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Coadjoint Orbits of the Virasoro Group

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Abstract. The coadjoint orbits of the Virasoro group, which have been investigated by Lazutkin and Pankratova and by Segal, should according to the Kirillov-Kostant theory be related to the unitary representations of the Virasoro group. In this paper, the classification of orbits is reconsidered, with an explicit description of the possible centralizers of coadjoint orbits. The possible orbits are $\mathrm{diff}(S^1)$ itself, $\mathrm{diff}(S^1)/S^1$, and $\mathrm{diff}(S^1)/SL^{(n)}(2,R)$, with $SL^{(n)}(2,R)$ a certain discrete series of embeddings of $SL(2,R)$ in $\mathrm{diff}(S^1)$, and $\mathrm{diff}S^1/T$, where T may be any of certain rather special one parameter subgroups of $\mathrm{diff}S^1$. An attempt is made to clarify the relation between orbits and representations. It appears that quantization of $\mathrm{diff}S^1/S^1$ is related to unitary representations with nondegenerate Kac determinant (unitary Verma modules), while quantization of $\mathrm{diff}S^1/SL^{(n)}(2,R)$ is seemingly related to unitary representations with null vectors in level n. A better understanding of how to quantize the relevant orbits might lead to a better geometrical understanding of Virasoro representation theory. In the process of investigating Virasoro coadjoint orbits, we observe the existence of left invariant symplectic structures on the Virasoro group manifold. As is described in an appendix, these give rise to Lie bialgebra structures with the Virasoro algebra as the underlying Lie algebra.

1. Introduction

The representation theory of finite dimensional compact semi-simple Lie groups is greatly clarified by the Borel-Weil-Bott theorem, whose content is roughly as follows. Let G be a compact semi-simple Lie group, and T a maximal torus. The quotient G/T (defined by the equivalence relation $g \sim gt$, with $g \in G$, $t \in T$) is a compact, complex manifold. The theorem in question interprets the irreducible unitary representations of G as spaces of holomorphic sections of certain holomorphic line bundles over G/T.

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More precisely, let R be a finite dimensional irreducible representation of G. The space of highest weight vectors is always a one dimensional vector space V, which furnishes a one dimensional representation (a character) of T. The product space $G \times V$ is a trivial line bundle over the G manifold. Its quotient by the action of T, i.e. by the equivalence relation $(g, v) \sim (gt, t^{-1}v)$ for $g \in G$, $v \in V$, $t \in T$, is a holomorphic line bundle L over G/T. L admits the G action $(g, v) \to (g'g, v)$ for $g' \in G$, so the space of holomorphic sections of L is naturally a G representation. One shows that this latter representation is in fact isomorphic to R. Among other things, this makes it possible to interpret the Weyl character formula in terms of fixed point formulas for the G action on G/T.

If one wishes to study the representation theory of a semi-simple Lie group which is not compact, this procedure must be generalized. A very illuminating approach has been the coadjoint orbit method of Kirillov and Kostant. In this method, one generalizes G/T to certain homogeneous spaces G/H which can be realized as coadjoint orbits; a coadjoint orbit is the orbit of a vector in the dual of the Lie algebra of G under the G action. One finds that a coadjoint orbit, say the orbit W_b of a coadjoint vector b, always has a natural symplectic structure. One attempts to quantize the symplectic variety W_b (i.e., to construct the quantum mechanical Hilbert space for a particle whose phase space is W_b). This will be impossible unless b obeys certain conditions, connected for instance with anomalies. For b such that W_b is quantizable, the quantum Hilbert space will furnish a unitary representation of the symmetry group G of W_b. One attempts to construct all unitary representations from quantization of suitable orbits. In case G is compact, it suffices to consider orbits of the form G/T, and quantization of those orbits amounts to constructing the spaces of sections of holomorphic line bundles as in the Borel-Weil-Bott theorem.

Let us now consider some of the infinite dimensional Lie groups which seem to be important in physics. Let G be a compact Lie group, LG the corresponding loop group, and $\widehat{LG}$ its universal central extension. The theory of highest weight representations of $\widehat{LG}$ is very similar in flavor to the representation theory of finite dimensional compact Lie groups, and in particular it is known that these representations can be constructed as sections of holomorphic line bundles over $\widehat{LG}/T$, with T a maximal torus in $\widehat{LG}$ [1].

On the other hand, the group $\mathrm{diff}S^1$ of diffeonorphisms of the circle, and its central extension $\widehat{\mathrm{diff}}S^1$, are in many ways very unlike compact Lie groups. $\mathrm{diff}S^1$ is perhaps more similar to its finite dimensional non-compact subgroup $SL(2, R)$ than to a compact Lie group. It therefore seems appropriate to study $\widehat{\mathrm{diff}}S^1$ by the general method of coadjoint orbits. In fact, coadjoint orbits of $\widehat{\mathrm{diff}}S^1$ have been studied by Lazutkin and Pankratova [2] and Segal [3]. Here we will pursue this study further. In the process we will be led into considerations of symplectic geometry somewhat analogous to the considerations of Kahler geometry considered recently by Bowick and Rajeev [4] and also by Kirillov and Yurtzev [6].

In Sect. two of this paper we present a brief introduction to the coadjoint orbit method. In Sect. three we go on to classify the coadjoint orbits of the Virasoro group, i.e. the central extension of $\mathrm{diff}S^1$. The possible orbits are as follows. For zero central charge, one possible orbit is the $\mathrm{diff}S^1$ manifold itself -- a surprising

result which does not have a finite dimensional analogue [1]. The other possible orbit for zero central charge is $\mathrm{diff}S^1/S^1$ (S^1 being the group of rotations of the circle, i.e. the subgroup of $\mathrm{diff}S^1$ generated by L_0). For non-zero central charge, in addition to $\mathrm{diff}S^1/S^1$, possible orbits are $\mathrm{diff}S^1/SL^{(n)}(2,R)$, with $SL^{(n)}(2,R)$ the subgroup of the Virasoro group which is generated by L_0, L_n, and L_{-n}. One also finds as possible orbits $\mathrm{diff}S^1/T$, where T is a one parameter subgroup of $\mathrm{diff}S^1$ generated by a vector field f which must obey certain conditions (f must have only single zeros or only double zeros, and there are certain restrictions on the derivatives of f at the zeros). These orbits have already been classified in [2, 3], and in a rather economical way, but I hope the treatment given here, and the explicit descriptions of centralizers of orbits, will be useful.

In the last section, we attempt to make the correspondence between orbits and representations. Certain features of this correspondence seem to be clear. The orbit $\mathrm{diff}S^1/S^1$ is related to Verma modules (that is, to those Verma modules which are unitary, namely in the region of central charge $c>1$), while the orbits $\mathrm{diff}S^1/SL^{(n)}(2,R)$ are related to degenerate representations with a null state on level n. (The significance of such representations for quantum field theory was first pointed out by Belavin et al. [6]. See also [7].) The only orbits which are quantizable by standard methods are $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$, for which the Kahler structure makes quantization possible. The semiclassical region in quantization of Virasoro orbits is $c\gg 1$. The only unitary representations of the Virasoro group for $c>1$ are indeed those which can be obtained by quantizing $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$. The discrete series of unitary Virasoro representations with $c<1$ [8], which are a subclass of the more general class of degenerate representations, must be related to quantization of appropriate Virasoro orbits, especially $\mathrm{diff}S^1/SL^{(n)}(2,R)$, in the strong coupling domain. $\mathrm{diff}S^1/T$ for certain T may also play a role. The fact that $\mathrm{diff}S^1/SL^{(n)}(2,R)$ (for $n>1$) cannot be quantized by standard methods very likely means that it cannot be quantized at all for large c, and if so this would be related to the existence of a discrete Virasoro series which only arises at $c<1$.

If one could actually quantize the Virasoro coadjoint orbits it would be possible to verify the above statements and perhaps to get a much richer understanding of Virasoro representation theory. Unfortunately, quantizing the Virasoro coadjoint orbits appears to be a very difficult problem. These orbits are somewhat similar to those orbits of finite dimensional semi-simple Lie groups which are related to the so-called unipotent representations, and quantizing those orbits has proved to be a difficult problem even in finite dimensions. For this reason, we will achieve in this paper only a limited degree of geometrical insight concerning Virasoro representation theory.

Poisson brackets for the "classical limit of the Virasoro algebra" have been studied in [7, 9]. They proved in [9] to have remarkable properties. Coadjoint orbits are really the systematic way of constructing the irreducible classical systems whose Poisson brackets are the classical limit of a given Lie algebra. In

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[1] This is equivalent to the existence on the Virasoro group manifold of left invariant symplectic structures - many inequivalent ones, as we will see. Their interpretation in terms of Lie bialgebras is discussed in the appendix

effect, therefore, we are studying the same systems studied in [7, 9], but decomposing those systems into their irreducible components, which are individual coadjoint orbits.

While coadjoint orbits have often been very useful in illuminating representation theory, the connection between orbits and representations is in other cases much less close than one might hope, so in exploring the Virasoro coadjoint orbits one must have a spirit of adventure.

2. Orbits and Group Representations

To keep this paper self-contained, we begin with an introduction to the relation between group representations and coadjoint orbits. The reader may wish to consult, e.g., [11].

Let G be a Lie group and $\mathcal{G}$ its Lie algebra. Elements of g will be denoted u, v, w, etc. Let $\mathcal{G}^*$ be the dual of the adjoint representation of $\mathcal{G}$, also known as the coadjoint representation. It is the space of all linear functionals on $\mathcal{G}$. Elements of $\mathcal{G}^*$ will be denoted as a, b, c. Elements of $\mathcal{G}$ and $\mathcal{G}^*$ will be called adjoint and coadjoint vectors, respectively. If G is such that there is an invariant quadratic form $(\ |\ )$ on the Lie algebra $\mathcal{G}$, then the adjoint and coadjoint representations are isomorphic. Indeed, in that case an adjoint vector v gives a linear functional on $\mathcal{G}$ by $u \to v(u) = (v|u)$. For finite dimensional semi-simple Lie groups like $SU(2)$ or $SL(2, R)$, there is such an invariant quadratic form $(v|u) = \mathrm{Tr}\, uv$, the trace taken in, say, the fundamental representation], so adjoint and coadjoint vectors can be identified in that case. [The invariant quadratic form in not positive definite in the case of $SL(2, R)$, but that is not relevant here.] As we will discuss in more detail in the next section, the adjoint and coadjoint representations are inequivalent for the Virasoro group.

The adjoint representation of a Lie algebra is explicitly the following. If v is an adjoint vector, then a generator u of the Lie algebra acts on v by

$$v \to u(v) = [u, v]\,. \tag{1}$$

The coadjoint representation can be described explicitly as follows. A coadjoint vector b is a linear functional on $\mathcal{G}$ which we may write as $v \to b(v)$, for v in $\mathcal{G}$. An element u of $\mathcal{G}$ in the coadjoint representation maps a coadjoint vector a to the coadjoint vector $u(a)$ defined by

$$(u(a))(v) = -a([u, v])\,. \tag{2}$$

The minus sign is needed so that the pairing v, $a \to a(v)$ is invariant under an infinitesimal transformation of a and v by u, i.e.,

$$(u(a))(v) + a(u(v)) = 0\,. \tag{3}$$

Now, fixing a coadjoint vector b, let W_b be its orbit under the action of G. We would like to define a natural G-invariant symplectic structure on W_b. In particular, we must define an antisymmetric bilinear form ω on tangent vectors to W_b at b. Thus, for tangent vectors a, a', we must define in a natural way $\omega(a, a')$.

a and a', being displacements of b, are naturally coadjoint vectors, not adjoint vectors. However, we require that a and a' should be not arbitrary coadjoint

vectors, but vectors which are tangent to the orbit W_b at b. Since W_b consists by definition of coadjoint vectors that can be reached from b by the G action, the statement that the particular coadjoint vectors a and a' are tangent to W_b at b means that b changes in the a or a' direction under some infinitesimal G transformation. Thus, there must be adjoint vectors u, u' with

$$u(b)=a\,,\qquad u'(b)=a'\,. \tag{4}$$

u and u' are not uniquely determined; the ambiguity is $u\to u+v$, $u'\to\to u'+v'$, where v and v' leave b invariant,

$$v(b)=v'(b)=0\,. \tag{5}$$

We now define the symplectic structure ω by

$$\omega(a,a')=b([u,u'])\,. \tag{6}$$

This is obviously antisymmetric and G invariant. It is also well defined, since if we shift u to $u+v$, where v obeys (5), then $\omega(a,a')$ is shifted by $b([v,u'])$ $=-(v(b))(u')=0$; here we have used (2) and (5).

To show that (6) defines a symplectic structure, we must show first of all that it is nondegenerate, and second that as a two form ω is closed. The nondegeneracy means that given a tangent vector to the orbit W_b, represented say by an adjoint vector u, we must find another tangent vector to the orbit, represented say by the adjoint vector u', such that

$$b([u,u'])\neq 0\,. \tag{7}$$

More exactly, we must prove the existence of such a u' only if u is such that $u(b)\neq 0$, for if $u(b)=0$, then the tangent vector to W_b generated by u is zero at b. If $u(b)\neq 0$, there must be u' such that $(u(b))(u')\neq 0$, and the definition of $u(b)$ shows at once that (7) is obeyed.

To show that the two form ω is closed is another exercise in tracking down definitions. Given any two form λ, the three form $d\lambda$ is defined by saying that to three vector fields u, v, w it assigns the value

$$\begin{aligned} d\lambda(u,v,w)=&(u\cdot\nabla)\lambda(v,w)+(v\cdot\nabla)(w,u)+(w\cdot\nabla)(u,v)\\ &-\omega([u,v],w)-\omega([v,w],u)-\omega([w,u],v)\,. \end{aligned} \tag{8}$$

Here $[u,v]$, for instance, is the commutator of the two vector fields u and v, defined by $[u^i\partial_i,v^j\partial_j]=[u,v]^k\partial_k$. In the case at hand, to determine if ω as defined in (6) is closed, we pick three adjoint vectors u, v, w and (denoting the vector fields they generate on W_b by the same name) we compute

$$\begin{aligned} d\omega(u,v,w)=&(u\cdot\nabla)b([v,w])+(v\cdot\nabla)b([w,u])+(w\cdot\nabla)b([u,v])\\ &-b([[u,v],w])-b([[v,w],u])-b([[w,u],v])\,. \end{aligned} \tag{9}$$

Here $(u\cdot\nabla)b([v,w])$ is the infinitesimal change in $b([v,w])$ under transformation by u, and is zero by (3) (with $u\to u$, $a\to b$, $v\to[v,w]$). The next two terms in (9) likewise vanish. As for the last three terms, they add up to zero because of the Jacobi identity,

$$[[u,v],w]+[[v,w],u]+[[w,u],v]=0\,. \tag{10}$$

This completes the demonstration that ω defines a symplectic structure.

In any quantum theory, one wants to know "what is the coupling constant." In the case at hand, given a coadjoint vector b, we can rescale it by any positive number t to get a new coadjoint vector tb. The new orbit W_{tb} is then isomorphic to the old one W_b, but the definition of the symplectic structure shows that it is now bigger by a factor of t. For large t, the symplectic structure becomes "large," so that a large "area" is attributed to any region of phase space; thus large t is the semiclassical domain, and $1/t$ plays the role of Planck's constant. Thus, "most" coadjoint orbits come in families with a weak coupling or semiclassical domain for large t. We have tacitly assumed a uniformity of the large t limit; this assumption fails for the special cone-like orbits associated with the unipotent representations; these orbits are invariant under scaling, and cannot be studied semiclassically.

Having constructed a symplectic structure on the orbit W_b, the next step is to understand the Poisson brackets. Let us recall the definitions.[2] Given a symplectic manifold M with symplectic form ω_{ij}, one defines the inverse matrix ω^{ij}. The Poisson bracket of any two functions A and B is then defined as

$$\{A, B\} = \omega^{ij}\partial_i A \partial_j B . \tag{11}$$

In the coadjoint orbit case, what are the functions whose Poisson brackets we would like to compute? The coadjoint space $\mathscr{G}^*$ is simply a linear space, and the most basic functions in a sense are the linear functions on the coordinates b_i of the coadjoint space. Indeed, every adjoint vector u gives a linear function

$$b \to \Phi_u(b) = b(u) \tag{12}$$

on coadjoint vectors. We will show that the Poisson brackets of these functions are

$$\{\Phi_u, \Phi_v\} = \Phi_{[u,v]} . \tag{13}$$

All Poisson brackets can be reduced to (13); for instance, an arbitrary polynomial in the coordinates is a polynomial in the Φ_u, and Poisson brackets of such polynomials follow immediately from (13).

To prove (13) we will show that the Φ_u are the "conserved charges" that generate the G symmetry. The argument is short if one keeps in mind the definitions. Given a general symplectic manifold M and a "Hamiltonian" H, the vector field V^i generated by H is

$$V^i = \omega^{ij}\partial_j H . \tag{14}$$

This formula is occasionally awkward to use, since it involves the inverse matrix ω^{ij}. An equivalent one, more convenient if V^i is given and one wishes to discover H, is

$$\partial_j H = \omega_{ij} V^i . \tag{15}$$

This can be written more abstractly as

$$dH = i_V(\omega) , \tag{16}$$

where dH is the exterior derivative of H, and i_V is the operation of contracting the first index of a differential form (here ω) with V. Equation (16) is equivalent to the

[2] Some elementary symplectic geometry can be found in, e.g., [12]

statement that for any vector field W,

$$i_W(dH) = i_W(i_V(\omega)). \tag{17}$$

In the case at hand, we are given several vector fields on $\mathcal{G}^*$. Indeed, for every adjoint vector u there is an infinitesimal transformation

$$\delta b = u(b) \tag{18}$$

of $\mathcal{G}^*$. [$u(b)$ was defined earlier. We will use the same symbol u to denote an adjoint vector and the vector field on $\mathcal{G}^*$ that it generates.] We want to show that the Hamiltonian that generates the transformation (18) is just $H = \Phi_u$. It is sufficient to verify (17), and we may consider the case in which W is the vector field generated by some other adjoint vector w (these are a basis for all vector fields). First we evaluate the left-hand side of (17). Here $i_w(d\Phi_u(b))$ is just the infinitesimal change in $\Phi_u(b)$ when b is transformed by w. In other words, it is

$$i_w(d\Phi_u(b)) = \Phi_u(w(b)). \tag{19}$$

Recalling the definitions of $\Phi_u(b)$ and $w(b)$, this is

$$i_w(d\Phi_u(b)) = (w(b))(u) = b([u, w]). \tag{20}$$

Now let us compare this with the right side of (17). We defined the symplectic structure ω by saying that at a point b in $\mathcal{G}^*$, the contraction of ω with the vector fields generated by group generators u and w is

$$i_w(i_u(\omega)) = b([u, w]). \tag{21}$$

Since this agrees with (20), we have shown that the functions $\Phi_u(b)$ are indeed the generators of the infinitesimal transformations $\delta b = u(b)$. Since those transformations generate the Lie algebra of $\mathcal{G}$, the claimed Poisson brackets (13) follow.

