

# Coalition Formation in $n$ -Person Stackelberg Games\*

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**Abstract.** Stackelberg games and their resulting nonconvex programming problems have been used to model multilevel economic systems. These formulations have suggested that independent players, acting sequentially, may not produce Pareto-optimal decisions. Such systems naturally encourage the introduction of an  $n$ -player abstract game which permits coalitions of players to form. This paper examines the mathematical characteristics of these imbedded games and the implications of their solution for the overall problem.

## 1 Introduction

The problem of hierarchical management has existed ever since people have attempted to organize. Today, hierarchies are found controlling our government, places of work, schools, churches, and even our families. By understanding them, we can perhaps improve their effect and eliminate their inherent inefficiencies.

This paper will examine a class of multilevel management control problems, and will extend the application of a model proposed by Bialas and Karwan [1]–[2] and many others. The original motivation for this work was to formulate multilevel decision-making problems found in economics and management. However, as one can quickly see, the mathematical foundations are based on the theory of Stackelberg games (see Simaan and Cruz [3]). Hence, the results here should help strengthen the links between optimal control theory and economic decision making.

There are two important limitations in earlier works. Although Bialas and Karwan [1] discuss a general  $n$ -player model, the geometric and algorithmic results of those authors and others (Bard and Falk [4], Candler and Townsley [5], and Fortuny and McCarl [6], for example) are chiefly limited to the two-player, linear case. A second limitation was that the solution to such problems need not be Pareto-optimal (see Bialas and Chew [7]). That is, there may exist feasible solutions (e.g., Nash points) which are of greater benefit to at least one participant, without reducing the benefits to any participant when compared to the Stackelberg solution (see Başar [8]). This is not a shortcoming of the formulation. Instead, it suggests that for those systems which can be modeled as a hierarchy of planners, each executing his policies rationally and in sequence, the resulting behavior for the entire system may be inadmissible.

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The purpose of this paper is two-fold. Its first aim is to provide a framework in which coalition formation in multilevel systems can be modeled. The second goal of this exposition is to develop a methodology for predicting coalition formation in hierarchical systems. In so doing, the mathematical model might be used to suggest modifications to the system structure to encourage or dissuade players from banding together.

## 2 Definitions

This section will briefly reintroduce the problem as presented by Bialas and Karwan [1]. For a more detailed discussion of the problem definition, the reader is referred to that paper.

Suppose the vector  $x \in \mathbb{R}^N$  is partitioned as  $(x^a, x^b)$ . Then let

$$\max\{f(x) : (x^a | x^b)\}$$

denote the maximum of the function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  by varying  $x^a$  for a fixed  $x^b$ .

**Definition 2.1** Let  $S \subset \mathbb{R}^N$  be compact, and let  $f : S \rightarrow \mathbb{R}$  be bounded. Then

$$\Psi_f(S) \equiv \{\hat{x} \in S \mid f(\hat{x}) = \max\{f(x) : (x^a | \hat{x}^b)\}\}$$

is the **set of rational reactions** of  $f$  over  $S$ .

To define the  $n$ -level ( $n$ -player) optimization problem, let the vector of decision variables  $x \in \mathbb{R}^N$  be partitioned among  $n$  players with

$$x^k \equiv (x_1^k, x_2^k, \dots, x_{N(k)}^k) \in \mathbb{R}^{N(k)} \quad (k = 1, 2, \dots, n)$$

where  $\sum_{k=1}^n N(k) = N$ . Furthermore, suppose  $S^1 \subset \mathbb{R}^N$  is compact and let

$$f_1(x), f_2(x), \dots, f_n(x)$$

be a sequence of bounded functions ( $f_i(x) : S^1 \rightarrow \mathbb{R}$ , for all  $i$ ).

