Coarse Ricci curvature of weighted Riemannian manifolds

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Abstract

We show that the generalized Ricci tensor of a weighted complete Riemannian manifold can be retrieved asymptotically from a scaled metric derivative of Wasserstein 1-distances between normalized weighted local volume measures. As an application, we demonstrate that the limiting coarse curvature of random geometric graphs sampled from Poisson point process with non-uniform intensity converges to the generalized Ricci tensor.

Contents

1 Introduction and result

We revisit the notion of coarse Ricci curvature studied by Ollivier [Oll09]. With the application to curvature of random geometric graphs in mind, we consider the case of a Riemannian manifold with a potential. As a novel element, we consider the Wasserstein distances of weighted volume measures on small geodesic balls to account for a smooth potential on the manifold and recover the smooth generalized Ricci curvature of Riemannian manifolds from a scaled metric derivative of Wasserstein distances of such measures.

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The approach to curvature using optimal transport distances of probability measures has been referred to in the literature as "coarse curvature" or "Wasserstein curvature" of the underlying space.

Let M be a complete, n-dimensional Riemannian manifold. For any $x \in M$ and $\varepsilon, \delta > 0$ we denote the uniform probability measure supported on the geodesic ball $B_\varepsilon(x)$ as

$$d\mu_x^\varepsilon(z) := \frac{\mathbb{1}_{B_\varepsilon(x)}(z)}{\text{vol}(B_\varepsilon(x))} d\text{vol}(z)$$

where vol is the standard Riemannian volume measure.

Let $V : M \to \mathbb{R}$ be a smooth potential and $e^{-V(z)}d\text{vol}(z)$ the corresponding weight measure on M. For any $x \in M$, define the (non-uniform) probability measure supported on the geodesic ball $B_\varepsilon(x)$,

$$d\nu_x^\varepsilon(z) := \mathbb{1}_{B_\varepsilon(x)}(z) \frac{e^{-V(z)}}{\int_{B_\varepsilon(x)} e^{-V(z')}dz'} d\text{vol}(z).$$

Our main results are Theorem 1.1 and Theorem 4.7. The first allows to extract the generalized Ricci tensor from 1-Wasserstein distances of such measures:

Theorem 1.1. *For any point $x_0 \in M$, vector $v \in T_{x_0}M$ with $\|v\| = 1$ and $\delta, \varepsilon > 0$ sufficiently small,*

$$W_1(\nu_{x_0}^\varepsilon, \nu_y^\varepsilon) = \delta \left(1 - \frac{\varepsilon^2}{2(n+2)} \left(Ric_{x_0}(v,v) + 2Hess_{x_0}V(v,v) \right) \right) + O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3) \tag{1.1}$$

where $y := \exp_{x_0}(\delta v)$.

Denoting the coarse curvature at scale ε as

$$\kappa_\varepsilon(x_0, y) := 1 - \frac{W_1(\nu_{x_0}^\varepsilon, \nu_y^\varepsilon)}{d(x_0, y)} \tag{1.2}$$

we deduce upon rearrangement of the expansion (1.1) and taking the limit that

$$\lim_{\varepsilon,\delta \to 0} \frac{2(n+2)}{\varepsilon^2} \kappa_\varepsilon(x_0, y) = \text{Ric}_{x_0}(v,v) + 2\text{Hess}_{x_0}(v,v). \tag{1.3}$$

Theorem 1.1 extends a result of Ollivier [Oll09] and we present a detailed proof.

As an application, in Theorem 4.7 we extend a result of Hoorn et al. [vdHCL+21] showing that the coarse curvature of random geometric graphs sampled from a Poisson point process with increasing intensity, proportional to the non-uniform measure $e^{-V(z)}\text{vol}(dz)$, converges to the smooth Ricci curvature modified by the Hessian of V. Our method allows to deal with the non-uniformity of the intensity of the Poisson process as well as the non-uniformity incurred by the exponential mapping.

Remark 1.2. We qualify the term "sufficiently small" for δ, ε in Theorem 1.1. In all arguments, we will assume that δ, ε are as small as needed in a way only dependent on a compact neighbourhood of $x_0 \in M$. The need for such restriction is for two reasons:

- in manifold distance estimates of Section 2 to ensure that all geodesics in the variations used are length-minimizing and unique. This can be done by restricting δ, ε to some small enough fraction of the uniform injectivity radius at x_0.
- in Wasserstein distance estimates of Section 3, to apply the Inverse Function Theorem for the transport map T which has non-zero determinant at x_0

We will assume throughout this work that such restrictions are in place and are covered implicitly by the "sufficiently small" assumption for δ, ε.

Remark 1.3. Ollivier [Oll09, Example 5] presented a similar result by using uniform measures shifted in the direction of $-\nabla V(x)$ for a uniform measure centered at $x \in M$. Nonetheless, to obtain this, the correct magnitude for this shift is chosen a posteriori. This is in contrast to our method of non-uniform measures, which does not allow this degree of freedom and can therefore be seen as more intrinsic to the weighted manifold. We emphasize the distinction in that our method employs non-uniform measures rather than uniform. Moreover, our approach is more suitable for our application to random geometric graphs sampled from a Poisson point process with non-uniform intensity.

We now follow through with two ingredients needed for proving Theorem 1.1, manifold distance and Wasserstein distance estimates, presented in Sections 2 and 3, respectively. The application to random geometric graphs constitutes Section 4. The clearly presented intermediate geometric estimates could be of independent interest.

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2 Manifold distance estimates

The 1-Wasserstein distance of measures is by definition the minimum of an average of distances of pairs of points on the manifold. Variations of geodesics are a standard tool for local estimation of distances on Riemannian manifolds. We introduce preliminary notation for this section.

Notation 2.1. Let $v \in T_{x_0}M$ be a unit vector. The maps $c : [0, \varepsilon] \times [0, 1] \to M$ will denote various smooth variations, to be specified, of the geodesic $\gamma : [0, 1] \to M$,

$$\gamma(t) := \exp_{x_0}(t\delta v)$$

so that $c(0, t) = \gamma(t)$. Denote the Jacobi field corresponding to the variation c as

$$J(t) := \left.\frac{\partial}{\partial s}\right|_{s=0} c(s, t)$$

and the covariant derivatives of J with respect to $\dot\gamma$ along γ by $\frac{D}{dt}J(t)$, $\frac{D^2}{dt^2}J(t)$. Recall J satisfies the Jacobi equation

$$\frac{D^2}{dt^2}J(t) = -R(J(t), \dot\gamma(t))\dot\gamma(t)$$

where R is the Riemann curvature tensor. For any vector field X defined along γ, we denote the part perpendicular to $\dot\gamma$ as

$$X^\perp := X - \frac{1}{\delta^2}\langle X, \dot\gamma\rangle \dot\gamma.$$

Denote by $/\!/_t\colon T_{x_0}M \to T_{\gamma(t)}M$ the parallel transport along the geodesic γ with respect to the Levi-Civita connection and denote by $\mathrm{Hess}_{x_0}V^\# : T_{x_0}M \to T_{x_0}M$ the unique linear map such that

$$\left\langle \mathrm{Hess}_{x_0}V^\#(v), w\right\rangle = \mathrm{Hess}_{x_0}V(v, w) \quad \forall v, w \in T_{x_0}M.$$

For the distance estimates we will make use of the standard formulas for first and second derivatives of the length

$$L(\varepsilon, \delta) := \int_0^1 \left\| \frac{\partial c(\varepsilon, t)}{\partial t} \right\| dt$$

in terms of J (see [Jos17, Chap. 6]):

Lemma 2.2. *The first two derivatives of the length L in the first variable at 0 are*

$$\frac{\partial}{\partial s}\bigg|_{s=0} L(s,\delta) = \frac{1}{\delta} \left\langle J(t), \dot\gamma(t) \right\rangle\Big|_{t=0}^{t=1}, \tag{2.1}$$

$$\begin{aligned} \frac{\partial^2}{\partial s^2}\bigg|_{s=0} L(s,\delta) &= \frac{1}{\delta}\left[\int_0^1 \left\langle \frac{D}{dt}J(t)^\perp, \frac{D}{dt}J(t)^\perp \right\rangle - \left\langle R(J(t)^\perp, \dot\gamma(t))\dot\gamma(t), J(t)^\perp \right\rangle dt\right] \\ &\quad + \frac{1}{\delta}\left\langle \frac{D}{\partial s}\frac{\partial}{\partial s}\bigg|_{s=0} c(s,t), \dot\gamma(t)\right\rangle\bigg|_{t=0}^{t=1} \\ &= \frac{1}{\delta}\left\langle \frac{D}{dt}J(t)^\perp, J(t)^\perp\right\rangle\bigg|_{t=0}^{t=1} + \frac{1}{\delta}\left\langle \frac{D}{\partial s}\frac{\partial}{\partial s}\bigg|_{s=0} c(s,t), \dot\gamma(t)\right\rangle\bigg|_{t=0}^{t=1}. \end{aligned} \tag{2.2}$$

We will consider two different variations c_1 and c_2 of the geodesic $\gamma(t) = \exp_{x_0}(t\delta v)$ which will afford key distance estimates for the optimal transport problem in Theorem 1.1.

2.1 Pointwise transport distance estimate

We proceed with defining c_1, the purpose of which is to approximate pointwise transport distance by a certain transport map defined later. Using these pointwise distance estimates, we will be able to conclude an upper bound for the Wasserstein distance in Theorem 1.1.

Notation 2.3. For any $v, w \in T_{x_0}M$ with $\|v\| = 1$ and $\|w\| \leqslant 1$ we introduce the transport vector

$$w' := w - \frac{\varepsilon}{2}(1 - \|w\|^2)(/\!\!/_1^{-1} \nabla V(y) - \nabla V(x_0)) = w + O(\delta\varepsilon), \tag{2.3}$$

where we abuse the notation $O(\delta\varepsilon)$ to denote a vector of magnitude of order $\delta\varepsilon$.
Consider the geodesics

$$\theta(s) := \exp_y(s /\!\!/_1 w'), \quad \eta(s) := \exp_{x_0}(sw) \tag{2.4}$$

where $y = \exp_{x_0}(\delta v)$ and the family indexed by s of geodesics parametrized by t starting from $\eta(s)$ and reaching $\theta(s)$ at $t = 1$, i. e. the map $c_1 : [0, \varepsilon] \times [0, 1] \to M$ defined by

$$c_1(s,t) = \exp_{\eta(s)}(t \exp^{-1}_{\eta(s)}(\theta(s)))$$

which is a variation of the geodesic $t \mapsto \exp_{x_0}(t\delta v)$. See Fig. 1 for an illustration of this variation. Note that

$$\theta(s) = c_1(s, 1), \quad \eta(s) = c_1(s, 0)$$

are the bottom and the top curves in Fig. 1 and for every $s \in [0, \varepsilon]$, $t \mapsto c_1(s,t)$ is a geodesic. Moreover, $\gamma(t) = \exp_{x_0}(t\delta v) = c_1(0,t)$ is the leftmost geodesic in the figure.

The aim of this variation is to estimate the distance of the two corners $\theta(\varepsilon)$ and $\eta(\varepsilon)$ given by the variation c_1 of the geodesic γ from η to θ. To emphasize the dependency on both δ and ε, we will denote the length

$$L_1(\varepsilon, \delta) := \int_0^1 \left\| \frac{\partial c_1(\varepsilon, t)}{\partial t} \right\| dt = d(c_1(\varepsilon, 1), c_1(\varepsilon, 0)) = d(\theta(\varepsilon), \eta(\varepsilon)). \tag{2.5}$$

We begin with preliminary estimates on the Jacobi field of the variation c_1.

Lemma 2.4. *The Jacobi field $J(t) = \frac{\partial}{\partial s}\big|_{s=0} c_1(s,t)$ satisfies*

$$\left\| \frac{D}{dt} J(t) \right\| = O(\delta\varepsilon) + O(\delta^2) \tag{2.6}$$

and

$$/\!/_t^{-1}\, J(t) = J(0) + O(\delta\varepsilon) + O(\delta^2) \quad \forall t \in [0,1]. \tag{2.7}$$

Proof. Recall $\dot\gamma(t) = \delta \,/\!/_t\, v$ and hence the Jacobi equation implies

$$\left\| \frac{D^2}{dt^2} J(t) \right\| = \|R(J(t), \dot\gamma, (t))\dot\gamma(t)\| \leqslant C\delta^2 \|J(t)\| \tag{2.8}$$

where $C = \sup_{t\in[0,1]} \|R_{\gamma(t)}(\cdot, /\!/_t\ v)\ /\!/_t\ v\| < \infty$ in the sense of operator norm. The expansion for J up to $t = 1$ with integral remainder is

$$/\!/_1^{-1}\, J(1) = J(0) + \frac{D}{dt}\bigg|_{t=0} J(t) + \int_0^1 (1-u)\, /\!/_u^{-1}\, \frac{D^2}{dt^2} J(u) du.$$

Since $/\!/_1^{-1}\, J(1) - J(0) = w' - w$ has magnitude of order $\delta\varepsilon$, this implies

$$\left\| \frac{D}{dt}\bigg|_{t=0} J(t) \right\| \leqslant O(\delta\varepsilon) + C\delta^2 \|J(t)\|$$

and hence

$$\|J(t)\| \leqslant \|J(0)\| + \int_0^t \left\| \frac{D}{dt} J(u) \right\| du \leqslant 1 + O(\delta\varepsilon) + C\delta^2 \int_0^t \|J(u)\| du$$

which yields by Grönwall's lemma that

$$\|J(t)\| \leqslant (1 + O(\varepsilon\delta)) e^{C\delta^2 t} = 1 + O(\delta^2) + O(\delta\varepsilon).$$

As a consequence, we deduce from (2.8) that

$$\frac{D^2}{dt^2} J(t) = O(\delta^2)$$

and thus

$$/\!/_t^{-1}\, \frac{D}{dt} J(t) = \frac{D}{dt}\bigg|_{t=0} J(t) + \int_0^t /\!/_u^{-1}\, \frac{D^2}{dt^2} J(u) du = O(\delta\varepsilon) + O(\delta^2)$$

and finally

$$/\!/_t^{-1}\, J(t) = J(0) + \int_0^t /\!/_u^{-1}\, \frac{D}{dt} J(u) du$$

where the second term has norm of the required order. □

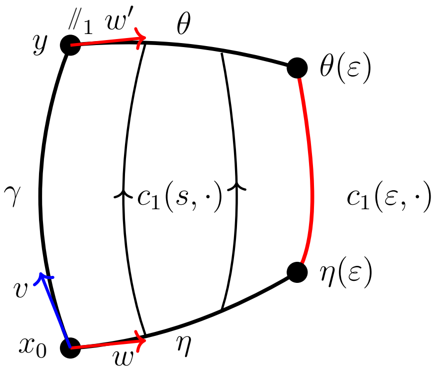

Figure 1: Geodesic variation c_1 (with positive sectional curvature in the v, w-plane)

Proposition 2.5. *For any $v, w \in T_{x_0}M$ with $\|v\| = 1, \|w\| \leqslant 1$ and δ, ε sufficiently small, we have the estimate for the distance between $\exp_{x_0}(\varepsilon w)$ and $\exp_{\exp_{x_0}(\delta v)}(\varepsilon /\!/_1\, w')$ where w' is given by (2.3), expressed by the geodesic length*

$$L_1(\varepsilon,\delta) = \delta\left(1 - \frac{\varepsilon^2}{2}\left[K_{x_0}(v,w)(\|w\|^2 - \langle v,w\rangle^2) + \mathit{Hess}_{x_0}V(v,v)(1-\|w\|^2)\right]\right) + O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3) \tag{2.9}$$

where the $O(\varepsilon^2\delta^2) + O(\delta\varepsilon^3)$ terms are uniformly bounded in v and w.

Proof. We expand the length $L_1(\varepsilon,\delta)$ in the first variable,

$$L_1(\varepsilon,\delta) = L_1(0,\delta) + \varepsilon \left.\frac{\partial}{\partial s}\right|_{s=0} L_1(s,\delta) + \frac{\varepsilon^2}{2}\left.\frac{\partial^2}{\partial s^2}\right|_{s=0} L_1(s,\delta) + O(\varepsilon^3) \tag{2.10}$$

and compute the first and second order coefficients using the variation $c_1(s,\cdot)$.