Now that we have understood the Poisson brackets, we would like to attempt quantization. If this can be achieved, the quantum Hilbert space will automatically be a unitary representation of G. The goal of the coadjoint method is to relate unitary representations of G to orbits in this way, and ideally to classify the unitary representations in terms of orbits.

Here, however, we meet the predicament of geometric quantization. Given a symplectic variety W, there is no general way to carry out "quantization" and obtain a quantum Hilbert space. The most familiar obstacle is perhaps that which arises if the symplectic form ω is topologically non-trivial as an element of the second cohomology group of W. In this case ω must be an integral element of the second cohomology group, to avoid global anomalies in the quantum theory. The essence of the matter is that the quantum Hilbert space should be the space of sections of a line bundle whose first Chern class is ω, and this can only exist if ω is integral.

The above cited situation is, however, by no means the only or typical example of an obstacle to quantization. Quantization can indeed be very difficult or impossible even in a topologically trivial situation. For instance, let D be a two dimensional disc in the plane, endowed with a symplectic structure ω.[3] The only

[3] I owe the following remarks to D. Kazhdan

invariant of the symplectic structure is the area

$$A = \int_D \omega, \tag{22}$$

which we will suppose is finite. Despite the seeming absence of a topological obstruction, there is no natural way to quantize this problem. One way to look at it is that the dimension of the quantum Hilbert space should be roughly $A/2\pi$, since according to the WKB theory, there should be roughly one quantum state for every 2π of phase space area. It is clearly impossible to quantize the disc in a way which varies smoothly with A and also respects this principle. More generally, even if one is willing to quantize the disc only for special values of A, or in a way that does not vary smoothly with A, there is no natural way to carry out quantization without being given a special class of functions whose quantum commutators are to be almost the same as their classical Poisson brackets.

Generally speaking, standard methods of quantization will work only for very special classes of symplectic varieties W. The usual case in most physical applications of quantum mechanics is that in which W is the total space of the cotangent bundle of a manifold M; then there is a standard rule for quantization, the quantum Hilbert space being the space of functions on M. Alternatively, if W is Kahler, and if there is a holomorphic line bundle L with a hermitian metric and the symplectic form ω as its curvature form, then the space of holomorphic sections of L can be interpreted as the quantum Hilbert space. (This case was one of the starting points for Bowick and Rajeev [4].) There are a few other situations which can be reduced to one of these by use of symmetries, and a physicist might note that there are also problems that reduce to one of the standard situations in perturbation theory even if they do not reduce to it exactly. But in general, there is no rule for quantizing a symplectic variety. This is a significant point, since we will meet some enigmatic symplectic varieties later in this paper.

To make the above ideas concrete, we will now work out the coadjoint orbit method in two examples – the compact Lie group $SU(2)$, and the non-compact one $SL(2, R)$. We will be rather brief, the intent being merely to illustrate the flavor of the subject.

(a) $SU(2)$

In the case of $SU(2)$, the adjoint and coadjoint representations coincide, because of the existence of an invariant (and positive definite) quadratic form.

The adjoint or coadjoint representation of $SU(2)$ is a three dimensional vector space with coordinates that we will call x, y, and z. The invariant quadratic form we can take to be $x^2+y^2+z^2$. $SU(2) \sim SO(3)$ acts on the vector (x, y, z) by rotations. Apart from the trivial orbit $x=y=z=0$, every orbit is a sphere, say

$$x^2+y^2+z^2=R^2 \tag{23}$$

for some radius R. The sphere is a complex manifold, isomorphic to CP^1. (It can be identified with G/T in the Borel-Weil-Bott theorem.) Quantization is carried out by noting that the round metric on the sphere is an $SU(2)$ invariant Kahler metric. The symplectic form ω can be regarded as the curvature form of a hermitian line bundle if R is an integer or half integer. This quantization of R is from the point of

view of the coadjoint method (or the Borel-Weil-Bott theorem) the origin of quantization of angular momentum.

(b) $SL(2, R)$

More subtle and for our purposes in this paper more illuminating is a non-compact group such as $SL(2, R)$. The initial stages of the analysis are similar to the above. Identifying $SL(2, R)$ with $SO(2, 1)$, the adjoint (or coadjoint) representation is again a three dimensional real space which we may describe with coordinates x, y, z and an invariant quadratic form $z^2 - x^2 - y^2$. The fact that this quadratic form is not positive definite will have a big influence, however. The orbits (apart from $x = y = z = 0$) are of the form

$$z^2 - x^2 - y^2 = R^2, \tag{24}$$

but now this gives three kinds of orbit, namely $R^2 > 0$, $R^2 < 0$, and $R^2 = 0$ (see Fig. 1). It will be worth our while to discuss the various kinds of orbit in turn.

(i) $R^2 > 0$. We first consider the hyperboloid $R^2 > 0$. It decomposes in fact into two components, $z > 0$ and $z < 0$.

A typical point on the orbit $R^2 > 0$, $z > 0$ is the point $(x, y, z) = (0, 0, R)$. The subgroup of $SL(2, R)$ that leaves this point fixed is the group $U(1)$ of rotations around the z axis. This is also a maximal torus of $SL(2, R)$. The hyperboloid can thus be identified as the quotient $SL(2, R)/U(1)$. This is as close an analogue as there is to the orbits G/T that one meets in studying compact Lie groups. The representations obtained by quantizing the hyperboloid are known as the discrete series.

A variety of remarks about this series of representations will be helpful. First of all, let us interpret the $U(1)$ that leaves fixed the point $(0, 0, R)$ as the "energy." Indeed, if $SL(2, R)$ is embedded in the Virasoro group as the subgroup generated by L_0 and $L_{\pm 1}$, then this $U(1)$ corresponds to L_0. A representation of $SL(2, R)$ will be

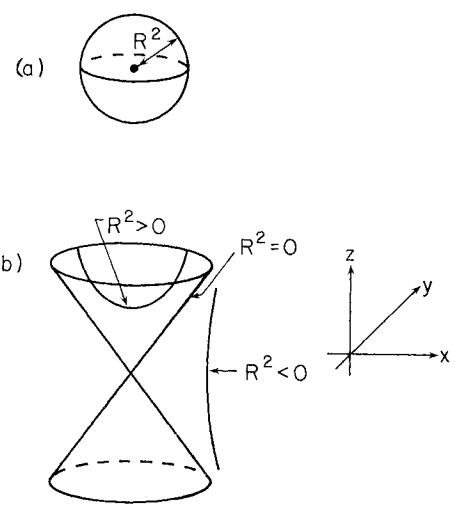

Fig. 1a and b. The coadjoint orbits that arise for **a** $SU(2)$; **b** $SL(2, R)$

called highest weight or lowest weight if this "L_0" is bounded below or above.[4] Now, according to our general discussion of the coadjoint orbit method, the "Hamiltonian" H which via Poisson brackets generates rotation around the z axis is simply one of the coordinates in the coadjoint space; in fact, it is $H=z$. Inspection of Fig. 1 shows that on the orbits of $R^2>0$, the function z is bounded above or below. Indeed, on the branch $z>0$, z is bounded below, so quantization of this orbit will correspond to a highest weight representation of $SL(2,R)$, while on the branch $z<0$, z is bounded above, and this will give a lowest weight representation of $SL(2,R)$. There is an obvious symmetry between the two, and for brevity we will consider the orbit $z>0$.

Next we observe that $D=SL(2,R)/U(1)$ is a complex manifold, which can be identified with the upper half plane or alternatively with the disc $|w|<1$ in the complex w plane. The $U(1)$ in question is $w\to e^{i\theta}w$. D is in fact a Kahler manifold, with an $SL(2,R)$ invariant Kahler metric. We interpret the Hilbert space of quantum mechanics on D as the space of holomorphic sections of a line bundle L with the symplectic form ω as its curvature form.

Before discussing line bundles, let us discuss holomorphic functions on D. We can choose a basis of holomorphic functions consisting of the L_0 eigenstates $1, w, w^2, \ldots$. On w^k, L_0 has eigenvalue k, so the "partition function" $\mathrm{Tr}q^{L_0}$ is

$$1+q+q^2+\ldots=\frac{1}{1-q}. \tag{25}$$

$SL(2,R)$ acts on holomorphic functions on D by

$$f(w)\to f\left(\frac{aw+b}{cw+d}\right), \tag{26}$$

with $ad-bc=1$, and a, b, c, d restricted so that this maps D to itself. (Thus, $|a|^2-|c|^2=|d|^2-|b|^2=1$, $ab^*=cd^*$.) The $SL(2,R)$ invariant measure on D is

$$d\mu=\frac{dwdw^*}{(1-w^*w)^2}, \tag{27}$$

and of course D has infinite volume with this measure. The usual metric on complex functions is

$$(f,g)=f^*g. \tag{28}$$

The formula

$$|f|^2=\int d\mu(f,f)=\int d\mu f^*f \tag{29}$$

is the obvious candidate for an $SL(2,R)$ invariant Hilbert space structure on the holomorphic functions on D. There aren't any square integrable functions in that sense, i.e., holomorphic functions on D with $|f|^2<\infty$, but if there were, as the definition of $|f|^2$ is $SL(2,R)$ invariant, they would automatically give a unitary representation of $SL(2,R)$.

[4] The nomenclature may seem backwards, and this indeed reflects a seemingly incurable clash in terminology between physics and representation theory

Now let us think in terms of quantization of coadjoint orbits. We are supposed to find a line bundle L with a hermitian form whose curvature form is the symplectic form ω of L; and the quantum Hilbert space should consist of holomorphic sections of L. Now, L will be trivial (topologically and holomorphically), since there are no other line bundles on D, but we must choose the correct metric on L. Up to a scalar multiple, the only invariant symplectic structure on D is

$$\omega_0 = -i\frac{dw \wedge dw^*}{(1-w^*w)^2}. \tag{30}$$

For the orbit of "radius" R, the symplectic form must be a multiple of this, and we will take

$$\omega_R = R \cdot \omega_0. \tag{31}$$

Picking a hermitian metric on L means choosing a real-valued function h and replacing (29) and (28) with

$$(f, g)_h = e^h f^* g, \tag{32}$$

and

$$|f|_h^2 = \int d\mu (f, f)_h = \int d\mu e^h f^* f. \tag{33}$$

The connection on L compatible with this metric has curvature form

$$F = -i\bar{\partial}\partial h. \tag{34}$$

To get $F = \omega_R$, we evidently need $h = 4R\ln(1-w^*w)$, so the inner product on sections of L is actually

$$|f|_R^2 = \int d\mu (1-w^*w)^{4R} f^* f = \int_D dw dw^* (1-w^*w)^{4R-4} f^* f. \tag{35}$$

With this hermitian structure, the $SL(2, R)$ action must be the one such that (35) is invariant. It is

$$f(w) \to \left(\frac{1}{cw+d}\right)^{2R} f\left(\frac{aw+b}{cw+d}\right). \tag{36}$$

This amounts to saying that

$$f(w)(dw)^R \to f(w')(dw')^R, \tag{37}$$

with $w' = (aw+b)/(cw+d)$; i.e., $f(w)$ is best regarded as a differential form of degree R.

Now, we wish to answer several questions. In what sense is the "discrete series" of $SL(2, R)$ representations discrete? To what extent are they unitary? What are the characters of these representations, and how could we have determined them without such a detailed analysis?

First of all, in contrast to the case of $SU(2)$, where the discreteness comes from quantizing the first Chern class of a line bundle, for $SL(2, R)$ the discreteness comes entirely from requiring that the group $SL(2, R)$ (rather than some covering group thereof) should act in a single-valued way. Thus, (36) makes sense at the Lie algebra level for any R, but if one wishes a single-valued action of the *group* $SL(2, R)$, then R

must be an integer of half-integer, because of the $2R$ power in (36). If one is interested in the $SL(2, R)$ Lie algebra rather than the group, the discrete series is not really discrete.

Second, as regards unitarity, let H_R be the space of holomorphic functions on D with inner product (35). With the $SL(2, R)$ action (36), this gives a unitary representation of $SL(2, R)$ as long as H_R is non-trivial, i.e., as long as there are holomorphic functions on D which are square-integrable in the sense of (35). A brief inspection reveals that this is so if and only if

$$R > 1/2 . \tag{38}$$

Thus, expressing things in terms of the $SL(2, R)$ Lie algebra, the "discrete series" is a continuous family of representations of $SL(2, R)$ which is defined for any R but for which the unitary structure that we have described breaks down at $R = 1/2$. This phenomenon seems to have a counterpart for certain Virasoro representations which are defined for any value of c (the central charge) but are unitary only if $c \geqq 1$. [In the Virasoro case, it is the unitarity of the representations, and not just the description of their unitary structure, which breaks down at a critical value of c. For $SL(2, R)$, it is, rather surprisingly, possible to give an alternative unitary structure for $0 < R \leqq 1/2$.]

Finally, let us discuss the characters of the discrete series, i.e., the values of $\mathrm{Tr} q^{L_0}$. The $U(1)$ generated by L_0 is $w \to e^{i\theta} w$, which we express as $w \to (aw + b)/(cw + d)$ with $a = e^{i\theta/2}$, $d = e^{-i\theta/2}$, $b = c = 0$. [This is consistent with $ad - bc = 1$, one of the defining relations of $SL(2, R)$.] From (36) we see that the transformation of f is

$$f(w) \to e^{i\theta R} f(e^{i\theta} w) . \tag{39}$$

Thus, the function $f = w^k$, $k = 0, 1, 2, \ldots$, has $L_0 = R + k$, and the partition function is

$$\mathrm{Tr} q^H = q^R + q^{R+1} + \ldots = \frac{q^R}{1 - q} . \tag{40}$$

In particular, the ground state energy is $L_0 = R$. This could have been guessed as follows. Recall that the representation that we are discussing is obtained by quantizing the orbit $z^2 - x^2 - y^2 = R^2$, $z > 0$, and that L_0 is the quantum operator corresponding to quantization of the "Hamiltonian" $H = z$. The minimum of the classical Hamiltonian is R, which is the value of the Hamiltonian at $(x, y, z) = (0, 0, R)$. At least for small R, with Planck's constant effectively small and the quantum fluctuations small, the ground state energy must be at least approximately the minimum of the classical Hamiltonian.

The character formula (40) could have been obtained without any detailed analysis as follows. L_0 is the quantum operator which implements a rotation around the z axis. Infinitesimally, this is generated by the vector field

$$\delta x = y , \quad \delta y = -x , \quad \delta z = 0 . \tag{41}$$

This vector field – call it V – acts holomorphically, i.e., it preserves the holomorphic structure of D, and generates a compact group ($U(1)$) of isometries. The only fixed point, i.e., the only point at which $V = 0$, is $x = y = 0$, which amounts to $D = 0$. What is more, while the $SL(2, R)$ representation under study consists of

the holomorphic sections of a line bundle L, the higher cohomology groups of this bundle are zero (since D is so simple topologically).

In such a situation, the fixed point version of the Atiyah-Singer index theorem [13] gives a formula for the character of the representation in terms of the zeros of V. For brevity, I will state the formula in a somewhat simplified way, assuming that the zeros of V are isolated points. (For more detail, and a "physical" proof, the reader may consult [14].) Near an isolated zero P of V, V can always be put in the form

$$V = \sum n_k \cdot w_k \frac{\partial}{\partial w_k} \tag{42}$$

with some integers n_k and local complex coordinates w_k. If there is only one fixed point, then the character will be

$$\mathrm{Tr}\, q^{L_0} = (-1)^s q^h \prod_k \frac{1}{1-q^{|n_k|}}, \tag{43}$$

where s is the number of negative n_k and h in the classical limit is the value of the classical Hamiltonian at the fixed point. In general h is the eigenvalue of V acting on the fiber of L at the fixed point. If there is more than one fixed point, then (43) must be summed over fixed points, and if the higher cohomology groups are not zero then (43) gives the "Lefschetz number" of V rather than the character of the representation furnished by the holomorphic sections of L.

In our case, there is only one fixed point, $s=0$, $h=R$, the only n_k is equal to 1, and (43) clearly agrees with the explicit computation of (40).

(ii) $R^2<0$. We now turn our attention to the $SL(2,R)$ representations which come from coadjoint orbits with $R^2<0$. They are known as the continuous series, and they deserve this name since even if one works at the group (and not just Lie algebra) level, there is a continuous family of unitary $SL(2,R)$ representations which arise from these orbits. We will be brief in discussing them, since they do not seem to be particularly relevant to our later work.

An orbit W with $R^2<0$ is topologically a cylinder, $S^1 \times R$, with S^1 being the intersection of the orbit with the plane $z=0$, and z playing the role of time. Thus, the orbit can be identified as the total space of the cotangent bundle of the circle $x^2+y^2=1$, $z=0$. This is a happy state of affairs, since cotangent bundles can be quantized. The quantum Hilbert space is simply the space of functions on the circle. Such functions certainly do not give a highest or lowest weight representation of $SL(2,R)$, since they have a Fourier expansion $\sum_n a_n e^{in\theta}$ with both positive and negative n. This reflects the fact that classically, the "Hamiltonian" z is not bounded above or below on the $R^2<0$ orbit. As in the holomorphic case which we discussed in more detail above, these functions must be taken to transform under $SL(2,R)$ with a "weight" which depends on R^2. One obtains unitary $SL(2,R)$ representations for all negative R^2.

(iii) $R^2=0$. Finally, we come to the cone $R^2=0$, which again decomposes under $SL(2,R)$ into the components of $z>0$ and $z<0$. The cone is in many ways the most mysterious coadjoint orbit of $SL(2,R)$.

While the hyperboloids $R^2>0$ and $R^2<0$ depend on the free parameter R^2, the cone $R^2=0$ is a unique orbit depending on no free parameter. For a higher

dimensional semi-simple Lie group there are a finite number of analogous exceptional orbits. Thus, while the analogues of $R^2>0$ and $R^2<0$ always give infinite families of orbits, a non-compact semi-simple Lie group always has just a finite number of special representations associated with cone-like orbits. They are known as the unipotent representations, the name simply meaning that they are finite in number. (We mustn't take the name too literally, though; an infinite dimensional Lie algebra might prove to have infinitely many "unipotent" representations.)

For $R^2>0$ or $R^2<0$, $|R^2|^{-1}$ plays in the coadjoint method the role of Planck's constant, and if quantization were possible in no other way one could always appeal to perturbation theory in the semi-classical domain. The uniqueness of the $R^2=0$ orbit means that attempting to quantize the cone is like trying to solve a physics problem in which there is no adjustable coupling constant.

Adding to the mystery of the cone is the following. The analogues for any semi-simple Lie group to the $R^2>0$ orbits are always Kahler manifolds which can be quantized as such. The analogues of the $R^2<0$ orbits are always cotangent bundles which again can be quantized as such. But the cone-like orbits are of neither type, and they cannot be quantized by any standard method.

