).

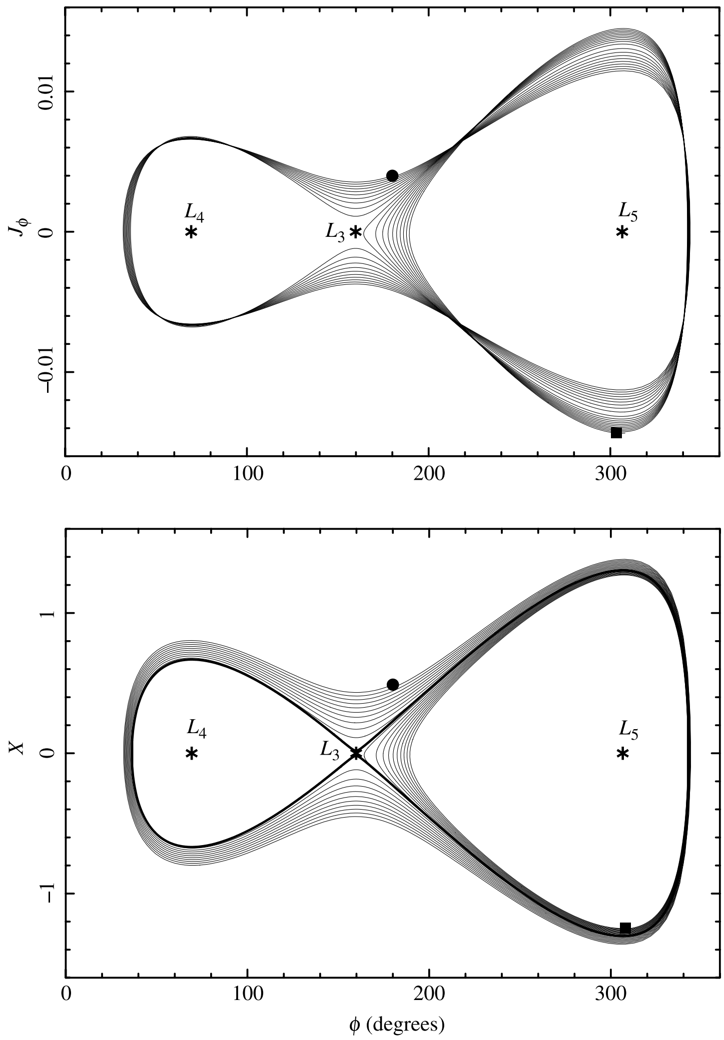

Figure 5. Example of co-orbital motion in the presence of satellite migration. The particle is launched at $t = 0$ with $\phi = 180°$ and $J_\phi = 0.004$ (solid circle), with the satellite orbiting at $a_s = 1$ around a planet of mass $M_0 = 1$. The motion is integrated till $t = 20000$, with $m_s = 10^{-4}$ and $\epsilon_s = \text{constant} = 3 \times 10^{-5}$, so that $\alpha = 0.3$. At the end of the integration (solid square), the satellite has nearly doubled its distance to the planet. Upper panel: motion of the particle in the phase space (ϕ, J_ϕ), illustrating the adiabatic invariance of the action $I_\phi = \oint J_\phi d\phi$. Lower panel: motion of the particle in the space (ϕ, X), showing the slow change of C, and the capture into a tadpole orbit around L_5. See the text for comments.

The upper panel shows the orbit in the phase space (ϕ, J_ϕ). Since the motion is Hamiltonian, and the evolution of J_s is slow, the area inside the trajectory, that is, the action I_ϕ:

$$I_\phi = \oint J_\phi d\phi \tag{25}$$

is adiabatically conserved, which explains the expansion of the orbit along the J_ϕ direction, as the amplitude in ϕ shrinks.

The lower panel shows the same orbit in the (ϕ, X) space. The thicker line delineates the separatrix between tadpole and horseshoe orbits. Since the asymmetry parameter α does not change with time in this integration, the separatrix does not change either. The trajectory is first a horseshoe orbit, and because $\epsilon_s > 0$, C decreases with time, see Equation (27). Thus, the area:

$$A = \oint X d\phi \tag{26}$$

inside the trajectory shrinks, and near the middle of the integration, the separatrix is crossed. Subsequently, the particle is captured into the attraction basin of L_5.

Finally, the physical area S inside the orbit:

$$S = \iint r dr d\phi \approx a_s \oint \Delta r(\phi) d\phi,$$

(see for instance the shaded area in Figure 4) is still different from A and I_ϕ. From $J_\phi / J_s \approx \Delta r/(2a_s)$, we derive:

$$S \approx 2\frac{a_s^2}{J_s} I_\phi \propto \frac{a_s^{3/2}}{M^{1/2}}$$

In the case where only J_s varies (with M constant), we have:

$$S \propto a_s^{3/2}.$$

In the case where only M_0 or m_s change (leaving J_s constant), we have $M \propto 1/a_s$, so that:

$$S \propto a_s^2.$$

With a mixture of the two processes, only the invariance of $a_s^{3/2}/M^{1/2}$ is insured. These results generalize the results of Fleming and Hamilton (2000) to any libration amplitude of the particle.