Let the **level- $k$  feasible region**,  $S^k$ , be recursively defined as

$$S^k \equiv \Psi_{f_{k-1}}(S^{k-1})$$

for  $k = 2, 3, \dots, n$ . The set  $S^k$  represents the feasible outcomes resulting from the rational reactions of players at levels

$1, 2, \dots, k-1$ . The optimization problem which must be solved by the player at level  $k$  is then

$$(L^k) : \quad \max\{f_k(x) : (x^k | x^{k+1}, \dots, x^n)\}$$

$$\text{st: } \quad x \in S^k.$$

This establishes a collection of nested mathematical programming problems  $\{L^1, L^2, \dots, L^n\}$  representing a hierarchical decision-making process. The problem  $L^n$  is called an  **$n$ -level mathematical programming problem**.

The results in this paper will be restricted to the linear case where

$$f_k(x) = c^k x \quad (k = 1, 2, \dots, n)$$

and

$$S^1 = \{x \in \mathbb{R}^N | Ax \leq b, x \geq 0\}.$$

Problem  $L^n$  is then sometimes referred to as the  **$n$ -level linear resource control problem**. For exposition, the problem may be more clearly represented as

$$\begin{aligned} & \max\{c^n x : (x^n)\} \\ \text{st: } & \max\{c^{n-1} x : (x^{n-1} | x^n)\} \\ & \vdots \\ \text{st: } & \max\{c^1 x : (x^1 | x^2, \dots, x^n)\} \\ \text{st: } & Ax \leq b \\ & x \geq 0. \end{aligned} \quad (1)$$

Bard and Falk [4], Bialas and Karwan [2], Candler and Townsely [5], and Fortuny and McCarl [6] have provided solution algorithms for the problem (1) when  $n = 2$ . As of yet, no efficient solution procedure exists for the case  $n > 2$ .

### 3 A Special Case

Although an efficient algorithm for the  $n$ -level linear resource control problem is not available, an efficient algorithm can be devised for a special subclass of such problems. This will permit one to illustrate the results on coalition formation. However, it must be emphasized that those results hold for any  $n$ -level linear resource control problem, not simply this special, illustrative case.

The special case we want to consider is the **multilevel continuous knapsack problem** of the form

$$\begin{aligned} K(n) \quad & \max\{c^n x : (x^n)\} \\ \text{st: } & \max\{c^{n-1} x : (x^{n-1} | x^n)\} \\ & \vdots \\ \text{st: } & \max\{c^1 x : (x^1 | x^2, \dots, x^n)\} \\ \text{st: } & \sum_{i=1}^n x_i \leq b \\ & x_i \geq 0 \quad \text{for } i = 1, \dots, n \end{aligned}$$

where  $b > 0$ ,  $N = n$ , and  $x^i = (x_i)$  are single component vectors for  $i = 1, 2, \dots, n$ .

The algorithm to solve  $K(n)$  requires  $c_1^1 > 0$ . The procedure merely inspects the cost coefficients of the  $n$  objective functions to obtain the solution:

**Step 0:** Initialize  $i=1$  and  $j=1$ . Set  $\hat{x}_1 = b$  and  $\hat{x}_k = 0$  for  $k \neq i$ . Go to *step 1*.

**Step 1:** If  $i = n$ , stop.  $\hat{x}$  is the solution. Otherwise, go to *step 2*.

**Step 2:** Set  $i = i + 1$ . If  $c_i^i > c_j^j$ , then set  $\hat{x}_i = b$  and  $\hat{x}_k = 0$  for  $k \neq i$ . Go to *step 3*. Otherwise, go to *step 1*.

**Step 3:** Set  $j = i$ . Go to *step 1*.

If no ties occur in *step 2* (i.e.,  $c_i^i \neq c_j^j$ ), then it can be shown that the above procedure solves the problem  $K(n)$  (see Chew [9]).

As an example, consider a decision-making game with three players, named 1, 2 and 3, each of whom controls a different commodity. Their task is to jointly fill a container (or knapsack) of unit size ( $b = 1$ ) with an amount of their respective commodities, never exceeding the capacity of the container. This will be performed in a sequential fashion, with player 3 taking his turn first. Suppose that at the end of the sequence, a referee pays each player one dollar for each unit of his own commodity which has been placed in the container. Since player 3 has preemptive control over the container, he will fill it completely with his commodity, and collect one dollar.