For $SL(2,R)$, these statements need some slight modification, as follows. (The following facts are central in one approach to the quantization of the conelike orbits [15].) In the case of $SL(2,R)$, the two types of hyperboloid and the cone are all two dimensional. The cone can be reached as the limit of a hyperboloid of negative R^2 as $R^2\to 0$. For higher dimensional semi-simple Lie groups, this is not so in general. The cone-like orbits have lower dimension than the hyperboloids, and cannot be reached as limits of hyperboloids.

In the coadjoint space of $SL(2,R)$, essentially the only $SL(2,R)$ invariant function is $R^2=z^2-x^2-y^2$. It corresponds to the quadratic Casimir operator in the Lie algebra. In specifying an orbit by the value of R^2, we are essentially labeling the orbit and thus (after quantization) the representation by the value of the quadratic Casimir. Determining which orbits can be quantized amounts to learning the possible values of this Casimir operator.

More generally, for a semi-simple Lie group G of rank k, there will be k Casimir operators, corresponding to k G invariant functions C_j, $j=1\ldots k$ in the coadjoint space. A typical orbit is labeled by the values of the C_j, say

$$C_j=c_j, \quad j=1\ldots k. \tag{44}$$

It is natural to specify an orbit in this way by the value of the Casimirs. The peculiarity of the cone-like orbits is that they cannot be specified by the values of G invariant functions, because they are of anomalously low dimension; their codimension is greater than k, and it takes more than k equations to define them. The cone-like orbits for general semi-simple Lie groups require for their specification a statement not just of the values of invariant functions C_j but also of some other functions, say F_r, $r=1\ldots s$, which are *not* G-invariant. Thus, the cone-like orbit is

$$C_j=c_j, \quad j=1\ldots k, \quad F_r=f_r, \quad r=1\ldots s. \tag{45}$$

In a successful quantization of a system which classically obeys (45), the C_j and F_r should be c-numbers and in particular should commute with the group generators.

This is not strange in the case of the Casimirs C_j, which indeed should be c-numbers in an irreducible representation of the group. But it is rather strange that the non-invariant functions F_r should be c-numbers in an infinite dimensional representation of G. This is the peculiarity of the unipotent representations.

In classifying coadjoint representations of the Virasoro group, we will likewise find exceptional orbits, somewhat analogous to the cones, whose specification requires giving the values of some non-invariant functions. Thus, some Virasoro representations have some properties in common with the unipotent representations. Actually, it seems that the theory of highest weight Virasoro representations combines features of the discrete and unipotent representations of finite dimensional non-compact semi-simple Lie groups.

(iv) The Origin. In surveying coadjoint orbits of $SL(2, R)$, there is one more which we must not omit. It is the origin,

$$x=y=z=0. \tag{46}$$

This illustrates, already at the level of $SL(2, R)$, an orbit whose description requires specifying the values of functions which are not $SL(2, R)$ invariant. Although the $SL(2, R)$ invariant function R^2 vanishes at the origin, the equation $R^2=0$ does not define the origin but rather the cone discussed above. To specify the origin one must call for the vanishing of certain non-invariant functions, such as x, y, and z in (46). In this sense, the origin has already at the $SL(2, R)$ level the characteristic property of the cone-like orbits for higher groups. However, the unitary $SL(2, R)$ representation which should be attached to the origin is the trivial representation – which is the only finite dimensional unitary representation. Surely, the trivial representation is the only one that we can expect to get by quantizing a classical system whose phase space consists only of a single point.

3. Classification of the Virasoro Orbits

In this section, we will classify the coadjoint orbits of the Virasoro group. As was noted in the introduction, this has already been carried out in [2, 3]. However, we will be more explicit on several points. Remarks about the implications for the representation theory will be found in the next section.

For practice, we first consider the case of diffS^1 without a central extension. This Lie algebra consists of vector fields $f(\theta)\dfrac{d}{d\theta}$ on a circle S^1. Here θ is an angular parameter, $0\leqq\theta\leqq 2\pi$. The commutation relations are of course

$$|f, g|=fg'-gf'. \tag{47}$$

A coadjoint vector is a quadratic differential $b(\theta)(d\theta)^2$. The pairing between vectors and quadratic differentials is

$$(g, b)=\int_0^{2\pi} d\theta g b. \tag{48}$$

Let us verify that this is invariant and at the same time determine the transformation law for b. Equation (47) says that under infinitesimal transformation by the vector field f, the change in g is

$$\delta g=fg'-f'g. \tag{49}$$

The transformation law for quadratic differentials is

$$\delta b = 2f'b + fb'. \tag{50}$$

Using these formulas, we see that indeed

$$\delta \int d\theta(gb) = \int d\theta((\delta g)b + g(\delta b))) = 0. \tag{51}$$

Of utmost importance for what follows, the fact that the adjoint representation consists of vector fields while the coadjoint representation consists of quadratic differentials means that these diffS^1 representations are not isomorphic – a fact that is also evident in the asymmetry of (49) and (50).

Now, given a coadjoint vector b, let us determine the subgroup H of diffS^1 which leaves it invariant. The orbit W_b of b will then be isomorphic to the homogeneous space diffS^1/H. H is generated by vectors f which are such that

$$0 = \delta b = 2f'b + fb'. \tag{52}$$

This immediately implies

$$f = b^{-1/2}. \tag{53}$$

Thus, for f to be non-singular, b must have no zeroes. The function b is thus either positive definite or negative definite. We may as well assume that b is positive definite. But then, by a diffS^1 transformation, we can set $b = b_0 = \text{const}$. To do this, define the invariant b_0 of the quadratic differential b by

$$\sqrt{b_0} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \sqrt{b(\theta)}. \tag{54}$$

Then define σ by

$$\sqrt{b_0}\, d\sigma = \sqrt{b(\theta)}\, d\theta. \tag{55}$$

Then σ is a monotonically increasing function of θ with $\sigma(\theta + 2\pi) = \sigma(\theta) + 2\pi$, and $\theta \to \sigma$ is a diffeomorphism of the circle that sets $b = b_0$. Therefore, the diffS^1 orbit of any positive definite $b(\theta)$ is the same as that of some constant b_0.

For $b = b_0 = \text{const}$, (53) says that the only vector field leaving b invariant is $f = \text{const}$. The subgroup of diffS^1 generated by this vector field is the group of rigid rotations of S^1; we will refer to this group as S^1. The orbit of b_0 under diffS^1 can thus be identified as a copy of diffS^1/S^1.

What if b has zeroes? Then according to (53), there is no non-trivial vector field which leaves b invariant. The orbit of b under diffS^1 is then all of diffS^1 (or in special cases, it is the quotient of diffS^1 by a finite cyclic group that permutes zeros of b). In particular, the fact that the diffS^1 manifold itself appears as a coadjoint orbit means that there are left invariant symplectic structures on the diffS^1 manifold. There are many such symplectic structures, in fact; they are classified by the conjugacy classes of quadratic differentials which have zeros and so are not stabilized by any vector field. To gain some insight about what these conjugacy classes are, let $\theta_1, \theta_2, \ldots, \theta_k$ be the zeros of b. The numbers

$$a_n = \int_{\theta_{n-1}}^{\theta_n} \sqrt{|b|} \cdot d\theta \tag{56}$$

are $\mathrm{diff}S^1$ invariants up to cyclic permutation. Thus, the left invariant symplectic structures on the $\mathrm{diff}S^1$ manifold come in families depending on the a_n.

For a group G to have a coadjoint orbit that is (as a left homogeneous space) isomorphic to G itself is a somewhat unusual state of affairs. It can never happen if the adjoint and coadjoint representations are isomorphic, because then a coadjoint vector can be regarded as an adjoint vector, and is always left invariant by at least one adjoint vector, namely itself. The possible significance of the left invariant symplectic structures on the Virasoro group manifold is discussed in the appendix.

From the Kahler structure on $\mathrm{diff}S^1/S^1$ (exploited in [4]) one might think that the $\mathrm{diff}S^1/S^1$ manifold is even dimensional and $\mathrm{diff}S^1$ itself odd dimensional. But the possibility of having a symplectic structure on the $\mathrm{diff}S^1$ manifold means that in some respects, this is similar to an even dimensional space.

We now consider instead of $\mathrm{diff}S^1$ the Virasoro group $\widehat{\mathrm{diff}}S^1$, which is the universal central extension of $\mathrm{diff}S^1$. The Lie algebra of $\widehat{\mathrm{diff}}S^1$ consists of vector fields $f(\theta)\dfrac{d}{d\theta}$ together with a central element c. The non-trivial commutators are[5]

$$\left[f\frac{d}{d\theta}, g\frac{d}{d\theta}\right] = (fg' - f'g)\frac{d}{d\theta} - \frac{ic}{48\pi}\int_0^{2\pi} d\theta(fg''' - f'''g). \tag{57}$$

The normalization is such that if $L_m = ie^{im\theta}\dfrac{d}{d\theta}$, then

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m^3\delta_{m+n}. \tag{58}$$

One often adds a constant to L_0 to replace m^3 in (58) by $m^3 - m$, but we will prefer for the time being to work with the cocycle chosen in (57).

A general element of the Virasoro algebra is of the form $g(\theta)\dfrac{d}{d\theta} - iac$ with g a vector field and a a real number; for brevity let us refer to this pair as (g, a). From (57) we see that the change in (g, a) under infinitesimal transformation by a vector field f is

$$\delta(g, a) = \left(fg' - f'g, \frac{1}{48\pi}\int d\theta(f'''g - fg''')\right). \tag{59}$$

Since $\mathrm{diff}S^1$ has been supplemented with a one dimensional center, the coadjoint representation contains in addition to the quadratic differentials b also a central element $\tilde{c}$ which is dual to c, i.e., $\tilde{c}(c) = 1$. A general coadjoint vector is then $b(\theta)d\theta^2 + it\tilde{c}$, with t a real number. For brevity we will call this (b, t). The pairing of an adjoint vector (f, a) with a coadjoint vector (b, t) is then

$$\langle (g, a), (b, t)\rangle = -i\left(\int (gb)d\theta + at\right). \tag{60}$$

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[5] The factor of i in the central charge term, which necessitates various similar factors later on, is conventional. Otherwise, it would have been more natural to absorb this factor in the definition of c

This is invariant under transformation by (f, a) if

$$\delta b = 2f'b + fb' - \frac{f'''t}{24\pi}, \tag{61}$$

and

$$\delta t = 0. \tag{62}$$

These last two equations define the coadjoint representation of the Virasoro group.

Before going on, let us try to understand what these formulas mean in ordinary language. Suppose, for instance, that we wish to study a classical orbit with $b = b_0 = \text{const}$, and some value of t. Let us calculate the value of L_0 for such a coadjoint vector at the classical level. L_0 is the generator of the Virasoro algebra corresponding to $g = i$, $a = 0$, so (60) tells us that a coadjoint vector (b_0, t) assigns to L_0 the value

$$L_0^{(0)} = (-i) \int i \cdot b_0 = 2\pi b_0. \tag{63}$$

On the other hand, the Virasoro generator c corresponds to $g = 0$, $a = i$, so according to (60) a Virasoro orbit of given b and t has central charge

$$c = t. \tag{64}$$

The reason for the superscript in (63) is that L_0 is usually defined so that the m^3 in (58) is replaced by $m^3 - m$. This involves a shift $L_0 \to L_0 + c/24$. Consequently, with the standard definition of L_0, the value of L_0 for the coadjoint vector (b_0, t) is

$$L_0 = 2\pi b_0 + c/24 = 2\pi b_0 + t/24. \tag{65}$$

Thus, in plain language, the significance of the mysterious parameters t and b_0 is that they determine c and L_0 according to (64) and (65).

Now let us proceed with the job of classifying the coadjoint orbits. Given b, we wish to study the f's for which

$$0 = \delta b = 2f'b + fb' - \frac{tf'''}{24\pi}. \tag{66}$$

Before plunging into details, let us make some qualitative remarks about (66). If b is given, (66) is a third order differential equation for f. Picking an arbitrary point on the circle, say $\theta = 0$, and picking the values of f, f', and f'' at $\theta = 0$, (66) then uniquely determines f for all θ. Integrating (66) from $\theta = 0$ to $\theta = 2\pi$, the question is then whether f turns out to be periodic in θ. The centralizer of the coadjoint vector b consists of such periodic solutions, and there can be at most three since f is uniquely determined by initial values of f, f', and f''.

To be more concrete about this, let $g = f'$, $h = g'$, and

$$F = \begin{pmatrix} f \\ g \\ h \end{pmatrix}. \tag{67}$$

Also, introduce the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 24\pi b'/t & 48\pi b/t & 0 \end{pmatrix}. \tag{68}$$

Being traceless, this matrix can be regarded as a generator of the group $SL(3, R)$. Then (66) is equivalent to

$$\frac{dF}{d\theta} = AF. \tag{69}$$

Clearly, A is playing the role of an $SL(3, R)$ gauge field on the circle. The solution of (69) is

$$F(\theta) = U(\theta) \cdot F(0), \tag{70}$$

with

$$U(\theta) = P \exp \int_0^\theta A(\theta). \tag{71}$$

To find a solution of the equation $F(2\pi) = F(0)$, we must choose $F(0)$ to be an eigenvector of the matrix $U(2\pi)$ with eigenvalue 1.

A priori, the number of such eigenvectors may be 0, 1, 2, or 3. The following simple argument, however, shows that at $t \neq 0$, the number of vector fields stabilizing a given coadjoint orbit is always 1 or 3. Let f and g be two vector fields, and let $M(f, g) = -M(g, f)$ be the antisymmetric bilinear form

$$M(f, g) = \int d\theta \left(g \left(-t \frac{d^3}{d\theta^3} + b \frac{d}{d\theta} + \frac{d}{d\theta} b \right) f \right). \tag{72}$$

A vector f is said to be in the kernel of M if and only if $M(f, g) = 0$ for all g; it is easy to see that this is so if and only if f obeys (66). Therefore, the number of linearly independent solutions of (66) is the same as the dimension of the kernel of M. On the other hand, it is well known that for an antisymmetric bilinear form such as M, the dimension of the kernel is a topological invariant modulo two. For an antisymmetric form such as M in an infinite dimensional function space, "topological invariant" means invariant under deformations that do not affect the topological class of the term with the most derivatives. The third order term in (72) is proportional to t, so we are not allowed to set $t = 0$ from nonzero t. But for $t \neq 0$, b multiplies lower order terms in (72), so the dimension of the kernel of M is independent of b modulo 2. Setting $b = 0$, the kernel of M is one dimensional [the only periodic solution of (66) being $f = 1$], so for all b the kernel of M has odd dimension. Therefore, for $t \neq 0$, every coadjoint orbit is stabilized by 1 or 3 vector fields.

The zeros of f will play an important role, for the following reason. We are interested in classifying f and b up to conjugation in $\mathrm{diff}S^1$, and it is clear that the number of zeros of f and the orders of the zeros are invariants. What other invariants does f have? If f has no zeros, it is clear that $1/f$ transforms like a

differential form and therefore

$$a^{-1} = \frac{1}{2\pi} \cdot \int \frac{d\theta}{f} \tag{73}$$

is an invariant. Solving the equation

$$\frac{d\theta}{f} = \frac{d\theta'}{a} \tag{74}$$

we find a reparametrization $\theta \to \theta'$ which replaces f by the constant a. So up to scaling and conjugation in diffS^1, any vector field without zeros is equivalent to a constant.

If indeed f is a constant, then (66) shows that $b = \text{const}$. Running this in reverse, suppose that b is a constant, say $b = b_0$. What solutions does (66) then have? Clearly, f can be a constant, and for generical b_0 this is the only solution of (66). Thus, generical constant b_0 is left invariant only by the subgroup of diffS^1 consisting of rigid rotations of the circle, and so the orbit of a generical b_0 is a copy of diffS^1/S^1 – the same orbit that we found as the only possible non-trivial orbit in the absence of a central charge. It is easy to see, however, that for special values of b_0, one gets exceptional orbits. Indeed, if

$$b_0 = \frac{-n^2 t}{48\pi}, \tag{75}$$

for $n = 1, 2, 3, \ldots$, then in addition to the solution $f = \text{const}$, (66) has the two additional solutions

$$f = e^{\pm in\theta}. \tag{76}$$

This means that the particular coadjoint vector (75) is left invariant by the three Virasoro generators L_0, L_n, and L_{-n} (plus the center of the Virasoro group). These generate a group which we will call $SL^{(n)}(2, R)$; it is isomorphic to an n-fold cover of $SL(2, R)$. The orbit of the coadjoint vector (75) is isomorphic to diff$S^1/SL^{(n)}(2, R)$.

Let us look a little bit more explicitly at the orbit containing the constant (b_0, t). If we make an infinitesimal transformation by the vector field

$$f = i \sum_n s_n e^{-in\theta}, \tag{77}$$

then b will be transformed into

$$b = b_0 + \frac{1}{2\pi} \sum_n \beta_n e^{-in\theta}, \tag{78}$$

with

$$\beta_n = 2\pi \left(2nb_0 + \frac{n^3 t}{24\pi} \right) s_n. \tag{79}$$

Unless $b_0 = -k^2 t/48\pi$ for some integer k, the coefficients of the s_n's are all non-zero for $n \neq 0$, and the β_n's of $n \neq 0$ can be adopted as coordinates for the orbit of (b_0, t).

Suppose that we want to find the orbit of an almost constant coadjoint vector (b, t), with

$$b = b_0 + \delta b, \tag{80}$$

where δb is an arbitrary small perturbation, for which we write a Fourier expansion

$$\delta b = \frac{1}{2\pi} \sum \beta_n e^{-in\theta}. \tag{81}$$

Except for $b_0 = -n^2 t/48\pi$, we will in fact obtain no new orbits in this way. The reason for this is that the β_n for $n \neq 0$ parametrize the orbit of (b_0, t), as we saw at the end of the last paragraph, and β_0 can be absorbed in shifting b_0. However, there are new orbits of the form

$$b = \frac{-n^2 t}{48\pi} + \delta b(\theta), \tag{82}$$

with δb a small perturbation, and we would now like to classify these. Before plunging into the analysis, it is possible to guess what we will find. Before turning on δb, the coadjoint vector (b_0, t) is stabilized by $SL^{(n)}(2, R)$. In studying a small perturbation, $\mathrm{diff} S^1$ is irrelevant; it is enough to think about the "unbroken symmetry group" $SL^{(n)}(2, R)$. Think of the $SL^{(n)}(2, R)$ invariant vector $(-n^2 t/48\pi, t)$ as the analogue of the origin in Fig. 1. There are three types of perturbation, amounting to $R^2 > 0$, $R^2 = 0$, and $R^2 < 0$. The first of these amounts to a shift in b_0, but the latter two will give new types of orbit.