7.2. GLOBAL BEHAVIOR OF THE TRAJECTORY

From the definition of C (Equation (22)), it follows that the variations $\dot{C}$ only arise from the explicit variations of the parameters with time, and not from the variations of J_ϕ and ϕ. It can be shown easily that:

$$\dot{C} = -\epsilon_{s\mu} n_s X^2 - \dot{\alpha}\phi, \tag{27}$$

where:

$$\epsilon_{s\mu} = 2\epsilon_s + \epsilon_\mu = \frac{1}{n_s} \cdot \frac{1}{m_s a_s} \cdot \frac{d(m_s a_s)}{dt}.$$

We have seen that in general the asymmetry parameter α can be of order unity, and that $\dot{\alpha}$ is of order $\alpha n_s \epsilon_0 \sqrt{\mu}$ (Equation (20)). Thus, the two terms in the right-hand side of Equation (27) are *a priori* comparable.

This complicates the problem because the function $g(\phi)$ then depends explicitly upon time, through the factor α. Thus, the graph of $g(\phi)$ (Figure 3) evolves with time, and C is not simply related to the geometry of the orbit.

However, we can still derive a general result concerning the evolution of the Jacobi constant C, using the adiabatic invariance of the action I_ϕ mentioned before (Equation (25)). From Equation (22), it follows that $X \propto J_\phi/\sqrt{m_s a_s}$. Thus, A is given by:

$$A = \frac{I_\phi}{\sqrt{m_s a_s}}. \tag{28}$$

If the particle does not cross the separatrix, then I_ϕ is adiabatically conserved, so that

$$A = A_0 \sqrt{\frac{m_{s0} a_{s0}}{m_s a_s}}, \tag{29}$$

where the subscript 0 denotes quantities at the initial time $t = 0$.

This means that the evolution of the particle orbit depends *only* on the variation of $m_s a_s$, once the initial conditions A_0, m_{s0} and a_{s0} have been specified. In practice, however, it is not simple to relate A to the shape of the trajectory, since this correspondence depends on the particular value of α, see examples in Figure 2.

7.3. CASES WITH α NEGLIGIBLE

A simpler situation can nevertheless be considered by assuming that α has some prescribed values at the beginning and at the end of the transformation, these two values being possibly different. Then, there is a unique correspondence between the area A and the shape of the trajectory at the beginning and at the end of the transformation.

We will consider here an even simpler case where these two prescribed values are equal and are close to zero at the beginning ($t = 0$) and at the end ($T \sim T_{\rm lib}/\epsilon_0$) of the evolution, while it can suffer some prescribed slow and smooth variations in between (with possible episodes where it can reach values of order unity).

At the beginning and at the end of the transformation ($\alpha = 0$), the Jacobi constant C is a univocal function of the area A enclosed in the trajectory $[\phi(t), X(t)]$. Thus, the variation of C between 0 and T only depends on the ratio $(m_s a_s)/(m_{s0} a_{s0})$, once A_0 is specified. We can actually define the quantity:

$$\tau = \log\left(\frac{m_s a_s}{m_{s0} a_{s0}}\right),$$

which is a convenient dimensionless measure of the variation of $m_s a_s$, with $\tau = 0$ by definition at $t = 0$.

Thus, C is a unique function of $\tau - \tau_0$, where $\tau_0 = 2 \log A_0$ accounts for the initial value of A. For $\alpha = 0$, the minimum value reachable by C is 1/2. We will see that in this case it is more convenient to study the variations of $\log(C - 1/2)$, instead of C, as a function of $\tau - \tau_0$.

In summary, we have shown that to within an arbitrary translation of τ_0, there exists a unique function $\mathcal{F}$ such that:

$$\log(C - 1/2) = \mathcal{F}(\tau - \tau_0). \tag{30}$$

Thus, the global evolution of the orbit only depends on the change of $m_s a_s$ (as measured by $\tau - \tau_0$) between the times 0 and T.

The graph of $\mathcal{F}$ is shown in Figure 6, where we use for convenience decimal logarithms along the axes, instead of the natural logarithms used in the text.

All the particles will follow the same graph $\mathcal{F}(\tau - \tau_0)$, the only difference being the speed of displacement of the particle along the graph. This speed is of order $\epsilon_{s\mu}$ (Equation (27)), so that the slower the variations of $m_s a_s$, the slower the evolution of the particle along the graph of $\mathcal{F}$.