The Jacobi field $J(t) := \left.\frac{\partial}{\partial s}\right|_{s=0} c_1(s,t)$ satisfies the boundary conditions

$$J(0) = \left.\frac{d}{ds}\right|_{s=0}\eta(s) = w, \qquad J(1) = \left.\frac{d}{ds}\right|_{s=0}\theta(s) = /\!/_1\, w'$$

where w' is given by Eq. (2.3). Then the first order coefficient in the expansion (2.10) is

$$\begin{aligned}\left.\frac{\partial}{\partial s}\right|_{s=0} L_1(s,\delta) &= \frac{1}{\delta}(\langle J(1),\dot\gamma(1)\rangle - \langle J(0),\dot\gamma(0)\rangle)\\ &= \langle /\!/_1\, w', /\!/_1\, v\rangle - \langle w, v\rangle\\ &= -\frac{\delta\varepsilon}{2}(1-\|w\|^2)\mathrm{Hess}_{x_0}V(v,v)\end{aligned}$$

by the formula (2.1) and plugging in for w'.

The second order variation of length formula (2.2) reduces to

$$\left.\frac{\partial^2}{\partial s^2}\right|_{s=0} L_1(s,\delta) = \frac{1}{\delta}\int_0^1 \left\langle \frac{D}{dt}J(t)^\perp, \frac{D}{dt}J(t)^\perp\right\rangle - \left\langle R(J(t)^\perp,\dot\gamma(t))\dot\gamma(t), J(t)^\perp\right\rangle dt$$

since $\frac{D}{\partial s}\frac{\partial}{\partial s}c(s,0) = \frac{D}{\partial s}\frac{\partial}{\partial s}c(s,1) = 0$ as $s \mapsto c(s,0), s \mapsto c(s,1)$ are geodesics. By the estimate (2.6), the first term is

$$\int_0^1 \left\langle \frac{D}{dt}J(t)^\perp, \frac{D}{dt}J(t)^\perp\right\rangle dt = O(\delta^2\varepsilon^2) + O(\delta^3\varepsilon) + O(\delta^4).$$

Moreover, by smoothness of R in the base point, we have $\| /\!/_t^{-1} \circ R \circ /\!/_t - R\| = O(\delta)$ as the parallel transport is along a geodesic of length δ. Then by multi-linearity of R and the estimate (2.7),

$$\int_0^1 \left\langle R(J(t)^\perp, \dot\gamma(t))\dot\gamma(t), J(t)^\perp \right\rangle dt = \delta^2 \int_0^1 \left\langle R(/\!/_t^{-1} J(t)^\perp, v)v, /\!/_t^{-1} J^\perp(t) \right\rangle dt + O(\delta^3)$$
$$= \delta^2 \left\langle R(w,v)v, w \right\rangle + O(\delta^3)$$

noting that $\left\langle R(w^\perp, v)v, w^\perp \right\rangle = \left\langle R(w,v)v,w \right\rangle$ by anti-symmetry of the curvature tensor and since $w^\perp - w$ is parallel to v by definition.

Therefore, coming back to the second order coefficient and plugging in,

$$\left.\frac{\partial^2}{\partial s^2}\right|_{s=0} L_1(s,\delta) = \delta \left\langle R(w,v)v,w \right\rangle + O(\delta^2) = \delta K(v,w)(1 - \langle v, w\rangle^2) + O(\delta^2).$$

Finally, the $O(\varepsilon^3)$ term in (2.10) is in fact $O(\delta\varepsilon^3)$ since $L_1(\varepsilon, \delta) = O(\delta)$ and L_1 is smooth. □

Remark 2.6. We will use repeatedly the following fact. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a function smooth at the origin. If $\phi(x,y) = O(x^a \wedge y^b)$ then $\phi(x,y) = O(x^a y^b)$. For $a = b = 1$, this can be seen by expanding up to first order,

$$\phi(x,y) = \phi(0,0) + \partial_x\phi(0,0)x + \partial_y\phi(0,0)y + R_{11}(x,y)x^2 + R_{12}(x,y)xy + R_{22}(x,y)y^2$$

for some smooth remainder functions $R_{11}, R_{12}, R_{21} \in C^\infty(\mathbb{R}^2)$. Then $\phi(x,y) = O(y)$ implies $\phi(0,0) = \partial_x\phi(0,0) = 0$ and $R_{11}(x,y) = O(y)$, and $\phi(x,y) = O(x)$ implies $\partial_y\phi(0,0) = 0$ and $R_{22}(x,y) = O(x)$. Then $\phi(x,y) = R'(x,y)xy$ for some $R' \in C^\infty(\mathbb{R}^2)$. The argument is similar for arbitrary $a, b \in \mathbb{N}$.

Proposition 2.5 is similar to the following distance estimate on geodesic triangles, see e. g. [Mey04] [LV10].

Lemma 2.7. *For any $w_1, w_2 \in T_{x_0}M$ linearly independent and $\varepsilon > 0$ sufficiently small,*

$$d(\exp_{x_0}(\varepsilon w_1), \exp_{x_0}(\varepsilon w_2)) = \varepsilon\|w_1 - w_2\| - \frac{1}{6}\frac{\left\langle R(w_1,w_2)w_2, w_1\right\rangle}{\|w_1 - w_2\|}\varepsilon^3 + O(\varepsilon^4).$$

2.2 Projection distance estimate

We construct the variation c_2 for the purpose of estimating the projection distance to a specific submanifold. This projection distance will serve as a suitable 1-Lipschitz function for establishing a lower bound on the Wasserstein distance in Theorem 1.1 by the Kantorovich-Rubinstein duality.

Let E be a smooth embedded submanifold of the Riemannian manifold M.

Definition 2.8. An open neighbourhood U of E in M is said to be a tubular neighbourhood if there exists an open subset of the normal bundle $W \subset TE^\perp$ such that $E \subset W$ and $\exp : W \to U$ is a diffeomorphism.

The following is due to R. Foote [Foo84]:

Lemma 2.9. *For every compact, embedded smooth submanifold E of M, there is a tubular neighbourhood U of E in M such that the shortest distance projection map*

$$p(x) := \mathit{argmin}_{z\in E} d(x,z). \tag{2.11}$$

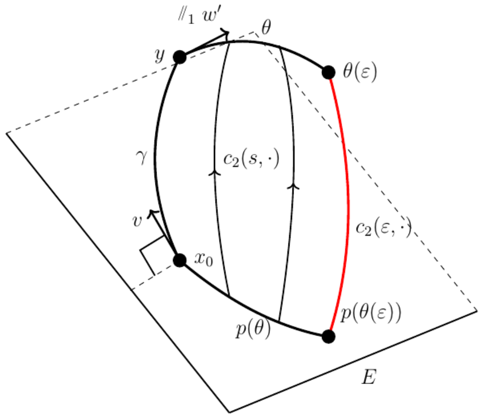

Figure 2: Geodesic variation c_2

is well-defined on U and smooth on $U \setminus E$, and the distance to projection $z \mapsto d(z, p(z))$ is also smooth on $U \setminus E$.

Moreover, if E is a codimension 1 submanifold with $\nu \in \Gamma(TE^\perp)$ a unit vector field normal to the submanifold then the signed distance to projection defined by

$$f(z) := \operatorname{sign}(\langle \exp_{p(z)}^{-1}(z), \nu(z)\rangle) d(z, p(z)) \tag{2.12}$$

is smooth on all of U.

We now consider the concrete submanifold of codimension 1,

$$E := \exp_{x_0}\{v^\perp\} := \{\exp_{x_0}(w) : w \in T_{x_0}M, \langle w, v\rangle = 0\} \subset M \tag{2.13}$$

and recall that the second fundamental form for E is the 2-covariant tensor field, defined at x_0 for two vectors $u, v \in T_{x_0}E$ as

$$\Pi(u, v) := \langle \nabla_u \nu, v\rangle = -\langle \nu, \nabla_u V\rangle$$

where V on the right is an arbitrary local vector field extending v (see [Jos17, Chap. 5]) and $\nu \in \Gamma(TE^\perp)$ a unit normal vector field with $\nu(x_0) = v$.

Remark 2.10. We shall make use of the fact that Π vanishes at $x_0 \in M$ for this submanifold, which we prove for the readers' convenience. Using the normal coordinates $(x^1, \dots, x^n)$ of M at $x_0 \in M$, we may assume $(x^1, \dots, x^{n-1})$ are the normal coordinates of E at x_0. Then $\Gamma_{ij}^k(x_0) = 0$ (see [Jos17, Chap. 1.4]) and for arbitrary vector fields $X = \sum_{i=1}^{n-1} X^i \frac{\partial}{\partial x^i}, Y = \sum_{i=1}^{n-1} Y^i \frac{\partial}{\partial x^i} \in \Gamma(TE)$, it holds that

$$\begin{aligned}\nabla_X Y(x_0) &= \sum_{i,j=1}^{n-1} X^i(x_0)\Gamma_{ij}^k(x_0)Y^j(x_0)\frac{\partial}{\partial x^k}(x_0) + X^i(x_0)\frac{\partial Y^j}{\partial x^j}(x_0)\frac{\partial}{\partial x^j}(x_0)\\ &= \sum_{i,j=1}^{n-1} X^i(x_0)\frac{\partial Y^j}{\partial x^j}(x_0)\frac{\partial}{\partial x^j}(x_0) \in T_{x_0}E\end{aligned}$$

and thus $\Pi(X, Y)(x_0) = -\langle \nu(x_0), \nabla_X Y(x_0)\rangle = 0$ since ν is normal to E.

We now define the geodesic variation c_2, with its depiction in Fig. 2. As before, denote for a vector $w \in T_{x_0}M$ with $\|w\| \leqslant 1$ the geodesic

$$\theta(s) := \exp_y(s /\!/_1\, w')$$

where w' is defined by (2.3). Let $c_2 : [0,\varepsilon] \times [0,1] \to M$ be the variation of the geodesic $\gamma(t) = \exp_{x_0}(t\delta v)$ defined by

$$c_2(s,t) = \exp_{p(\theta(s))}(t \exp^{-1}_{p(\theta(s))}(\theta(s)))$$

where p is the projection map given by (2.11), and denote the length

$$L_2(\varepsilon,\delta) := \int_0^1 \left\| \frac{\partial c_2(\varepsilon,t)}{\partial t} \right\| dt = d(c_2(\varepsilon,1), c_2(\varepsilon,0)) = d(\theta(\varepsilon), p(\theta(\varepsilon)))$$

and the signed length

$$\tilde{L}_2(\varepsilon,\delta) := \operatorname{sign}(\langle \exp^{-1}_{x_0}(c_2(\varepsilon,\delta)), v\rangle) L_2(\varepsilon,\delta), \tag{2.14}$$

which is smooth by Lemma 2.9.

Proposition 2.11. *For any vectors $v, w \in T_{x_0}M$ with $\|v\| = 1, \|w\| \leqslant 1$ and δ, ε sufficiently small, the signed distance between $\exp_{\exp_{x_0}(\delta v)}(\varepsilon \mathbin{/\!/}_1 w')$ and its projection to E has expansion*

$$\begin{aligned}\tilde{L}_2(\varepsilon,\delta) = \delta + \varepsilon \langle v, w\rangle - \frac{\varepsilon^2\delta}{2}\left[K_{x_0}(v,w)(\|w\|^2 - \langle v,w\rangle^2) + \mathit{Hess}_{x_0}V(v,v)(1-\|w\|^2)\right] \\ + O(\varepsilon^2\delta^2) + O(\varepsilon^3)\end{aligned} \tag{2.15}$$

where the $O(\varepsilon^2\delta^2) + O(\varepsilon^3)$ terms are uniformly bounded in v and w.

Proof. Since $\tilde{L}_2(s,\delta) = L_2(s,\delta)$ for small s, their derivatives at $s = 0$ also agree, and we may apply again the method of geodesic variations to compute $\frac{\partial}{\partial s}\big|_{s=0} L_2(s,\delta)$ and $\frac{\partial^2}{\partial s^2}\Big|_{s=0} L_2(s,\delta)$ in order to obtain the coefficients of the expansion of $\tilde{L}_2(\varepsilon,\delta)$.

The Jacobi field $J(t) = \frac{\partial}{\partial s}\big|_{s=0} c_2(s,t)$ satisfies boundary condition $J(1) = \mathbin{/\!/}_1 w'$. Moreover, $J(0) \in T_{x_0}E$ because $c_2(s,0) \in E$ for all $s \in [0,\varepsilon]$, hence in particular $\dot\gamma(0) \perp J(0)$ and $J^\perp(0) = J(0)$. By the formula (2.1), the order ε coefficient is thus

$$\begin{aligned}\frac{\partial}{\partial s}\bigg|_{s=0} L_2(s,\delta) &= \frac{1}{\delta}(\langle J(1), \dot\gamma(1)\rangle - \langle J(0), \dot\gamma(0)\rangle) \\ &= \frac{1}{\delta}\left\langle \mathbin{/\!/}_1^{-1} J(1), \mathbin{/\!/}_1^{-1} \dot\gamma(1)\right\rangle \\ &= \langle v, w\rangle - \frac{\delta\varepsilon}{2}\mathrm{Hess}_{x_0}(v,v)(1-\|w\|^2).\end{aligned} \tag{2.16}$$

By the formula (2.2), the order ε^2 coefficient is

$$\begin{aligned}\frac{\partial^2}{\partial s^2}\bigg|_{s=0} L_2(s,\delta) = \frac{1}{\delta}\left(\left\langle \frac{D}{dt}\bigg|_{t=1} J^\perp(t), J^\perp(1)\right\rangle - \left\langle \frac{D}{dt}\bigg|_{t=0} J^\perp(t), J^\perp(0)\right\rangle\right) \\ + \frac{1}{\delta}\left(\left\langle \frac{D}{\partial s}\frac{\partial}{\partial s}\bigg|_{s=0} c_2(s,1), \dot\gamma(1)\right\rangle - \left\langle \frac{D}{\partial s}\frac{\partial}{\partial s}\bigg|_{s=0} c_2(s,0), \dot\gamma(0)\right\rangle\right).\end{aligned} \tag{2.17}$$

We show that all terms on the right vanish except for the first one which we then further estimate. For any $u \in T_{x_0}E$,

$$\left\langle \frac{D}{dt}\bigg|_{t=0} J^\perp(t), u\right\rangle = \left\langle \left(\frac{D}{\partial s}\frac{\partial}{\partial t}\bigg|_{t=s=0} c_2(s,t)\right)^\perp, u\right\rangle = \delta\Pi\left(\frac{\partial}{\partial s}\bigg|_{s=0} c_2(s,0), u\right) = 0$$

as the second fundamental form vanishes at x_0. Since $\frac{\partial}{\partial s}\big|_{s=0} c(s,0) \in T_{x_0}E$, this implies $\frac{D}{dt}\big|_{t=0} J^\perp(t) = 0$. Similarly,

$$\left\langle \frac{D}{\partial s}\frac{\partial}{\partial s}\bigg|_{s=0} c(s,0), \dot\gamma(0)\right\rangle = -\delta\Pi\left(\frac{\partial}{\partial s}\bigg|_{s=0} c(s,0), \frac{\partial}{\partial s}\bigg|_{s=0} c(s,0)\right) = 0.$$

Since $s \mapsto \theta(s)$ is a geodesic, $\frac{D}{\partial s}\frac{\partial}{\partial s} c(s,1) = \frac{D}{ds}\theta(s) = 0$, the third term is also 0.

We now estimate the first term in (2.17). For any $t \in [0,1]$ we have the expansion with integral remainder,

$$\begin{aligned}
/\!\!/_t^{-1}\, J^\perp(t) &= J^\perp(0) + \int_0^t (t-u)\, /\!\!/_t^{-1}\, \frac{D^2}{dt^2} J^\perp(t) dt \\
&= J^\perp(0) - \delta^2 \int_0^t (t-u)\, /\!\!/_t^{-1}\, R(J^\perp(t), /\!\!/_t\, v)\, /\!\!/_t\, v\; dt \\
&= J^\perp(0) + O(\delta^2).
\end{aligned}$$

This in particular also holds for $t = 1$ so we deduce that

$$/\!\!/_1^{-1}\, J^\perp(1) -\, /\!\!/_t^{-1}\, J^\perp(t) = O(\delta^2).$$

Moreover, the Jacobi equation gives

$$/\!\!/_1^{-1}\, \frac{D}{dt}\bigg|_{t=1} J^\perp(t) = -\delta^2 \int_0^1 /\!\!/_t^{-1}\, R(J^\perp(t), /\!\!/_t\, v)\, /\!\!/_t\, v\; dt$$

so the second derivative is

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}\bigg|_{s=0} L_2(s,\delta) &= \frac{1}{\delta}\left\langle \frac{D}{dt}\bigg|_{t=1} J^\perp(t), J^\perp(1)\right\rangle \\
&= -\delta \int_0^1 \left\langle /\!\!/_t^{-1}\, R(J^\perp(t), /\!\!/_t\, v)\, /\!\!/_t\, v, /\!\!/_1^{-1}\, J^\perp(1)\right\rangle dt \\
&= -\delta \int_0^1 \left\langle R(/\!\!/_t^{-1}\, J^\perp(t), v)v, /\!\!/_1^{-1}\, J^\perp(1)\right\rangle dt + O(\delta^2) \\
&= -\delta \int_0^1 \left\langle R(/\!\!/_1^{-1}\, J^\perp(1) + O(\delta^2), v)v, /\!\!/_1^{-1}\, J^\perp(1)\right\rangle dt + O(\delta^2) \\
&= -\delta \left\langle R(w', v)v, w'\right\rangle + O(\delta^2) \\
&= -\delta \left\langle R(w, v)v, w\right\rangle + O(\delta^2)
\end{aligned}$$

using that $\|\, /\!\!/_t^{-1} \circ R \circ /\!\!/_t - R\| = O(\delta)$ and $\|w' - w\| = O(\delta\varepsilon)$.