To carry out the analysis, and also for our purposes in the next section, it is useful to work out an explicit formula for the matrix $U(2\pi)$ [defined in (71)] for an arbitrary coadjoint vector of the form (82); recall that stabilizers of orbits correspond to solutions of $UF = F$. So, recalling the definition of A in (68), we write $A = A_0 + \delta A$, with

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -n^2 & 0 \end{pmatrix}, \tag{83}$$

and

$$\delta A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 24\delta b'/t & 48\pi\delta b/t & 0 \end{pmatrix}. \tag{84}$$

Then we have

$$U(2\pi) = P \exp \int_0^{2\pi} A \, d\theta = 1 + \delta U(2\pi) + O(\delta b^2) \tag{85}$$

with

$$\delta U(2\pi) = \int_0^{2\pi} d\theta' \, P \exp \int_{\theta'}^{2\pi} A_0 d\theta' \cdot \delta A(\theta') \cdot P \exp \int_0^{\theta'} A_0 d\theta'. \tag{86}$$

After a straightforward but mildly tedious computation, we find

$$\delta U(2\pi)=\begin{pmatrix} -y/n & (-z+x)/n^2 & 0 \\ x & 0 & (-z+x)/n^2 \\ ny & z & y/n \end{pmatrix}, \tag{87}$$

where we have defined

$$z=24\pi\beta_0/t\,,\qquad x=12\pi(\beta_n+\beta_{-n})/t\,,\qquad y=-12\pi i(\beta_n-\beta_{-n})/t\,. \tag{88}$$

The reason that δU in (87) comes out to depend only on $\beta_0, \beta_{\pm n}$ is that the other β's (as we saw earlier) lie on the orbit of (b_0, t) and thus give $U=1$. x, y, and z transform in the adjoint representation of $SL^{(n)}(2, R)$ [a fact which is more or less obvious in (88) though not in (87)] and correspond to the coordinate axes in Fig. 1. In the characteristic polynomial

$$\det(\lambda-\delta U)=\lambda^3+\frac{\lambda}{n^2}\cdot(z^2-x^2-y^2) \tag{89}$$

the invariant quadratic form $R^2=z^2-x^2-y^2$ appears. Just as in Fig. 1, there are three possible types of perturbations that can be added to the starting point (b_0, t).

For $R^2>0$, we have, e.g., $z\neq 0$, $x=y=0$. From the definition of x, y, z, we see that this perturbation is just a constant shift in b_0. Inspection of (87) reveals that for this perturbation, δU has one zero eigenvalue, corresponding to the eigenvector

$$F=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{90}$$

From the definition of F, we see that this corresponds to the vector field $f=1$. Thus, we recover the statement that for generic b_0, the vector (b_0, t) is stabilized only by rigid rotations of S^1.

For $R^2<0$, we have, e.g., $x\neq 0$, $y=z=0$. Inspection of (87) reveals that δU has one zero eigenvalue, corresponding to

$$F=\begin{pmatrix} 1 \\ 0 \\ -n^2 \end{pmatrix}, \tag{91}$$

or in other words to the vector field

$$f_n=\cos(n\theta)\,. \tag{92}$$

Thus, a perturbation of $R^2<0$ gives a new type of coadjoint orbit, stabilized by a one parameter subgroup generated by f_n. More precisely, we have determined in (92) the generator of the subgroup in question only to lowest order in δb. Let us call the one parameter subgroup in question $T_{(n,\Delta)}$, with Δ a parameter proportional to R^2. A useful characterization of Δ will appear later.

Finally, we consider a perturbation with $R^2=0$, e.g., $x=z\neq 0$, $y=0$. Again, inspection of (87) reveals that δU has precisely one zero eigenvalue, namely

$$F=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{93}$$

This corresponds to the vector field

$$\bar{f}_n = 1 - \cos(n\theta). \tag{94}$$

The stabilizer of an $R^2 = 0$ orbit is the one parameter group generated by $\bar{f}_n$, or more exactly by a vector field which goes over to $\bar{f}_n$ as the perturbation is removed. In considering a perturbation of $R^2 = 0$, the magnitude of z and x does not matter [they can be rescaled by a suitable $SL^{(n)}(2, R)$ transformation], but the sign of z does matter. We will refer to the one parameter subgroups of diffS^1 which stabilize the $R^2 = 0$ orbits as $\tilde{T}_{n,\pm}$, with the sign corresponding to the sign of z.

By now we have discovered certain classes of coadjoint orbits, the stabilizers being S^1, $SL^{(n)}(2, R)$, $T_{(n,A)}$, and $\tilde{T}_{(n\pm)}$. The orbits stabilized by these latter groups are also stabilized by a cyclic group of order n, consisting of rigid rotations of S^1 by an angle which is a multiple of $2\pi/n$. We will now analyze (66) more systematically, and show that no new orbits turn up.

Though we really want to determine f for given b, it is convenient to first suppose f is known and determine what b must be. As a preliminary, we let

$$b = \frac{t}{24\pi} f^{-2} h, \tag{95}$$

whence (66) becomes

$$h' = ff''' \tag{96}$$

or equivalently

$$\frac{dh}{d\theta} = \frac{d}{d\theta}(ff'' - \tfrac{1}{2}(f')^2). \tag{97}$$

It follows that

$$h - d = ff'' - \tfrac{1}{2}(f')^2 \tag{98}$$

with some integration constant d.

We will have to systematically study the possible zeros of f. We already know that f cannot have a triple zero, since the third order equation (66) would give f identically zero if f had a triple zero. However, the numbers of simple and double zeros of f and the cyclic order in which these are arranged on the circle are invariants. In addition, at a simple zero of f, f' is an invariant, independent of the choice of a local parametrization.[6] If we are only interested in f up to scaling [and that is the case here since (66) is homogeneous in f], then it is the ratios $f'(\theta_1)/f'(\theta_2)$ (with θ_1 and θ_2 being two zeros of f) which are invariants. In our problem we can quickly determine most of these invariants. Indeed, (95) shows that h must vanish at a zero of f if b is to be regular. At a zero of f, (98) thus reduces to

$$d = \tfrac{1}{2}(f')^2. \tag{99}$$

[6] Let the simple zero of f be at $\theta = 0$. Suppose f is of the form $f(\theta) = a_1\theta + a_2\theta^2 + \ldots$. Under the change of parameter $\theta \to \tilde{\theta} = c_1\theta + c_2\theta^2 + \ldots$, with $f \to \tilde{f} = (\partial\tilde{\theta}/\partial\theta) f$, it is clear that $f'(0) = a_1$ is invariant

If f has even one double zero, i.e. one zero at which f' also vanishes, then (99) shows $d=0$ and therefore that f' must also vanish at all other zeros of f. So if f has even one double zero, all of its zeros are double zeros; f has either all single zeros or all double zeros. And if the zeros are single zeros, then $|f'|$ has a common value $\sqrt{2d}$ at all zeros.

At a double zero of a vector field f, the ratio $(f''')^2/(f'')^3$ is an invariant.[7] Again, we can readily determine these invariants. From (95), we see that b is regular if and only if h has a fourth order zero at every double zero of f. From (98) we see that at a double zero of f ($f=f'=0$), $h'''=f''f'''$, so for $h'''=0$ (so that h has a zero of fourth order) we require $f'''=0$.

So we conclude that the stabilizer of a coadjoint orbit is a vector field with either (i) only single zeros and the same value of $|f'|$ at each zero; or (ii) only double zeros and $f'''=0$ at each double zero. Moreover, any vector field of one of those two types does stabilize a coadjoint orbit which is unique (up to choice of t). To find it, simply determine h from (98) (picking d so $h=0$ at zeros of f) and then use (95) to determine b. To complete the classification of coadjoint orbits it is, therefore, clearly sufficient to classify vector fields of the two types just stated.

We consider first the case in which f has only simple zeros and a common value of $|f'|=1$ at the zeros; by scaling we take this common value to be 1. Let $\theta_1, \theta_2, \ldots, \theta_r$ be the zeros of f. (There are an even number of them since f changes sign at each simple zero.) Let g be another vector field with the same properties – precisely r simple zeros, $|g'|=1$ at each zero. We wish to determine under what conditions g is conjugate to f in $\mathrm{diff}S^1$. As a preliminary, we can certainly conjugate g so that its zeros are the same position as those of f and so that g' and f' have the same sign at θ_1 and hence at each of the θ_k. Let Δ_k be the region $\theta_k<\theta<\theta_{k+1}$. We will first conjugate f into g in the region Δ_k and then try to match the resulting solutions. First we find τ_k to solve the equation

$$\frac{d\theta}{f(\theta)}=d\tau_k \tag{100}$$

in Δ_k. This is a first order equation, so τ_k is only determined up to an integration constant, which we fix by requiring

$$\tau_k\sim -|\ln(\theta-\theta_k)| \tag{101}$$

for $\theta\to\theta_k$. (The symbol $\sim$ means that the difference between left- and right-hand sides vanishes for $\theta\to\theta_k$.) It will then be the case that

$$\tau_k\sim +|\ln(\theta_{k+1}-\theta)|+a_k \tag{102}$$

for $\theta\to\theta_{k+1}$. Here a_k is an integration constant which can be determined by solving (100). Likewise, in the region Δ_k, we solve

$$\frac{d\theta'}{g(\theta')}=d\tau_k', \tag{103}$$

[7] If $f=a_2\theta^2+a_3\theta^3+\ldots$, then a_3^2/a_2^3 is invariant under $\theta\to\tilde{\theta}=c_1\theta+c_2\theta^2+\ldots$

subject to

$$\tau_k' \sim -|\ln(\theta' - \theta_k)| \tag{104}$$

for $\theta \to \theta_k$. For $\theta \to \theta_{k+1}$ we will then have

$$\tau_k' \sim |\ln(\theta_{k+1} - \theta')| + a_k' \tag{105}$$

with some constant a_k'. For any constant e_k, the equation

$$\tau_k = \tau_k' + e_k \tag{106}$$

then defines – when combined with the above equations expressing τ_k and τ_k' in terms of θ and θ' – a reparametrization of Δ_k which turns $f(\theta)(d/d\theta)$ into $g(\theta')(d/d\theta')$. This means that in each Δ_k, f is conjugate to g, but not uniquely since e_k is arbitrary. It remains to attempt to pick the e_k so that the above operations, defined separately on each Δ_k, fit together smoothly at the endpoints θ_k.

Collating (101), (104), and (106), we learn that for $\theta \to \theta_k$ from above,

$$(\theta - \theta_k) \sim (\theta' - \theta_k) e^{e_k}. \tag{107}$$

On the other hand, collating (102), (105), and (106), we learn that for $\theta \to \theta_k$ from below,

$$(\theta_k - \theta) e^{-a_{k-1}} = (\theta_k - \theta') e^{-a_{k-1}' - e_{k-1}}. \tag{108}$$

Comparing (107) to (108), and requiring that the limiting behavior of $\theta'(\theta)$ as θ_k is approached from below should match smoothly that which is obtained as θ_k is approached from above, we require

$$e_{k-1} + e_k = a_{k-1} - a_{k-1}'. \tag{109}$$

These equations, with $k = 1 \ldots r$, have a solution if and only if

$$\sum_{k=1}^{r} (-1)^k (a_k - a_k') = 0, \tag{110}$$

or equivalently only if f and g have the same value of

$$\Delta = \sum_{k=1}^{r} (-1)^k a_k. \tag{111}$$

Therefore, two vector fields with only simple zeros, the same number of zeros, and the same derivatives at the zeros are conjugate in $\mathrm{diff}S^1$ if and only if they have the same value of Δ. Incidentally, an alternative and perhaps more transparent definition of Δ is

$$\Delta = \lim_{\varepsilon \to 0} \int_{|\theta - \theta_k| > \varepsilon} \frac{1}{f} d\theta. \tag{112}$$

Here the θ_k are the zeros of f. This is invariant under reparametrization of θ because the non-invariance of the part of the integral with θ just above θ_k cancels that with θ just below θ_k. This definition of Δ is valid for any vector field with only simple zeros (i.e., there is no restriction to $|f'| = 1$ at the zeros). Clearly, Δ is a regularized version of the invariant (73) for vector fields without zeros.

We find, then, that vector fields of type (i) (only single zeros and $|f'|=1$ at each zero) are classified by the number of zeros and by Δ. In classifying orbits, the sign of Δ is irrelevant, since f and $-f$ stabilize the same orbit. The number of zeros is even, say $2n$, since f changes sign at a simple zero. In (92) we have given a particular vector field of type (i) with $2n$ zeros which is easily seen (on grounds of symmetry) to have $\Delta=0$. This vector field stabilizes the orbit of $(-n^2t/48\pi, t)$, which of course is also stabilized by $SL^{(n)}(2, R)$. In the discussion leading to (92) we described a perturbation of that orbit which breaks $SL^{(n)}(2, R)$ down to a subgroup $T_{n,\Delta}$. The generator of $T_{n,\Delta}$ is a vector field $f_{n,\Delta}$ which goes over to (92) as $\Delta\to 0$ [and which has $\Delta\neq 0$, though we did not demonstrate this explicitly, since as we have seen Δ is the only invariant describing a perturbation of a type (i) vector field]. Thus the orbits whose stabilizers are generated by vector fields of type (i) are precisely our friends $\mathrm{diff}S^1/T_{n,\Delta}$.

We now will consider vector fields of type (ii); the analysis is rather similar. Let $f(\theta)\dfrac{\partial}{\partial\theta}$ and $g(\theta')\dfrac{\partial}{\partial\theta'}$ be two vector fields of type (ii). We wish to know when some identification of the coordinates $\theta'=s(\theta)$ will turn f into g. Again, it is clearly necessary that f and g should have the same number of zeros. We can clearly always redefine θ' so that the zeros of f or g are at the same values $\theta_1, \ldots, \theta_n$ of θ and θ', and by reparametrizing θ and θ' near zeros, we can assume that $f\sim(\theta-\theta_k)^2$, $g\sim(\theta'-\theta_k)^2$ near a zero. Again, we let Δ_k be the interval from θ_k to θ_{k+1}, and in Δ_k we solve the equations

$$\frac{d\theta}{f}=d\tau_k\,,\qquad \frac{d\theta'}{g}=d\tau_k'\,. \tag{113}$$

We normalize τ_k, τ_k' so that as θ, θ' approach θ_k from above,

$$\tau_k\sim-\frac{1}{\theta-\theta_k}\,,\qquad \tau_k'\sim-\frac{1}{\theta'-\theta_k} \tag{114}$$

with errors that vanish as θ, $\theta'\to\theta_k$. It is here that we use the hypothesis that $f'''=g'''=0$ at θ_k; otherwise, in (114) there would be terms proportional to $\ln(\theta-\theta_k)$ and $\ln(\theta'-\theta_k)$. The functions τ and τ' which obey (114) will then behave as

$$\tau_k\sim-\frac{1}{\theta-\theta_{k+1}}+a_k\,,\qquad \tau_k'\sim-\frac{1}{\theta'-\theta_{k+1}}+a_k' \tag{115}$$

for θ, $\theta'\to\theta_{k+1}$. After finding the functions τ_k and τ_k' as above, in each region Δ_k, we define a transformation $\theta\to\theta'(\theta)$ by relating τ_k and τ_k',

$$\tau_k=\alpha\tau_k'+e_k\,. \tag{116}$$

Here α and e_k are adjustable constants; α must be positive so that the transformation $\theta\to\theta'$ preserves orientation of S^1. The map (116) turns $f\dfrac{d}{d\theta}$ into $g\dfrac{d}{d\theta'}$ in each Δ_k, but we must determine whether the limiting behavior as θ and θ' approach θ_k from above matches that when they approach θ_k from below. The necessary condition is

$$e_k=e_{k-1}+\alpha a_{k-1}'-a_{k-1}\,. \tag{117}$$

It is possible to find e_k obeying these conditions if and only if

$$\alpha \sum_k a'_k = \sum_k a_k, \tag{118}$$

so this is the condition for being able to conjugate f into g. What this means is that if we define

$$U(f) = \sum_k a_k, \qquad U(g) = \sum_k a'_k, \tag{119}$$

then f and g are conjugate if $U(f)$ and $U(g)$ have the same sign, or both vanish.

In (94), we have given a vector field $\tilde{f}_n$ of type (ii) with n zeros. It has $U=0$. We now know that up to conjugacy it has two types of perturbation preserving the type (ii) condition, namely $U>0$ and $U<0$. In the discussion surrounding (94), we described two types of perturbation of that vector field and the orbit it stabilizes, corresponding to the positive and negative portions of the cone or to the stabilizers $\tilde{T}_{n,\pm}$. We now see that these perturbations exhausted the possible invariants of vector fields of type (ii). This therefore completes the determination of the possible stabilizers of coadjoint orbits of the Virasoro group.

4. Quantization and Virasoro Representations

Finally, in this section we will attempt to establish the relation between coadjoint orbits of the Virasoro group and unitary Virasoro representations. The organization of the discussion will be as follows. First we will work out formulas for the Poisson brackets and the Virasoro generators (including the "Hamiltonian" L_0). We will find, in particular, that the role of Planck's constant is played by c^{-1}, the inverse of the Virasoro central charge. If we regard Virasoro orbits $\mathrm{diff}S^1/H$ (with various possible H's classified in the last section) as classical many-body systems, then quantizing such an orbit is similar to trying to solve a problem in quantum field theory; the coupling constant in quantum mechanical perturbation theory is c^{-1}. Then we will study the behavior of the classical Hamiltonian for various orbits, and find that the only orbits for which the energy is bounded below in the classical approximation are $\mathrm{diff}S^1/S^1$, $\mathrm{diff}S^1/SL^{(1)}(2,R)$, and $\mathrm{diff}S^1/\tilde{T}_{1,+}$. These are therefore the only orbits which could be expected to give highest weight representations in the semiclassical (large c) regime. Also, it turns out that $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$ are the only orbits for which there is a stable minimum of the "Hamiltonian" L_0 about which one might expand semiclassically. Thus, these orbits are the only candidates for giving highest weight representations for large c. The next issue is to ask which orbits can be quantized by standard methods. $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$ are Kahler and can be quantized as such. The orbits $\mathrm{diff}S^1/SL^{(n)}(2,R)$ with $n>1$, though apparently important in Virasoro representation theory, are not Kahler, nor (apparently) are they cotangent bundles. I believe that the same is true for the $\mathrm{diff}S^1/T_{n,\Delta}$ and $\mathrm{diff}S^1/\tilde{T}_{n,\pm}$, though this question requires more study. If an orbit is not Kahler and is not a cotangent bundle, this means that there is no general way to quantize it, and it is logical to think that this means that that orbit can only be quantized for special values of c. Therefore, the only orbits from which we can expect to construct representations in a whole range of c are $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$; these are also the only orbits which have

semiclassical expansions, for reasons which we have just alluded to. Our next topic will therefore be the Kahler quantization of $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$. We will find that in the region of $c>1$, all unitary Virasoro representations come from quantization of these orbits. The known theory of Virasoro representations indicates that the Kahler quantization of $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$ must break down at $c=1$. To understand this probably requires methods of differential geometry as opposed to the methods of symplectic geometry pursued in this paper. Finally, we will investigate the discrete series of Virasoro representations of $c\geqq 1$ [8], and argue that these are to be understood in terms of quantization of the orbits $\mathrm{diff}S^1/SL^{(n)}(2,R)$, $n\geqq 1$. (It is conceivable that $\mathrm{diff}S^1/\tilde{T}_{n,+}$ also plays a role.) We will argue – and it is perhaps the main result of this paper – that any quantization of $\mathrm{diff}S^1/SL^{(n)}(2,R)$ will give a Virasoro representation with a null vector on level n. In this way, we will obtain, I believe, an insight into the geometrical meaning of these "null vectors," and a partial explanation of the properties of the discrete series. A fuller understanding of the discrete series might emerge upon learning for which values of c the orbit $\mathrm{diff}S^1/SL^{(n)}(2,R)$ can be quantized, but this is a problem we will not be able to solve.