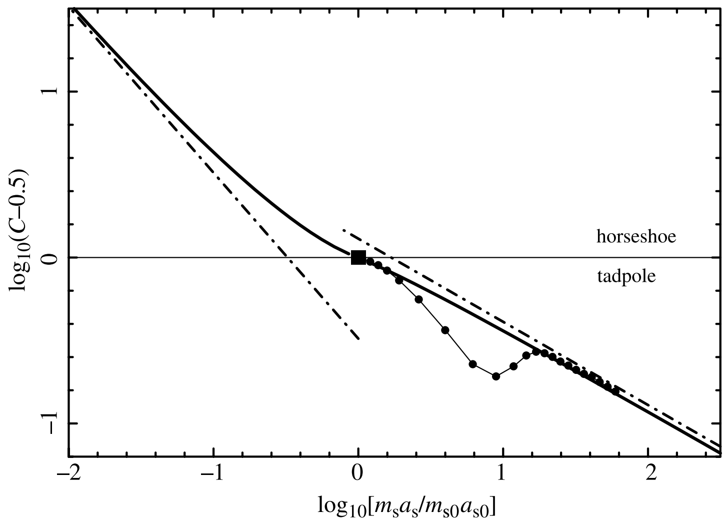

Figure 6. Solid curve: graph of the function $\mathcal{F}$ (Equation (30)), giving the Jacobi constant C of the particle as a function of $m_s a_s$, in the case $\alpha = 0$. Note that we use decimal logarithms in this diagram, while natural logarithms are considered in the text. Note also that the initial condition ($m_s a_s = m_{s0} a_{s0}$, solid square) is arbitrarily chosen so that the particle is at the point L_3, where $C = 3/2$, at $t = 0$. This corresponds to the separating point between horseshoe orbits [with $\log_{10}(C - 0.5) > 0$] and tadpole orbits [with $\log_{10}(C - 0.5) < 0$]. The two dash-dotted straight lines of slopes -1 (at the left) and -0.5 (at the right) are the asymptotic behaviors of $\mathcal{F}$ for very small and very large values of $m_s a_s / m_{s0} a_{s0}$, respectively, see Equations (32) and (31). Bullets connected by the thin curve: example of evolution of C when $\alpha \neq 0$. In this example, α varies smoothly starting from 0 at $t = 0$, when the particle is at the L_3 point of a satellite of mass $m_s = 10^{-4}$, up to $\alpha \approx 0.17$ at $t = 10^5$, and back to $\alpha = 0$ at $t = 2.8 \times 10^5$. Each bullet is plotted at time intervals of $\Delta t = 1.25 \times 10^4$.

Actually, τ can be viewed as a 'slow time', so that all the particles will follow the graph of $\mathcal{F}$ at a speed of order unity, when using the slow time τ as variable.

Note that in this analysis, all the parameters M_0, m_s, a_s and the variations ϵ_M, ϵ_μ, ϵ_s have now completely disappeared. Thus, the function $\mathcal{F}$ gives a kind of 'universal' behavior of co-orbital evolution with slow variations of M_0, m_s and/or a_s, starting and ending with $\alpha = 0$.

7.4. CASE WITH α SIGNIFICANT

For information, a case where α deviates significantly from zero between 0 and T is shown in Figure 6, see the bullets connected by the thin line. In this case, α (and also m_s and a_s) has some prescribed smooth variations with $\alpha = 0$ at the beginning and the end of the integration ($T = 2.8 \times 10^5$), with a peak $\alpha \approx 0.17$ at $t = 10^5$.

At the beginning, when α is small, the particle follows the graph of $\mathcal{F}$, then it deviates significantly for larger α's, and comes back to the graph when α vanishes again. Thus Equation (30) can be used to obtain C, but only at the boundary times $t = 0$ and $t = T$.

7.5. ASYMPTOTIC BEHAVIOR FOR SMALL AND LARGE LIBRATIONS

The Jacobi constant C is a monotonous decreasing function of $m_s a_s$, whatever the individual variations of m_s and a_s may be. For instance, increasing $m_s a_s$ will always tend to decrease C, that is, to reduce the libration amplitude of the particle from a horseshoe orbit to a tadpole one, the particle being eventually attracted towards one of the points L_4 or L_5.

The variation of C as a function of $m_s a_s$ takes simple expressions in the limits where $\Delta C = C - 1/2$ becomes large or small (while keeping $\alpha = 0$).

For small values of ΔC, the particle tends towards, say, the equilateral Lagrange point L_4, where $\Delta C = 0$. Then:

$$\Delta C \approx X^2 + \frac{f_4''}{2}(\phi - \pi/3)^2,$$

where $f_4'' = f''(\pi/3)$.

Near L_4, the quantities X and $\phi - \pi/3$ have harmonic variations, whose amplitudes will be denoted ΔX_m and $\Delta\phi_m$, respectively. Thus $\Delta C \approx \Delta^2 X_m \approx f_4'' \Delta^2\phi_m/2$. The area enclosed in the trajectory (ϕ,X) is then:

$$A \approx \pi \Delta X_m \cdot \Delta\phi_m \approx \pi\sqrt{\frac{2}{f_4''}}\Delta C.$$

Finally, we can express A in terms of $m_s a_s$ and the initial area A_0 (Equation (29)). In the case shown in Figure 6, the particle is initially at L_3, so A_0 is the area enclosed in the separatrix surrounding the point L_4, see one of the shaded zones in the upper left panel of Figure 2. Numerical calculations show that $A_0 = 3.57\ldots$. Furthermore, $f_4'' = 9/4$, so that:

$$\Delta C \approx 1.21 \times \exp(-\tau/2) = 1.21\sqrt{\frac{m_{s0}a_{s0}}{m_s a_s}} \tag{31}$$

for small ΔC, (i.e. here for large $m_s a_s/m_{s0}a_{s0}$) and assuming that the particle is at L_3 at $t = 0$.