However, suppose that the rules are changed slightly so that, in addition, player 3 could collect five dollars for each unit of *player one's* commodity which is placed in the container. Since player 2 does not reap such a benefit from player one's commodity, player 2 would fill the container with his own commodity on his turn, if given the opportunity. For this reason, player 3 has no choice but to fill the container with his commodity and collect only one dollar.

The three-level continuous knapsack problem which models this second scenario is (dropping the  $(\cdot | \cdot)$  notation):

$$\begin{aligned} K(3) \quad & \max 5x_1 + x_3 \\ \text{st: } & \max x_2 \\ \text{st: } & \max x_1 \\ \text{st: } & x_1 + x_2 + x_3 \leq 1 \\ & x_i \geq 0, \quad i = 1, 2, 3 \end{aligned}$$

Note that there are six dollars available to the three players, which simply divided equally could benefit each of them. However, because of the sequential and independent nature of the decisions, such a solution cannot be attained. The solution to the above problem is not Pareto-optimal.

### 4 Coalition Formation

As suggested by the previous example, the formation of coalitions among subsets of the players could provide a means to achieve Pareto-optimality. The members of each coalition act for the benefit of the coalition as a whole. The question immediately raised is "which coalitions will tend to form, and what will be the resulting benefits?"

The abstract game and the solution concept proposed by Shenoy [10]<sup>1</sup> provides the foundation for answering this question.

**Definition 4.1** An **abstract game** is a pair  $(X, \text{dom})$  where  $X$  is a set whose members are called **outcomes** and  $\text{dom}$  is a binary relation on  $X$  called **domination**.

<sup>1</sup>See also the earlier work of Lucas and Thrall [13].

Let  $G = \{1, 2, \dots, n\}$  denote the set of  $n$  players in a linear resource control problem. Let  $2^G$  denote the set of all coalitions of  $G$  and let  $\mathcal{P} = \{P_1, P_2, \dots, P_M\}$  denote a **coalition structure** (c.s.) or **partition** of  $G$  into coalitions where  $P_i \neq \emptyset, P_i \cap P_j = \emptyset$  for all  $i \neq j$  and  $P_1 \cup P_2 \cup \dots \cup P_M = G$ . Let  $\Pi$  denote the set of all coalition structures. Let  $\mathcal{P}_0 \equiv \{\{1\}, \{2\}, \dots, \{n\}\}$  denote the coalition structure where no coalitions have formed and let  $\mathcal{P}_G \equiv \{G\}$  denote the **grand coalition**. The linear resource control problem corresponding to  $\mathcal{P}_0$ , say  $L(\mathcal{P}_0)$ , is precisely problem (1).

Consider the coalition structure  $\mathcal{P} = \{P_1, P_2, \dots, P_M\}$  and suppose  $P_j \in \mathcal{P}$ . Let  $P(i)$  denote that  $P_j \in \mathcal{P}$  such that player  $i \in P_j$ . Assuming utility is additive and transferable, the objective function of each player in coalition  $P_j$  is the sum of the coalition members' individual objective functions under  $\mathcal{P}_0$ . The corresponding linear resource control problem can then be written as

$$\begin{aligned} L(\mathcal{P}) : & \max \{ \sum_{i \in P(n)} c^i x : (x^n) \} \\ & \text{st: } \max \{ \sum_{i \in P(n-1)} c^i x : (x^{n-1} | x^n) \} \\ & \quad \vdots \\ & \text{st: } \max \{ \sum_{i \in P(1)} c^i x : (x^1 | x^2, \dots, x^n) \} \\ & \text{st: } Ax \leq b \\ & \quad x \geq 0. \end{aligned}$$

Let the solution to  $L(\mathcal{P})$  be denoted by  $\hat{x}(\mathcal{P})$ .

**Definition 4.2** The **value** of (or **payoff** to) coalition  $P_j \in \mathcal{P}$  denoted by  $v(P_j, \mathcal{P})$ , is equal to

$$\sum_{i \in P_j} c^i \hat{x}(\mathcal{P}).$$

Bialas and Chew [7] have shown that  $v(\cdot)$  need not be superadditive. Hence, one must take great care when applying some of the traditional game theory results which require superadditivity to this class of problems.