Highest weight Virasoro representations are labeled by the central charge c and the ground state energy h. The basic mystery about the discrete series of unitary representations is what singles out the discrete set of values of c and h at which these representations appear. From the point of view of the present paper, a partial answer to this question is that the Virasoro coadjoint orbits that seem to be related to the discrete series are $\mathrm{diff}S^1/SL^{(n)}(2,R)$, which depend only on one adjustable parameter, the central charge. Thus, one relation between h and c is fixed already at the classical level. Classically, this relation is simply $h=-(n^2-1)c/48\pi$; at the quantum level, it is the statement that there should be a null vector at level n. (The difference between the two statements reflects the quantum corrections, which indeed are large for small c.) To complete the picture and obtain the second quantization condition on h and c, one would have to learn how to quantize the orbits $\mathrm{diff}S^1/SL^{(n)}(2,R)$ in the strongly coupled domain, and this appears to be a difficult problem.

Generalities About Quantization

We must begin by working out the general formulas of the coadjoint orbit method, as they apply in the case of the Virasoro algebra.

A coadjoint orbit, say the orbit W_β of the vector $\beta=(b_0(\theta),t)$, consists of all coadjoint vectors into which β can be transformed by the action of the Virasoro group. It is somewhat awkward to express the condition on a coadjoint vector which says that it is in the same orbit as β. However, W_β is a homogeneous space of the Virasoro group; every β' in W_β can be reached from β by some reparametrization of the circle

$$\theta\to s(\theta)\,. \tag{120}$$

Of course, s may not be uniquely determined; the whole idea of the coadjoint orbit method depends on the fact that given β may be invariant under (120) for certain special $s(\theta)$, classified in the last section. Nevertheless, it is convenient to parametrize orbits by the $s(\theta)$. Of course, the condition that (120) should be a

diffeomorphism of the circle means in particular that it is invertible. Thus, instead of viewing $s=s(\theta)$ as a function of θ, we can view $\theta=\theta(s)$ as a function of s.

Under the diffeomorphism $\theta\to s(\theta)$, t is invariant; indeed, t plays the role of the central charge associated with an orbit. The Virasoro central charge is usually denoted as c; our viewpoint until now has been that c is the abstract central element of the Virasoro Lie algebra, and t is the value of c on a particular orbit. Henceforth, we will be less pedantic, and identity t and c.

Under $\theta\to s(\theta)$, b_0 transforms to a quadratic differential which we will call $\tilde{b}_s$. The latter has at a point s' on the circle the value

$$\tilde{b}_s(s')=\left(\frac{b_0(\theta(s))}{(ds/d\theta)^2}+\frac{c}{24\pi}\frac{\{s,\theta\}}{(ds/d\theta)^2}\right)_{s=s'}. \tag{121}$$

Here $\{s,\theta\}$ is the Schwarzian derivative,

$$\{s,\theta\}=\frac{d^3s/d\theta^3}{ds/d\theta}-\frac{3}{2}\left(\frac{d^2s/d\theta^2}{ds/d\theta}\right)^2. \tag{122}$$

It obeys the important identity

$$\frac{\{s,\theta\}}{(ds/d\theta)^2}=-\{\theta,s\}. \tag{123}$$

Equation (121) is an integrated form of (61).

A generator $\widehat{\mathrm{diff}}S^1$ consisting of a vector field u acts on s by

$$\delta s=u(s), \tag{124}$$

or more explicitly $\delta s(\theta)=u(s(\theta))$. According to the general prescription reviewed in Sect. 2, the symplectic structure of the orbit W_β is defined by the formula

$$\omega(u,v)=\langle(\tilde{b}_s,ic),[(u,0),(v,0)]\rangle. \tag{125}$$

In that somewhat abstract looking formula, $\langle\ ,\ \rangle$ denotes the pairing of the coadjoint vector $(\tilde{b}_s,ic)$ with the commutator of the adjoint vectors $(u,0)$ and $(v,0)$. Remembering the anomaly, that commutator is $(uv'-u'v,(-ic/48\pi)\int(uv'''-u'''v))$, so (125) is really

$$\omega(u,v)=\int ds'(uv'-u'v)\left(\frac{b_0(\theta(s))}{(ds/d\theta)^2}+\frac{c}{24\pi}\{\theta,s\}\right)_{s=s'}+\frac{c}{48\pi}\int ds'(uv'''-u'''v). \tag{126}$$

Thus, u and v are regarded as vector fields on the orbit W_β, and $\omega(u,v)$ is a real-valued function on the orbit; the value of this function at a point on the orbit defined by the diffeomorphism $\theta\to s(\theta)$ is given in (126).

From (126), we can draw an important conclusion. The semiclassical region (the region in which perturbation theory is valid) is the region in which ω is large, so that a large volume is associated with any portion of classical phase space, and quantum fluctuations are small. From (126), we see, therefore, that the semiclassical region is the region of large b_0 and c. In practice, b_0 is proportional to c for most of the interesting orbits, so effectively the semiclassical region is that of large c, i.e., large central charge. The expansion parameter in quantum mechanical perturbation theory will be $1/c$.

Now, we wish to work out the formulas for the Virasoro generators. The conserved charge which generates the transformation $\delta s = u(s)$ is, according to our discussion in Sect. 2, simply

$$\Phi_u(s) = \int ds'(u(s')\tilde{b}_s(s')) = \int ds' \left(\frac{u(s')b_0(\theta(s'))}{(ds/d\theta)^2} - \frac{c}{24\pi}\{\theta, s\} \right)_{s=s'}. \tag{127}$$

Let us be a little more explicit for the orbits $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(n)}(2,R)$ for which we can assume $b_0 = \text{constant}$. To obtain formulas for the Virasoro generators we substitute $u(s) = e^{ins}$ in (127) and get (for $n \neq 0$)

$$L_n(s) = b_0 \int ds' \left(\frac{u(s)}{(ds/d\theta)^2} \right)_{s=s'} - \frac{c}{24\pi} \int ds'(u(s)\{\theta, s\})_{s=s'}. \tag{128}$$

For L_0 it is usual to shift $L_0 \to L_0 + c/24$, giving

$$L_0(s) = b_0 \int ds' \left(\frac{1}{(ds/d\theta)^2} \right)_{s=s'} + \frac{c}{24} + \frac{c}{48\pi} \int ds' \left(\frac{d^2\theta/ds^2}{d\theta/ds} \right)^2. \tag{129}$$

We have integrated by parts in the last term in (129), to exhibit the fact that this term makes a positive definite contribution to L_0, for $c > 0$.

We have argued above that the region of validity of quantum mechanical perturbation theory should be large c. Let us now work out the rudiments of the perturbative formulas which are valid in this region. We will limit ourselves to the orbits $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(n)}(2,R)$ which can be represented as orbits of coadjoint vectors (b_0, c) for b_0 constant. (We will later see that there is no perturbative expansion for other orbits.) We take

$$s(\theta) = \theta + \delta s = \theta + \frac{1}{2\pi} \sum_{n \neq 0} s_n e^{-in\theta}. \tag{130}$$

Then from (128), we see that to lowest order in δs,

$$L_n(s) = 2in\left(b_0 + \frac{n^2 c}{48\pi} \right) s_n + O((\delta s)^2). \tag{131}$$

More exactly, this is the lowest order formula for L_n except for certain special values of n, which are $n = 0$ for $\mathrm{diff}S^1/S^1$ and $n = 0, \pm k$ for $\mathrm{diff}S^1/SL^{(k)}(2,R)$. These are the cases for which s_n in (130) leaves (b_0, c) invariant, and therefore for which $L_n(s)$ vanishes in the approximation of (131). For those special cases, the appropriate formulas are as follows. For L_0 we have

$$\begin{aligned} L_0 &= 2\pi b_0 + \frac{c}{24} + \frac{1}{\pi} \sum_{n=1}^{\infty} s_{-n} s_n \left(b_0 n^2 + \frac{cn^4}{48\pi} \right) \\ &= 2\pi b_0 + \frac{c}{24} + \frac{1}{\pi} \sum_{n=1}^{\infty} |s_n|^2 \left(b_0 n^2 + \frac{cn^4}{48\pi} \right) + O((\delta s)^3). \end{aligned} \tag{132}$$

This completes the picture except at $b_0 = -k^2 c/48\pi$, where the leading formula for $L_{\pm k}$ is

$$L_{\pm k} = \frac{c}{96\pi^2} \sum_{n=-\infty}^{\infty} s_{-n} s_{\pm k+n} \cdot (n(n \pm k)(n^2 \pm nk - k^2)). \tag{133}$$

Let us now work out very explicit formulas for the Poisson brackets. If $s=\theta+\sum s_n e^{-in\theta}/2\pi$, and $s'=\theta+\sum s'_n e^{-in\theta}/2\pi$, then from the definition of ω we get

$$\omega(s,s')=\frac{1}{\pi}\sum_n s_n s'_{-n}\cdot\left(inb_0+\frac{in^3c}{48\pi}\right). \tag{134}$$

This means that in the basis of Fourier modes s_n, the matrix elements of ω are

$$\omega_{m,n}=-\frac{1}{\pi}\delta_{m+n}\left(inb_0+\frac{in^3t}{48\pi}\right). \tag{135}$$

The Poisson bracket of two functions A and B on phase space is

$$\{A,B\}=\omega^{-1}_{n,m}\frac{\partial A}{\partial s_n}\frac{\partial B}{\partial s_m}. \tag{136}$$

Here $\omega^{-1}_{n,m}$ are the Fourier modes of the inverse matrix of ω, namely

$$\omega^{-1}_{n,m}=i\pi\delta_{n+m}\left(nb_0+\frac{n^3c}{48\pi}\right)^{-1}. \tag{137}$$

If, for instance, $A=s_n$, $B=s_m$, then (136) gives

$$\{s_n,s_m\}=i\pi\delta_{m+n}\left(nb_0+\frac{n^3c}{48\pi}\right)^{-1}+O(\delta s). \tag{138}$$

Thus, to lowest order in δs, the s_n obey the Poisson brackets of free bosons. In (138) we see that the Poisson brackets of the s_n are of order $1/c$, which means upon quantization that the quantum fluctuations in the s_n will be of order

$$\Delta s_n\sim\frac{1}{\sqrt{c}}. \tag{139}$$

It is the smallness of (139) for large c that justifies the $1/c$ expansion. In (138), we have computed the Poisson brackets only to lowest order in δs or in other words only to lowest order in c^{-1}. Actually, a general principle of symplectic geometry guarantees that locally the symplectic structure can always be put in a standard form $\omega=\sum dp^i\wedge dq^i$; in the present context, this means that by adding suitable $1/c$ corrections in the definition of the s_n, one could avoid corrections to (138) (at least in a finite region of field space or in other words to all finite orders in $1/c$). This is, however, not true of the formulas for the L_n, which will have $1/c$ corrections.

Let us finally check explicitly the Virasoro commutation relations, to lowest order in $1/c$. This question needs to be formulated with some care. The Virasoro algebra (allowing for the shift $L_0\to L_0+c/24$ which was made above to obtain the standard form of the cocycle) is

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{\delta_{m+n}(m^3-m)}{24}\cdot c. \tag{140}$$

Equation (140) is compatible with the ansatz

$$L_n\sim\sqrt{c},\qquad n\neq 0,\qquad L_0=h+O(1), \tag{141}$$

where h is a c-number (the ground state energy) which may be of order c, and the q-number part of L_0 is only of order one. A look back to (131) and (132) will reveal that (with $s_n \sim 1/\sqrt{c}$) Eq. (141) is indeed the correct scaling law for the large c behavior. (There is an exception to this if $b_0 = -k^2 c/24\pi$, whereupon $L_{\pm k}$ are of order 1 rather than of order $\sqrt{c}$.) The scaling law (141) shows that the dominant terms on the right-hand side of (140) are the anomaly term and the c-number part of L_0; working to lowest order in $1/c$, these are the only terms one should obtain.

Indeed, from (138) and (131) we can calculate the Poisson brackets

$$\{L_n, L_m\} = \delta_{n+m} \cdot (4i\pi) \cdot \left(nb_0 + \frac{n^3 c}{48\pi} \right). \tag{142}$$

Looking back to the formula (132) for L_0, we see that (142) agrees with the Virasoro algebra (140) to lowest order in $1/c$. [The quantum commutator in the Lie algebra (140) is $-i$ times the classical Poisson bracket.] The higher order terms are of course guaranteed to work out from the general considerations in Sect. 2.

Behavior of the Classical Hamiltonian

We now want to approach a little closer to the physical properties of Virasoro coadjoint orbits. As a first step we would like to search for stable critical points of the "Hamiltonian" L_0, for which we have given various formulas in the last subsection. [The general formula is (127); for $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(n)}(2, R)$ there is some simplification given in (129); and the perturbative formula is (132).] It might appear formidable to look for critical points of the highly nonlinear expressions (127), (129), but in fact locating all possible critical points is a short exercise in Hamiltonian mechanics. We recall that the whole idea of Hamiltonian mechanics is that a function H on phase space generates the transformation

$$\delta\phi^k = \omega^{kl} \frac{\partial H}{\partial \phi^l} \tag{143}$$

of the phase space coordinates ϕ^k. Critical points of H (points at which $\partial_l H = 0$) correspond exactly to points with $\delta\phi^k = 0$, that is, points which are invariant under the transformation generated by H.

Specializing this to the case of Virasoro coadjoint orbits, with L_0 playing the role of the "Hamiltonian," we know that the transformation generated by L_0 is a rotation of the circle, $\theta \to \theta + \mathrm{const}$. A critical point of L_0 thus must correspond to a coadjoint vector $(b(\theta), c)$ which is invariant under rotation of the circle; thus, it must correspond to a constant coadjoint vector (b_0, c). We have seen in Sect. 3 which coadjoint orbits contain such constant vectors; they are $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(n)}(2, R)$ (and $\mathrm{diff}S^1$ itself in the case of zero central charge, which we will not explore). The other Virasoro coadjoint orbits, namely $\mathrm{diff}S^1/T_{n,\Delta}$ and $\mathrm{diff}S^1/\tilde{T}_{n,\pm}$, do not contain any critical point of the Hamiltonian at all.

For the orbits $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(n)}(2, R)$ which do contain critical points of L_0, let us determine whether the unique such critical point [the constant (b_0, t)] is stable. From (132), we see that the requirement is

$$b_0 n^2 + \frac{cn^4}{48\pi} > 0, \qquad n = 1, 2, 3, \ldots. \tag{144}$$

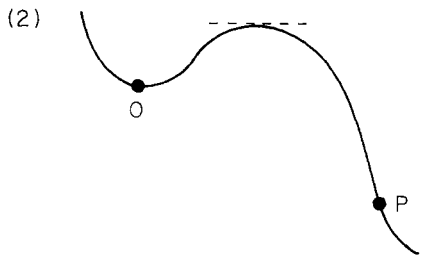

Fig. 2. If a stable critical point O of the Hamiltonian is not the absolute minimum, i.e., if there is a point P at which the energy is less than it is at O, there should be another critical point of the Hamiltonian "between" O and P

For $\mathrm{diff}S^1/S^1$, this will be true provided

$$b_0 > \frac{-c}{48\pi}, \tag{145}$$

which according to (132) means that h, the c-number piece of L_0, must be positive. This is the classical limit of a statement that the Virasoro highest weight should be positive for quantization of $\mathrm{diff}S^1/S^1$. (The significance of positive h in Virasoro representation theory will be discussed later.) For $\mathrm{diff}S^1/SL^{(n)}(2,R)$, we see from (132) that the unique critical point $(-n^2c/48\pi, c)$ is stable for $n=1$ but not for $n>1$. Thus, the only Virasoro coadjoint orbits with stable stationary points of the Hamiltonian are $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$. On this grounds alone, $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$ are the only orbits which should be quantizable in the semiclassical or large c limit; after all, the semi-classical limit requires having a stable critical point of the Hamiltonian around which to expand.

It is instructive to pursue the analysis of the classical Hamiltonian further and ask for what orbits the classical Hamiltonian is bounded below. We observe immediately that this will only happen for positive c, since for negative c the second term in (129) – which dominates at high frequencies – has the wrong sign. We will now argue (somewhat heuristically) that $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$ have energy bounded below. The basic idea is to show that the known unique critical point of L_0 – let us call it O – must be the absolute minimum. Indeed, if there were a point P with lower energy than the energy at O, then "between" O and P there should be a second critical point, as indicated heuristically in Fig. 2. This would contradict the fact that O is known to be the unique critical point of the energy, so we conclude that P cannot exist and O is the absolute minimum. To make this argument precise, one requires some information about the topology of $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$; if for instance, we were allowed to cut out a hole between O and P containing the second critical point in Fig. 2, it would clearly be possible to avoid the extra critical point while having a point P with energy less than that at O. In fact, $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$ are contractible spaces topologically, so the argument above should go through, though I will not try to make it rigorous.[8]

[8] The argument is patterned on an attempt by Brill and Deser to prove positivity of the energy in classical general relativity [16], which is however a case in which the phase space is far from being contractible, so that justifying the argument would be far more difficult

I will now try to show that for most of the other Virasoro coadjoint orbits, L_0 is *not* bounded below at the classical level. Let us consider first the case of $\mathrm{diff}S^1/SL^{(n)}(2,R)$ for $n>1$. This is the orbit of the vector $(-n^2c/48\pi, c)$, which we write as $\beta_1+\beta_2$, with

$$\beta_1=(-c/48\pi, c), \qquad \beta_2=((1-n^2)c/48\pi, 0). \tag{146}$$

The point of this decomposition is that β_1 is invariant under the subgroup $SL(2,R)$ of $\mathrm{diff}S^1$ generated by L_0, L_1, and L_{-1}, while β_2 is not invariant under this group and makes a negative contribution to the energy. By a suitable $SL(2,R)$ transformation, without changing β_1, we could transform β_2 into a coadjoint vector that would make an arbitrarily negative contribution to the general formula (129) for the energy. Indeed, β_2 can be conjugated in $SL(2,R)$ to the vector

$$((p+q\cos\theta)(1-n^2c/48\pi), 0) \tag{147}$$

with any p, q such that $p>0$ and $p^2-q^2=1$. The contribution of (147) to (129) is $p(1-n^2)c/24$, and this is unbounded below, for $n>1$.