When ΔC is large, the particle has a roughly rectangular trajectory in the (X, ϕ) space with a constant $|X|$ most of the time, and a rapid reversal when it approaches the satellite (see for instance the outermost trajectory in the upper left panel of Figure 2). In this case:

$$C \approx X^2$$

$$A \approx 4\pi|X|.$$

Using again Equation (29), we arrive at:

$$\Delta C \approx \left(\frac{A_0}{4\pi}\right)^2 \left(\frac{m_{s0}a_{s0}}{m_s a_s}\right),$$

where now A_0 is the area enclosed inside the separatrix surrounding both L_4 and L_5, see the sum of the two shaded areas in the upper left panel of Figure 2. Thus we now have $A_0 = 2 \times 3.57\ldots = 7.15\ldots$ and:

$$\Delta C \approx 0.324 \times \exp(-\tau) = 0.324\frac{m_{s0}a_{s0}}{m_s a_s} \tag{32}$$

for large ΔC, (i.e. here for small $m_s a_s/m_{s0}a_{s0}$), assuming again that the particle is at L_3 at $t = 0$.

The asymptotic behaviors described in Equations (31) and (32) are plotted as straight lines of slopes $-1/2$ and -1, respectively, in the log–log diagram in Figure 6.

7.6. SMALL LIBRATIONS NEAR L_4 OR L_5

Close to one of the points L_4 or L_5, we saw that $\Delta C \approx f_4''\Delta^2\phi_m/2$ and $\Delta C \propto 1/(m_s a_s)^{1/2}$, thus the amplitude of libration around one of these points is such that:

$$\Delta\phi_m \propto \frac{1}{\sqrt{f_4''}}\frac{1}{(m_s a_s)^{1/4}}, \tag{33}$$

a result already derived by Fleming and Hamilton (2000). This result is valid as long as the asymmetry parameter α is small enough so that the points L_4 or L_5 are not shifted too much.

For large α's, the expression above would be the same, except that f_4'' should be replaced by $g''(\phi_4)$ or $g''(\phi_5)$, the second derivative of $g = f - \alpha\phi$ with respect to ϕ near the new equilibrium point L_4 or L_5. In particular, for the critical value $\alpha = 0.7265\ldots$ (resp. $\alpha = -0.7265\ldots$) the singular point reduces to a cusp in the (ϕ, X) space, see Figure 2, and $g''(\phi_4)$ [resp. $g''(\phi_5)$] tends to zero. In this case, whatever the variations of $m_s a_s$ are, the amplitude $\Delta\phi_m$ diverges, so that the particle crosses the separatrix and escapes the libration region surrounding the equilibrium point.

Note that Equation (33) predicts a rather weak dependence of $\Delta\phi_m$ upon $m_s a_s$. For instance, dividing the libration amplitude $\Delta\phi_m$ by 2 requires multiplying $m_s a_s$ by a large factor of 16. This weak dependence is more generally apparent in Figure 6: significant changes of C require large (by orders of magnitude) variations of $m_s a_s$.

We shall see in the next section that much efficient processes can affect the particle trajectory.

7.7. LARGE HORSESHOE LIBRATIONS

Large horseshoe librations corresponds to large values of C, for which $C \propto 1/(m_s a_s)$ (Equation (32)). If $\phi_{\min}$ denotes the minimum angular distance between the particle and the satellite, then large C's correspond to small values of $\phi_{\min}$. Then $C = f(\phi_{\min}) = -\cos(\phi_{\min}) + 1/|2\sin(\phi_{\min}/2)| \approx 1/\phi_{\min}$, where $\phi_{\min}$ is expressed in radian. Consequently:

$$\phi_{\min} \propto m_s a_s. \tag{34}$$

This behavior will be respected as long as the particle stays away from the satellite Hill's sphere, that is, as long as C remains smaller than $\mu^{-1/3}$, see Section 5.

8. Effects of a Torque on the Particle

Up to now, we have considered that only the satellite exchanges angular momentum with the exterior. However, the particle can also suffer a variation of angular momentum, $\dot{J}_{\rm torque}$, due to a torque. Again, we will assume that this specific torque causes a slow change of the particle semi-major axis, but does not excite its eccentricity.

As we shall see, the effect of the torque can be split into two contributions: one is due to the *average* value of the torque near a_s, while the other one is due to the *spatial variations* of the torque around that average value. Thus, the specific torque acting on the particle, $\dot{J}_{\rm torque}$, can be put under the non-dimensional and normalized form:

$$T(r) = \frac{1}{n_s} \cdot \frac{\dot{J}_{\rm torque}}{J_s} = \epsilon_p [1 + G(\Delta r)],$$

where ϵ_p is a small dimensionless parameter and $G(\Delta r)$ is a dimensionless function of $\Delta r = r - a_s$ which remains of order unity. We can also assume without loss of generality that $G(\Delta r)$ has a mean value of zero around $\Delta r = 0$. The function G thus describes the local behavior of the torque, such that a gradient or more complicated variations due for instance to a sharply defined resonance with another body.

Then, the variation of J_ϕ caused by the external torques acting on the satellite and on the particle reads:

$$\dot{J}_{\phi,\text{torque}} = \dot{J}_{\text{torque}} - \dot{J}_s = (\epsilon_p - \epsilon_s) n_s J_s + \epsilon_p n_s J_s G(\Delta r).$$

The term $(\epsilon_p - \epsilon_s) n_s J_s$ is independent of ϕ, and can thus be absorbed in the Hamiltonian, as before. This merely consists in replacing the asymmetry coefficient $\alpha = \epsilon_s/\mu$ by:

$$\alpha = \frac{\epsilon_s - \epsilon_p}{\mu}$$

in Equation (17). In particular, if the satellite and the particle receive the same specific torque, then $\alpha = 0$ and the phase space is symmetrical. This is easily understood by the fact that in this case the satellite and the particle migrate (due to these torques) at the same rates.