**Definition 4.3** A **solution configuration** (s.c.) is a pair  $(r, \mathcal{P})$  where  $r$  is an  $n$ -dimensional vector called an **imputation**, whose elements represent the payoff to each player  $i$  under coalition structure  $\mathcal{P}$ .

Let  $SC(LR)$  denote the set of all solution configurations which are feasible for the linear resource control problem under consideration. The we can define the binary relation  $\text{dom}$  as follows:

**Definition 4.4** Let  $(r, \mathcal{P}_r), (s, \mathcal{P}_s) \in SC(LR)$ . Then  $(r, \mathcal{P}_r)$  **dominates**  $(s, \mathcal{P}_s)$ , denoted by  $(r, \mathcal{P}_r) \text{ dom } (s, \mathcal{P}_s)$ , if and only if there exists a nonempty  $R \in \mathcal{P}_r$  such that

1.  $r_i > s_i$  for all  $i \in R$ , and
2.  $\sum_{i \in R} r_i \leq v(R, \mathcal{P}_R)$ .

Condition (1) implies that each decision maker in  $R$  prefers coalition structure  $\mathcal{P}_r$  to coalition structure  $\mathcal{P}_s$ . Condition (2) ensures that  $R$  is a feasible coalition in  $\mathcal{P}_r$ . That is,  $R$  does not demand more from imputation  $r$  than its value,  $v(R, \mathcal{P}_R)$ .

**Definition 4.5** The **core**,  $\mathcal{C}$ , of an abstract game is the set of undominated solution configurations.

When the core is nonempty, each of its elements represents an enforceable solution configuration which may offer increased benefits to the multilevel system. Once the players have negotiated an outcome within the core, no further negotiations or outcomes are possible.

**Proposition 4.1** If  $(z, \mathcal{P}) \in \mathcal{C} \neq \emptyset$ , then

$$\sum_{i=1}^n z_i = \sum_{i=1}^n c^i \hat{x}(\mathcal{P}_G).$$

*Proof:* 1.  $(\sum_{i=1}^n z_i \leq \sum_{i=1}^n c^i \hat{x}(\mathcal{P}_G))$ . Let

$$\mathcal{P} = \{P_1, P_2, \dots, P_M\}.$$

Then

$$\sum_{i=1}^n z_i \leq \sum_{k=1}^M v(P_k, \mathcal{P}) = \sum_{k=1}^M \sum_{j \in P_k} c^j \hat{x}(\mathcal{P}) = \sum_{i=1}^n c^i \hat{x}(\mathcal{P}).$$

Hence,

$$\sum_{i=1}^n z_i \leq \sum_{i=1}^n c^i \hat{x}(\mathcal{P}) \leq \sum_{i=1}^n c^i \hat{x}(\mathcal{P}_G)$$

since  $\mathcal{P}_G$  maximizes  $\sum_{i=1}^n c^i \hat{x}(\mathcal{P})$  over all  $\mathcal{P} \in \Pi$ .

2.  $(\sum_{i=1}^n z_i \geq \sum_{i=1}^n c^i \hat{x}(\mathcal{P}_G))$ . Suppose

$$\sum_{i=1}^n z_i < \sum_{i=1}^n c^i \hat{x}(\mathcal{P}_G).$$

Then  $(z, \mathcal{P})$  must be dominated by some s.c. involving  $\mathcal{P}_G$ . Hence,  $(z, \mathcal{P}) \notin \mathcal{C}$ , a contradiction. ■

**Proposition 4.2** If  $\mathcal{C} \neq \emptyset$ , then there exists an imputation  $z$  such that  $(z, \mathcal{P}_G) \in \mathcal{C}$ .