It remains to consider the orbits $\mathrm{diff}S^1/T_{n,A}$ and $\mathrm{diff}S^1/\tilde{T}_{n,\pm}$. Let us recall how these orbits arose in Sect. 3. They are orbits of the coadjoint vector $\beta=\beta_0+\delta\beta$, where β_0 is $SL^{(n)}(2,R)$ invariant, and $\delta\beta$ is a small perturbation. The analysis now proceeds just as in the analysis of $SL(2,R)$ (recall Fig. 1). $\mathrm{diff}S^1/T_{n,A}$ corresponds to an $R^2<0$ perturbation in the figure, say $z=y=0, x\neq 0$; such a perturbation can be conjugated in $SL^{(n)}(2,R)$ to a perturbation with arbitrarily negative z or in other words arbitrarily negative energy. As for $\mathrm{diff}S^1/\tilde{T}_{n,\pm}$, these correspond to perturbations of $R^2=0$ and positive or negative z. In the negative z case, the energy can again be made arbitrarily negative by an $SL^{(n)}(2,R)$ transformation, so the classical Hamiltonian is unbounded below, but this will not occur for positive z.

In the last paragraph, to sharpen the analogy with $SL(2,R)$, we have tried conjugating with $SL^{(n)}(2,R)$. However, as in our discussion of $\mathrm{diff}S^1/SL^{(n)}(2,R)$, we could (even for $n\neq 1$) explore the Hamiltonian of $\mathrm{diff}S^1/\tilde{T}_{n,+}$ by conjugating by $SL^{(1)}(2,R)$. Since the orbits $\mathrm{diff}S^1/\tilde{T}_{n,+}$ can actually be obtained by an arbitrarily small perturbation of $\mathrm{diff}S^1/SL^{(n)}(2,R)$, conjugation by $SL^{(1)}(2,R)$ will give the same answer for $\mathrm{diff}S^1/\tilde{T}_{n,+}$ that it gave for $\mathrm{diff}S^1/SL^{(n)}(2,R)$: the energy is unbounded below for $n>1$. However, the energy is presumably bounded below for $\mathrm{diff}S^1/T_{1,+}$, and this, along with $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$, completes the list Virasoro coadjoint orbits for which the classical Hamiltonian is bounded below.

Quantizability of Orbits

As was discussed in Sect. 2, the only symplectic varieties for which one has a general framework for quantization are cotangent bundles and Kahler manifolds. We will now discuss the extent to which Virasoro coadjoint orbits are of these types. $\mathrm{diff}S^1/S^1$ has a Kahler structure, recently investigated in [4, 5]; we will see that $\mathrm{diff}S^1/SL^{(1)}(2,R)$ likewise has a Kahler structure, but $\mathrm{diff}S^1/SL^{(n)}(2,R)$ does not for $n>1$. I believe that the other orbits of the form $\mathrm{diff}S^1/T$ likewise do not have Kahler structures, but I will not attempt to prove this. It seems likely, as well, that none of the Virasoro coadjoint orbits are cotangent bundles, but I will not try to prove this. Modulo the latter conjectures, our considerations about Kahler

structures will single out $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$ as the only Virasoro coadjoint orbits for which there is a standard framework for quantization; we have seen in the last subsection that these are also the only good candidates for semiclassical quantization (stable minimum of the Hamiltonian and energy bounded below).

Let us first discuss the issue of a Kahler structure in general. Consider a Lie group G and a subgroup H; we want to determine whether the homogeneous space G/H has a G-invariant Kahler structure. We recall that G/H consists of equivalence classes of elements of G, with g and g' considered equivalent if $g=g'h$ for some $h\in H$. Let 1 denote the identity of G and $\bar{1}$ the corresponding equivalence class. What is the subground of G that leaves $\bar{1}$ invariant? The equation $g\cdot\bar{1}=\bar{1}$ means that there is $h\in H$ with $g\cdot 1=1\cdot h$, and this clearly is true if and only if $g\in H$. Thus, $\bar{1}$ is left fixed precisely by the subgroup $H\subset G$.

Let J be a G invariant complex structure on G/H. The restriction J_1 of J to the point $\bar{1}\in G/H$ is just a matrix (of square -1) on tangent vectors. What is the condition for J to be invariant under the action of a given element $g\in G$? Since G/H is homogeneous, we may as well study this question at the point $\bar{1}$ in G/H. For $g\notin H$, g maps $\bar{1}$ to some other point $g\bar{1}$ in G/H, and g invariance determines the value of J at $g\bar{1}$ in terms of its value at $\bar{1}$. Clearly, as g ranges over G, a G invariant complex structure on G/H (or any of a wide class of local geometric structures on G/H) is uniquely determined by its restriction to $\bar{1}$ (or in general by the restriction to infinitesimal neighborhoods thereof). What if $g\in H$? Then g maps $\bar{1}$ to itself, so G invariance of J requires that J_1 (the complex structure at $\bar{1}$) should be H invariant. Thus, G invariant complex structures on G/H correspond precisely to H invariant complex structures at the origin $\bar{1}\in G/H$.

To analyze the possible structures at the origin is a problem that can be studied by means of Lie algebras rather than Lie groups. Let $\mathcal{G}$ and $\mathcal{H}$ be the Lie algebras of G and H. Let $\mathcal{K}$ be the complement of $\mathcal{H}$ in $\mathcal{G}$, i.e.,

$$\mathcal{G}=\mathcal{H}\oplus\mathcal{K}\,. \tag{148}$$

(The choice of $\mathcal{K}$ is not unique; we choose it to be invariant under conjugation by H.) $\mathcal{K}$ can be regarded as the tangent space to G/H at $\bar{1}$. An almost complex structure at $\bar{1}$ is a decomposition of the tangent space as

$$\mathcal{K}=\mathcal{K}_+\oplus\mathcal{K}_-\,, \tag{149}$$

with $\mathcal{K}_-$ the complex conjugate of $\mathcal{K}_+$. Elements of $\mathcal{K}_+$ and $\mathcal{K}_-$ correspond, respectively, to holomorphic and antiholomorphic directions on G/H, in this almost complex structure. The decomposition (149) of the tangent space at $\bar{1}$ is H invariant if it is closed under commutation with elements of $\mathcal{H}$, or more exactly if

$$[h,k_+]\in\mathcal{H}\oplus\mathcal{K} \tag{150}$$

for all $h\in\mathcal{H}$, $k_+\in\mathcal{K}_+$. An almost complex structure is said to be integrable, and to give a complex structure, if the holomorphic tangent vectors are closed under commutation, or in the situation under discussion if

$$[k_+,\tilde{k}_+]\in\mathcal{H}\oplus\mathcal{K}\,. \tag{151}$$

Let us now take for G the Virasoro group and consider some of the H's which correspond to stabilizers of coadjoint vectors. First consider $H=S^1$, generated by

L_0, corresponding to the coadjoint orbit $\mathrm{diff}S^1/S^1$. Thus, $\mathscr{H}$ is spanned by L_0, and for $\mathscr{K}$ we can choose the basis $\{L_n | n \neq 0\}$. Clearly, we can decompose this in an $\mathscr{H}$ invariant way as $\mathscr{K} = \mathscr{K}_+ \oplus \mathscr{K}_-$, where $\mathscr{K}_+$ is spanned by L_n, $n > 0$. Then $\mathscr{K}_+$ is a Lie algebra, so (151) is obeyed, and we obtain a Virasoro invariant complex structure on $\mathrm{diff}S^1/S^1$. In fact, it has been recently studied in [4, 5].

Now let us study the orbit $\mathrm{diff}S^1/SL^{(1)}(2, R)$. $\mathscr{H}$ is spanned by L_0, L_1, L_{-1}, and we can pick $\mathscr{K}$ to be spanned by the L_n with $n > 1$ or $n < -1$. An $\mathscr{H}$ invariant splitting of $\mathscr{K}$ immediately presents itself, with $\mathscr{K}_+$ spanned by L_n, $n > 1$. Again, $\mathscr{K}_+$ is a Lie algebra, so (151) is obeyed, and we get a Virasoro invariant complex structure on $\mathrm{diff}S^1/SL^{(1)}(2, R)$.

Finally, we consider $\mathrm{diff}S^1/SL^{(n)}(2, R)$, with $n > 1$. $\mathscr{H}$ is spanned by L_0, L_n, L_{-n}. We can choose $\mathscr{K}$ to be spanned by L_m, $m \neq 0$, n, $-n$. A little reflection will show that $\mathrm{diff}S^1/SL^{(n)}(2, R)$, with $n > 1$ is completely different from $\mathrm{diff}S^1/SL^{(1)}(2, R)$. Consider the adjoint vector L_k with $0 < k < n$. If it is included in $\mathscr{K}_+$, then (150) shows that $\mathscr{K}_+$ must include the repeated commutators of L_k with L_n or L_{-n}, so in this case $\mathscr{K}_+$ must include all generators L_{k+tn} with arbitrary positive or negative integer n. Thus, $\mathscr{K}_+$ cannot consist of L_k with only positive or only negative k. Indeed, if n is even, there is no Virasoro invariant almost complex structure on $\mathrm{diff}S^1/SL^{(n)}(2, R)$, since for $k = n/2$ the requirement that L_k and L_{k-n} should be both in $\mathscr{K}_+$ or both in $\mathscr{K}_-$ contradicts the requirement that $\mathscr{K}_-$ should be the complex conjugate of $\mathscr{K}_+$. For n odd but greater than one, there are invariant almost complex structures on $\mathrm{diff}S^1/SL^{(n)}(2, R)$, obtained by making an arbitrary assignment of L_k to $\mathscr{K}_+$ or to $\mathscr{K}_-$ for $0 < k < n$ and $k = 2n$ and then imposing (150) and the requirement that $\mathscr{K}_\pm$ are complex conjugates of each other. Although one gets in this way a variety of invariant almost complex structures on $\mathrm{diff}S^1/SL^{(n)}(2, R)$, for n odd and greater than one, these are seemingly rather uncanonical; in any case, they cannot be chosen to obey (151), so there are no invariant complex structures on $\mathrm{diff}S^1/SL^{(n)}(2, R)$ for $n > 1$.

For quantization we wish not just a complex structure but a Kahler structure on $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2, R)$. Kahler structures on $\mathrm{diff}S^1/S^1$ were described in detail in [4]. From our standpoint in this paper, the following way of looking at these structures is natural. On a complex manifold, a Kahler form is the same as a symplectic form ω which is of type (1, 1) (one holomorphic and one anti-holomorphic index). Symplectic structures on coadjoint orbits are exactly what we have been constructing in this paper. In the case of $\mathrm{diff}S^1/S^1$, regarded as the orbit of the coadjoint vector (b_0, c), there is a two parameter family of symplectic structures, labeled by the values of b_0 and c. These have already been described explicitly in (126) and the perturbative expansion in (138) gives the explicit form of the symplectic structure at the particular point (b_0, c). To obtain Kahler structures on $\mathrm{diff}S^1/S^1$, it is enough to obtain symplectic structures of type (1, 1), and because of homogeneity, it suffices to check the (1, 1) property at one point, say (b_0, c). The (1, 1) property at the point (b_0, c) amounts to the statement that the Poisson bracket of s_n with s_m vanishes if n and m have the same sign (i.e., if s_n and s_m are both holomorphic or both antiholomorphic coordinates). This is clearly true according to (138). Likewise, regarding $\mathrm{diff}S^1/SL^{(1)}(2, R)$ as the coadjoint orbit of the vector $(-c/48\pi, c)$ the Kirillov-Kostant method gives a one parameter family of symplectic structures on $\mathrm{diff}S^1/SL^{(1)}(2, R)$, labeled by c. These are again described

in (126) and (138), simply on setting $b_0 = -c/48\pi$, and from (138) the (1, 1) property is again obvious. This description makes it obvious that in a sense the Kahler structures on $\mathrm{diff}S^1/SL^{(1)}(2,R)$ are $h \to 0$ limits of the Kahler structures on $\mathrm{diff}S^1/S^1$.

We thus conclude (modulo a few unproved conjectures at the beginning of this subsection) that the only Virasoro coadjoint orbits for which there is a general geometric strategy for quantization are $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$. We also saw in the last subsection that these orbits are the only ones for which one can reasonably expect to carry out quantization in quantum mechanical perturbation theory. Thus, there is no general strategy for quantizing, say, $\mathrm{diff}S^1/SL^{(n)}(2,R)$, $n>1$, which is not Kahler and presumably not a cotangent bundle, and which as we saw in the last subsection also cannot be expected to have a perturbative expansion for large c. The combination of these factors, and especially the last, strongly suggests that in fact $\mathrm{diff}S^1/SL^{(n)}(2,R)$, with $n>1$, is unquantizable for sufficiently large c. Whether it can be quantized for small c, in a way not tied to perturbation theory or to the usual geometric strategies for quantization, is another story. Similar remarks could be made for the other orbits $\mathrm{diff}S^1/T$; of these, the one closest to being quantizable is perhaps $\mathrm{diff}S^1/\tilde{T}_{1,+}$, which has energy bounded below (but no perturbation expansion since the Hamiltonian has no extremum) and can be regarded as a perturbation of the quantizable spaces $\mathrm{diff}S^1/SL^{(1)}(2,R)$ or $\mathrm{diff}S^1/S^1$. Incidentally, the properties of $\mathrm{diff}S^1/\tilde{T}_{1,+}$ are quite analogous to those of the Liouville quantum field theory [17, 7], and these may be related.

Review of Virasoro Representation Theory

Before plunging into a discussion of quantization of orbits, let us pause for a (rather brief and sketchy) review of the basic facts about Virasoro representation theory which one wishes to explain.

The Virasoro representations of interest are those of "highest weight." This simply means that the spectrum of L_0 is bounded below. Highest weight representations can be constructed by assuming an arbitrary lowest eigenvalue h of L_0 with corresponding eigenvector $|\Omega\rangle$, which necessarily obeys

$$L_m|\Omega\rangle = 0, \quad m>0. \tag{152}$$

The Virasoro algebra can then be represented on the "Verma module," which consists of the states

$$\prod_{n=1}^{\infty} L_{-n}^{a_n}|, |\Omega\rangle. \tag{153}$$

Here the a_m are integers, all but finitely many of them zero. We will sometimes abbreviate (153) as $L_{-I}|\Omega\rangle$.

If we assume that the states (153) are linearly independent, we can readily work out the partition function $\mathrm{Tr}q^{L_0}$. Equation (153) has $L_0 = h + \sum na_n$. The counting of the a_n amounts to studying the partition function of an ideal bose gas, and the result is simply

$$\mathrm{Tr}q^{L_0} = q^h \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{154}$$

It is important to determine when the Verma module [with all states (153) linearly independent] has a unitary structure (i.e., a positive definite inner product) compatible with

$$L_n = L^*_{-n}. \tag{155}$$

The basic idea is to work out the norm of the state $L_{-I}|\Omega\rangle$ from

$$|L_{-I}|\Omega\rangle|^2 = \langle\Omega|L_I L_{-I}|\Omega\rangle. \tag{156}$$

One evaluates the right-hand side of (156) by moving the L_m to the right until they annihilate $|\Omega\rangle$, using the Virasoro commutation relations along the way. For an important example, consider the state $L_{-1}|\Omega\rangle$. Its norm is

$$|L_{-1}|\Omega\rangle|^2 = \langle\Omega|[L_1, L_{-1}]|\Omega\rangle = 2\langle\Omega|L_0|\Omega\rangle = 2h. \tag{157}$$

Therefore, a highest weight Virasoro representation with $h<0$ cannot be unitary.

For $h=0$ the situation is more delicate. In a hypothetical unitary representation with $h=0$, (157) clearly implies that

$$L_{-1}|\Omega\rangle = 0. \tag{158}$$

This then is the simplest situation in which one may not assume that the states (153) are linearly independent. We can readily work out the partition function of a representation in which the states (153) are subject to no relations of linear dependence except those that follow from (158). By moving L_{-1} to the right in (153), using the commutation relations when they arise, until L_{-1} annihilates $|\Omega\rangle$, one can reexpress (153) without L_{-1}. Thus, a highest weight representation of $h=0$ in which (158) is valid is spanned by the states

$$\prod_{n=2}^{\infty} L_{-n}^{a_n}|\Omega\rangle. \tag{159}$$

If there are no relations of linear dependence in (153) except the consequences of (158), then the states (159) are linearly independent. Clearly, in this case the partition function is

$$\prod_{n=2}^{\infty} \frac{1}{1-q^n}. \tag{160}$$

It is a non-trivial fact that the Verma module with partition function (154) is unitary if $c \geqq 1$ and $h>0$, and a degenerate representation with partition function (160) is unitary if $h=0$ and $c \geqq 1$. All other unitary representations have $c \leqq 1$, and their description is more subtle. We will say that a representation has a null vector on level n if there is a relation

$$\left(L_{-n} + \sum \alpha^I \cdot \prod_{k=1}^{n-1} L_{-k}^{a_k^I}\right)|\Omega\rangle = 0 \tag{161}$$

which is not a consequence of a similar relation on a previous level. (Here I is a partition of n into positive integers less than n, in the form $n = \sum k a_k^I$.) Kac [18] determined the conditions on c and h under which one can form a representation with a null vector on level n. (For further developments, see [19].) These

representations were used in quantum field theory in [6], and their unitarity was investigated in [8].

A representation with only a null vector on level n, that is, all linear dependences in (153) are consequences of a single relation (161), would by analogy with (160) have the partition function

$$\mathrm{Tr} q^{L_0} = q^h \prod_{k \geq 1, k \neq n} \frac{1}{1-q^k}. \tag{162}$$

This is the formula that one obtains if one moves L_{-n} to the right in (153), and eliminates it via (161), so as to express the states as linear combinations of

$$\prod_{k \geq 1, k \neq n} L_{-k}^{a_k} |\Omega\rangle. \tag{163}$$

Assuming that all linear dependences in the Verma module are consequences of (161), the states (163) are linearly independent, and therefore the partition function is that of (162). It is known that a Virasoro representation with these properties is unitary if and only if $c=1$ and $h=(n-1)^2$.

There remains the possibility of a representation with more than one independent null vector, i.e., a representation in which the linear dependences in (153) do not follow from a single relation (161). The mysterious discrete series of unitary representations of $c<1$ possess this property [8]. Their characters are more complicated than (154) or (160), and have been computed by Rocha-Caridi [20].

Quantization of $\mathrm{diff}S^1/S^1$ *and* $\mathrm{diff}S^1/SL^{(1)}(2,R)$

In this section, we will discuss the quantization of the only Virasoro coadjoint orbits which can be straightforwardly quantized, namely the Kahler manifolds $\mathrm{diff}S^1/S^1$ and $\mathrm{diff}S^1/SL^{(1)}(2,R)$. These are also the only orbits which have perturbative expansions, and can be quantized for large c.