After the elimination of $(\epsilon_p - \epsilon_s) n_s J_s$, we are left with the term

$$\dot{J}_{\phi,\text{torque}} = \epsilon_p n_s J_s G(\Delta r),$$

due to the local spatial variation of the torque near a_s.

Because $\dot{J}_{\phi,\text{torque}}$ now depends on J_ϕ through Δr, it destroys the Hamiltonian nature of the motion. Furthermore, the changes of Δr now occurs on a period which matches (by definition) the libration period of the particle. Consequently, the condition (18) is not satisfied and the action I_ϕ is *no more* conserved, so that the arguments used in Section 7 to follow the evolution of the orbit cannot be used any more.

In this new situation, Equation (27) is replaced by:

$$\dot{C} = -\epsilon_{s\mu} n_s X^2 + 2\epsilon_p n_s \sqrt{\frac{3}{2\mu}} X G(\Delta r), \tag{35}$$

where we assume $\dot{\alpha} = 0$.

Two extreme cases can arise at that point: (1) the torque varies only slightly over the radial excursion of the particle, that is, over a radial distance of order $\Delta r \sim a_s\sqrt{\mu}$. In this case, $G' = dG/dr \ll 1/(a_s\sqrt{\mu})$. Another extreme case occurs when (2) the torque varies significantly over that distance, that is, $G' \gtrsim 1/(a_s\sqrt{\mu})$.

8.1. SMALL TORQUE GRADIENTS

In the first case, we can write $G(r) \approx G'\Delta r$, by remembering that the average value of G around a_s is zero. Using $\Delta r \approx 2(a_s J_\phi / J_s)$ and the definition of X, we obtain:

$$\dot{C} \approx -\epsilon_{s\mu} n_s X^2 + 4\epsilon_p n_s (a_s G') X^2. \quad (36)$$

Consequently, a gradient G' such that $|a_s G'|$ is of order unity can be more efficient in modifying C than are the variation of $m_s a_s$ discussed previously (for comparable $\epsilon_{s\mu}$ and ϵ_p). A torque gradient acting on the particle can thus alone confine the particle near L_4 or L_5 ($\epsilon_p G' < 0$), or on the contrary pull it away from these stable libration regions ($\epsilon_p G' > 0$).

This gradient need not be very strong for this domination to appear. For instance a smooth tidal torque, $T(r) \propto (a_s/r)^3$, yields $a_s G' = -3$, so that the second term in the right-hand side of Equation (36) is one order of magnitude larger than the first one, again for comparable $\epsilon_{s\mu}$ and ϵ_p. Since $\epsilon_p G' < 0$ in this case, the tidal torque will act to confine the particle near L_4 or L_5.

8.2. LARGE TORQUE GRADIENTS

We now consider cases where the torque has a significant variation over Δr, that is, that $G(r)$ has variations of order unity. Then, the second term in the right-hand side of Equation (35) largely dominates, by a factor $\sim 1/\sqrt{\mu}$, the term.

An extreme case of such a situation arises when T is a step function, with for instance $G(\Delta r) = 1$ [that is, $T(r) = 2\epsilon_p$] for $\Delta r < 0$ and $G(\Delta r) = -1$ [that is, $T(r) = 0$] for $\Delta r > 0$. This can happen when a resonance sharply define the location where the torque is applied, see for instance the case of Janus and Prometheus in resonance 7:6 with the outer edge of Saturn's A ring (Lissauer et al., 1985).

Then $XG(r) = -|X|$ and:

$$\dot{C} \approx -2\epsilon_p n_s \sqrt{\frac{3}{2\mu}} |X|. \quad (37)$$

The comparison of Equations (36) and (37) shows that a sharp torque gradient is much more efficient, by the large factor $1/\sqrt{\mu}$, than a slowly varying torque to change C. In the example examined here, $\dot{C} < 0$, and the particle will be very efficiently confined near one of the equilibrium points L_4 or L_5 (it can even reach this equilibrium in a finite time).

An illustration of such an evolution is shown in Figure 7. In this case the particle is submitted to a normalized torque $T(r) = \epsilon_p$ for $r > a_s$ and $T(r) = 0$ for $r < a_s$, with $\epsilon_p < 0$. We have again $\dot{C} < 0$ and we can see the very rapid confinement of the particle near L_4.