*Proof:* If  $\mathcal{C} \neq \emptyset$ , then there exists a s.c.,  $(z, \mathcal{P})$ , such that  $(z, \mathcal{P}) \in \mathcal{C}$ . From Proposition 4.1, we have

$$\sum_{i=1}^n z_i = \sum_{i=1}^n c^i \hat{x}(\mathcal{P}_G).$$

Hence,  $(z, \mathcal{P}_G)$  is a solution configuration which cannot be dominated by any other solution configuration. Therefore,  $(z, \mathcal{P}_G) \in \mathcal{C}$ . ■

**Proposition 4.3** A linear resource control problem produces an embedded game  $(SC(LR), \text{dom})$  with an empty core if there exist coalition structures  $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_\ell$  and coalitions  $S_i \in \mathcal{P}_i$  ( $i = 1, 2, \dots, \ell$ ) with  $S_j \cap S_k = \emptyset$  for all  $j \neq k$  such that

$$\sum_{j=1}^{\ell} \sum_{i \in S_j} c^i \hat{x}(\mathcal{P}_j) > \sum_{i=1}^n c^i \hat{x}(\mathcal{P}_G).$$

*Proof:* We will show that for any solution configuration,  $(z, \mathcal{P}_G)$ , there exists a solution configuration,  $(y, \mathcal{P}_j)$ , such that, for some coalition  $S_j \in \mathcal{P}_j$

$$\sum_{i \in S_j} y_i > \sum_{i \in S_j} z_i \quad (j = 1, \dots, \ell). \quad (2)$$

Hence, for any feasible choice of  $z$ , there exists a solution configuration,  $(y, \mathcal{P}_j)$ , such that

$$(y, \mathcal{P}_j) \text{ dom } (z, \mathcal{P}_G).$$

To prove this, suppose that for all  $(y, \mathcal{P}_j)$ ,

$$\sum_{i \in S_j} y_i \leq \sum_{i \in S_j} z_i. \quad (3)$$

In particular, (3) will be true for a solution configuration,  $(y, \mathcal{P}_j)$ , with

$$\sum_{i \in S_j} y_i = \sum_{i \in S_j} c^i \hat{x}(\mathcal{P}_j).$$

Summing both sides of (3) over  $j$  will then yield

$$\sum_{j=1}^{\ell} \sum_{i \in S_j} c^i \hat{x}(\mathcal{P}_j) \leq \sum_{j=1}^{\ell} \sum_{i \in S_j} x_i = \sum_{i=1}^n c^i \hat{x}(\mathcal{P}_G),$$

which is a contradiction.

Therefore, from inequality (2),  $(z, \mathcal{P}_G) \notin \mathcal{C}$  for any feasible  $z$ . Using Proposition 4.2, this implies that  $\mathcal{C} = \emptyset$ . ■

## 5 Examples

**Example 1:** The following problem is a three-level continuous knapsack problem with no core:

$$\begin{aligned} K(3) \quad & \max 2x_1 + 5x_2 + x_3 \\ & \text{st: } \max x_1 + 3x_3 \\ & \text{st: } \max 4x_1 + x_2 + 4x_3 \\ & \text{st: } x_1 + x_2 + x_3 \leq 1 \\ & \quad x_i \geq 0, \quad i = 1, 2, 3. \end{aligned}$$

Under coalition structure  $\mathcal{P}_0$ , the payoffs are  $v(\{1\}, \mathcal{P}_0) = 4$ ,  $v(\{2\}, \mathcal{P}_0) = 1$ , and  $v(\{3\}, \mathcal{P}_0) = 2$ . Under the grand coalition,  $\mathcal{P}_G$ , the payoff is  $v(\{1, 2, 3\}, \mathcal{P}_G) = 8$ .

Now consider the coalition structure  $\mathcal{P} = \{\{1\}, \{2, 3\}\}$ . The payoffs are  $v(\{1\}, \mathcal{P}) = 1$ , and  $v(\{2, 3\}, \mathcal{P}) = 5$ . Note that coalitions  $\{1\}$  and  $\{2, 3\}$  are disjoint and that

$$v(\{1\}, \mathcal{P}_0) + v(\{2, 3\}, \mathcal{P}) > v(\{1, 2, 3\}, \mathcal{P}_G).$$

Therefore, from Proposition 4.3, the embedded game has an empty core.