Hence the coefficients of the expansion of $\tilde{L}_2(\varepsilon,\delta)$ in ε are as required. □

We may deduce the expansion in ε of the difference of signed lengths:

Corollary 2.12. *For any vectors $v, w \in T_{x_0}M$ with $\|v\| = 1$, $\|w\| \leqslant 1$ and δ, ε sufficiently small,*

$$\begin{aligned}
&\tilde{L}_2(\varepsilon,\delta) - \tilde{L}_2(\varepsilon,0) \\
&\qquad = \delta\left(1 - \frac{\varepsilon^2}{2}\left[K_{x_0}(v,w)(\|w\|^2 - \langle v,w\rangle^2) + \mathit{Hess}_{x_0}V(v,v)(1-\|w\|^2)\right]\right) \\
&\qquad\quad + O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3) \\
&\qquad = L_1(\varepsilon,\delta) + O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3)
\end{aligned}$$

where the $O(\varepsilon^2\delta^2) + O(\delta\varepsilon^3)$ terms are uniformly bounded in v and w.

Proof. From the preceding proposition, subtracting the two lengths we first obtain almost the expansion above, except the last term is at first only $O(\varepsilon^3)$, but since the difference vanishes is also $O(\delta)$, it must in fact be $O(\delta\varepsilon^3)$, see Remark 2.6. The second equality in the statement of the corollary follows by comparison with Proposition 2.5 since the expansions agree up to the term $O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3)$. □

3 Wasserstein distance approximations

The following two preliminary lemmas relate the Ricci curvature to the sectional curvature by integral averages, thereby providing a bridge between the distance estimates of the previous section and the Wasserstein distance estimates that will follow.

The standard Ricci curvature is usually defined as the contraction of the Riemann curvature tensor and expressed equivalently by the sectional curvature,

$$\mathrm{Ric}(v,v) := \sum_{i=1}^{n} \langle R(v,e_i)e_i, v\rangle = K(v,e_i)(1-\langle v,e_i\rangle^2).$$

The Ricci curvature can equivalently be expressed as an average over a sphere or ball of arbitrary non-zero radius. We shall denote by B_r the ball of radius $r > 0$ in $T_{x_0}M$ centered at the origin and σ the uniform surface measure on the sphere ∂B_ε.

Lemma 3.1 (Ricci curvature as average over a sphere)**.** *For any $x_0 \in M, v \in T_{x_0}M$ with $\|v\| = 1$ and $\varepsilon > 0$, it holds that*

$$\frac{\varepsilon^2}{n} Ric(v,v) = \fint_{\partial B_\varepsilon} K(v,w)(\varepsilon^2 - \langle v,w\rangle^2) d\sigma(w). \tag{3.1}$$

Proof. The standard definition of Ricci curvature

$$\mathrm{Ric}(v,v) := \sum_{i=1}^{n} \langle R(v,e_i)e_i, v\rangle = \sum_{i=1}^{n} K(v,e_i)(1-\langle v,e_i\rangle^2) \tag{3.2}$$

is independent of the choice of orthonormal basis (e_i) of T_pM [Jos17, Chap. 4.3]. We exploit this by integrating the expression on the right over all orthogonal transformations in $SO(n)$ with respect to the Haar measure on $SO(n)$. Denote S^{n-1} the unit sphere and note that it is a homogeneous space with the transitive group action

$$SO(n) \times S^{n-1} \to S^{n-1}, \quad (A,v) \mapsto Av$$

and moreover $S^{n-1} \cong SO(n)/SO(n-1)$.

Let $(e_1, \dots e_n)$ be any orthonormal basis of $T_{x_0}M$. The average integral of the right-hand side of (3.2) over $SO(n)$ is

$$\fint_{SO(n)} \sum_{i=1}^{n} K(v,Ae_i)(1-\langle v,Ae_i\rangle^2) dA = n \fint_{S^{n-1}} K(v,w)(1-\langle v,w\rangle^2) d\sigma(w).$$

The equality follows from the fact that the mapping

$$SO(n) \to S^{n-1}, A \mapsto Ae$$

is surjective for any $e \in S^{n-1}$ and the pushforward of the Haar measure on $SO(n)$ is a multiple of the Riemannian volume measure on S^{n-1}, where we assume that S^{n-1} is equipped with the homogeneous Riemannian structure. This multiplicative constant vanishes when taking the average integral and we obtain (3.1) for $\varepsilon = 1$. The change of variable $\tilde{w} = \varepsilon w$ gives the formula for arbitrary $\varepsilon > 0$. □

Lemma 3.2 (Ricci curvature as average over a ball)**.** *For any $x_0 \in M$, $v \in T_{x_0}M$, $\|v\| = 1$ and $\varepsilon > 0$ it holds that*

$$\frac{\varepsilon^2}{n+2}Ric(v,v) = \fint_{B_\varepsilon} K(v,w)\left(\|w\|^2 - \langle v,w\rangle^2\right) dw \tag{3.3}$$

where dw denotes the Euclidean volume measure on $T_{x_0}M$.

Proof. Let σ denote the standard surface measure on ∂B_r and write the integral on the right as

$$\begin{aligned} &\frac{1}{|B_\varepsilon|}\int_0^\varepsilon \int_{\partial B_r} K(v,w)(\|w\|^2 - \langle v,w\rangle^2) d\sigma(w) dr \\ &= \int_0^\varepsilon \frac{\sigma(\partial B_r)}{|B_\varepsilon|} r^2 \fint_{\partial B_r} K(v,w)\left(1 - \frac{\langle v,w\rangle^2}{\|w\|^2}\right) d\sigma(w) dr \\ &= n\int_0^\varepsilon \varepsilon^{-n} r^{n+1} dr \fint_{\partial B_1} K(v,w)(1 - \langle v,w\rangle^2) d\sigma(w) \\ &= \frac{\varepsilon^2}{n+2}\mathrm{Ric}(v,v) \end{aligned} \tag{3.4}$$

where the second equation follows from the fact that the integrand of the $d\sigma(w)$ integral is independent of r, so the average can be taken over ball of any radius, and the integral over r follows from the fact that

$$|B_r| = \left(\frac{r}{\varepsilon}\right)^n |B_\varepsilon|, \quad \sigma(\partial B_r) = \frac{\partial |B_r|}{\partial r}$$

which gives $\frac{\sigma(\partial B_r)}{|B_\varepsilon|} = n\varepsilon^{-n} r^{n-1}$. The last line follows from the identity (3.1). $\square$

3.1 Density estimates

Lemma 3.2 represented the Ricci curvature as an average over a ball in the tangent space. Since it's more natural to consider measures on the manifold, we will apply Lemma 3.8 to follow to compare uniform measures on geodesic balls in the manifold and the push-forwards of the uniform measure on the ball in the tangent space by the exponential map. The density estimates of this section are auxilliary in proving the Wasserstein distance approximations of the next section.

We shall make repeated use of the following simple observation about determinants, which will appear when using the change of variable formula.

Lemma 3.3. *Let $A(\varepsilon) = (a_{ij}(\varepsilon)) = \delta_{ij} + b_{ij}(\varepsilon)$ be a smooth $n \times n$ matrix-valued function where $b_{ij}(\varepsilon) = O(\varepsilon^2)$. Then*

$$\det A(\varepsilon) = 1 + \sum_i b_{ii}(\varepsilon) + O(\varepsilon^4).$$

Proof. Recall

$$\det A = \sum_{\sigma \in P_n} \mathrm{sgn}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}$$

and consider the permutations σ by two cases:

- if $\sigma \neq \mathrm{id}$ then $\exists i,j$ such that $\sigma(i) \neq i, \sigma(j) \neq j$. Then $a_{i\sigma(i)}a_{j\sigma(j)} = b_{i\sigma(i)}b_{j\sigma(j)} = O(\varepsilon^4)$ and thus also $a_{1\sigma(1)} \dots a_{n\sigma(n)} = O(\varepsilon^4)$.

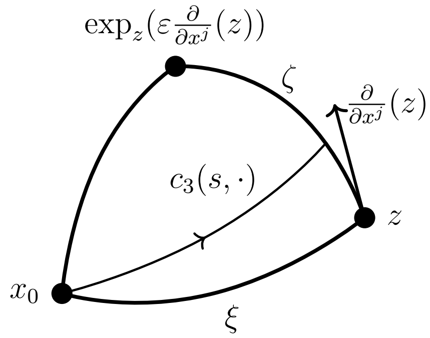

Figure 3: Geodesic variation c_3

- if $\sigma = \mathrm{id}$ then $a_{1\sigma(1)} \dots a_{n\sigma(n)} = a_{11} \dots a_{nn} = \prod_i (1 + b_{ii}(\varepsilon))^n = 1 + \sum_i b_{ii}(\varepsilon) + O(\varepsilon^4)$.

Summing over all permutations σ, we see that the constant term is 1, the $O(\varepsilon)$ term vanishes, the ε^2 term is $\sum_i b_{ii}(\varepsilon)$ and the rest is of order ε^4. $\square$

Remark 3.4. Similarly, if $A(\varepsilon, \delta) = (a_{ij}(\varepsilon, \delta)) = \delta_{ij} + b_{ij}(\varepsilon, \delta)$ is such that $b_{ij} = O(\delta\varepsilon)$ then

$$\det A(\varepsilon, \delta) = 1 + \sum_i b_{ii}(\varepsilon, \delta) + O(\delta^2\varepsilon^2).$$

Lemma 3.5. *In normal coordinates $(x^1, \dots, x^n)$ at $x_0 \in M$, it holds that*

$$g_{ij}(z) = \delta_{ij} + O(\varepsilon^2) \quad \forall z \in B_\varepsilon(x_0)$$

and hence $\det(g_{ij}(z)) = 1 + O(\varepsilon^2)$.

Proof. Writing $z = \exp_{x_0}(sw)$ for some $w \in T_{x_0}M$, $|w| = 1$ and $s > 0$, we expand

$$\begin{aligned} g_{ij}(z) &= g_{ij}(\exp_{x_0}(sw)) \\ &= g_{ij}(x_0) + s\frac{\partial}{\partial x^k} g_{ij}(x_0) w^k + r_{ij}(z, x_0) \\ &= \delta_{ij} + r_{ij}(z, x_0) \end{aligned}$$

where $r_{ij}(\cdot, x_0) = O(\varepsilon^2)$ is the remainder and we applied the property of normal coordinates that $g_{ij}(x_0) = \delta_{ij}$ and $\frac{\partial}{\partial x^k} g_{ij}(x_0) = 0$ (see [Jos17, Chapter 1.4]). $\square$

Lemma 3.6. *In local coordinates $(x^1, \dots, x^n)$ at $x_0 \in M$, it holds that*

$$\frac{\partial(\exp_{x_0}^{-1})^i}{\partial x^j}(z) = \delta_{ij} + O(\varepsilon^2) \quad \forall z \in B_\varepsilon(x_0)$$

and hence

$$\det\left(\frac{\partial(\exp_{x_0}^{-1})^i}{\partial x^j}(z)\right)_{ij} = 1 + O(\varepsilon^2) \quad \forall z \in B_\varepsilon(x_0)$$

Proof. Let $z \in B_\varepsilon^M(x_0)$ be fixed and define the two geodesics

$$\xi(t) := \exp_{x_0}\left(t \exp_{x_0}^{-1}(z)\right), \quad \zeta(s) := \exp_z\left(s\frac{\partial}{\partial x^j}(z)\right).$$

To compute $\frac{\partial(\exp_{x_0}^{-1})^i}{\partial x^j}$ we consider the variation of the geodesic ξ given by

$$c_3(s,t) := \exp_{x_0}\left(t\exp_{x_0}^{-1}(\zeta(s))\right)$$

(see Fig. 3) with the associated Jacobi field along ξ,

$$J(t) := \frac{\partial}{\partial s}\bigg|_{s=0} c_3(s,t)$$

which satisfies the Jacobi equation with boundary conditions,

$$\frac{D^2}{dt^2}J(t) = -R(J(t),\dot{\xi})\dot{\xi}, \quad J(0) = 0, \quad J(1) = \frac{\partial}{\partial x^j}(z).$$

Denote $E_1(\xi(t)) := \dot{\xi}(t)$ and consider $E_2, \dots, E_n$ a completion of E_1 to an orthonormal frame field along ξ. Then $J(t) = J^i(t)E_i(t)$ for some smooth $J^i : [0,1] \to \mathbb{R}$. Denoting $\dot{J}^i$, $\ddot{J}^i$ the derivatives in t, the Jacobi equation reads

$$\ddot{J}(t)^i = -J^j(t)\|\dot{\xi}\|^2 \langle R(E_j,E_1)E_1,E_i\rangle = -J^j(t)\|\dot{\xi}\|^2 R^i_{1j1}. \tag{3.5}$$

It is clear that the initial and final conditions with respect to the frame (E_i) are $J^i(0) = 0$ and $J^i(1) = \delta^{ij}$.

Moreover, for any fixed $z \in B_\varepsilon(x_0)$,

$$\begin{aligned}
\frac{D}{dt}\bigg|_{t=0} J(t) &= \frac{D}{\partial t}\frac{\partial}{\partial s}\bigg|_{s=0,t=0} c_3(s,t) = \frac{D}{\partial s}\bigg|_{s=0}\frac{\partial}{\partial t}\bigg|_{t=0} c_3(s,t) \\
&= \frac{D}{ds}\bigg|_{s=0} \exp_{x_0}^{-1}(c_3(s,1)) = \frac{d}{ds}\bigg|_{s=0} \exp_{x_0}^{-1}(c_3(s,1)) \\
&= D\exp_{x_0}^{-1}\left(\frac{d}{ds}\bigg|_{s=0} c_3(s,1)\right) = D\exp_{x_0}^{-1}\left(\frac{\partial}{\partial x^j}(z)\right) \\
&= \frac{\partial \exp_{x_0}^{-1}}{\partial x^j}(z).
\end{aligned}$$

Integrating the Jacobi equation (3.5) twice yields

$$J^i(1) - J^i(0) - \dot{J}^i(0) = -\int_0^1\int_0^t J^j(s)\|\dot{\xi}(s)\|^2 R^i_{1j1}(\xi(s))dsdt$$

and hence

$$\frac{\partial(\exp_{x_0}^{-1})^i}{\partial x^j}(z) = \dot{J}^i(0) = \delta^{ij} + \int_0^1\int_0^t J^j(s)\|\dot{\xi}(s)\|^2 R^i_{1j1}(\xi(s))dsdt = \delta^{ij} + O(\varepsilon^2)$$

because $\|\dot{\xi}\| = O(\varepsilon)$. $\square$

Notation 3.7. For $x \in M$, denote

$$\tilde{B}_\varepsilon(x) \subset T_xM, \quad B_\varepsilon(x) := \exp_x\left(\tilde{B}_\varepsilon(x)\right) \subset M$$

the ε-balls in T_xM and M, respectively, and the probability measures

$$\begin{aligned}
d\tilde{\mu}_x^\varepsilon(w) &= \frac{\mathbb{1}_{\tilde{B}_\varepsilon(x)}(w)}{|\tilde{B}_\varepsilon(x)|}dw && \text{on } T_xM, \\
\bar{\mu}_x^\varepsilon(z) &= (\exp_x)_*\tilde{\mu}_x^\varepsilon(z) && \text{on } M, \\
d\tilde{\nu}_x^\varepsilon(w) &= \frac{e^{-V(\exp_x w)}}{\int_{\tilde{B}_\varepsilon(x)} e^{-V(\exp_x w')}d\tilde{\mu}_x^\varepsilon(w')}d\tilde{\mu}_x^\varepsilon(w) && \text{on } T_xM, \\
\bar{\nu}_x^\varepsilon(z) &= (\exp_x)_*\tilde{\nu}_x^\varepsilon(z) && \text{on } M, \\
d\nu_x^\varepsilon(z) &= \frac{e^{-V(z)}}{\int_{B_\varepsilon(x)} e^{-V(z')}d\mu_x^\varepsilon(z')}d\mu_x^\varepsilon(z) && \text{on } T_xM.
\end{aligned}$$

The auxilliary measures $\bar{\mu}_x^\varepsilon, \bar{\nu}_x^\varepsilon$ will be instrumental in approximating $W_1(\nu_{x_0}^\varepsilon, \nu_y^\varepsilon)$.