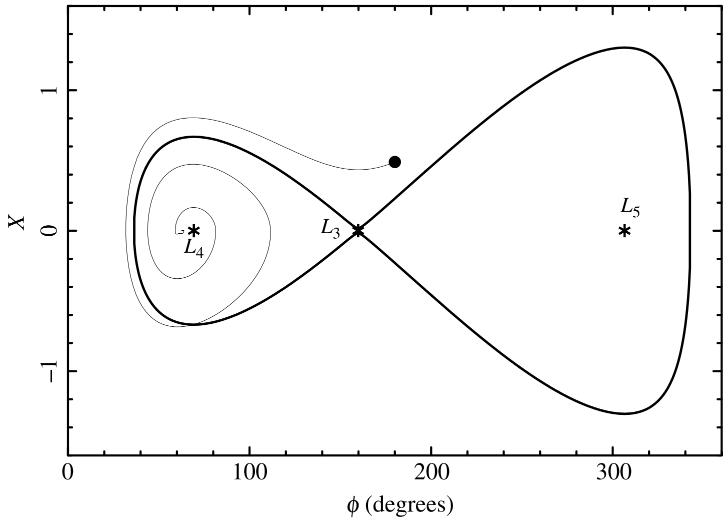

Figure 7. The same integration as in Figure 5, but with $\epsilon_s = 0$, while a negative torque is now applied to the particle, with $\epsilon_p = -3 \times 10^{-5}$. Furthermore this torque is applied only on one side of the orbit, when $X > 0$, which mimics the effects of a sharply defined resonance. Note the very rapid confinement towards L_4, as compared to the slow evolution observed in Figure 5.

The large value of $\dot{C}$ obtained in this situation explains the short time scales associated with the orbital evolution of the Janus–Prometheus system (Lissauer et al., 1985).

9. Conclusions

We have investigated in this paper the effects of slow variations of the parameters on the evolution of a test particle co-orbital with a satellite, in the frame of the restricted circular and planar three-body problem. (The case where the satellite and the particle have comparable masses can actually be reduced to the restricted problem, as shown in the Appendix.)

With no variations of the parameters, the motion of the test particle is ruled by the Hamiltonian of Equation (11), which tells how the angle ϕ between the satellite and the particles evolves with time. This Hamiltonian can have three slowly varying parameters: (1) the mass of the primary, M_0, (2) the mass of the satellite, m_s (with $m_s \ll M_0$) (3) the satellite angular momentum $J_s = \sqrt{\mathcal{G}(M_0 + m_s)a_s}$, where a_s is the satellite orbital radius. Equivalently, the varying parameters can be written as $M = M_0 + m_s$, $\mu = m_s/M$ and $J_s = \sqrt{\mathcal{G}Ma_s}$. We also assume at this stage that no external torque is applied to the particle.

A change of J_s causes an explicit time dependence in the equations of motions of the particle, and thus introduces in the new Hamiltonian (Equation (15)) an asymmetry coefficient $\alpha = \epsilon_s/\mu$, where $\epsilon_s = \dot{J}_s/n_s J_s$ is a non-dimensional measure of the torque applied to the satellite. The coefficient α entirely describes the asymmetry of the phase space with respect to $\phi = \pi$ (where ϕ is the angular separation between the satellite and the particle), caused by the torque ϵ_s. Examples of such asymmetries are given in Figures 2–5.

Note that because $\alpha = \epsilon_s/\mu$ is the ratio of two small quantities, it is not necessarily small itself, and note also that α is itself a varying parameter of the Hamiltonian of Equation (15).

We point out that this asymmetry is caused by an explicit variation of J_s (i.e. by a torque applied to the satellite), and not by a variation of a_s *per se*. For instance, if the satellite migrates because of a variation of M_0 or m_s only, no asymmetry with respect to $\phi = \pi$ will be observed in the motion of the particle.

The long term evolution of the particle motion can be traced thanks to adiabatic invariant arguments, for any libration amplitude of the particle in horseshoe or tadpole orbits.

It then turns out that the evolution of the orbit only depends on the variations of $m_s a_s$, where the variation of a_s can be caused indifferently by a variation of M_0, m_s or J_s. For practical purposes, the follow up of the orbit can make use of Equation (29). This equation yields at any moment a quantity (the area $A = \oint X d\phi$, see Equation (26)) which is uniquely related to the shape of the particle orbit, once α is specified. If α varies, this dependence is complicated, but it becomes simple if we assume that $\alpha = 0$ at the beginning and at the end of the transformation.

In this case, the particle Jacobi constant C (as defined in Equation (22)) depends only on $m_s a_s$, through a function $\mathcal{F}$, see Equation (30) and the graph of $\mathcal{F}$ in Figure 6.

We obtain asymptotic behaviors of $\Delta C = C - 1/2$ versus $m_s a_s$ when ΔC becomes small: $\Delta C \propto (m_s a_s)^{-1/2}$, or large: $\Delta C \propto (m_s a_s)^{-1}$. This allows us to retrieve the dependence of the small libration amplitude $\Delta\phi_m$ near the equilibrium points L_4 and L_5 (i.e. for small ΔC's): $\Delta\phi_m \propto (m_s a_s)^{-1/4}$, see Fleming and Hamilton (2000).

At the other end (ΔC large), we obtain the dependence of the closest angular approach ϕ_{min} between the satellite and the particle: $\phi_{min} \propto m_s a_s$.

The dependence of C versus $m_s a_s$ is actually rather weak, see Figure 6: moderate variations of C require large variations of $m_s a_s$. Now, we show in Section 8 that even moderate torque radial *gradients* applied to the particle can affect its co-orbital motion significantly more than the variations of the parameters M_0, m_s or J_s.

