Int. J. Contemp. Math. Sciences, Vol. 3, 2008, no. 32, 1557 - 1567

Co-ordinated s-Convex Function in the First Sense with Some Hadamard-Type Inequalities

Mohammad Alomari [1] and Maslina Darus [2]

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor, Malaysia

Abstract

In this paper a Hadamard's type inequality of s–convex function in first sense and s–convex function of 2–variables on the co–ordinates are given. A monotonic nondecreasing mapping connected with the Hadamard's inequality for Lipschitzian s–convex mapping in the first sense of one variable is established.

Keywords: Hadamard's inequality, s–Convex function, Co–ordinated s–convex function.

1 Introduction

Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex mapping defined on the interval I of real numbers and a, $b \in I$, with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as Hadamard's inequality for convex mappings.

[1]First author: alomari@math.com
[2]Corresponding author: maslina@pkrisc.cc.ukm.my

In [9], Orlicz introduced two definitions of s–convexity of real valued functions. A function $f : \mathbf{R}^+ \to \mathbf{R}$, where $\mathbf{R}^+ = [0, \infty)$, is said to be s–convex in the first sense if

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y) \tag{2}$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \ge 0$ with $\alpha^s + \beta^s = 1$ and for some fixed $s \in (0, 1]$. We denote this class of functions by K_s^1.

Also, a function $f : \mathbf{R}^+ \to \mathbf{R}$, where $\mathbf{R}^+ = [0, \infty)$, is said to be s–convex in the second sense if

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y) \tag{3}$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. We denote this class of functions by K_s^2.

These definitions of s–convexity, for so called φ–functions, was introduced by Orlicz in [9] and was used in the theory of Orlicz spaces (see [7], [8], [10]). A function $f : \mathbf{R}^+ \to \mathbf{R}^+$ is said to be φ–function if $f(0) = 0$ and f is non–decreasing and continuous. Its easily to check that the both s–convexity mean just the convexity when $s = 1$.

In [4], Hudzik and Maligrada considered among others the class of functions which are s–convex in the first sense. This class is defined in the following way:

A function $f : [0, \infty) \to \mathbf{R}$ is said to be *s–convex in the first sense* if

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y) \tag{4}$$

holds for all $x, y \in [0, \infty)$, $\alpha, \beta \ge 0$ with $\alpha^s + \beta^s = 1$ and for some fixed $s \in (0, 1]$. It can be easily seen that every 1–convex function is convex.

Also, in [4], Hudzik and Maligrada proved a variant properties of s–convex function in the first and in the second sense, let us take the following theorem.

Theorem 1.1 *Let $0 < s \le 1$. If $f \in K_s^2$ and $f(0) = 0$ then $f \in K_s^1$.*

In [5] Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s–convex functions in the first sense.

Theorem 1.2 *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s–convex function in the first sense, where $s \in (0, 1)$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1[0, 1]$, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a) + sf(b)}{s+1}. \tag{5}$$

The above inequalities are sharp.

Also, in [5], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s–convex functions in the second sense.

Theorem 1.3 *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s–convex function in the second sense, where $s \in (0, 1)$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1[0, 1]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a) + f(b)}{s+1} \tag{6}$$

the constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3). The above inequalities are sharp.

After that, in [6], Dragomir established the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbf{R}^2$.

Theorem 1.4 *Suppose that $f : \Delta \to \boldsymbol{R}$ is co-ordinated convex on Δ. Then one has the inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y)\,dydx \\ &\le \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \end{aligned} \tag{7}$$

The above inequalities are sharp.

In this paper we will point out a Hadamard–type inequality of s–convex function in first sense and s–convex functions of 2–variables on the co–ordinates. A monotonic nondecreasing mapping connected with the Hadamard's inequality for Lipschitzian s–convex mapping in the first sense of one variable is given.

For refinements, counterparts, generalizations and new Hadamard's–type inequalities see [1–6].

2 Hadamard's Inequality

In [2], Alomari and Darus established the definition of s–convex function in the second sense on co–ordinates. Similarly, one can define the s–convex function in the first sense on co–ordinates, as follows:

Definition 2.1 *Consider the bidimensional interval $\Delta := [a,b] \times [c,d]$ in $[0,\infty)^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \to \boldsymbol{R}$ is s–convex in the first sense on Δ if*

$$f(\alpha x + \beta z, \alpha y + \beta w) \leq \alpha^s f(x,y) + \beta^s f(z,w),$$

holds for all $(x,y),(z,w) \in \Delta$ with $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$ and for some fixed $s \in (0,1]$.