Lemma 3.8. *For δ, ε sufficiently small, $\bar{\mu}_x^\varepsilon$ and μ_x^ε are equivalent measures on M, supported on $B_\varepsilon(x)$ with mutual density*

$$\frac{d\bar{\mu}_x^\varepsilon}{d\mu_x^\varepsilon}(z) = 1 + h(z,x) \tag{3.6}$$

where $h : M \times M \to \mathbb{R}$ is an a. e. smooth function such that $h(\cdot, x) = O(\varepsilon^2)$ for every $x \in M$ and $\int h(z,x) d\mu_x(z) = 0$.

Proof. The uniform measures on $\tilde{B}_\varepsilon(x)$ and $B_\varepsilon(x)$ can be represented as differential forms,

$$d\tilde{\mu}_x^\varepsilon := \tilde{C}\mathbb{1}_{B_\varepsilon(0)} de^1 \wedge \ldots \wedge de^n, \quad d\mu_x^\varepsilon := C\mathbb{1}_{B_\varepsilon^M(x)} \sqrt{\det(g_{ij})} dx^1 \wedge \ldots \wedge dx^n, \tag{3.7}$$

where $(e^1, \ldots, e^n)$ are Euclidean coordinates at $0 \in T_xM$, $(x^1, \ldots, x^n)$ are normal coordinates at $x_0 \in M$ and $\tilde{C} := \left|\tilde{B}_\varepsilon(x)\right|^{-1}$, $C := \mathrm{vol}(B_\varepsilon(x))^{-1}$ are the normalizing constants. Denoting $(\exp_x^{-1})^i = e^i \circ \exp_x^{-1}$, the push-forward $\bar{\mu}_x^\varepsilon$ is then

$$\begin{aligned} d\bar{\mu}_x^\varepsilon(z) = (\exp_x)_* d\tilde{\mu}_x^\varepsilon(z) &= \tilde{C}\mathbb{1}_{B_\varepsilon(x)}(z) d(e^1 \circ \exp_x^{-1}) \wedge \ldots \wedge d(e^n \circ \exp_x^{-1}) \\ &= \tilde{C}\mathbb{1}_{B_\varepsilon(x)}(z) \det\left(\frac{\partial(\exp_x^{-1})^i}{\partial x^j}(z)\right) dx^1 \wedge \ldots \wedge dx^n. \end{aligned} \tag{3.8}$$

See [Lee13, Chap. 9] for details on this local coordinate expression. Comparing (3.7) and (3.8), since the determinants are non-vanishing we see that the two measures are equivalent and their mutual density is given by

$$\frac{d\bar{\mu}_x^\varepsilon}{d\mu_x^\varepsilon}(z) = \frac{\tilde{C}}{C} \det\left(\frac{\partial(\exp_x^{-1})^i}{\partial x^j}(z)\right) \times \det(g_{ij}(z))^{-\frac{1}{2}}. \tag{3.9}$$

Smoothness in x_0 is clear and we know from Lemmas 3.5 and 3.6 that the determinants on the right are $1 + O(\varepsilon^2)$ for every $x \in M$. The expansion $(1+a)^{-\frac{1}{2}} = 1 - \frac{1}{2}a + O(a^2)$ implies also $\det(g_{ij})^{-\frac{1}{2}} = 1 + O(\varepsilon^2)$.

It remains to see the ratio of the normalizing constants satisfies $\frac{\tilde{C}}{C} = 1 + O(\varepsilon^2)$ as a smooth function of the point $x \in M$. Integrating both sides of (3.9) over $B_\varepsilon(x)$ with respect to μ_x^ε we obtain

$$\bar{\mu}_x^\varepsilon(B_\varepsilon(x)) = \frac{\tilde{C}}{C} \mu_x^\varepsilon(B_\varepsilon(x))(1 + O(\varepsilon^2)). \tag{3.10}$$

The probability measures $\bar{\mu}_x^\varepsilon, \mu_x^\varepsilon$ are both supported on $B_\varepsilon(x)$ so $\frac{\bar{\mu}_x^\varepsilon(B_\varepsilon^M(x))}{\mu_x^\varepsilon(B_\varepsilon^M(x))} = 1$ and rearrangement of (3.10) gives $\frac{C}{\tilde{C}} = 1 + O(\varepsilon^2)$ as a smooth function of x. We obtain the same for the inverse ratio $\frac{\tilde{C}}{C}$ by the expansion $\frac{1}{1+z} = 1 - z + O(z^2)$.
Since all factors in (3.9) are $1 + O(\varepsilon^2)$, the same holds for their product, i.e.

$$\frac{d\bar{\mu}_x^\varepsilon}{d\mu_x^\varepsilon}(u) = 1 + O(\varepsilon^2). \tag{3.11}$$

Labelling the $O(\varepsilon^2)$ term as $h(\cdot, x)$, we have

$$1 = \int \frac{d\bar{\mu}_x^\varepsilon}{d\mu_x^\varepsilon}(z) d\mu_x^\varepsilon(z) = 1 + \int h(z,x) d\mu_x^\varepsilon(z) \tag{3.12}$$

implying that $\int h(z,x) d\mu_x^\varepsilon(z) = 0$. $\square$

Similarly to Lemma 3.8, we have:

Lemma 3.9. *For every $x \in M$,*

$$\frac{d\bar{\nu}_x^\varepsilon}{d\nu_x^\varepsilon}(z) = 1 + h(z,x)$$

where $h : M \times M \to \mathbb{R}$ is an a. e. smooth function such that $h(\cdot,x) = O(\varepsilon^2)$ for every $x \in M$ and $\int h(z,x)d\bar{\mu}_x(z) = \int h(z,x)d\bar{\nu}_x(z) = 0$.

Proof. By Lemma 3.8,

$$\frac{d\bar{\nu}_x^\varepsilon}{d\nu_x^\varepsilon}(z) = \frac{d\bar{\mu}_x^\varepsilon}{d\mu_x^\varepsilon}(z) = 1 + h(z,x).$$

□

Lemma 3.10. *For any $x \in M$, the mutual densities expand as*

$$\frac{d\bar{\nu}_x^\varepsilon}{d\bar{\mu}_x^\varepsilon}(z) = \mathbb{1}_{B_\varepsilon(x)}(z)(1 - \langle \nabla V(x), \exp_x^{-1}(z) \rangle + h_1(z,x)),$$
$$\frac{d\nu_x^\varepsilon}{d\mu_x^\varepsilon}(z) = \mathbb{1}_{B_\varepsilon(x)}(z)(1 - \langle \nabla V(x), \exp_x^{-1}(z) \rangle + h_2(z,x))$$

where $h_i : M \times M \to \mathbb{R}$ are a.e. smooth functions such that $h_i(\cdot,x) = O(\varepsilon^2)$ for every $x \in M$ and $\int h_i(z,x)d\bar{\mu}_x^\varepsilon(z) = 0$ for $i = 1,2$.

Proof. From Notation 3.7,

$$\frac{d\bar{\nu}_x^\varepsilon}{d\bar{\mu}_x^\varepsilon}(z) = \frac{d(\exp_x)_*\tilde{\nu}_x^\varepsilon}{d(\exp_x)_*\tilde{\mu}_x^\varepsilon}(z) = \frac{d\tilde{\nu}_x^\varepsilon}{d\tilde{\mu}_x^\varepsilon}(\exp_x^{-1} z) = \frac{e^{-V(z)}}{\int_{B_\varepsilon(x)} e^{-V(z)}d\bar{\mu}_x^\varepsilon(z)}. \tag{3.13}$$

Expand the numerator in 3.13 as

$$\begin{aligned} e^{-V(z)} &= e^{-V(x)}e^{V(x)-V(z)} \\ &= e^{-V(x)}(1 - \langle \nabla V(x), \exp_x^{-1}(z) \rangle + O(\varepsilon^2)) \end{aligned}$$

Similarly for the denominator,

$$e^{-V(x)} \int_{B_\varepsilon(x)} (1 - \langle \nabla V(x), \exp_x^{-1}(z') \rangle + O(\varepsilon^2))d\bar{\mu}_x^\varepsilon(z') = e^{-V(x)} \left(1 + O(\varepsilon^2)\right)$$

where the ∇V term vanishes because

$$\int_{B_\varepsilon(x)} \langle \nabla V(x), \exp_x^{-1}(z') \rangle \, d\bar{\mu}_x^\varepsilon(z') = \langle \nabla V(x), \int_{B_\varepsilon(0)} w d\tilde{\mu}_x^\varepsilon(w) \rangle = 0. \tag{3.14}$$

Cancelling the factor $e^{-V(x)}$, the density (3.13) can thus be written as

$$\frac{d\bar{\nu}_x^\varepsilon}{d\bar{\mu}_x^\varepsilon}(z) = \frac{1 - \langle \nabla V(x), \exp_x^{-1}(z) \rangle + O(\varepsilon^2)}{1 + O(\varepsilon^2)} = 1 - \langle \nabla V(x), \exp_x^{-1}(z) \rangle + O(\varepsilon^2)$$

Writing the $O(\varepsilon^2)$ term explicitly as $h(\cdot,x)$, we have

$$\begin{aligned} 1 = \int \frac{d\bar{\nu}_x^\varepsilon}{d\bar{\mu}_x^\varepsilon}(z)d\bar{\mu}_x(z) &= \int (1 - \langle \nabla V(x), \exp_x^{-1}(z) \rangle + h(z,x))d\bar{\mu}_x(z) \\ &= 1 + \int h(z,x)d\bar{\mu}_x(z) \end{aligned}$$

since the ∇V term vanishes again, and this implies $\int h(z,x)d\bar{\mu}_x(z) = 0$.

The expansion of the density $\frac{d\nu_x^\varepsilon}{d\mu_x^\varepsilon}(z)$ is obtained the same way, starting from

$$\frac{d\nu_x^\varepsilon}{d\mu_x^\varepsilon}(z) = \frac{e^{-V(z)}}{\int_{B_\varepsilon(x)} e^{-V(z)}d\mu_x^\varepsilon(z)}.$$

□

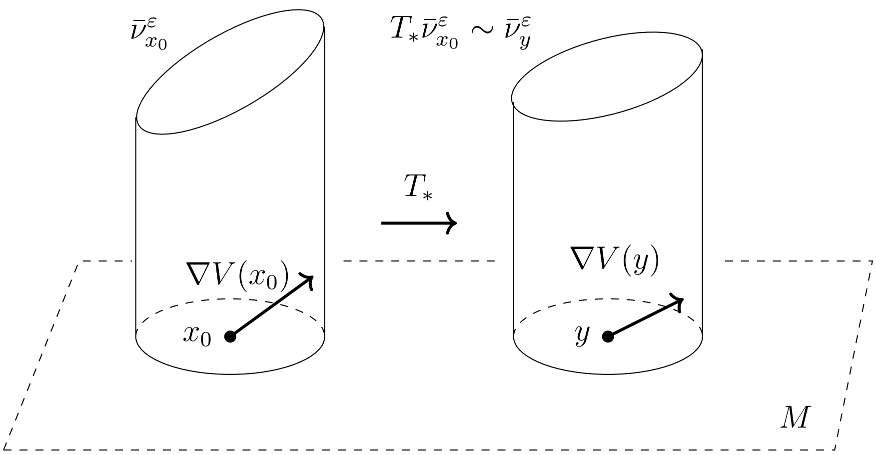

Figure 4: For small $\varepsilon > 0$, the density of the non-uniform measures $\bar{\nu}^{\varepsilon}_{x_0}, \bar{\nu}^{\varepsilon}_y$ resemble cylinders with a slant top, the slanting being described by the gradient of V. This motivates the choice of T as a scaled difference of the gradients of the tops, together with parallel translation from $B_\varepsilon(x_0)$ to $B_\varepsilon(y)$.

3.2 An approximate transport plan

Recall we wish to obtain an expansion of $W_1(\nu^{\varepsilon}_{x_0}, \nu^{\varepsilon}_y)$ in ε and δ. For this we propose an "approximate" transport map $T : B_\varepsilon(x_0) \to B_\varepsilon(y)$ which realizes the distance from Proposition 2.5 in the sense that $d(z, Tz) = L_1(\varepsilon, \delta)$. We first define such a map using a map between tangent spaces $\tilde{T} : \tilde{B}_\varepsilon(x_0) \to \tilde{B}_\varepsilon(y)$.

Definition 3.11. Define the map $\tilde{T} : \tilde{B}_\varepsilon(x_0) \to \tilde{B}_\varepsilon(y)$ as

$$\tilde{T}(w) := /\!\!/_1\, w - \frac{1}{2}(\varepsilon^2 - \|w\|^2)(\nabla V(y) - /\!\!/_1\, \nabla V(x_0))). \tag{3.15}$$

Remark 3.12. The choice of this map is motivated by Lemma 3.10, which says that for small $\varepsilon > 0$, the densities of $\bar{\nu}^{\varepsilon}_{x_0}$ and $\bar{\nu}^{\varepsilon}_y$ are well approximated by affine functions on the respective supports. See Fig. 4 for illustration. Any near-optimal transport of mass from $\bar{\nu}^{\varepsilon}_{x_0}$ to $\bar{\nu}^{\varepsilon}_y$ should thus consist of a translation from $B_\varepsilon(x_0)$ to $B_\varepsilon(y)$ together with a realignment of mass according to the difference of the gradients $/\!\!/_1^{-1}\, \nabla V(y) - \nabla V(x_0)$, scaled with the distance from the centre of the support. This is because points on the boundary of $B_\varepsilon(x_0)$ should only be translated, while points near x_0 should be translated as well as moved in the direction of $-(/\!\!/_1^{-1}\, \nabla V(y) - V(x_0))$. The above scaling by the factor $\frac{1}{2}(\varepsilon^2 - \|w\|^2)$ turns out to define a good enough approximate transport map.

Remark 3.13. In the following, the determinant of a linear map will always be understood with respect to a pair of orthonormal bases of the domain and codomain.

Lemma 3.14. *For $0 < \varepsilon < 1$ and $\delta > 0$ sufficiently small, the map $\tilde{T}$ is a well-defined diffeomorphism.*

Proof. Define $\hat{T} : \tilde{B}_\varepsilon(x_0) \to \tilde{B}_\varepsilon(x_0)$ as

$$\hat{T}(w) := w - \frac{1}{2}(\varepsilon^2 - \|w\|^2)(/\!\!/_1^{-1}\, \nabla V(y) - \nabla V(x_0)))$$

so that $\tilde{T} = /\!\!/_1 \circ \hat{T}$. Since $/\!\!/_1$: $\tilde{B}_\varepsilon(x_0) \to \tilde{B}_\varepsilon(y)$ is a diffeomorphism, it is sufficient to show that $\hat{T}$ is a diffeomorphism. We show the latter by finding the smooth inverse $\hat{T}^{-1}$. To further simplify notation we write

$$\hat{T}(w) = w - \frac{1}{2}(\varepsilon^2 - \|w\|^2)\alpha\mathbf{e}$$

with

$$\mathbf{e} := \frac{/\!/_1^{-1} \nabla V(y) - \nabla V(x_0)}{\| /\!/_1^{-1} \nabla V(y) - \nabla V(x_0))\|}, \quad \alpha := \| /\!/_1^{-1} \nabla V(y) - \nabla V(x_0))\| = O(\delta) > 0.$$

Decompose any vector $w \in \tilde{B}_\varepsilon(x_0)$ as $w = u + r\mathbf{e}$ with $u \perp \mathbf{e}$ and $-\sqrt{\varepsilon^2 - \|w\|^2} \leqslant r \leqslant \sqrt{\varepsilon^2 - \|w\|^2}$ so that

$$\hat{T}(u + r\mathbf{e}) = u + r\mathbf{e} - \frac{1}{2}(\varepsilon^2 - \|u\|^2 - r^2)\alpha\mathbf{e} = u + h_u(r)\mathbf{e}$$

where $h_u(r) := r - \frac{1}{2}(\varepsilon^2 - \|u\|^2 - r^2)\alpha$. It is clear that

$$h_u(\pm\sqrt{\varepsilon^2 - \|u\|^2}) = \pm\sqrt{\varepsilon^2 - \|u\|^2}, \quad \frac{dh_u}{dr}(r) = 1 + \alpha r$$

Hence as long as $\alpha\varepsilon > -1$, the map

$$h_u : [-\sqrt{\varepsilon^2 - \|u\|^2}, \sqrt{\varepsilon^2 - \|u\|^2}] \to [-\sqrt{\varepsilon^2 - \|u\|^2}, \sqrt{\varepsilon^2 - \|u\|^2}]$$

is a diffeomorphism. Since the decomposition $w = u + r\mathbf{e}$ is unique and $(u, r) \mapsto h_u(r)$ is smooth, we conclude that $\hat{T}$ is a diffeomorphism. A fortiori, solving $h_u(r) = s$ for r we may obtain the inverse explicitly as

$$\hat{T}^{-1}(u + s\mathbf{e}) = u + \frac{1}{\alpha}\left(-1 + \sqrt{1 + \alpha^2(\varepsilon^2 - \|u\|^2) + 2\alpha y}\right)\mathbf{e},$$

completing the proof. □

Lemma 3.15. *The Jacobian of $\tilde{T}$ satisfies for any $w \in \tilde{B}_\varepsilon(x_0)$*

$$\det D_w\tilde{T} = 1 + \left\langle /\!/_1^{-1} \nabla V(y) - \nabla V(x_0), w\right\rangle + O(\delta^2\varepsilon^2) \tag{3.16}$$

where the $O(\delta^2\varepsilon^2)$ term is uniformly bounded in w.