This effect is exacerbated when there is a strong torque gradient over the radial excursion of the particle. Then, this gradient can dominate by orders of magnitude any other effects due to the variations of the parameters. One can compare for instance Figures 5 and 7, where the torque strengths are comparable, but where the torque radial gradients behave very differently.

A consequence of this study is to show that some care must be taken when studying the co-orbital motion with slow variations of the parameters. If no torque radial gradient is exerted on the particle, then all the results presented in the first sections of this paper can be used for any libration amplitude of the particle. If a torque gradient appears on the particle, then Equations (36) or (37) in Section 8 can quantify the importance of these torque gradients.

Note finally that we have considered here only co-rotation sites which share the satellite orbit. All our results are valid, with straightforward modifications, when applied to other co-rotation resonances outside the satellite orbit.

Acknowledgements

This work benefited from discussions with M.J. Duncan, P. Goldreich, D.P. Hamilton and P.D. Nicholson. Part of this research was conducted while B.S. was at Queen's University in Kingston, as a Cave Lecturer. We also thank an anonymous reviewer for several suggestions which improved the presentation of this article.

Appendix

We have considered in the paper that the particle has a zero mass. All the results derived therein can be generalized to the case where the co-orbital satellites have comparable masses, say m_1 and m_2. Each satellite, with polar co-ordinates (r_1, ϑ_1) and (r_2, ϑ_2), is assumed to have zero orbital eccentricity. The angular momenta of the satellites with respect to the planet are denoted $\mathcal{J}_1$ and $\mathcal{J}_2$, while their specific angular momenta are denoted $J_1 = \mathcal{J}_1/m_1$ and $J_2 = \mathcal{J}_2/m_2$.

The derivation of the equations of motion for co-orbital massive satellites is briefly summarized here. Classical calculations (see for instance Yoder et al., 1983, 1989) show that the system can now be described by the Hamiltonian:

$$\mathcal{H}_5(\vartheta_1, \mathcal{J}_1; \vartheta_2, \mathcal{J}_2) = -m_1^3 \frac{(\mathcal{G}M)^2}{2\mathcal{J}_1^2} - m_2^3 \frac{(\mathcal{G}M)^2}{2\mathcal{J}_2^2} - r_a^2 n_a^2 \frac{m_1 m_2}{M} f(\vartheta_2 - \vartheta_1), \tag{A.1}$$

where r_a and n_a are the average orbital radius and average mean motion of the two satellites, see below, and $M = M_0 + m_1 + m_2$. Note that contrarily to $\mathcal{H}_2$ in Equation (9), the Hamiltonian $\mathcal{H}_5$ has now the dimension of an energy (and not of a specific energy). Note also that we neglect again the terms arising from the orbital eccentricities and inclinations of the co-orbital satellites (Yoder et al., 1983, 1989).

Because $\mathcal{H}_5$ depends only on the angular variable $\phi = \vartheta_2 - \vartheta_1$, one can use as angular variables and conjugate momenta:

$$\vartheta_a = \frac{m_1\vartheta_1 + m_2\vartheta_2}{m_1 + m_2} \quad \mathcal{J}_a = \mathcal{J}_1 + \mathcal{J}_2 \tag{A.2}$$

$$\phi = \vartheta_2 - \vartheta_1 \quad \mathcal{J}_\phi = \frac{m_1\mathcal{J}_2 - m_2\mathcal{J}_1}{m_1 + m_2}. \tag{A.3}$$

Since $\mathcal{H}_5$ does not depends on ϑ_a, the sum of the angular momenta of the two satellites, $\mathcal{J}_a$, is conserved. This is not true in general, since the conservation of angular momentum concerns the total system of the two satellites *and* the planet, not the two satellites only. The conservation of $\mathcal{J}_a$ is true on the average only, and stems from the approximation $r_1 \approx r_2$.

Instead of $\mathcal{J}_a$ and $\mathcal{J}_\phi$, one can work with the specific momenta:

$$J_a = \frac{\mathcal{J}_a}{m_1 + m_2} = \frac{m_1 J_1 + m_2 J_2}{m_1 + m_2} \tag{A.4}$$

$$J_\phi = \frac{\mathcal{J}_\phi}{m_r} = J_2 - J_1, \tag{A.5}$$

where m_r is the reduced mass $m_r = m_1 m_2/(m_1 + m_2)$ of the two satellites.

Because $r_1 \approx r_2$, we have $|J_\phi| \ll |J_a|$, so that the average mean motion of the two satellites, $n_a = d\vartheta_a/dt = \partial\mathcal{H}_5/\partial\mathcal{J}_a = (\partial\mathcal{H}_5/\partial J_a)/(m_1 + m_2)$, depends only on J_a and is thus constant. We can define the average orbital radius of the two satellites, r_a, by Kepler's third law $r_a^2 n_a^2 = \mathcal{G}M$. Then:

$$r_a \approx \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

We are thus left with the variations of (ϕ, $\mathcal{J}_\phi$) as described by the Hamiltonian $\mathcal{H}_5$, or also, with the variations of (ϕ,J_ϕ), as described by $\mathcal{H}_5/m_r$. As in Section 4.1, we can expand $\mathcal{H}_5/m_r$ near J_a, using $|J_\phi| \ll J_a$. Elementary calculations on the expression of $\mathcal{H}_5$ in Equation (A1) then show that ϕ and J_ϕ are conjugates for the Hamiltonian:

$$\mathcal{H}_6 = -\left(\frac{\mathcal{G}M}{J_a}\right)^2 \left[\frac{3}{2}\left(\frac{J_\phi}{J_a}\right)^2 + \mu f(\phi)\right], \tag{A.6}$$

where $\mu = (m_1 + m_2)/(M_0 + m_1 + m_2)$. This is similar to the expression obtained in Equation (11), except that J_s has been replaced by J_a.