Therefore, one can talk about co–ordinated s–convex function in the first sense, as follows:

A function $f : \Delta \to \mathbf{R}$ is s–convex in the first sense on Δ is called co–ordinated s–convex in the first sense on Δ if the partial mappings $f_y : [a,b] \to \mathbf{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbf{R}$, $f_x(v) = f(x,v)$, are s–convex in the first sense for all $y \in [c,d]$ and $x \in [a,b]$ such that $s \in (0,1]$, i.e, the partial mappings f_y and f_x s–convex with same fixed $s \in (0,1]$.

The following inequalities is considered the Hadamard–type inequalities for s–convex function in the first sense on the co–ordinates.

Theorem 2.2 *Suppose that $f : \Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to [0,\infty)$ is s–convex function on the co–ordinates in the first sense on Δ. Then one has the inequalities:*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \qquad (8)\\
\leq\ & \frac{1}{2(s+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{s}{b-a}\int_a^b f(x,d)\,dx\right. \\
& \left. + \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{s}{d-c}\int_c^d f(b,y)\,dy\right] \\
\leq\ & \frac{f(a,c) + sf(b,c) + sf(a,d) + s^2 f(b,d)}{(s+1)^2}.
\end{aligned}$$

The above inequalities are sharp.

Proof. Since $f : \Delta \to \mathbf{R}$ is co–ordinated s–convex in first sense on Δ it follows that the mapping $g_x : [c,d] \to [0,\infty)$, $g_x(y) = f(x,y)$ is s–convex on $[c,d]$ for all $x \in [a,b]$. Then by s–Hadamard's inequality (5) one has:

$$g_x\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d g_x(y)\,dy \leq \frac{g_x(c)+sg_x(d)}{s+1}, \quad \forall x \in [a,b].$$

That is,

$$f\left(x,\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d f(x,y)\,dy \leq \frac{f(x,c)+sf(x,d)}{s+1}, \quad \forall x \in [a,b].$$

Integrating this inequality on $[a,b]$, we have

$$\begin{aligned} \frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\ &\leq \frac{1}{s+1}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{s}{b-a}\int_a^b f(x,d)\,dx\right]. \end{aligned} \tag{9}$$

A similar arguments applied for the mapping $g_y : [a,b] \to [0,\infty)$, $g_y(x) = f(x,y)$, we get

$$\begin{aligned} \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy &\leq \frac{1}{(d-c)(b-a)}\int_c^d\int_a^b f(x,y)\,dxdy \\ &\leq \frac{1}{s+1}\left[\frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{s}{d-c}\int_c^d f(b,y)\,dy\right]. \end{aligned} \tag{10}$$

Summing the inequalities (9) and (10), we get the second and the third inequalities in (8).

Therefore, by s–Hadamard's inequality (5), we also have:

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \tag{11}$$

and

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \tag{12}$$

which give, by addition the first inequality in (8).

Finally, by the same inequality we can also state:

$$\frac{1}{b-a}\int_a^b f(x,c)\,dx \le \frac{f(a,c)+sf(b,c)}{s+1}$$

$$\frac{1}{b-a}\int_a^b f(x,d)\,dx \le \frac{f(a,d)+sf(b,d)}{s+1}$$

$$\frac{1}{d-c}\int_c^d f(a,y)\,dy \le \frac{f(a,c)+sf(a,d)}{s+1}$$

and

$$\frac{1}{d-c}\int_c^d f(b,y)\,dy \le \frac{f(b,c)+sf(b,d)}{s+1}$$

which give, by addition the last inequality in (8).

Remark 2.3 *In (8) if $s=1$ then the inequality reduced to inequality (7).*

Corollary 2.4 *Suppose that $f:\Delta=[a,b]\times[a,b]\subseteq[0,\infty)^2\to[0,\infty)$ is s–convex function on the co–ordinates in the first sense on Δ. Then one has the inequalities:*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{a+b}{2}\right) \\
\le\ & \frac{1}{2(b-a)}\int_a^b\left\{f\left(x,\frac{a+b}{2}\right)+f\left(\frac{a+b}{2},x\right)\right\}dx \\
\le\ & \frac{1}{(b-a)^2}\int_a^b\int_a^b f(x,y)\,dydx \qquad (13)\\
\le\ & \frac{1}{(s+1)(b-a)}\int_a^b\{f(x,a)+f(a,x)+s[f(x,b)+f(b,x)]\}\,dx \\
\le\ & \frac{f(a,a)+sf(b,a)+sf(a,b)+s^2f(b,b)}{(s+1)^2}.
\end{aligned}$$

The above inequalities are sharp.