We first discuss the quantization from the point of view of ordinary quantum mechanical perturbation theory, valid for large enough c. First we consider $\mathrm{diff}S^1/S^1$. According to (132), the leading large c approximation to L_0 is

$$L_0 = h + \sum_{n=1}^{\infty} c_n |s_n|^2, \tag{164}$$

with formulas given in (132) for the ground state energy h and the c_n. Also, according to (138), the s_n have the Poisson brackets of free bosons. Let $|\Omega\rangle$ be the ground state annihilated by the s_n for $n>0$, and with $L_0=h$ [with a possible quantum correction to the classical formula that follows from (132) expressing h in terms of b_0 and t]. The Poisson bracket relation

$$\{L_0, s_n\} = -ins_n \tag{165}$$

shows that the quantum state

$$\prod_{n=1}^{\infty} s_{-n}^{a_n} |\Omega\rangle. \tag{166}$$

has

$$L_0 = h + \sum n a_n . \tag{167}$$

The partition function of this system is clearly

$$\mathrm{Tr} q^H = q^h \prod_{n=1}^{\infty} \frac{1}{1-q^n} . \tag{168}$$

Now, let us consider the quantization of $\mathrm{diff} S^1/SL^{(1)}(2, R)$ from the stand-point of perturbation theory. First, setting $b_0 = -c/48\pi$, we see that the constant term drops out of the formula in (132) for L_0. Also, s_1 and s_{-1} are to be dropped; their coefficient vanishes in (132), and more fundamentally they are not independent parameters on the $\mathrm{diff} S^1/SL^{(1)}(2, R)$ orbit. The formula for L_0 thus reduces to

$$L_0 = \sum_{n=2}^{\infty} c_n |s_n|^2 , \tag{169}$$

and computing the partition function of this restricted set of bosons, we immediately get

$$\mathrm{Tr} q^H = \prod_{n=2}^{\infty} \frac{1}{1-q^n} . \tag{170}$$

Comparing back to the discussion in the last subsection, we see immediately that (169) is the partition function of a Verma module, and unitary in the range $c > 1$, $h > 0$. As for our derivation of (169), it is limited to $c \gg 1$ (for validity of perturbation theory) and $h > 0$ (the region in which the classical Hamiltonian is bounded below). Again comparing to the last subsection, we note that (170) is the correct partition function for a representation with $h = 0$, $L_{-1}|\Omega\rangle = 0$, and otherwise generical; such representations are unitary for $c \geqq 1$. Again, our derivation of (170) is valid for $c \gg 1$, and does not illuminate the question of why this formula should be exact in a whole finite range of c.

Not only do the formulas (169) and (170) coincide with some discussed in the last section; the derivations are also clearly analogous, with a substitution $L_n \to s_n$. The "reason" for this analogy is clearly that according to (131), in the large c limit L_n and s_n are actually proportional.

Now, let us try to rederive (169) and (170) in a way that will have a somewhat broader range of validity, using the Kahler structure of the orbits in question and the fixed point theorem whose application to $SL(2, R)$ was discussed in Sect. 2 [see Eq. (43)].[9] In Kahler quantization, one is to find a suitable line bundle L and then construct the space of holomorphic sections. Homogeneous line bundles on a homogeneous space G/H are in one to one correspondence with one dimensional representations of H (by an argument somewhat similar to the one we used above in studying homogeneous complex structures). For present purposes, $\mathrm{diff} S^1/S^1$ is best regarded as $\widehat{\mathrm{diff}} S^1/S^1 \times \tilde{S}^1$, with $\widehat{\mathrm{diff}} S^1$ the central extension of $\mathrm{diff} S^1$, S^1 generated by L_0, and $\tilde{S}^1$ the center of $\widehat{\mathrm{diff}} S^1$. One dimensional representations of S^1

[9] This has been investigated independently by I. Singer, with the motivation of obtaining a more geometric understanding of the "ghosts" of string quantization

$\times \tilde{S}^1$ are labeled by the eigenvalues of the generators of the two S^1's; these are usually called h and c. Thus, there is a two parameter family of homogeneous line bundles $L_{h,c}$ on $\mathrm{diff}S^1/S^1$.[10] Thus, one can attempt Kahler quantization for any h, c.

To use (43) to study $\mathrm{diff}S^1/S^1$, we need to identify the "rotation angles" which appear in that formula. Clearly, near the unique fixed point of L_0 (the minimum of the Hamiltonian), L_0 generates the transformation $\delta s_n \sim n s_n$, which corresponds to the holomorphic vector field

$$V = \sum_{n=1}^{\infty} n s_n \frac{\partial}{\partial s_n}. \tag{171}$$

The rotation angles to be used in (43) are thus $1, 2, 3, \ldots$. Consequently the fixed point formula gives

$$\mathrm{Tr} q^H = q^h \prod_{n=1}^{\infty} \frac{1}{1-q^n} \tag{172}$$

for the partition function to be obtained by Kahler quantization of $\mathrm{diff}S^1/S^1$.

Similarly, we can apply the fixed point formula to $\mathrm{diff}S^1/SL^{(1)}(2, R)$. This is to be regarded as $\widehat{\mathrm{diff}}S^1/SL(2, R) \times \tilde{U}(1)$. As $SL(2, R)$ has no non-trivial one dimensional representations, the homogeneous line bundles now form merely a one dimensional family L'_c, labeled by the eigenvalue of $\tilde{U}(1)$. This is really the reason that quantization of $\mathrm{diff}S^1/SL^{(1)}(2, R)$ gives automatically $h = 0$. As for the rotation angles, in $\mathrm{diff}S^1/SL^{(1)}(2, R)$ the coordinates $s_{\pm 1}$ are missing, and L_0 behaves near its minimum like the holomorphic vector field

$$V' = \sum_{n=2}^{\infty} n s_n \frac{\partial}{\partial s_n}. \tag{173}$$

Applying the fixed point formula, we now find that Kahler quantization of $\mathrm{diff}S^1/SL^{(1)}(2, R)$ should give the partition function

$$\mathrm{Tr} q^H = \prod_{n=2}^{\infty} \frac{1}{1-q^n}. \tag{174}$$

Now that we have derived (172) and (174) for the third time, one may have a feeling of satisfaction. But a new problem appears. While in the perturbation theory derivation of (169) and (170), it appeared that the range of validity of these formulas might be rather limited, the fixed point theorem that we have used to get (172) and (174) is so general that it might make one wonder why these formulas only hold for $c \geqq 1$. An understanding of this is beyond the scope of the present paper, but a clue may very probably come from the comparison with our earlier

[10] This should be regarded as a conceptual explanation for the existence of a two parameter family of homogeneous Kahler structures on $\mathrm{diff}S^1/S^1$, as shown in [4]. The two parameter family of homogeneous Kahler forms obtained there can be regarded as curvature forms of homogeneous connections on the two parameter family of line bundles $L_{h,c}$. The computations of curvature in [4, 5] can be understood as answering the question of which of the $L_{h,c}$ is to be regarded as the canonical line bundle of $\mathrm{diff}S^1/S^1$

discussion of $SL(2,R)$ [see Eq. (38)]. In that discussion, we found that for any R, the fixed point formula gave correctly the partition function of the space of holomorphic sections of the relevant line bundle, but these ceased to be square integrable at $R=1/2$.[11] It is very plausible that (172) and (174) give correctly the partition functions of the spaces of holomorphic sections of the line bundles $L_{h,c}$ and L'_c for all c, but that these spaces cease at $c=1$ to give unitary Virasoro representations, because of a breakdown of square-integrability. Methods of *differential* geometry (as opposed to the *symplectic* geometry of this paper) are needed to answer this and other questions. Such methods are under study elsewhere [21].

Axiomatic Approach to Quantization of $\mathrm{diff}S^1/SL^{(n)}(2,R)$

It remains to attempt to come to grips with some of the "unquantizable" orbits. We will concentrate on $\mathrm{diff}S^1/SL^{(k)}(2,R)$, which seems the most promising.

In Sect. 3, we classified Virasoro coadjoint orbits by studying a third order differential equation $0=2f'b+fb'-tf'''/24\pi$. In studying this equation, it was convenient to introduce a certain $SL(3,R)$ "gauge field" A [see Eq. (68)], and to define the global holonomy

$$U_0=P\exp\int_0^{2\pi}A(\theta)d\theta\,. \tag{175}$$

[which was called $U(2\pi)$ in Sect. 3]. From its definition in terms of the holonomy associated with a Virasoro invariant differential equation, it is clear that U_0 is invariant under diffeomorphisms of the circle that leave $\theta=0$ invariant [more precisely, it is invariant under diffeomorphisms $\theta\to\theta'$ such that near $\theta=0$, $\theta'=\theta+O(\theta^3)$]. It is equally clear that under a diffeomorphism ϕ that does *not* fix $\theta=0$, U_0 is not invariant. If $\phi(0)=\theta$, then conjugation by ϕ will turn (175) into

$$U_\theta=P\exp\int_\theta^{\theta+2\pi}A(\theta')d\theta'\,. \tag{176}$$

U_θ will not equal U_0, in general, but they are conjugate; in fact

$$U_\theta=MU_0M^{-1}\,, \tag{177}$$

with $M=P\exp\int_0^\theta A(\theta')d\theta'$.

Let $U(\theta)^i_j$ denote the matrix elements of the matrix U_θ. The U^i_j are not invariant under the Virasoro algebra, in view of the non-trivial transformation law (177). However, in view of our earlier remarks, the $U^i_j(\theta)$ are invariant under diffeomorphisms that leave fixed the point θ (up to third order). This means that the $U^i_j(\theta)$ are local operators (though not conformal fields, because of the conditions on the first two derivatives at θ). This in itself is remarkable. The $U^i_j(\theta)$ are non-polynomial expressions in the coadjoint vector $b(\theta)$. In the coadjoint method, the

[11] Of course, the local fixed point formula cannot see a global phenomenon like the breakdown of square integrability at $R=1/2$. But a proper proof of the fixed point formula for noncompact manifolds like $SL(2,R)/S^1$ would somehow see this

linear functionals of $b(\theta)$ are the Virasoro generators, and polynomials in $b(\theta)$ make up the enveloping algebra of the Virasoro algebra. Thus, instead of thinking of $U^i_j(\theta)$ as a non-polynomial expression in the mysterious quadratic differential b, we can more incisively think of it as a non-polynomial expression in the Virasoro generators. What we have found is that in a suitable completion of the Virasoro enveloping algebra, it is possible to find non-trivial, non-polynomial objects $U^i_j(\theta)$ that transform as local operators.

So far, we have not specialized this discussion to any particular coadjoint orbit. However, in Sect. 3, we learned that the orbits $\mathrm{diff}S^1/SL^{(k)}(2,R)$ correspond to $U_0=1$, or in other words to

$$U(\theta)^i_j-\delta^i_j=0\,. \tag{178}$$

Certainly this is an unusual state of affairs. On the left-hand side of (178) we have certain non-polynomial expressions in the Virasoro generators which miraculously transform as local operators, and we find that the classical mechanical system consisting of the symplectic orbit $\mathrm{diff}S^1/SL^{(k)}(2,R)$ is defined by the vanishing of these operators. This should presumably mean that if one succeeds in quantizing the orbits $\mathrm{diff}S^1/SL^{(k)}(2,R)$, the local operators in (178) will be zero, or at least c-numbers. The vanishing of the local operators (178) ought to be the defining property of representations that are obtained by quantizing the $\mathrm{diff}S^1/SL^{(k)}(2,R)$. An understanding of (178) at the quantum mechanical level would very likely shed a great deal of light on Virasoro representation theory, but here we will only manage some simple comments by way of introduction.

Let us think about the problem in the large c semiclassical regime where things should be simple. Equation (178) was already investigated to lowest order in $1/c$ in Sect. 3. There we considered an almost constant coadjoint vector $(-n^2c/48\pi+\delta b,c)$, corresponding to the orbit $\mathrm{diff}S^1/SL^{(k)}(2,R)$. With

$$\delta b=\frac{1}{2\pi}\sum\beta_n e^{-in\theta}\,, \tag{179}$$

we found in (87) that to lowest order in $1/c$, (178) is equivalent to

$$\beta_0=\beta_k=\beta_{-k}=0\,. \tag{180}$$

We wish to reexpress this as the vanishing of certain functions of the L_n's. This in fact is not difficult. Since (180) does not restrict the β_n for $n\neq 0,\ \pm k$, we can take those as independent coordinates for the $\mathrm{diff}S^1/SL^{(k)}(2,R)$ orbit. To lowest order in $1/c$, the β_n's are simply proportional to the L_n's [see Eq. (131), expressing the L_n in terms of s_n, and (79), relating β_n to s_n]. Thus, as independent coordinates on the $\mathrm{diff}S^1/SL^{(k)}(2,R)$ manifold we can at the classical level take L_n for $n\neq 0,\ \pm k$. Equation (180) means that it is not necessary to introduce additional coordinates corresponding to L_0 and $L_{\pm k}$, and therefore that it must be possible in representations that come by quantizing $\mathrm{diff}S^1/SL^{(k)}(2,R)$ to solve for L_0 and $L_{\pm k}$ in terms of the other L_n's.

This idea of solving for some Virasoro generators in terms of others may seem rather bizarre at first sight, but in essence we have already carried it out in Eqs. (132) and (133), where L_0 and $L_{\pm k}$ are expressed in terms of the s_n, which in turn are

proportional to the L_n according to Eq. (131). Expressed as a formula for L_0 in terms of the L_n of $n \neq 0, \pm k$, and after setting $b = -n^2 c/48\pi$ as is appropriate for the orbit $\mathrm{diff}S^1/SL^{(k)}(2,R)$, (132) becomes

$$L_0 = \frac{(1-k^2)c}{24} + \frac{12}{c} \sum_{n \geq 1, n \neq k} \frac{L_{-n}L_n}{n^2-k^2}. \tag{181}$$

With $L_n \sim \sqrt{c}$, (181) should be viewed as the first two terms in an expansion of L_0 in powers of c^{-1}. The next term will be of the general form $LLL/c^2 \sim c^{-1/2}$. Likewise we can reinterpret (133) in the form

$$L_{\pm k} = \frac{6}{c} \sum \frac{L_{-n}L_{\pm k+n}}{n(n \pm k)}. \tag{182}$$

This, again, is the first term in a process of "solving for $L_{\pm k}$ in terms of the other L_n's."

Now that we have understood that quantizing $\mathrm{diff}S^1/SL^{(k)}(2,R)$ means in part finding a Virasoro representation in which L_0 and $L_{\pm k}$ can be expressed in terms of the other generators, we can study this process without reference to a $1/c$ expansion. Consider then any highest weight representation of the Virasoro algebra. Let us try to "solve for L_0" in the form

$$L_0 = h + \sum_{n=1}^{\infty} a_n L_{-n}L_n + \sum_{i \leq j \leq k;\, i,j,k \neq 0;\, i+j+k=0} a_{ijk} L_i L_j L_k + \ldots. \tag{183}$$

The restriction $i,j,k \neq 0$ is imposed because we wish to solve for L_0 in terms of the other L_n. The restriction $i \leqq j \leqq k$ is just a normalization convention; any "solution for L_0" can be taken (in a unique way) into this form by using the Virasoro algebra to rearrange the terms. The condition $i+j+k=0$ ensures that the right-hand side of (183) commutes with L_0, which is certainly a necessary condition for validity of that equation. Of course, the condition $i+j+k=0$ and its analogues for higher order terms in (183) means that "creation operators" L_{-n} are always accompanied by "annihilation operators" L_{+m}.

Before discussing whether (183) will have a solution, let us discuss in what sense such a solution will be considered to converge. In our large c analysis, convergence was taken to mean a successive smallness of the higher order terms, with $h \sim c$, $a_n \sim 1/c$, $a_{ijk} \sim 1/c^2$, etc. We now wish to look at things in a way which is valid even for small c. To this end we need a different notion of convergence. We observe that acting of a given Fock state of $L_0 = h+n$, only finitely many terms on the right-hand side of (183) give a non-zero result, since sufficiently high order terms have annihilation operators of excessively high number or high order. As a result, the right-hand side of (183) has a well-defined meaning for each Fock space state. We simply consider (183) to be true if it is true acting on each state in Fock space.

It is now rather easy to see that (183) always has a solution. Suppose, inductively, that we have chosen the right-hand side of (183) so that this equation is valid in acting on any Fock space state of energy at most $h+n-1$. Consider states of energy $h+n$. These are of the general form

$$L_{-I}|\Omega\rangle, \tag{184}$$

where L_{-I} is short-hand for some expression

$$\prod L_{-n_i}^{a_i}, \tag{185}$$

with

$$\sum a_i n_i = n. \tag{186}$$

We may suppose that a basis of possible L_{-I} of level n has been chosen. The states

$$L_{-I}|\Omega\rangle \tag{187}$$

span the states of energy $h+n$, but may not be linearly independent. By adding a suitable expression

$$\sum_{IJ} \alpha_{IJ} L_{-I} L_J \tag{188}$$

to the right-hand side of (183), the matrix elements of that right-hand side among states of energy $h+n$ can be adjusted in an arbitrary fashion. Carrying this out repeatedly, energy level by energy level, we find inductively a "solution for L_0" of the form (183).

If the states (187) are linearly independent, then there is no arbitrariness in the inductive step – the choice of (188) is uniquely determined, since the operators $L_{-I}L_J$ are linearly independent when regarded as matrices in the space of linearly independent states $L_{-K}|\Omega\rangle$ of energy $h+n$. Thus, in this case there is a unique formula (183) for the "solution for L_0." In particular, all of the Virasoro generators L_n, $n \neq 0$ will (presumably) appear in the unique formula (183). It cannot in this situation be possible to find another non-trivial formula solving for, say, L_k in terms of L_n with $n \neq 0, \pm k$, since substituting this solution for L_k in (183) would give a new formula contradicting the uniqueness of (183). Thus, the Virasoro representations of $c > 1, h > 0$ are representations in which one can solve for L_0 but not for any other L_n. They cannot be interpreted in terms of quantization of the $\mathrm{diff}S^1/SL^{(k)}(2, R)$. This is just as well, since we have interpreted those representations in terms of quantization of $\mathrm{diff}S^1/S^1$. On the other hand, the most fascinating unitary representations of the Virasoro algebra, especially those with $c < 1$, are representations for which the states (187) are not linearly independent, so (183) is not unique. This suggests that in those representations it may be possible to solve for Virasoro generators other than L_0, and that these representations might have something to do with quantization of $\mathrm{diff}S^1/SL^{(k)}(2, R)$. We will now investigate this point.