In summary, the system of two satellites can be described by an equivalent, reduced system composed of an 'averaged satellite', with mass $m_1 + m_2$, specific angular momentum J_a and orbital radius r_a, interacting with a co-orbital test particle of longitude $\vartheta_a + \phi$ and specific angular momentum $J_\phi = J_a + J_\phi$. The inversion of Equations (A.2) and (A.3) then yields the individual motion of each satellite from the motion of the average satellite and the test particle.

Equation (A.6) shows that eventually we are led to the same problem as before: M and μ can be varied slowly according to small parameters $\epsilon_M = \dot{M}/(n_a M)$ and $\epsilon_\mu = \dot{\mu}/(n_a \mu)$, with exactly the same results as before.

Furthermore, we can describe the specific torques $\dot{J}_{1,\mathrm{torque}}$ and $\dot{J}_{2,\mathrm{torque}}$ applied to the satellites 1 and 2 by two small parameters ϵ_1 and ϵ_2, and two dimensionless functions T_1 and T_2 of zero mean near r_a, such that:

$$\frac{1}{n_s} \cdot \frac{\dot{J}_{1,\text{torque}}}{J_a} = \epsilon_1 \left[1 + G_1(\Delta r_1)\right] \tag{A.7}$$

$$\frac{1}{n_s} \cdot \frac{\dot{J}_{2,\text{torque}}}{J_a} = \epsilon_2 \left[1 + G_2(\Delta r_2)\right], \tag{A.8}$$

where $\Delta r_1 = r_1 - r_a$ and $\Delta r_2 = r_2 - r_a$ (so that $m_1 \Delta r_1 + m_2 \Delta r_2 \approx 0$). These torques cause a general drift of J_a at a rate characterized by:

$$\epsilon_a = \frac{1}{n_s} \cdot \frac{\dot{J}_a}{J_a} = \frac{m_1 \epsilon_1 + m_2 \epsilon_2}{m_1 + m_2}, \tag{A.9}$$

which replaces the parameter ϵ_s defined in Equation (14).

Then it is easy to see that the variations of $(\phi,\ J_\phi)$ are described by the Hamiltonian

$$\mathcal{H}_7(\phi, J_\phi) = -\left(\frac{\mathcal{G}M}{J_a}\right)^2 \left[\frac{3}{2}\left(\frac{J_\phi}{J_a}\right)^2 + \mu g(\phi)\right], \tag{A.10}$$

where $g(\phi) = f(\phi) - \alpha\phi$, with the new asymmetry coefficient:

$$\alpha = \frac{\epsilon_1 - \epsilon_2}{\mu}, \tag{A.11}$$

yielding the same results as in Section 6.

Finally, the Jacobi constant of the reduced problem, $C = X^2 + g(\phi)$, where $X = \sqrt{3/2\mu}(J_\phi/J_a)$ now suffers variations:

$$\dot{C} = -\epsilon_{a\mu} n_s X^2 + 2n_a \sqrt{\frac{3}{2\mu}} X[\epsilon_2 G_2(\Delta r_2) - \epsilon_1 G_1(\Delta r_1)], \tag{A.12}$$

where $\epsilon_{a\mu} = 2\epsilon_a + \epsilon_\mu$, with results similar to those of Section 8, after replacing $\epsilon_p G(\Delta r)$ by $\epsilon_2 G_2(\Delta r_2) - \epsilon_1 G_1(\Delta r_1)$. From $J_\phi \approx J_a(\Delta r_1 - \Delta r_2)/2r_a$ and $m_1 \Delta r_1 + m_2 \Delta r_2 \approx 0$, we can express Δr_1 and Δr_2 in terms of J_ϕ, and then, X. Consequently, in the approximation of small torque gradients, the equation above can be written:

$$\dot{C} \approx -\epsilon_{a\mu} n_s X^2 + 4n_a r_a \frac{\epsilon_2 G_2' m_1 + \epsilon_1 G_1' m_2}{m_1 + m_2} X^2, \tag{A.13}$$

identical to Equation (36), except that $\epsilon_p G'$ has been replaced by $(\epsilon_2 G_2' m_1 + \epsilon_1 G_1' m_2)/(m_1 + m_2)$.

In the approximation of large torque gradients, in particular when the torque is a step function around r_a, Equation (37) is replaced by:

$$\dot{C} \approx -2(\epsilon_1 + \epsilon_2) n_s \sqrt{\frac{3}{2\mu}} |X|, \tag{A.14}$$

that is, the coefficient ϵ_p is replaced by $\epsilon_1 + \epsilon_2$.

Note that the coefficients ϵ_1 and ϵ_2 enter through their difference in Equation (A.11) (i.e. when the *torques* are concerned), while they appear in an additive manner in Equations (A.13) and (A.14), when the effects of the *gradients* of the torques are considered.

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