Corollary 2.5 *In Corollary 2.4 if in addition f is symmetric,* i.e, $f(x,y) = f(y,x)$ *for all* $(x,y) \in [a,b] \times [a,b]$, *we have*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \\
\leq\ & \frac{1}{(b-a)} \int_a^b f\left(x, \frac{a+b}{2}\right) dx \\
\leq\ & \frac{1}{(b-a)^2} \int_a^b \int_a^b f(x,y)\, dy dx \\
\leq\ & \frac{2}{(s+1)(b-a)} \int_a^b \{f(x,a) + sf(x,b)\}\, dx \\
\leq\ & \frac{f(a,a) + 2sf(a,b) + s^2 f(b,b)}{(s+1)^2}.
\end{aligned} \tag{14}$$

The above inequalities are sharp.

Now, the following inequality is considered the mapping connected with the inequalities in (5) and (6), as follows:

Theorem 2.6 *Let $f : [a,b] \subseteq [0,\infty) \to [0,\infty)$ be s–convex function in the second sense on $[a,b]$ and $f(0) = 0$. Define a function $H : [0,1] \to \boldsymbol{R}$ be such that*

$$H(t) = \begin{cases} \frac{1}{s+1}\left[f(tb + (1-t)a) + sf(ta + (1-t)b)\right], & 0 \leq t \leq s \\ \\ \frac{1}{s+1}\left[f(tb + (1-t)a) + f(a)\right], & s \leq t \leq 1. \end{cases}$$

Then,

(1) H is s–convex in the first sense on $[0,1]$.

(2) H is non–decreasing function on $[0,1]$.

(3) We have the bounds:

$$\begin{aligned}
\inf_{t\in[0,1]} H(t) = \frac{f(a) + sf(b)}{s+1} = H(0) \quad & \leq \quad H(t) \\
& \leq \quad H(1) = \frac{f(a) + f(b)}{s+1} = \sup_{t\in[0,1]} H(t).
\end{aligned}$$

Proof. Suppose that $f : [a,b] \subseteq [0,\infty) \to [0,\infty)$ be s–convex function in the second sense on $[a,b]$ and $f(0) = 0$. Then by Theorem 1.1 f is s–convex function in the first sense on $[a,b]$.

1. Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$. To show that H is s–convex we have three non-trivial cases:

   (a) For $t_1, t_2 \in [0, s]$

$$\begin{aligned}
& H(\alpha t_1 + \beta t_2) \\
= & \frac{1}{s+1}\left[f\left(\left(\alpha t_1 + \beta t_2\right) b + \left(1 - \alpha t_1 + \beta t_2\right) a\right)\right. \\
& \left. + sf\left(\left(\alpha t_1 + \beta t_2\right) a + \left(1 - \alpha t_1 + \beta t_2\right) b\right)\right] \\
= & \frac{1}{s+1}\left[f\left(\alpha\left(t_1 b + \left(1 - t_1\right) a\right) + \beta\left(t_2 b + \left(1 - t_2\right) a\right)\right)\right. \\
& \left. + sf\left(\alpha\left(t_1 a + \left(1 - t_1\right) b\right) + \beta\left(t_2 a + \left(1 - t_2\right) b\right)\right)\right] \\
\leq & \frac{1}{s+1}\left[\alpha^s f\left(t_1 b + \left(1 - t_1\right) a\right) + \beta^s f\left(t_2 b + \left(1 - t_2\right) a\right)\right. \\
& \left. + s\left(\alpha^s f\left(t_1 a + \left(1 - t_1\right) b\right) + \beta^s f\left(t_2 a + \left(1 - t_2\right) b\right)\right)\right] \\
= & \alpha^s \frac{f\left(t_1 b + \left(1 - t_1\right) a\right) + sf\left(t_1 a + \left(1 - t_1\right) b\right)}{s+1} \\
& + \beta^s \frac{f\left(t_2 b + \left(1 - t_2\right) a\right) + sf\left(t_2 a + \left(1 - t_2\right) b\right)}{s+1} \\
= & \alpha^s H(t_1) + \beta^s H(t_2)
\end{aligned}$$