We will show that a highest weight Virasoro representation in which one can solve for $L_{\pm k}$ as well as L_0 is precisely one in which there is a null vector at level k. Recall that this means that there is a relation

$$L_{-k}|\Omega\rangle = \sum_{0<i,\, j;\, i+j=k} u_{ij} L_{-i} L_{-j}|\Omega\rangle + \ldots, \tag{189}$$

where ... are terms multilinear in L_{-j}, $0 < j < k$. Dual to (189) is a relation

$$\langle\Omega|L_k = \langle\Omega|\sum u_{ij}^* L_j L_i + \ldots. \tag{190}$$

We wish to see that the relations (189), (190) make it possible to choose the right-hand side of (183) so as not to contain $L_{\pm k}$. It is of course sufficient to show that at

the inductive step, where one adds (188) to the right-hand side of (183), the coefficients in (188) can be chosen so that $L_{\pm k}$ do not appear. To this end, write out the factors L_J and L_{-I} in the product $L_{-I}L_J$ more explicitly. We have, say,

$$L_J = L_{j_1}L_{j_2}\dots L_{j_k}, \qquad L_{-I} = L_{-i_1}L_{-i_2}\dots L_{-i_m}. \tag{191}$$

In choosing the coefficients α_{IJ} in (188), the crucial thing is to get the right matrix elements of (188) between states $|U\rangle$ and $|V\rangle$ of energy $h+n$. These matrix elements factorize,

$$\langle V|\alpha_{IJ}L_{-I}L_J|U\rangle = \alpha_{IJ}\langle V|L_{-I}|\Omega\rangle\langle\Omega|L_J|U\rangle, \tag{192}$$

since $|\Omega\rangle$ is the only possible intermediate state of energy h. Now, by moving L_k to the left in L_J and to the right in L_{-I}, and then replacing them with the expressions which appear on the right-hand sides of (190) and (189), we can eliminate $L_{\pm k}$ from the operator which appears in (192) without changing the matrix elements. This shows that the inductive step in the derivation of (183) can be carried out with a choice of (188) that does not contain $L_{\pm k}$.

What is the virtue of eliminating $L_{\pm k}$ from (183)? We can now write

$$L_k = \frac{1}{k}[L_k, L_0] = \frac{1}{k}[L_k, \mathrm{RHS}] \tag{193}$$

and likewise

$$L_{-k} = -\frac{1}{k}[L_{-k}, L_0] = -\frac{1}{k}[L_{-k}, \mathrm{RHS}], \tag{194}$$

with RHS the right-hand side of (183). From (193) and (194) we wish to obtain formulas "solving for $L_{\pm k}$" in terms of the L_n for $n \neq 0, \pm k$. In fact, since L_0 and $L_{\pm k}$ to not appear in RHS, L_0 and L_k do not appear on the right of (193). However, L_{-k} may appear in the right-hand side of (193), since there is L_{-2k} in RHS. Likewise, L_k but not L_0 or L_{-k} may appear on the right-hand side of (194). The basic idea is now to eliminate L_{-k} from the right-hand side of (193) by using (194), and eliminate L_k from the right-hand side of (194) by using (193). This process will then have to be repeated, since using (194) to eliminate L_{-k} from the right-hand side of (193) will reintroduce L_k, etc. It is necessary to show that this process, carried out infinitely many times, will converge to a formula for $L_{\pm k}$ in terms of the L_n for $n \neq 0, \pm k$. Convergence means of course that the formulas should be well-defined and valid when acting on any Fock space state.

To verify that this process converges requires only a little bit of care with operator ordering. We order the factors on the right-hand side of (193) in the form

$$L_{-j_i}\dots L_{-j_r}\cdot(L_{-k})^t\cdot L_{i_1}\dots L_{i_s}. \tag{195}$$

Thus, L_{-k} is put to the right of other annihilation operators. In (195) L_{-k} is raised to the t^{th} power, for some t. The point of this is that then when we replace L_{-k} with the right-hand side of (194), any terms in (195) that still contain unwanted factors of L_k will terminate on the right with a longer string of annihilation operators than we had previously. Likewise, if in (194) we use an operator ordering of the form

$$L_{-j_1}\dots L_{-j_r}\cdot(L_k)^u\cdot L_{t_1}\dots L_{i_s}, \tag{196}$$

then upon using (193) to eliminate L_k, any terms that contain unwanted factors of L_{-k} will terminate on the right with a longer string of annihilation operators than before. After each step in the stepwise elimination of $L_{\pm k}$ from the right-hand sides of (193) and (194), we order the factors like (195) or (196), as appropriate. Then in each step of the process, the still troublesome terms terminate on the right with longer and longer chains of annihilation operators. Since any given Fock space state is annihilated by any sufficiently long chain of annihilation operators, this shows that the stepwise elimination of $L_{\pm k}$ from the right-hand sides of (193) and (194) converges to an expression with a well-defined and therefore correct action on each Fock space state.

We have shown, then, that in a highest weight Virasoro representation with a null state on level k, it is possible to "solve for $L_{\pm k}$" as well as L_0. The converse is also true, and more trivial. Given a "solution for L_{-k}," that is a formula

$$L_{-k} = \sum_{i+j=-k, i \leqq j} u_{ij} L_i L_j + \ldots , \tag{197}$$

one sees immediately that acting on $|\Omega\rangle$ with (197) gives a relation of the form (189) asserting the existence of a null state on level k. The representations with a null state on level k are thus precisely the ones in which it is possible to solve for $L_{\pm k}$ as well as L_0.

Going back to the diff$S^1/SL^{(k)}(2, R)$, we expect that a successful quantization of diff$S^1/SL^{(k)}(2, R)$ will give a Virasoro representation in which it will be possible to solve for L_0 and $L_{\pm k}$. Thus, quantization of diff$S^1/SL^{(k)}(2, R)$ should give a Virasoro representation with a null state at level k.

In [18], Kac determined the condition on h and c for existence of a null state on level k. For $k=1$ the requirement is $h=0$, so representations of $h=0$ should correspond to quantization of diff$S^1/SL^{(1)}(2, R)$, as we indeed learned in the last subsection. For $k>1$ there is a more complicated (multi-component) curve in the h, c plane giving a null vector on level n. The corresponding degenerate representations should be related to quantization of diff$S^1/SL^{(k)}(2, R)$.

This is not the whole story of Virasoro representation theory, however. The only unitary representations with a null vector for only one value of k at $k>1$ are those with $c=1$, $h=(k-1)^2/4$. For $c<1$ unitarity is not possible for any value of h and c with a null state in level n, but only at a certain discrete series of values of h and c [8], corresponding among other things to the existence of independent null vectors at two different levels.

If quantization of diff$S^1/SL^{(k)}(2, R)$ is to give the unitary discrete series (and not the one parameter family of representations with a null vector at level k), then quantization of diff$S^1/SL^{(k)}(2, R)$ must require more than the mere ability to solve for L_0 and $L_{\pm k}$. So what is missing in our discussion? I believe that what is missing must be the *locality* of the mysterious operators on the left-hand side of (178).

Any expression for some Virasoro operators in terms of others, like (183) or (197), gives an expression in the enveloping algebra of the Virasoro algebra which is not zero universally but may nonetheless be zero in some representation. Such an expression can be multiplied by any element of the Virasoro enveloping algebra (i.e., any function of the L_n's) to give any of a variety of other formulas. Merely given the existence of a non-invariant function like (197) that is zero in some

representation, this does not guarantee that there is a *local* operator with those properties. I would conjecture, then, that although for any value of h and c that gives a null vector on level k, there exist non-invariant elements of the enveloping algebra which vanish in the corresponding representation, local elements with that property only exist at special values of h and c uncovered in [8].

Apart from the lack of insight about the last conjecture, the main gap in the present analysis is the failure to exhibit *a priori* why there are relations between the orbits $\mathrm{diff}S^1/SL^{(k)}(2, R)$ for different k. Yet from Virasoro representation theory, we know that there are. For instance, in the study of the discrete series, one finds that for $c<1$, at the values of c where there are unitary representations, several of these arise simultaneously with different values of h, connected presumably to quantization of different $\mathrm{diff}S^1/SL^{(k)}(2, R)$. What is more, the unitary representations of $c<1$ have simultaneous null vectors on two different levels, so they possibly can be regarded as quantizations of $\mathrm{diff}S^1/SL^{(k)}(2, R)$ for two different values of k. Why there should be such relations between different $\mathrm{diff}S^1/SL^{(k)}(2, R)$ is a question outside the scope of the general coadjoint orbit method; the answer probably must be found using concepts of local quantum field theory.

The germ of the relation between different Virasoro coadjoint orbits is, I believe, that the "k" in $\mathrm{diff}S^1/SL^{(k)}(2, R)$ is purely a global invariant, which cannot be defined if one "cuts" a point out of the circle. The quantization of $\mathrm{diff}S^1/SL^{(k)}(2, R)$ must give Virasoro representations in which it is possible to interpret

$$U(\theta_1, \theta_2) = P \exp \int_{\theta_1}^{\theta_2} A \cdot d\theta \tag{198}$$

as a quantum mechanical operator with correct commutation relations with the Virasoro algebra. The attempt to do so would involve normal ordering difficulties that depend on c but not on k. k arises when one forms the global object $U(\theta + 2\pi, \theta)$ (integrated, that is, all the way around the circle) and one tries to find quantum solutions of $U = 1$. The solutions of this equation are labeled, classically at least, by k, and k presumably should come in quantum mechanically as well when one tries to make sense of $U = 1$.

Appendix

In Sect. 3 of this paper, we observed that there are left invariant symplectic structures on the Virasoro group manifold. Although it played little role in the analysis above of Virasoro representations, it is nonetheless a very striking fact. The purpose of this appendix is to make a few remarks about the possible significance of the left invariant symplectic structures on $\mathrm{diff}S^1$ in terms of Lie bialgebras and quantum groups.

String theory is a remarkably rich system in which, to a large extent, the really core ideas remain elusive. Another remarkably rich and multi-faceted subject is the theory of integrable quantum systems. There are a variety of indications that these two subjects may be related. $1+1$ dimensional quantum field theory plays a pivotal role in both string theory and the theory of integrable quantum systems. A surprisingly large number of conformally invariant quantum field theories in $1+1$

dimensions can be reached as critical points of systems which are integrable, but not conformally invariant, away from the critical point.

In conjecturing the possible existence of a relation between integrable quantum systems and string theory, I do not mean in any sense that string theory itself should be integrable. The idea is more that some of the missing mathematical structures which are necessary ingredients in a proper geometrical formulation of string theory may be found in the realm of completely integrable systems.

As in string theory, so in the case of completely integrable systems, it is a difficult and perplexing problem to identify the really central ideas. This problem has been studied very intensively in the case of integrable quantum systems. One fascinating framework for thinking about integrable systems is the interpretation of the Yang-Baxter equation in terms of Hopf algebras or quantum groups [22].

Just as a Lie algebra is the infinitesimal counterpart of a Lie group, a Lie bialgebra is the infinitesimal counterpart of a Hopf algebra. A Lie bialgebra is a Lie algebra $\mathcal{G}$ together with a linear map

$$\phi : \mathcal{G} \to \wedge^2 \mathcal{G}, \tag{A1}$$

obeying the following two conditions. First, let $\mathcal{G}^*$ denote the dual of the vector space $\mathcal{G}$. Then the linear transformation ϕ has a dual

$$\phi^* : \wedge^2 \mathcal{G}^* \to \mathcal{G}^*, \tag{A2}$$

and one requires that this should obey the Jacobi identity, giving a Lie algebra structure to $\mathcal{G}^*$. Second, one requires that ϕ should be a one cocycle on $\mathcal{G}$ (with values in the $\mathcal{G}$ module $\wedge^2 \mathcal{G}$). This means that for $u, v \in \mathcal{G}$,

$$[u, \phi(v)] - [v, \phi(u)] = \phi([u, v]). \tag{A3}$$

Equation (A3) is to be seen as an equation of compatibility between the Lie algebra structures on $\mathcal{G}$ and $\mathcal{G}^*$. Although this is not immediately apparent, a little study of (A3) reveals that it is invariant under the duality $\mathcal{G} \leftrightarrow \mathcal{G}^*$.

The cocycle condition means that ϕ defines a cohomology class in $H^1(\mathcal{G}, \wedge^2 \mathcal{G})$. In practice, in many interesting examples this cohomology class is zero; the group $H^1(\mathcal{G}, \wedge^2 \mathcal{G}^*)$ may vanish, and even if it does not vanish ϕ may be zero as an element of this group. The cohomology class of ϕ is zero if and only if there is some element $r \in \wedge^2 \mathcal{G}$ such that for all $u \in \mathcal{G}$,

$$\phi(u) = [u, r]. \tag{A4}$$

A Lie bialgebra in which ϕ can be written as in (A4) is said to be a coboundary Lie bialgebra. The cocycle condition on ϕ is an automatic consequence of (A4). The other condition on ϕ – its dual (A2) must obey the Jacobi identity – turns out, rather amazingly, to be the statement that r must obey the so-called classical Yang-Baxter equation. That equation plays a central role in broad classes of integrable systems. It has its origins in the realm of factorizable scattering theory [23] and solvable lattice systems [24]. For references, see, e.g., [25–29]. In general, the classical Yang-Baxter equation is a highly non-linear equation, but there is a certain situation in which it linearizes. If $r \in \wedge^2 \mathcal{G}$ is invertible, its inverse r^{-1} is an

element of $\wedge^2\mathcal{G}^*$. This in turn can be interpreted as a linear functional on $\mathcal{G}$:

$$r^{-1}: \wedge^2\mathcal{G} \to R. \tag{A5}$$

(R denotes the real numbers.) Let G be the group with Lie algebra $\mathcal{G}$. (If G does not exist, what follows can be expressed in terms of a formal group.) In case r is invertible, the classical Yang-Baxter equations reduce [26, 22] to the statement that the left invariant two form on the G manifold whose value at the origin is r^{-1} is closed. In other words, if r is invertible, the classical Yang-Baxter equations say that r^{-1} determines a left invariant symplectic structure ω_L on the G manifold.[12] In this case, the inverse of r^{-1}, or in other words r, defines Poisson brackets of a classical dynamical system on the G manifold. Even if r is not invertible, the nonlinear classical Yang-Baxter equations amount to the associativity of certain Poisson brackets constructed from r. Quantization of these Poisson brackets leads to the notion of a "quantum group" [22] in which the quantum algebra of observables together with the group multiplication law make up a Hopf algebra.

Considering the case in which r is invertible and r^{-1} is closed, there are two possibilities. Closedness of r^{-1} means that r^{-1} defines a cohomology class in $H^2(\mathcal{G}, R)$. If this class is non-trivial, then r^{-1} defines a non-trivial central extension of $\mathcal{G}$. If, on the other hand, the cohomology class of r^{-1} is trivial, then there must exist a linear functional on $\mathcal{G}$, or in other words a coadjoint vector $b \in \mathcal{G}^*$, such that

$$r^{-1}(u, v) = b([u, v]) \tag{A6}$$

for any $u, v \in \mathcal{G}$. Invertibility of r^{-1} then means that b is such that for all u, there is v with $b([u, v]) \neq 0$. Recalling the definition of the coadjoint representation from Sect. 2, the action of an adjoint vector u on $b \in \mathcal{G}^*$ is defined by saying that $u(b)$ is the element of $\mathcal{G}^*$ which to $v \in \mathcal{G}$ assigns the value

$$u(b)(v) = -b([u, v]). \tag{A7}$$

Existence for all u of v with $b([u, v]) \neq 0$ is thus the statement that $u(b) \neq 0$ for all u. In this case, there is no adjoint vector that leaves b invariant, and the coadjoint orbit of b is a copy of G, perhaps divided by a discrete group.

Let us now take $\mathcal{G}$ to be the Lie algebra of $\mathrm{diff}S^1$ [13] and assess the above issues. In this case, it can be seen that $H^1(\mathcal{G}, \wedge^2\mathcal{G}) = 0$, so any Lie bialgebra structure is of coboundary type. There naturally then arises the problem of whether the object $r \in \wedge^2\mathcal{G}$ which was pivotal in the above discussion is invertible. Whether with $\mathcal{G} = \mathrm{diff}S^1$ there are solutions of the classical Yang-Baxter equations with r not invertible is an interesting question which I will not try to answer here. But we can

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[12] Or a right invariant one ω_R. It is really the Poisson bracket coming from $\omega_L^{-1} - \omega_R^{-1}$ that is compatible with the group multiplication law in a sense described in [22] and plays a pivotal role in the motion of a quantum group

[13] Without, that is, a central extension. If one takes $\mathcal{G}$ to be the central extension $\widehat{\mathrm{diff}S^1}$, then a Lie bialgebra structure with invertible r would correspond to a left invariant symplectic structure on the $\widehat{\mathrm{diff}S^1}$ manifold; there are none since (a) this group has no non-trivial central extensions; (b) a Lie bialgebra structure corresponding to a trivial central extension is impossible since every coadjoint vector is at least stabilized by the center of $\widehat{\mathrm{diff}S^1}$

immediately use our results from Sect. 3 to classify the solutions of the classical Yang-Baxter equation with invertible r.

First, can it be that the two cocycle $r^{-1} \in H^2(\mathcal{G}, R)$ is non-trivial? The classification of central extensions of $\mathrm{diff}S^1$ shows that a non-trivial two cocycle is necessarily of the form

$$r^{-1}(f,g) = \int d\theta \left(f \left(c \frac{d^3}{d\theta^3} + b \frac{d}{d\theta} + \frac{d}{d\theta} b \right) g \right) \tag{A8}$$

for some b and some nonzero c.[14] We have already studied the skew form (A8) in Sect. 3, [see Eq. (72)] and we learned that for any non-zero c and any b, this skew form has a non-trivial kernel. That non-trivial kernel would mean that r^{-1} is not invertible, contradicting the hypothesis by which we reduced the search for solutions of the classical Yang-Baxter equation to the search for left invariant symplectic structures of $\mathrm{diff}S^1$.

Therefore, we should consider the case in which r^{-1} is trivial cohomologically, and so can be written as (A6) for some b. Here we have more luck. As we learned at the beginning of Sect. 3, every quadratic differential b which has at least one zero has the property that the quadratic form (A6) on tangent vectors [which is (A8) with $c=0$] has no kernel. Thus, we have an embarrassment of riches, a Lie bialgebra structure on $\mathrm{diff}S^1$ (without central extension) for every conjugacy class of quadratic differentials except the positive or negative definite ones. The classification of these quadratic differentials is quite complicated; the simplest invariants were described in Sect. 3.

The next step would naturally be to ask whether these Lie bialgebra structures on $\mathrm{diff}S^1$ can be quantized in the sense of [22]. Perhaps at the level of quantization the plethora of Lie bialgebra structures for $\mathrm{diff}S^1$ can be reduced to more manageable proportions. I hope to return to this question elsewhere.

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[14] In studying central extensions of the Virasoro algebra, one shows usually that up to the addition to the energy momentum tensor of a c-number, any two cocycle can be put in the form (A8) with $b=0$. Adding a c number to the energy momentum tensor in the general form $T(0) \to T(0) + b(0)$, with b a coadjoint vector, gives a correction to the Virasoro commutation relations which precisely adds the b dependent terms to (A8)

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