Proof. Consider $(e_i)_{i=1}^n$ an orthonormal basis of $T_{x_0}M$ and $(/\!/_1\, e_i)_{i=1}^n$ the corresponding parallel-translated orthonormal basis of T_yM. With respect to these bases, the components of $D_w\tilde{T}$ at any $w \in \tilde{B}_\varepsilon(x_0)$ are expressed as

$$\begin{aligned}\left\langle D_w\tilde{T}(e_i), /\!/_1\, e_j\right\rangle &= \langle /\!/_1\, e_i, /\!/_1\, e_j\rangle + \langle /\!/_1\, w, /\!/_1\, e_i\rangle \langle \nabla V(y) - /\!/_1\, \nabla V(x_0), /\!/_1\, e_j\rangle \\ &= \delta_{ij} + \langle w, e_i\rangle \left\langle /\!/_1^{-1} \nabla V(y) - \nabla V(x_0), e_j\right\rangle.\end{aligned} \tag{3.17}$$

Then

$$\det D_w\tilde{T} = \det\left(\delta_{ij} + \langle w, e_i\rangle \left\langle /\!/_1^{-1} \nabla V(y) - \nabla V(x_0), e_j\right\rangle\right)_{ij} \tag{3.18}$$

where δ_{ij} stands for the Kronecker delta. Since $\langle w, e_i\rangle \left\langle /\!/_1^{-1} \nabla V(y) - \nabla V(x_0), e_j\right\rangle = O(\delta\varepsilon)$, we can deduce (see Remark 3.4) that

$$\begin{aligned}\det D_w\tilde{T} &= 1 + \sum_i \langle w, e_i\rangle \left\langle /\!/_1^{-1} \nabla V(y) - \nabla V(x_0), e_i\right\rangle + O(\varepsilon^2\delta^2) \\ &= 1 + \left\langle w, /\!/_1^{-1} \nabla V(y) - \nabla V(x_0)\right\rangle + O(\varepsilon^2\delta^2)\end{aligned} \tag{3.19}$$

as required. □

Remark 3.16. It follows from the definition (3.15) that

$$\tilde{T}w = /\!\!/_1\, w + O(\delta\varepsilon^2) \quad \forall w \in \tilde{B}_\varepsilon(x_0) \tag{3.20}$$

and

$$\tilde{T}^{-1}\tilde{w} = /\!\!/_1^{-1}\, \tilde{w} + O(\delta\varepsilon^2) \quad \forall \tilde{w} \in \tilde{B}_\varepsilon(y). \tag{3.21}$$

As a consequence of (3.16) and (3.21), we also have

$$\begin{aligned} \det D_{\tilde{w}}\tilde{T}^{-1} &= 1 - \left\langle \tilde{T}^{-1}\tilde{w}, /\!\!/_1^{-1}\, \nabla V(y) - \nabla V(x_0) \right\rangle + O(\varepsilon^2\delta^2) \\ &= 1 - \langle \tilde{w}, \nabla V(y) - /\!\!/_1\, \nabla V(x_0)\rangle + O(\delta^2\varepsilon^2). \end{aligned} \tag{3.22}$$

The exponential map is locally a diffeomorphism and $B_\varepsilon(x) = \exp_x(\tilde{B}_\varepsilon(x))$. This allows us to define the following diffeomorphism on the manifold, which we shall employ as our *approximate* transport map between ν_{x_0} and ν_y.

Definition 3.17. Define the map $T : B_\varepsilon(x_0) \to B_\varepsilon(y)$ as

$$Tz := \exp_y(\tilde{T} \exp_{x_0}^{-1}(z)) \tag{3.23}$$

Remark 3.18. We explain the relationship between the map T and the distance estimates of Section 2. In the notation introduced in the context of geodesic variations c_1, c_2, if for any unit vector $w \in T_{x_0}M$ we label $z := \exp_{x_0}(\varepsilon w)$ then

$$Tz = \exp_y(\varepsilon\tilde{T}w) = \exp_y(\varepsilon\, /\!\!/_1\, w')$$

where w' is given by (2.3). In the notation for geodesics of the variation (2.4) we have $z = \eta(\varepsilon)$ and $Tz = \theta(\varepsilon)$, and thus in particular

$$\begin{aligned} d(z, Tz) &= d(\eta(\varepsilon), \theta(\varepsilon)) = L_1(\varepsilon, \delta), \\ d(z, p(z)) &= d(p(\eta(\varepsilon)), \eta(\varepsilon)) = L_2(\varepsilon, 0), \\ d(Tz, p(Tz)) &= d(p(\theta(\varepsilon)), \theta(\varepsilon)) = L_2(\varepsilon, \delta). \end{aligned}$$

where $p : M \to E$ is the projection map (2.11).

While T is not an exact transport map from $\nu_{x_0}^\varepsilon$ to ν_y^ε, it turns out that the relevant Wasserstein distances may still be approximated using the pushforward measure $T_*\nu_{x_0}^\varepsilon$ in place of ν_y^ε, and the distance $W_1(\nu_{x_0}^\varepsilon, T_*\nu_{x_0}^\varepsilon)$ can in turn be approximated by $W_1(\bar{\nu}_{x_0}^\varepsilon, T_*\bar{\nu}_{x_0}^\varepsilon)$. This is formalized below in Proposition 3.23 as the proof requires preliminaries to follow. The merit of such approximation is that $W_1(\bar{\nu}_{x_0}^\varepsilon, T_*\bar{\nu}_{x_0}^\varepsilon)$ can be computed using the distance estimates obtained in the previous section. The upper bound for $W_1(\bar{\nu}_{x_0}^\varepsilon, T_*\bar{\nu}_{x_0}^\varepsilon)$ is established from the concrete transport plan T and the lower bound is computed by using a concrete 1-Lipschitz function.

Notation 3.19. Denote $z := \exp_{x_0}(\varepsilon w)$ so that $Tz = \exp_y(\varepsilon\, /\!\!/_1\, w')$ where w' is prescribed by (2.3), and let r_0 be the uniform injectivity radius on a fixed, compact neighbourhood of x_0. Recall that in the context of Proposition 2.11 we defined the submanifold

$$E := \{\exp_{x_0}(w) : w \in T_{x_0}M, \langle w, v\rangle = 0\} \subset M,$$

the projection $p : M \to E$, $p(x) := \operatorname{argmin}_{z\in E} d(x, z)$, $\nu \in \Gamma(TE^\perp)$ a unit normal vector field with $\nu(x_0) = v$ and the signed distance to projection, $f : B_{r_0/3}(x_0) \to \mathbb{R}$,

$$f(z) := \operatorname{sign}(\langle \exp_{p(z)}^{-1}(z), \nu(z)\rangle) d(z, p(z)) \tag{3.24}$$

Note that $f(Tz) = \tilde{L}_2(\varepsilon, \delta)$ and $f(z) = \tilde{L}_2(\varepsilon, 0)$ where $\tilde{L}_2$ was defined by (2.14), the signed length of the geodesic realizing the distance to projection.

Lemma 3.20. *The signed distance to projection is 1-Lipschitz and for any vectors $v, w \in T_{x_0}M$ with $\|v\| = 1, \|w\| \leqslant 1$ and $\delta, \varepsilon > 0$ sufficiently small,*

$$
\begin{aligned}
f(Tz) - f(z) = \delta \left(1 - \frac{\varepsilon^2}{2} K_{x_0}(v,w)(\|w\|^2 - \langle v,w\rangle^2) - \frac{\varepsilon^2}{2} \mathit{Hess}_{x_0} V(v,v)(1 - \|w\|^2)\right) \\
+ O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3)
\end{aligned}
\tag{3.25}
$$

where $O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3)$ is uniformly bounded in z.

Proof. We first show that f is 1-Lipschitz. Since $z \mapsto \langle \exp_{p(z)}^{-1}(z), \nu(z)\rangle$ is continuous, $E \cap B_{r_0/3}(x_0)$ is the boundary between the two connected components

$$
\{z \in B_{r_0/3}(x_0) : \langle \exp_{p(z)}^{-1}(z), \nu(z)\rangle > 0\}, \quad \{z \in B_{r_0/3}(x_0) : \langle \exp_{p(z)}^{-1}(z), \nu(z)\rangle < 0\}.
\tag{3.26}
$$

First, consider $z_1, z_2 \in B_{r_0/3}(x_0)$ such that $\left\langle \exp_{x_0}^{-1}(z_1), v\right\rangle \left\langle \exp_{x_0}^{-1}(z_2), v\right\rangle \geqslant 0$. This means that z_1, z_2 lie in the same connected component of $B_{r_0/3}(x_0)$. Then by the triangle inequality,

$$
|f(z_1) - f(z_2)| = |d(z_1, p(z_1)) - d(z_2, p(z_2))| \leqslant d(z_1, z_2)
$$

Second, consider $z_1, z_2 \in B_{r_0/3}(x_0)$ such that

$$
\left\langle \exp_{x_0}(z_1), v\right\rangle \left\langle \exp_{x_0}(z_2), v\right\rangle < 0
$$

meaning that z_1, z_2 lie in distinct components of $B_{r_0/3}(x_0)$. Suppose $\xi : [0,1] \to M$ is a length-minimizing geodesic connecting z_1 and z_2. Then

$$
\begin{aligned}
d(\xi(t), x_0) &\leqslant d(\xi(t), z_1) + d(z_1, x_0) \leqslant d(z_2, z_1) + d(z_1, x_0) \\
&\leqslant d(z_2, x_0) + 2d(z_1, x_0) < \frac{1}{3} r_0 + \frac{2}{3} r_0 = r_0.
\end{aligned}
$$

This means that the geodesic realising the distance between z_1 and z_2 lies in $B_{r_0/3}(x_0)$, and hence must pass through $E \cap B_{r_0/3}(x_0)$ because it is the boundary between the two connected components (3.26). Therefore, there exists $z_0 \in E \cap B_{r_0/3}(x_0)$ such that

$$
|f(z_1) - f(z_2)| = d(z_1, p(z_1)) + d(z_2, p(z_2)) \leqslant d(z_1, z_0) + d(z_2, z_0) = d(z_1, z_2).
$$

Finally, by Corollary 2.12,

$$
\begin{aligned}
f(Tz) - f(z) &= \tilde{L}_2(\varepsilon, \delta) - \tilde{L}_2(\varepsilon, 0) \\
&= \delta \left(1 - \frac{\varepsilon^2}{2} \left[K_{x_0}(v,w)(\|w\|^2 - \langle v,w\rangle^2) + \mathrm{Hess}_{x_0} V(v,v)(1 - \|w\|^2)\right]\right) \\
&\quad + O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3).
\end{aligned}
$$

□

The signed distance to projection leads to the correct lower bound for $W_1(\bar{\nu}^{\varepsilon}_{x_0}, T_* \bar{\nu}^{\varepsilon}_{x_0})$:

Lemma 3.21. *For f defined by (3.24) and all $v \in T_{x_0}M$ unit vectors and $\delta, \varepsilon > 0$ small enough,*

$$
\begin{aligned}
\int_M (f(Tz) - f(z)) d\bar{\nu}^{\varepsilon}_{x_0}(z) = \delta \left(1 - \frac{\varepsilon^2}{2(n+2)} (\mathit{Ric}_{x_0}(v,v) + 2\mathit{Hess}_{x_0} V(v,v))\right) \\
+ O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3)
\end{aligned}
\tag{3.27}
$$

where $O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3)$ is uniformly bounded in z.

Proof. Change the variable of integration to $w = \exp_{x_0}^{-1}(z)$ and recall $\bar{\mu}_{x_0}^{\varepsilon} := (\exp_{x_0})_* \tilde{\mu}_{x_0}^{\varepsilon}$. Then the integral on the left can be written by the preceding Lemma as

$$\int_{T_{x_0}M} \delta \left(1 - \frac{1}{2} K_{x_0}(v,w)(\|w\|^2 - \langle v,w\rangle^2) - \frac{1}{2}\mathrm{Hess}_{x_0} V(v,v)(\varepsilon^2 - \|w\|^2)\right) d\tilde{\mu}_{x_0}^{\varepsilon}(w) \\ + O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3).$$

Integrating the Hessian term in spherical coordinates, we have

$$\begin{aligned} \frac{1}{2}\int_{T_{x_0}M} (\varepsilon^2 - \|w\|^2) d\tilde{\mu}_{x_0}^{\varepsilon}(w) &= \frac{\varepsilon^2}{2} - \frac{1}{2|B_\varepsilon(x_0)|}\int_0^\varepsilon r^2 |\partial B_r(x_0)| dr \\ &= \frac{\varepsilon^2}{2} - \frac{n\varepsilon^2}{2(n+2)} = \frac{\varepsilon^2}{n+2} \end{aligned}$$

and the sectional curvature term integrates to give the Ricci term as per Lemma 3.2. $\square$

Theorem 3.22. *For any point $x_0 \in M$, vector $v \in T_{x_0}M$ with $\|v\| = 1$ and $\delta, \varepsilon > 0$ sufficiently small,*

$$W_1(\bar{\nu}_{x_0}^{\varepsilon}, T_*\bar{\nu}_{x_0}^{\varepsilon}) = \delta\left(1 - \frac{\varepsilon^2}{2(n+2)}\left(Ric_{x_0}(v,v) + 2Hess_{x_0}V(v,v)\right)\right) + O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3) \tag{3.28}$$

Proof. Using the transport map (3.15), we have the upper bound

$$\begin{aligned} W_1(\bar{\nu}_{x_0}^{\varepsilon}, T_*\bar{\nu}_{x_0}^{\varepsilon}) &\leqslant \int_M d(Tz,z) d\bar{\nu}_{x_0}^{\varepsilon}(z) \\ &= \int_{T_{x_0}M} \delta\left(1 - \frac{1}{2}K_{x_0}(v,w)(\|w\|^2 - \langle v,w\rangle^2) - \frac{1}{2}\mathrm{Hess}_{x_0}V(v,v)(\varepsilon^2 - \|w\|^2)\right) \\ &\quad \times (1 - \langle \nabla V(x_0), w\rangle + h(\exp_{x_0}(w), x_0)) d\tilde{\mu}_{x_0}^{\varepsilon}(w) + O(\delta\varepsilon^3) + O(\delta^2\varepsilon^2) \\ &= \delta\left(1 - \frac{\varepsilon^2}{2(n+2)}\left(\mathrm{Ric}_{x_0}(v,v) + 2\mathrm{Hess}_{x_0}V(v,v)\right)\right) + O(\delta\varepsilon^3) + O(\delta^2\varepsilon^2) \end{aligned} \tag{3.29}$$

where we eliminated the h and ∇V terms since

$$\int \langle \nabla V(x_0), w\rangle \, d\tilde{\mu}_{x_0}^{\varepsilon}(w) = 0, \qquad \int h(\exp_{x_0}(w), x_0) d\tilde{\mu}_{x_0}^{\varepsilon}(w) = 0$$

The Ricci curvature on the right appears by Lemma 3.2 and the Hessian term is obtained as in Lemma 3.21.