   (b) For $t_1, t_2 \in [s, 1]$

$$\begin{aligned}
& H(\alpha t_1 + \beta t_2) \\
= & \frac{1}{s+1}\left[f\left(\left(\alpha t_1 + \beta t_2\right) b + \left(1 - \alpha t_1 + \beta t_2\right) a\right) + f(a)\right] \\
= & \frac{1}{s+1}\left[f\left(\alpha\left(t_1 b + \left(1 - t_1\right) a\right) + \beta\left(t_2 b + \left(1 - t_2\right) a\right)\right) + f(a)\right] \\
\leq & \frac{1}{s+1}\left[\alpha^s f\left(t_1 b + \left(1 - t_1\right) a\right) + \beta^s f\left(t_2 b + \left(1 - t_2\right) a\right) + f(a)\right] \\
= & \alpha^s \frac{f\left(t_1 b + \left(1 - t_1\right) a\right) + f(a)}{s+1} + \beta^s \frac{f\left(t_2 b + \left(1 - t_2\right) a\right) + f(a)}{s+1} \\
= & \alpha^s H(t_1) + \beta^s H(t_2)
\end{aligned}$$

   (c) Without loss of generality, assume that $t_1 \in [0, s]$ and $t_2 \in [s, 1]$. Now, since $0 \leq t_1 \leq s$ and $s \leq t_2 \leq 1$, then $0 \leq \alpha t_1 \leq \alpha s$ and $\beta s \leq \beta t_2 \leq \beta$, therefore, $\beta s \leq s \leq \alpha t_1 + \beta t_2 \leq \alpha s + \beta$. Hence, $\alpha t_1 + \beta t_2 \in [s, 1]$ and by case (b) above we obtain

$$H(\alpha t_1 + \beta t_2) \leq \alpha^s H(t_1) + \beta^s H(t_2)$$

   which shows that H is s–convex in the first sense on $[0, 1]$.

2. Let $t_1, t_2 \in [0,1]$ and without loss of generality assume that $0 \le t_1 \le t_2$. Since f is s–convex in the first sense then f is non–decreasing on $(0,\infty)$.

   Now, If $0 \le t_1 \le t_2 \le s$, then it's easy to see that $H(t_1) \le H(t_2)$. Also, if $s \le t_1 \le t_2 \le 1$, then one can see that $H(t_1) \le H(t_2)$.

   It remains to check $t_1 \le s \le t_2$, to get that, it suffices to show that the function $g(s) = f(a) + sf(sa + (1-s)b)$ is non–decreasing on $[t_1, t_2]$. Therefore,

   $$g(t_1) = f(a) + t_1 f(t_1 a + (1-t_1) b) \le f(a) + t_2 f(t_2 a + (1-t_2) b) = g(t_2),$$

   with $g(t_1 = 0) = f(a)$ and $g(t_2 = 1) = 2f(a)$, which shows that H is non–decreasing on $[0,1]$.

3. It follows from (2) that, for all $t \in [0,1]$,

   $$H(0) = \frac{f(a) + sf(b)}{s+1} \le H(t) \le \frac{f(a) + f(b)}{s+1} = H(1)\ .$$

This completes the proof.

Corollary 2.7 *If $f : [a,b] \subseteq [0,\infty) \to [0,\infty)$ be s–convex function in the first sense on $[a,b]$. Then, the result above in Theorem 2.6 holds.*

Theorem 2.8 *Let $f : [a,b] \to \boldsymbol{R}$ satisfy Lipschitzian conditions. That is, for all $t_1, t_2 \in [0,1]$, we have*

$$|f(t_1) - f(t_2)| \le L|t_1 - t_2|$$

where, L is positive constant. Then

$$|H(t_1) - H(t_2)| \le \begin{cases} L(b-a)|t_1 - t_2| & ,\ 0 \le t_1 \le t_2 \le s \le 1 \\ \frac{L(b-a)}{s+1}|t_1 - t_2| & ,\ 0 < s \le t_1 \le t_2 \le 1 \\ \frac{L(b-a)}{s+1}\left(|t_1 - t_2| + |1 - t_1|\right) & ,\ 0 \le t_1 \le s \le t_2 \le 1 \end{cases} \quad (15)$$

Proof. For $t_1, t_2 \in [0,1]$, we have two cases:

1. If $0 \le t_1 \le t_2 \le s \le 1$, then

$$\begin{aligned} |H(t_1) - H(t_2)| &= \frac{1}{s+1}\left|f(t_1 b + (1-t_1) a) + sf(t_1 a + (1-t_1) b)\right. \\ &\quad \left.- \left[f(t_2 b + (1-t_2) a) + sf(t_2 a + (1-t_2) b)\right]\right| \\ &\le \frac{1}{s+1}\left|f(t_1 b + (1-t_1) a) - f(t_2 b + (1-t_2) a)\right| \\ &\quad + \frac{s}{s+1}\left|f(t_1 a + (1-t_1) b) - f(t_2 a + (1-t_2) b)\right| \\ &\le L(b-a)|t_1 - t_2| \end{aligned}$$

2. If $0 < s \leq t_1 \leq t_2 \leq 1$, then

$$\begin{aligned}
|H(t_1) - H(t_2)| &= \frac{1}{s+1} \left| f(t_1 b + (1-t_1) a) + f(a) \right. \\
&\quad \left. - [f(t_2 b + (1-t_2) a) + f(a)] \right| \\
&\leq \frac{1}{s+1} \left| f(t_1 b + (1-t_1) a) - f(t_2 b + (1-t_2) a) \right| \\
&\leq \frac{L}{s+1} (b-a) |t_1 - t_2|
\end{aligned}$$

3. Without loss of generality, if $0 \leq t_1 \leq s$ and $s \leq t_2 \leq 1$, then

$$\begin{aligned}
|H(t_1) - H(t_2)| &= \frac{1}{s+1} \left| f(t_1 b + (1-t_1) a) + s f(t_1 a + (1-t_1) b) \right. \\
&\quad \left. - [f(t_2 b + (1-t_2) a) + f(a)] \right| \\
&\leq \frac{1}{s+1} \left| f(t_1 b + (1-t_1) a) - f(t_2 b + (1-t_2) a) \right| \\
&\quad + \frac{1}{s+1} \left| s f(t_1 a + (1-t_1) b) - f(a) \right| \\
&\leq \frac{L}{s+1} (b-a) |t_1 - t_2| + \frac{1}{s+1} \left| f(t_1 a + (1-t_1) b) - f(a) \right| \\
&\leq \frac{L}{s+1} (b-a) |t_1 - t_2| + \frac{L}{s+1} (b-a) |1 - t_1| \\
&\leq \frac{L}{s+1} (b-a) (|t_1 - t_2| + |1 - t_1|)
\end{aligned}$$

This completes the proof.

Remark 2.9 *In (15) if we take $t_1 = 1$ and $t_2 = 0$, then (15) reduce to*

$$|H(1) - H(0)| = |f(b)| \leq \frac{L(b-a)}{1-s} \tag{16}$$

where, $0 < s < 1$.

The inequality (16) is the s–Hadamard–type inequality for Lipschitzian s–convex mapping in the first sense of one variable.

Acknowledgement. The work here is supported by the Grant: UKM–GUP–TMK–07–02–107.

References

[1] M. Alomari and M. Darus, A mapping connected with Hadamard–type inequalties in 4–variables, *Int. Journal of Math. Analysis,* **2** (13) (2008), 601–628.

[2] M. Alomari and M. Darus, The Hadamard's inequality for s–convex function of 2–variables On The co–ordinates, *Int. Journal of Math. Analysis,* **2** (13) (2008), 629-638.

[3] M. Alomari and M. Darus, The Hadamard's inequality for s–convex function, *Int. Journal of Math. Analysis,* **2** (13) (2008), 639-646.

[4] H. Hudzik, L. Maligranda, Some remarks on s–convex functions, *Aequationes Math.,* **48** (1994), 100-111.

[5] S.S. Dragomir, S. Fitzpatrick, The Hadamard's inequality for s-convex functions in the second sense, *Demonstratio Math.,* **32** (4) (1999), 687-696.

[6] S. S. Dragomir, On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese Journal of Mathematics*, **5** (2001), 775–788.

[7] W. Matuszewska and W. Orlicz, A note on the theorey of s–normed spaces of φ–integrable functions, *studia Math.,* **21** (1981), 107–115.

[8] J. Musielak, Orlicz soaces and modular spaces, *Lecture Notes in Mathematics, Vol. 1034*, Springer–Verlag, New York / Berlin, 1983.

[9] W. Orlicz, A note on modular spaces, *I, Bull. Acad. Polon. Sci. Math. Astronom. Phys.,* **9** (1961), 157–162.

[10] S. Rolewicz, Metric Linear Spaces, 2nd ed., PW N, Warsaw, 1984.

Received: May 5, 2008