**Example 2:** A three-level problem with a core is

$$\begin{aligned} K(3) \quad & \max 2x_1 + x_3 \\ & \text{st: } \max 2x_1 + 3x_2 + x_3 \\ & \text{st: } \max 3x_1 + x_2 + x_3 \\ & \text{st: } x_1 + x_2 + x_3 \leq 1 \\ & \quad x_i \geq 0, \quad i = 1, 2, 3. \end{aligned}$$

For a three-level problem, there are five possible coalition structures:

$$\begin{aligned} \mathcal{P}_0 &= \{\{1\}, \{2\}, \{3\}\} & \mathcal{P}_3 &= \{\{1\}, \{2, 3\}\} \\ \mathcal{P}_1 &= \{\{1, 2\}, \{3\}\} & \mathcal{P}_G &= \{\{1, 2, 3\}\} \\ \mathcal{P}_2 &= \{\{1, 3\}, \{2\}\} & & \end{aligned}$$

The corresponding payoffs are

$$\begin{aligned} v(\{1\}, \mathcal{P}_0) &= 1 & v(\{1, 3\}, \mathcal{P}_2) &= 2 \\ v(\{2\}, \mathcal{P}_0) &= 1 & v(\{2\}, \mathcal{P}_2) &= 1 \\ v(\{3\}, \mathcal{P}_0) &= 1 & v(\{1\}, \mathcal{P}_3) &= 3 \\ v(\{1, 2\}, \mathcal{P}_1) &= 5 & v(\{2, 3\}, \mathcal{P}_3) &= 4 \\ v(\{3\}, \mathcal{P}_1) &= 2 & v(\{1, 2, 3\}, \mathcal{P}_G) &= 7. \end{aligned}$$

From Proposition 4.1, all solution configurations within the core yield a payoff to the entire system equal to the amount realized by the grand coalition,  $\mathcal{P}_G$ . Hence, only solution configurations with coalition structures which yield this amount are possible candidates for the core. For this example, solution configurations  $(r, \mathcal{P}_1)$ ,  $(s, \mathcal{P}_3)$  and  $(t, \mathcal{P}_G)$ , where  $r$ ,  $s$ , and  $t$  are, as of yet, unspecified imputations, are possible elements of the core.

The next step in determining the core is to partition imputations  $r$ ,  $s$ , and  $t$  such that their respective solution configurations cannot be dominated. Note that it is sufficient to partition only imputation  $r$  since  $s$  and  $t$  will be equivalent. The imputation  $r = (r_1, r_2, r_3)$  must satisfy

$$\begin{aligned} r_1 &\geq 3 & r_1 + r_2 &\geq 5 \\ r_2 &\geq 1 & r_1 + r_3 &\geq 2 \\ r_3 &\geq 2 & r_2 + r_3 &\geq 4 \\ r_1 + r_2 + r_3 &= 7. \end{aligned}$$

In this case, there is only one feasible imputation,  $r = (3, 2, 2)$ . Hence

$$\mathcal{C} = \{((3, 2, 2), \mathcal{P}_1), ((3, 2, 2), \mathcal{P}_3), ((3, 2, 2), \mathcal{P}_G)\}.$$

In general, the set of imputation in the core may be a convex polyhedron rather than a single point. In such cases, the above procedure does not specify a final disbursement of payoffs among the players. This issue is examined extensively in the game theory literature (see, for example, Lucas [11], and Shapley [12]).

## 6 Conclusions

This paper has presented a method for evaluating the effects of coalition formation in a particular class of  $n$ -person Stackelberg games. Most of the results here should be amenable to extension into more general cases. The solutions to such problems might suggest ways in which a hierarchical system could be restructured to remove overall system inefficiencies. In addition, the formulation can help predict the additional benefits which would accrue from such modifications.

The methods developed here have characterized the  $n$ -level linear resource control problem as an abstract game. If a nonempty core exists for such a game, then enforceable coalitions exists which could gain additional benefits from the system. It has been shown that there exist models of three (and higher) level hierarchical systems where such cooperative arrangements do not tend to arise.

There are a number of issues which have yet to be addressed. One of these is the development of an efficient algorithm to construct the core if it is nonempty. Also, when the core is empty, it is uncertain which solution concept should be used to allow the players to obtain the increased payoffs which still may be available.

Of course, this approach will not be appropriate for all hierarchical system management problems. However, for those systems where it is applicable, it should assist in improving system performance.

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