For the converse direction, we use the Kantorovich-Rubinstein duality, choosing the 1-Lipschitz test function f defined by (3.24), so that by Lemma 3.21 we have the lower bound

$$\begin{aligned} W_1(\bar{\nu}_{x_0}^{\varepsilon}, T_*\bar{\nu}_{x_0}^{\varepsilon}) &\geqslant \int_M f(z)(dT_*\bar{\nu}_{x_0}^{\varepsilon}(z) - d\bar{\nu}_{x_0}^{\varepsilon}(z)) \\ &= \int_{B_\varepsilon(x_0)} (f(Tz) - f(z)) d\bar{\nu}_{x_0}^{\varepsilon}(z) \\ &= \delta\left(1 - \frac{\varepsilon^2}{2(n+2)}(\mathrm{Ric}_{x_0}(v,v) + 2\mathrm{Hess}_{x_0}V(v,v))\right) \\ &\quad + O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3). \end{aligned}$$

This shows the upper and lower bound agree up to terms $O(\delta\varepsilon^3) + O(\delta^2\varepsilon^2)$ and their values are as required. $\square$

We now prove the aforementioned approximation property and refer to Notation 3.7 for the measures involved.

Proposition 3.23. *For $x_0 \in M$ and $\delta, \varepsilon > 0$ sufficiently small,*

$$\begin{aligned} W_1(\nu^\varepsilon_{x_0}, \nu^\varepsilon_y) &= W_1(\nu^\varepsilon_{x_0}, T_*\nu^\varepsilon_{x_0}) + O(\delta\varepsilon^3) + O(\delta^2\varepsilon^2), \\ W_1(\nu^\varepsilon_{x_0}, T_*\nu^\varepsilon_{x_0}) &= W_1(\bar\nu^\varepsilon_{x_0}, T_*\bar\nu^\varepsilon_{x_0}) + O(\delta\varepsilon^3) + O(\delta^2\varepsilon^2) \end{aligned}$$

where $y = \exp_{x_0}(\delta v)$.

We need density estimates of the following two lemmas for the proof.

Lemma 3.24.

$$\frac{d(T_*\bar\nu^\varepsilon_{x_0})}{d\bar\nu^\varepsilon_y}(z) = \mathbb{1}_{B_y(\varepsilon)}(z)(1 + h'(z,y))$$

where $h' : M \times M \to \mathbb{R}$ is smooth a. e. and such that $h'(\cdot, y) = O(\delta\varepsilon^2)$ and $\int_{B_\varepsilon(y)} h'(z,y) d\bar\nu^\varepsilon_y(z) = 0$ for all $y \in M$.

Proof. We first show the density estimate

$$\frac{d(T_*\bar\nu^\varepsilon_{x_0})}{d\bar\mu^\varepsilon_y}(z) = (1 - \left\langle \nabla V(y), \exp_y^{-1}(z) \right\rangle + h(T^{-1}z, x_0))\mathbb{1}_{B_\varepsilon(y)}(z). \tag{3.30}$$

For any bounded Borel measurable $f : M \to \mathbb{R}$,

$$\begin{aligned} \int_M f(z) d(T_*\bar\nu^\varepsilon_{x_0})(z) &= \int_M f(Tz) d\bar\nu^\varepsilon_{x_0}(z) = \int_M f(Tz) \frac{d\bar\nu^\varepsilon_{x_0}}{d\bar\mu^\varepsilon_{x_0}}(z) d\bar\mu^\varepsilon_{x_0}(z) \\ &= \int_M f(Tz)(1 - \left\langle \nabla V(x_0), \exp_{x_0}^{-1}(z) \right\rangle + h(z, x_0)) d\bar\mu^\varepsilon_{x_0}(z) \\ &= \int_{T_{x_0}M} f(\exp_y \tilde T w)(1 - \langle \nabla V(x_0), w \rangle + h(\exp_{x_0}(w), x_0)) d\tilde\mu^\varepsilon_{x_0}(w) \end{aligned}$$

where we applied Lemma 3.10 on the second line. Substituting $\tilde w := \tilde T w$ and using the change of variable formula, this becomes

$$\begin{aligned} &\int_{T_yM} f(\exp_y \tilde w) \left(1 - \left\langle \nabla V(x_0), \tilde T^{-1}\tilde w \right\rangle + h(\exp_{x_0}(\tilde T^{-1}\tilde w), x_0)\right) \left|\det D_{\tilde w} \tilde T^{-1}\right| d\tilde\mu^\varepsilon_y(\tilde w) \\ &\quad = \int_M f(z) \left(1 - \left\langle \nabla V(x_0), \tilde T^{-1} \exp_y^{-1}(z) \right\rangle + h(T^{-1}z, x_0)\right) \\ &\qquad\qquad \times \left(1 - \left\langle \nabla V(y) - /\!\!/_1 \nabla V(x_0), \exp_y^{-1}(z) \right\rangle + O(\delta^2\varepsilon^2)\right) d\bar\mu^\varepsilon_y(z) \end{aligned}$$

having applied the determinant formula (3.22). Using that

$$\tilde T^{-1} \exp_y^{-1}(z) = /\!\!/_1^{-1} \exp_y^{-1}(z) + O(\delta\varepsilon^2)$$

and absorbing the $O(\delta\varepsilon^2)$ term into h, this simplifies to

$$\int_M f(z) \left(1 - \left\langle \nabla V(y), \exp_y^{-1}(z) \right\rangle + h(T^{-1}z, x_0)\right) d\bar\mu^\varepsilon_y(z)$$

as required to obtain the density (3.30).

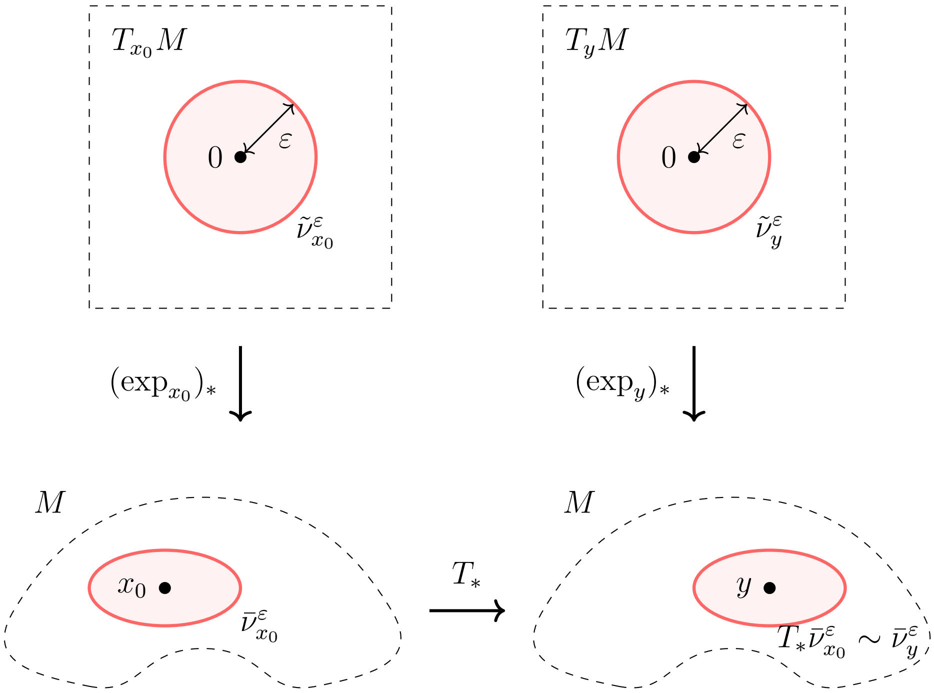

Figure 5: Illustration of the relationship between $T_*\bar{\nu}^\varepsilon_{x_0}$ and $\bar{\nu}^\varepsilon_y$. The two measures are equivalent with mutual density of the form $1 + O(\delta\varepsilon^2)$.

Finally, using Lemma 3.10 and (3.30) we obtain

$$
\begin{aligned}
\frac{dT_*\bar{\nu}^\varepsilon_{x_0}}{d\bar{\nu}^\varepsilon_y}(z) &= \frac{d\bar{\mu}^\varepsilon_y}{d\bar{\nu}^\varepsilon_y}(z)\frac{dT_*\bar{\nu}^\varepsilon_{x_0}}{d\bar{\mu}^\varepsilon_y}(z) = \left(\frac{d\bar{\nu}^\varepsilon_y}{d\bar{\mu}^\varepsilon_y}(z)\right)^{-1}\frac{dT_*\bar{\nu}^\varepsilon_{x_0}}{d\bar{\mu}^\varepsilon_y}(z) \\
&= \mathbb{1}_{B_y(\varepsilon)}(z)(1 + \langle \nabla V(y), \exp_y^{-1}(z)\rangle - h(z,y)) \\
&\qquad \times (1 - \langle \nabla V(y), \exp_y^{-1}(z)\rangle + h(T^{-1}z, x_0)) \\
&= \mathbb{1}_{B_y(\varepsilon)}(z)(h(T^{-1}z, x_0) - h(z,y)).
\end{aligned} \tag{3.31}
$$

The function $(y,z) \mapsto h(T^{-1}z, x_0) - h(z,y)$ is smooth a.e. on $M \times M$ and $z \mapsto h(T^{-1}z, x_0) - h(z,y)$ is of order ε^2 for every y and vanishing for $y = x_0$ (i.e. $\delta = 0$). Hence $h(T^{-1}z, x_0) - h(z,y) = O(\delta \wedge \varepsilon^2)$ and by smoothness of h thus of order $\delta\varepsilon^2$, see Remark 2.6. $\square$

Remark 3.25. The cancellation in (3.31) occurs because of the specific choice of T, and is essential for the $O(\delta\varepsilon^2)$ estimate.

The estimate from Lemma 3.24 carries over to ν_{x_0} and ν_y as we shall prove below.

Lemma 3.26.

$$\frac{d(T_*\nu^\varepsilon_{x_0})}{d\nu^\varepsilon_y}(z) = \mathbb{1}_{B_y(\varepsilon)}(z)(1 + h(z,y))$$

where $h' : M \times M \to \mathbb{R}$ is smooth a.e. and such that $h'(\cdot, y) = O(\delta\varepsilon^2)$ and

$$\int_{B_\varepsilon(y)} h'(z,y)d\bar{\nu}^\varepsilon_y(z) = 0$$

for all $y \in M$.

Proof. By Lemma 3.9 we may write

$$d\nu^{\varepsilon}_{x_0}(z) = (1 + h(x_0, z))d\bar{\nu}^{\varepsilon}_{x_0}(z), \quad d\nu^{\varepsilon}_{y}(z) = (1 + h(y, z))d\bar{\nu}^{\varepsilon}_{y}(z)$$

for a function $h : M \times M \to \mathbb{R}$ smooth a. e. and such that $h(\cdot, x) = O(\varepsilon^2)$ and $\int h(z, x)d\bar{\mu}_x(z) = 0$ for every $x \in M$. Then

$$\begin{aligned}
d(T_*\nu_{x_0})(z) &= T_*[(1 + h(x_0, z))d\bar{\nu}^{\varepsilon}_{x_0}(z)] \\
&= (1 + h(x_0, T^{-1}z))d(T_*\bar{\nu}^{\varepsilon}_{x_0})(z) \\
&= (1 + h(y, z) + O(\delta\varepsilon^2))d(T_*\bar{\nu}^{\varepsilon}_{x_0})(z) \\
&= (1 + h(y, z) + O(\delta\varepsilon^2))\frac{d(T_*\bar{\nu}^{\varepsilon}_{x_0})}{d\bar{\nu}^{\varepsilon}_{y}}(z)d\bar{\nu}^{\varepsilon}_{y}(z) \\
&= (1 + h(y, z) + O(\delta\varepsilon^2))(1 + O(\delta\varepsilon^2)d\bar{\nu}^{\varepsilon}_{y}(z) \\
&= (1 + h(y, z) + O(\delta\varepsilon^2))d\bar{\nu}^{\varepsilon}_{y}(z) \\
&= (1 + O(\delta\varepsilon^2))d\nu^{\varepsilon}_{y}(z).
\end{aligned}$$

On the third line we applied that $h(y, z) - h(x_0, T^{-1}z) = O(\delta\varepsilon^2)$ by the same argument as in the proof of Lemma 3.24. The mean zero property of $h'(\cdot, y)$ again follows from $1 + h(\cdot, y)$ being a density with respect to a probability measure. □

Proof of Proposition 3.23. For the first equality, $\forall f \in \mathrm{Lip}_1(M)$:

$$\begin{aligned}
\int f(z)(d(T_*\nu^{\varepsilon}_{x_0})(z) - d\nu^{\varepsilon}_{x_0}(z)) &= \int f(z)\left(\frac{d(T_*\nu^{\varepsilon}_{x_0})}{d\nu^{\varepsilon}_{y}}(z)d\nu^{\varepsilon}_{y}(z) - d\nu^{\varepsilon}_{x_0}(z)\right) \\
&= \int f(z)((1 + h'(y, z))d\nu^{\varepsilon}_{y}(z) - d\nu^{\varepsilon}_{x_0}(z)) \\
&= \int f(z)(d\nu^{\varepsilon}_{y}(z) - d\nu^{\varepsilon}_{x_0}(z)) \\
&\quad + \int (f(z) - f(y))h'(y, z)d\nu^{\varepsilon}_{y}
\end{aligned}$$

and the last term is of order $\delta\varepsilon^3$ as $h' = O(\delta\varepsilon^2)$ by the preceding lemma.

For the second equality, we know from Theorem 3.22 and the equality (3.29) that T satisfies

$$\int d(z, Tz)d\bar{\nu}^{\varepsilon}_{x_0}(z) = W_1(\bar{\nu}^{\varepsilon}_{x_0}, T_*\bar{\nu}^{\varepsilon}_{x_0}) + O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3)$$

and additionally from Lemma 3.21 that there exists $f \in \mathrm{Lip}_1(M)$ such that

$$\int f(z)(d(T_*\bar{\nu}_{x_0})(z) - d\bar{\nu}_{x_0}(z)) = W_1(\bar{\nu}^{\varepsilon}_{x_0}, T_*\bar{\nu}^{\varepsilon}_{x_0}) + O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3). \tag{3.32}$$

Proposition 2.5 gives $d(Tz, z) = d(y, x_0)(1 + O(\varepsilon^2))$, hence we obtain the upper bound

$$\begin{aligned}
W_1(\nu^{\varepsilon}_{x_0}, T_*\nu^{\varepsilon}_{x_0}) &\leqslant \int d(Tz, z)d\nu^{\varepsilon}_{x_0}(z) \\
&= \int d(Tz, z)(1 + h(x_0, z))d\bar{\nu}^{\varepsilon}_{x_0}(z) \\
&= \int d(Tz, z)d\bar{\nu}^{\varepsilon}_{x_0}(z) + \int d(y, x_0)(1 + O(\varepsilon^2))h(x_0, z)d\bar{\nu}^{\varepsilon}_{x_0}(z) \\
&= W_1(\bar{\nu}_{x_0}, T_*\bar{\nu}_{x_0}) + O(\delta^2\varepsilon^2) + O(\delta\varepsilon^3)
\end{aligned}$$

having applied the property $\int h(x_0,z)d\bar{\nu}^{\varepsilon}_{x_0}(z)=0$ on the last line.
By Proposition 2.11, for δ,ε sufficiently small the f from Lemma 3.21 satisfies

$$\begin{aligned}
f(Tz)-f(z) &= \tilde{L}_2(\varepsilon,\delta)-\tilde{L}_2(\varepsilon,0)\\
&= L_1(\varepsilon,\delta)+O(\delta^2\varepsilon^2)+O(\delta\varepsilon^3)\\
&= d(Tz,z)+O(\delta^2\varepsilon^2)+O(\delta\varepsilon^3)\\
&= d(x_0,y)(1+O(\varepsilon^2))+O(\delta^2\varepsilon^2)+O(\delta\varepsilon^3)
\end{aligned}$$

which leads to the lower bound

$$\begin{aligned}
W_1(\nu^{\varepsilon}_{x_0},T_*\nu^{\varepsilon}_{x_0}) &\geqslant \int f(z)(d(T_*\nu^{\varepsilon}_{x_0})(z)-d\nu^{\varepsilon}_{x_0}(z))\\
&= \int (f(Tz)-f(z))d\nu^{\varepsilon}_{x_0}(z)\\
&= \int (f(Tz)-f(z))(1+h(x_0,z))d\bar{\nu}^{\varepsilon}_{x_0}(z)\\
&= \int (f(Tz)-f(z))d\bar{\nu}^{\varepsilon}_{x_0}(z)+\int d(y,x_0)(1+O(\varepsilon^2))h(x_0,z)d\bar{\nu}^{\varepsilon}_{x_0}(z)\\
&= W_1(\bar{\nu}_{x_0},T_*\bar{\nu}_{x_0})+O(\delta^2\varepsilon^2)+O(\delta\varepsilon^3)
\end{aligned}$$

having applied the mean zero property of h again on the last line. □

Theorem 3.22 and Proposition 3.23 together yield Theorem 1.1.

4 Application to random geometric graphs

Hoorn et al. [vdHLTK20] showed that the coarse curvature of the random geometric graph sampled from a uniform Poisson point process on a Riemannian manifold converges in expectation to the smooth Ricci curvature as the intensity of the process increases.

Theorem 1.1 allows us to extend their result to manifolds equipped with a smooth potential $V : M \to \mathbb{R}$, sampling now from a Poisson point process with increasing intensity, proportional to the non-uniform measure $e^{-V(z)}\mathrm{vol}(dz)$. By sampling we mean using the empirical measures of the Poisson point process as the vertices of the random geometric graph. Our novel method allows to deal with this non-uniformity as well as the non-uniformity incurred by the exponential mapping.

We recall a few definitions in order to state our extended result, Theorem 4.7. We refer to [LP18] and [Pen03] for a background on Poisson point processes and random geometric graphs, respectively.

Definition 4.1. Let $(\mathcal{X},\mathcal{B},\mu)$ be a σ-finite measure space, $\mathcal{M}(\mathcal{X})$ the set of measures on $\mathcal{X}$ and $(\Omega,\mathcal{F},\mathbb{P})$ a probability space. A random measure $\mathcal{P}:\Omega\times\mathcal{B}\to[0,\infty]$, or equivalently $\mathcal{P}:\Omega\to\mathcal{M}(\mathcal{X})$, is said to be a Poisson point process on $\mathcal{X}$ with intensity μ if

(i) $\forall A\in\mathcal{B}$ with $\mu(A)<\infty$, $\mathcal{P}(\cdot,A)$ is a random variable with Poisson$(\mu(A))$ distribution,

(ii) $\forall A,B\in\mathcal{B}$ disjoint and μ-finite, the random variables $\mathcal{P}(\cdot,A)$ and $\mathcal{P}(\cdot,B)$ are independent,

(iii) $\forall\omega\in\Omega:\mathcal{P}(\omega,\cdot)$ is a measure on $(\mathcal{X},\mathcal{M})$.

Given a measure μ, the Poisson point process can be constructed as follows. Let $(W_k)_{k\in\mathbb{N}}$ be a countable partition of $\mathcal{X}$ such that $\mu(W_k) < \infty$ and let $(N_k)_{k\in\mathbb{N}}$ be independent random variables distributed as $N_k \sim \text{Poisson}(\mu(W_k))$. Further, for every $k \in \mathbb{N}$ let $(X_i^{(k)})_{i\in\mathbb{N}}$ be independent random variables which are also independent of $(N_k)_{k\in\mathbb{N}}$ and have distribution μ restricted to W_k and normalized to a probability measure. Define the random empirical measures

$$\mathcal{P}^{(k)} : \Omega \to \mathcal{M}(\mathcal{X}), \quad \omega \mapsto \sum_{i=1}^{N_k(\omega)} \delta_{X_i^{(k)}(\omega)}. \tag{4.1}$$

The sum $\mathcal{P} := \sum_{k=1}^{\infty} \mathcal{P}^{(k)}$ is well-defined by monotone convergence, satisfies the properties in Definition 4.1, and $\mathcal{P}(A) < \infty$ almost surely if and only if $\mu(A) < \infty$. It can be shown that all Poisson point processes with a σ-finite intensity measure take this form, see [LP18, Chap. 6].

The following transformation property of Poisson processes will be needed later:

Lemma 4.2. *[LP18] Let $(\mathcal{Y}, \mathcal{B}')$ be another measurable space and $\psi : \mathcal{X} \to \mathcal{Y}$ a measurable map. Then the push-forward process $\mathcal{P}' : \Omega \times \mathcal{B}' \to [0, \infty]$ defined by $\mathcal{P}'(\omega, \cdot) := \psi_* \mathcal{P}(\omega, \cdot)$ is a Poisson point process on $\mathcal{Y}$ with intensity $\psi_*\mu$. Writing the process $\mathcal{P}$ in the form (4.1), the push-forward process can be written as*

$$\mathcal{P}' = \sum_{k=1}^{\infty} \mathcal{P}'^{(k)}, \quad \mathcal{P}'^{(k)}(\omega, \cdot) = \sum_{i=1}^{N_k(\omega)} \delta_{\psi(X_i^{(k)}(\omega))}$$

Proof. The random variable $\mathcal{P}'(\cdot, A) = \mathcal{P}(\cdot, \psi^{-1}(A))$ has a Poisson$(\mu(\psi^{-1}(A)))$ distribution, meaning that $\mathcal{P}'$ has intensity $\psi_*\mu$. Properties (ii) and (iii) are preserved by the push-forward. □

Let $\mathcal{P}$ be a Poisson point process on the complete, orientable Riemannian manifold M with Riemannian distance d and $x_0, y \in M$ two points. We implicitly identify every Dirac measure δ_x from the Poisson point process with the point x.

Definition 4.3. A random geometric graph sampled from $\mathcal{P}$ with roots x_0, y and connectivity radius $\varepsilon > 0$ is the weighted graph denoted by $G(x_0, y, \varepsilon)$ with nodes given by

$$\mathcal{V}(\omega) = \{X_i^{(k)}(\omega) : 1 \leqslant i \leqslant N_k(\omega), k \in \mathbb{N}\} \cup \{x_0, y\}$$

where the variables $X_i^{(k)}$ originate from the Poisson process $\mathcal{P}$ by (4.1), and edges

$$\{(u, v) : u, v \in \mathcal{V}(\omega), d(u, v) < \varepsilon\}.$$

The edge weights are given by manifold distance $d(u, v)$ for every edge (u, v). Denote the graph distance by

$$d_G(x, z) := \inf \left\{ \sum_{i=0}^{m} d(x_i, x_{i+1}) : x_0 = x, x_m = z, (x_i, x_{i+1}) \text{ edge}, m \in \mathbb{N} \right\}.$$

Notation 4.4. Let $V : M \to \mathbb{R}$ be a smooth potential on M, fix $x_0 \in M$ and denote by $(\mathcal{P}_n)_{n\in\mathbb{N}}$ the sequence of Poisson point processes with intensity measures

$$n e^{-V(z)+V(x_0)} \text{dvol}(z). \tag{4.2}$$

For sequences $(\delta_n)_{n\in\mathbb{N}}, (\varepsilon_n)_{n\in\mathbb{N}}$ with $\delta_n, \varepsilon_n \to 0$ as $n \to \infty$, label the sequence of points $y_n := \exp_{x_0}(\delta_n v)$ approaching x_0 from a fixed direction $v \in T_{x_0}M$. This gives rise to a sequence of rooted random graphs $G_n(x_0, y_n, \varepsilon_n)$ according to Definition 4.3. For each $n \in \mathbb{N}$ denote the set of nodes

$$\mathcal{V}_n(\omega) := \{X_i^{(k,n)}(\omega) : 1 \leqslant i \leqslant N_{k,n}(\omega), k \in \mathbb{N}\} \cup \{x_0, y\}$$

where the variables $X_i^{(k,n)}$ come from the process $\mathcal{P}_n$ by (4.1) and are assumed to be independent for all $n \in \mathbb{N}$. For the graph distance d_{G_n} define the δ_n-neighbourhood of x in G_n as

$$B_{\delta_n}^{G_n}(x) := \{z \in \mathcal{V}_n : d_{G_n}(x, z) < \delta_n\}.$$

For any node $x \in \mathcal{V}_n$ denote the normalized empirical measures

$$\eta_x^{\delta_n}(z) := \begin{cases} \frac{1}{\#(B_{\delta_n}^{G_n}(x))}, & z \in B_{\delta_n}^{G_n}(x) \\ 0, & \text{otherwise} \end{cases} \tag{4.3}$$

Denote the graph curvature of $G_n(x_0, y_n, \varepsilon_n)$ at x_0 in the direction of $v = \delta_n^{-1} \exp_{x_0}^{-1}(y_n)$ as

$$\kappa_n(x_0, y_n) := 1 - \frac{W_1^{G_n}(\eta_{x_0}^{\delta_n}, \eta_{y_n}^{\delta_n})}{d_{G_n}(x_0, y_n)}. \tag{4.4}$$

Remark 4.5. For the sequence of graphs $G_n(x_0, y_n, \varepsilon_n)$ we thus have the corresponding collections of random empirical measures $(\eta_x^{\delta_n})_{x\in\mathcal{V}_n}$ and coarse curvatures $\kappa_n(x_0, y_n)$ at x_0, the limit of which we are looking to establish in the sequel. We emphasise that these are all random objects, i.e. dependent on $\omega \in \Omega$, although we suppress this from the notation.

Convergence of the coarse graph curvature to the generalized Ricci curvature can be proved under an assumption on the rate of convergence of the graph parameters δ_n, ε_n to zero. For two sequences $(a_n), (b_n)$ we denote $a_n \sim b_n$ as $n \to \infty$ if there exist $c, C > 0$ such that $cb_n \leqslant a_n \leqslant Cb_n$ for all $n \in \mathbb{N}$.

Assumption 4.6. Denoting $N = \dim M$, assume that $\varepsilon_n \leqslant \delta_n$ and

$$\varepsilon_n \sim \log(n)^a n^{-\alpha}, \quad \delta_n \sim \log(n)^b n^{-\beta}$$

where the constants α, β, a, b are such that

$$0 < \beta \leqslant \alpha, \quad \alpha + 2\beta \leqslant \frac{1}{N}$$

and in case of equality additionally $\begin{cases} a \leqslant b & \text{if } \alpha = \beta, \\ a + 2b > \frac{2}{N} & \text{if } \alpha + 2\beta = \frac{1}{N}. \end{cases}$

Below is the main result of the section, with the relevant objects set up in Notation 4.4.

Theorem 4.7. *Let $\kappa_n(x_0, y)$ be the graph curvatures corresponding to the rooted random graphs $G_n(x_0, y_n, \varepsilon_n)$ with $y_n = \exp_{x_0}(\delta_n v)$ generated by the sequence of Poisson processes with intensity measures*

$$ne^{-V(z)+V(x_0)} vol(dz)$$

Under Assumption 4.6, there exists $(c_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} c_n = 0$ and

$$\mathbb{E}\left[\left|W_1^{G_n}(\eta_{x_0}^{\delta_n}, \eta_{y_n}^{\delta_n}) - W_1(\nu_{x_0}^{\delta_n}, \nu_{y_n}^{\delta_n})\right|\right] \leqslant c_n \delta_n^3 \tag{4.5}$$

which implies

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{2(N+2)}{\delta_n^2}\kappa_n - (Ric_{x_0}(v) + 2Hess_{x_0}V(v))\right|\right] = 0. \tag{4.6}$$

Remark 4.8. The asymptotic bound (4.5) is indeed sufficient for (4.6), because in combination with Theorem 1.1 and by the triangle inequality we have the upper bound for the latter given by

$$\frac{2(N+2)}{\delta_n^2}\mathbb{E}\left[\left|\kappa_n(x_0,y_n)-\left(1-\frac{W_1(\nu_{x_0}^{\delta_n},\nu_{y_n}^{\delta_n})}{\delta_n}\right)\right|\right]$$
$$+\left|\frac{2(N+2)}{\delta_n^2}\left(1-\frac{W_1(\nu_{x_0}^{\delta_n},\nu_{y_n}^{\delta_n})}{\delta_n}\right)-(\mathrm{Ric}_{x_0}(v)+2\mathrm{Hess}_{x_0}V(v))\right|.$$

The second term converges to 0 by (1.3) and the first term can be written as

$$\frac{2(N+2)}{\delta_n^3}\mathbb{E}\left[\left|W_1^{G_n}(\eta_{x_0}^{\delta_n},\eta_{y_n}^{\delta_n})-W_1(\nu_{x_0}^{\delta_n},\nu_{y_n}^{\delta_n})\right|\right]$$

which vanishes as $n\to\infty$ if (4.5) holds.

Henceforth, we thus focus on establishing (4.5). We follow the methods laid out in [vdHCL+21], which consist in showing that the graph distances d_{G_n} can be extended to the manifold to give a close approximation of the Riemannian distance d, and then using such extension to estimate the difference (4.5).

Let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence given by

$$\lambda_n := (\log n)^{\frac{2}{N}} n^{-\frac{1}{N}}$$

which can be regarded in view of the following definition as a "distance extension radius".

Definition 4.9. Let $G_n = G_n(x_0,y_n,\delta_n)$ be the rooted random geometric graphs of Theorem 4.7 and for any $x,y\in M$ denote by $G_n\cup\{x,y\}$ the graph extended by nodes $\{x,y\}$ and extended by edges $\{(x,z) : z\in B_{\lambda_n}(x)\cap G_n\}\cup\{(y,z) : z\in B_{\lambda_n}(y)\cap G_n\}$ with extension radius λ_n. Further, let $(d_n)_{n\in\mathbb{N}}$ be the sequence of random functions $d_n : M\times M\to\mathbb{R}$ defined by

$$d_n(x,y) := d(x,\tilde{x}) + d(y,\tilde{y}) + d_{G_n}(\tilde{x},\tilde{y})$$

where

$$\tilde{x} = \operatorname*{argmin}_{z\in B_{\lambda_n}(x)\cap G_n} d(x,z),\qquad \tilde{y} = \operatorname*{argmin}_{z\in B_{\lambda_n}(y)\cap G_n} d(y,z).$$

if both $\tilde{x},\tilde{y}$ exist and set $d_n(x,y)=\infty$ otherwise.

While $\tilde{x},\tilde{y}$ may not exist for every $\omega\in\Omega$, rendering $d_n(x,y)=\infty$, the following states there is an event of high probability where this does not occur and where d_n is moreover a metric.

Lemma 4.10. *Under Assumption 4.6, for all $Q>0$ there exists a sequence of events $(\Omega_n)_{n\in\mathbb{N}}$ and a sequence of reals $(c_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} c_n = 0$, $\mathbb{P}(\Omega_n)\geqslant 1-c_n\delta_n^3$ and the following properties hold for all $n\in\mathbb{N}$ and $\omega\in\Omega_n$:*

(i) $(B_{\delta_n Q}(x_0), d_n)$ is a metric space,

(ii) for all $x,y\in B_{\delta_n Q}(x_0) : |d_n(x,y)-d(x,y)|\leqslant c_n\delta_n^3$,

(iii) for all $x,y\in B_{\delta_n Q}(x_0)$ there exists a path in $G_n\cup\{x,y\}$ connecting x and y.

Proof. The construction of the event Ω_n is the same as in the case of uniform measures, see Lemma 5.1 and Corollary 5.2 in [vdHLTK20]. □

Remark 4.11. The interpretation of these properties is that on the event Ω_n the random graph covers $B_{\delta_n Q}(x_0)$ in a way that allows to extend the graph distance to a distance on the manifold that closely approximates the original Riemannian distance with high probability.

In fact, instead of the Riemannian distance of connected vertices one could use any graph metric satisfying properties in Lemma 4.10. Such graph metrics are referred to in [vdHLTK20] as "δ_n-good approximations" to the Riemannian metric.

Denote by W, W^{d_n}, W^{G_n} the Wasserstein distances corresponding to the Riemannian distance d, approximating manifold distance d_n, and the graph distance d_{G_n}, respectively. The following states that the Wasserstein distances W^{d_n} and W are close on Ω_n and is a direct consequence of properties of Ω_n.

Lemma 4.12. *There exists a sequence $(c_n)_{n\in\mathbb{N}}$ such that $c_n \to 0$ and for all $\mu, \nu \in P(B_{\delta_n Q}(x_0))$,*

$$\mathbb{E}\left[|W_1(\mu,\nu) - W_1^{d_n}(\mu,\nu)|\Big|\Omega_n\right] \leqslant c_n\delta_n^3.$$

Proof. This is an easy consequence of property (ii) above, see [vdHLTK20, Lemma 5.4]. □

To prove Theorem 4.7, we now follow along the lines of [vdHLTK20]. On the event Ω_n, the integrand in (4.5) can be split into three parts as

$$\begin{aligned}\left(W_1^{G_n}(\eta_{x_0}^{\delta_n}, \eta_{y_n}^{\delta_n}) - W_1^{d_n}(\eta_{x_0}^{\delta_n}, \eta_{y_n}^{\delta_n})\right) + \left(W_1^{d_n}(\eta_{x_0}^{\delta_n}, \eta_{y_n}^{\delta_n}) - W_1^{d_n}(\nu_{x_0}^{\delta_n}, \nu_{y_n}^{\delta_n})\right)& \\ + \left(W_1^{d_n}(\nu_{x_0}^{\delta_n}, \nu_{y_n}^{\delta_n}) - W_1(\nu_{x_0}^{\delta_n}, \nu_{y_n}^{\delta_n})\right).&\end{aligned} \tag{4.7}$$

The first term is 0 since the measures $\eta_{x_0}^{\delta_n}, \eta_y^{\delta_n}$ are supported on the nodes of the graphs G_n and $d_n(x,y) = d_{G_n}(x,y)$ for all nodes $x, y \in \mathcal{V}_n$ as d_n extends d_{G_n} from the graph to the manifold. The last term is $o(\delta_n^3)$ on Ω_n by Lemma 4.12 if we take arbitrary $Q > 3$. We can further estimate the second term in conditional expectation as

$$\begin{aligned}&\mathbb{E}\left[|W_1^{d_n}(\eta_{x_0}^{\delta_n}, \eta_{y_n}^{\delta_n}) - W_1^{d_n}(\nu_{x_0}^{\delta_n}, \nu_{y_n}^{\delta_n})|\big|\Omega_n\right] \\ &\qquad\leqslant \mathbb{E}\left[W_1^{d_n}(\eta_{x_0}^{\delta_n}, \nu_{x_0}^{\delta_n}) + W_1^{d_n}(\eta_{y_n}^{\delta_n}, \nu_{y_n}^{\delta_n})\big|\Omega_n\right] \\ &\qquad\leqslant \mathbb{E}\left[W_1(\eta_{x_0}^{\delta_n}, \nu_{x_0}^{\delta_n}) + W_1(\eta_{y_n}^{\delta_n}, \nu_{y_n}^{\delta_n})\big|\Omega_n\right] + 2c_n\delta_n^3.\end{aligned} \tag{4.8}$$

For some sequence $(c_n')_{n\in\mathbb{N}}$ with $\lim_n c_n' = 0$, the total expectation is therefore

$$\begin{aligned}&\mathbb{E}\left[\left|W_1^{G_n}(\eta_{x_0}^{\delta_n}, \eta_y^{\delta_n}) - W_1(\nu_{x_0}^{\delta_n}, \nu_y^{\delta_n})\right|\right] \\ &= \mathbb{E}\left[\left|W_1^{G_n}(\eta_{x_0}^{\delta_n}, \eta_y^{\delta_n}) - W_1(\nu_{x_0}^{\delta_n}, \nu_y^{\delta_n})\right| |\Omega_n\right] \mathbb{P}(\Omega_n) \\ &\quad + \mathbb{E}\left[\left|W_1^{G_n}(\eta_{x_0}^{\delta_n}, \eta_y^{\delta_n}) - W_1(\nu_{x_0}^{\delta_n}, \nu_y^{\delta_n})\right| |\Omega \setminus \Omega_n\right] (1 - \mathbb{P}(\Omega_n)) \\ &\leqslant \mathbb{E}[W_1(\eta_{x_0}^{\delta_n}, \nu_{x_0}^{\delta_n})] + \mathbb{E}[W_1(\eta_y^{\delta_n}, \nu_y^{\delta_n})] + c_n'\delta_n^3.\end{aligned} \tag{4.9}$$

where we used that the second addend on the second line is uniformly bounded by $3\delta_n$. It remains to show that the terms on the right in (4.9) decay faster than $c_n''\delta_n^3$ as $n \to \infty$ for some sequence $(c_n'')_{n\in\mathbb{N}}$ with $\lim_n c_n'' = 0$, which requires an extension of the original argument of [vdHLTK20] to non-uniform measures. In particular, we formulate a generalization of Propositions 5.6-5.9 of the mentioned work.

Our method consists in deforming uniform measures into non-uniform measures with small deviations from uniformity, and then showing the corresponding perturbance of distance is small enough, so that the change in the Wasserstein distance is also small. The deformation maps we shall employ come from the next lemma.

For $U \subset \mathbb{R}^N$ an open subset, $k \in \mathbb{N}$ and $\alpha \in (0,1)$ denote by $C^{k,\alpha}(U)$ the Hölder space of functions with norm

$$\|g\|_{C^{k,\alpha}(U)} := \sum_{i=0}^{k} \sup_{x\in U} \|\nabla^i g(x)\| + \sup_{\substack{x,y\in U\\ x\neq y}} \frac{\|\nabla^k g(x) - \nabla^k g(y)\|}{\|x-y\|^\alpha}.$$

Combining Theorems 1 and 2 of [DM90] we note:

Lemma 4.13. *Let $U \subset \mathbb{R}^N$ be an open set with a $C^{3+k,\alpha}$ boundary and $g \in C^{k,\alpha}(U)$ such that $\int_U g(w)dw = 0$. Then there exists a vector field $F \in C^{k+1,\alpha}(U,\mathbb{R}^N)$ such that the map $\psi : U \to U$ given by $\psi(x) = x + F(x)$ is a diffeomorphism satisfying*

$$\begin{aligned} \det D\psi(x) &= 1 + g(x) \qquad \textit{in } U \\ \psi(x) &= x \qquad\qquad\quad \textit{on } \partial U \end{aligned} \tag{4.10}$$

and there is $C(U,k,\alpha) > 0$ such that $\|F\|_{C^{k+1,\alpha}(U)} \leqslant C\|g\|_{C^{k,\alpha}(U)}$.

For our application, let $k = 0$, $\alpha \in (0,1)$ arbitrary and $U = B_\delta(0) \subset \mathbb{R}^N$ with varying parameter $\delta > 0$. We emphasize that in general the constant $C(U,k,\alpha)$ does depend on U, but for $U = B_\delta(0)$ one can deduce the following estimate which is uniform in δ.

Lemma 4.14. *There exists a $C > 0$ such that for all $\delta > 0$ and every $g \in C^1(\mathbb{R}^N)$ with $\int_{B_\delta(0)} g(z)dz = 0$ there exists $F \in C^1(B_\delta(0),\mathbb{R}^N)$ such that $\psi : B_\delta(0) \to B_\delta(0)$ given by $\psi(w) = w + F(w)$ satisfies*

$$\begin{aligned} \det D\psi(w) &= 1 + g(w) \qquad \textit{in } B_\delta(0) \\ \psi(w) &= w \qquad\qquad\quad \textit{on } \partial B_\delta(0) \end{aligned} \tag{4.11}$$

and

$$\sup_{w\in B_\delta(0)} \|\nabla F(w)\| \leqslant C\Big(\sup_{w\in B_\delta(0)} |g(w)| + \delta \sup_{w\in B_\delta(0)} \|\nabla g(w)\|\Big). \tag{4.12}$$

Proof. Let C be the constant from Lemma 4.13 for $U = B_1(0)$, $k = 0$ and any $\alpha \in (0,1)$ and consider the shrinking diffeomorphism

$$\phi_\delta : B_1(0) \to B_\delta(0), \quad \phi_\delta(w) := \delta w$$

For any $g' \in C^{k,\alpha}(B_\delta(0))$ with $\int_{B_\delta(0)} g'(w)dw = 0$, we have $g := g' \circ \phi_\delta \in C^{k,\alpha}(B_1(0))$ and $\int_{B_1(0)} g'(\phi_\delta(w))dw = 0$. Hence there exists $F \in C^{k+1,\alpha}(B_1(0),\mathbb{R}^N)$ such that $\psi(w) = w + F(w)$ satisfies (4.10) with $g = g' \circ \phi_\delta$. Define $F' := \delta F(\frac{\cdot}{\delta}) \in C^{k,\alpha}(B_\delta(0))$ and

$$\psi' : B_\delta(0) \to B_\delta(0), \quad \psi'(w) := w + F'(w)$$

so that

$$D\psi'(\delta w) = D\psi(w) \quad \forall w \in B_1(0)$$

and thus

$$\det D\psi'(\delta w) = \det D\psi(w) = g'(\delta w) \quad \forall w \in B_1(0)$$

which is equivalent to

$$\det D\psi'(w) = g'(w) \quad \forall w \in B_\delta(0).$$

Moreover,

$$\begin{aligned}\sup_{w\in B_\delta(0)}\|\nabla F'(w)\| = \sup_{w\in B_1(0)}\|\nabla F(w)\| &\leqslant \sup_{w\in B_1(0)}|g(w)| + \sup_{\substack{w_1,w_2\in B_1(0)\\ w_1\neq w_2}}\frac{|g(w_1)-g(w_2)|}{\|w_1-w_2\|^\alpha}\\ &\leqslant \sup_{w\in B_1(0)}|g(w)| + \sup_{w\in B_1(0)}|\nabla g(w)|\\ &= \sup_{w\in B_\delta(0)}|g'(w)| + \delta\sup_{w\in B_\delta(0)}|\nabla g'(w)|.\end{aligned}$$

□

For any point $x\in M$ identify $T_xM\cong\mathbb{R}^N$.

Corollary 4.15. *Assume there is a $C>0$ such that for all $\delta>0$ and $x\in M$, $g_\delta\in C^1(\tilde{B}_\delta(x))$ satisfies $\int_{B_\delta(0)}g_\delta(w)dw=0$ and $\sup_{w\in\tilde{B}_\delta(x)}|g_\delta(w)|\leqslant C\delta$. Then for all $x\in M$ and $\delta>0$ there exists $F\in C^1(\tilde{B}_\delta(x),\mathbb{R}^N)$ such that $\psi:\tilde{B}_\delta(x)\to\tilde{B}_\delta(x)$ given by $\psi(w)=w+F(w)$ satisfies*

$$\begin{aligned}\det D\psi(w) &= 1+g_\delta(w) && \text{in } \tilde{B}_\delta(x)\\ \psi(w) &= w && \text{on } \partial\tilde{B}_\delta(x)\end{aligned}$$

and there exists $C'>0$ with

$$\sup_{w\in\tilde{B}_\delta(x)}\|\nabla F(w)\|\leqslant C'\delta\quad\forall x\in M,\delta>0. \tag{4.13}$$

Proof. The derivative of g_δ must satisfy

$$\sup_{w\in\tilde{B}_\delta(0)}\|\nabla g_\delta(w)\|\leqslant C''$$

for some constant C'' uniform for all $x\in M,\delta>0$, otherwise $\sup_{w\in\tilde{B}_\delta(x)}|g_\delta(w)|\leqslant C\delta$ may be violated by e.g. the mean value theorem. Hence the bound (4.12) becomes $C'\delta$ for some $C'>0$. □

Consider now the concrete function for our application defined by

$$g_\delta(w) := \left(\frac{d(\exp_x^{-1})_*\nu_x^{\delta_n}}{d\tilde{\mu}_x^{\delta_n}}(w)-1\right)\mathbb{1}_{\tilde{B}_\delta(x)}(w). \tag{4.14}$$

Similarly to Lemma 3.10, we have

$$\frac{d(\exp_x^{-1})_*\nu_x^{\delta_n}}{d\tilde{\mu}_x^{\delta_n}}(w) = \frac{d\nu_x^{\delta_n}}{d\bar{\mu}_x^{\delta_n}}(\exp_x w) = \frac{e^{-V(\exp_x w)}}{\int_{\tilde{B}_\delta(x)}e^{-V(\exp_x w')}d\tilde{\mu}(w')} = 1+O(\delta) \tag{4.15}$$

where the constant in $O(\delta)$ depends on a fixed compact neighbourhood of x_0 so Corollary 4.15 applies to g_δ.

Corollary 4.16. *The diffeomorphism $\psi:\tilde{B}_{\delta_n}(x)\to\tilde{B}_{\delta_n}(x)$ from Corollary 4.15 applied to g_δ given by (4.14) satisfies $(\exp_x)_*\psi_*\tilde{\mu}_x^{\delta_n}=\nu_x^{\delta_n}$.*

Proof. The change of variable formula and the definition of g_δ give

$$\begin{aligned}d(\psi_*\tilde{\mu}_x^{\delta_n})(w) &= |\det D_w\psi|\,d\tilde{\mu}_x^{\delta_n}(w)\\ &= (1+g_\delta(w))d\tilde{\mu}_x^{\delta_n}(w)\\ &= \frac{d(\exp_x^{-1})_*\nu_x^{\delta_n}}{d\tilde{\mu}_x^{\delta_n}}(w)d\tilde{\mu}_x^{\delta_n}(w)\\ &= d((\exp_x^{-1})_*\nu_x^{\delta_n})(w)\end{aligned}$$

and pushing forward by the exponential mapping yields the result. □

We now proceed with approximating the measures $\nu_x^{\delta_n}$ in optimal transport distance by the empirical measures coming from the Poisson point process $\mathcal{P}_n$, with the aim of finding an upper bound for (4.8). This was proved for Euclidean spaces, [vdHLTK20, Prop. 5.9], building up on the work of Talagrand [Tal92]:

Lemma 4.17. *Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be the sequence of Poisson point processes on $\mathbb{R}^N$ with intensity $n(1+b_n)$ vol(dx) for a sequence $(b_n)_{n\in\mathbb{N}}$ with $\lim_n b_n = 0$, $\tilde{\eta}^{\delta_n}$ the random empirical measures on $B_{\delta_n}(0) \subset \mathbb{R}^N$ corresponding to $\mathcal{P}_n$ and μ^{δ_n} the uniform probability measure on $B_{\delta_n}(0)$. Then there is a sequence (c_n) such that $\lim_n c_n = 0$ and*

$$\mathbb{E}[W_1(\tilde{\eta}^{\delta_n}, \mu^{\delta_n})] \leqslant c_n \delta_n^3.$$

The following states that if the change of distances incurred by a bijection ψ is small then the Wasserstein distances of any two measures pushed forward by ψ may only increase a little.

Lemma 4.18. *Let $\mathcal{X}, \mathcal{Y}$ be Polish spaces. For every $\delta > 0$ let $\psi : \mathcal{X} \to \mathcal{Y}$ be a measurable map such that for all $K \subset M$ compact there is a $C(K) > 0$ such that*

$$|d(\psi(x), \psi(y)) - d(x, y)| \leqslant \delta C(K) d(x, y) \quad \forall x, y \in K.$$

Then for every $\mu, \nu \in P(K)$,

$$W_1(\psi_*\mu, \psi_*\nu) \leqslant W_1(\mu, \nu)(1 + C(K)\delta).$$

Proof. Let π be an optimal coupling realising $W_1(\psi_*\mu, \psi_*, \nu)$. Then the pushforward $\psi_*\pi := \pi(\psi^{-1}\cdot, \psi^{-1}\cdot)$ is a (not necessarily optimal) coupling between μ and ν, therefore

$$\begin{aligned} W_1(\psi_*\mu, \psi_*\nu) &\leqslant \int d(x, y) d(\psi_*\pi)(x, y) = \int d(\psi(x), \psi(y)) d\pi(x, y) \\ &\leqslant \int d(x, y)(1 + C(K)\delta) d\pi(x, y) = W_1(\mu, \nu)(1 + C(K)\delta). \end{aligned}$$

□

Lemma 4.19. *There exists a sequence $(c_n)_{n\in\mathbb{N}}$ such that $\lim_n c_n = 0$ and*

$$\mathbb{E}[W_1(\eta_x^{\delta_n}, \nu_x^{\delta_n})] \leqslant c_n \delta_n^3 \quad \forall x \in B_{\delta_n}(x_0).$$

Proof. For every $n \in \mathbb{N}$ and $x \in B_{\delta_n}(x_0)$ let $F \in C^1(\tilde{B}_\delta(x), \mathbb{R}^N)$ be the vector field and $\psi(w) := \exp_x(w + F(w))$ the diffeomorphism such that $\det D_w\psi = 1 + g_\delta(w)$ and so in particular $(\exp_x)_*\psi_*\tilde{\mu}_x^{\delta_n} = \nu_x^{\delta_n}$ from Corollary 4.16. Denote the normalizing constants

$$Z_1 := \int_{\tilde{B}_{\delta_n}(x)} e^{-V(\exp_x w)+V(x)} dw, \quad Z_2 := \int_{B_{\delta_n}(x)} e^{-V(z)+V(x)} \mathrm{dvol}(z)$$

and let $\tilde{\eta}_x^{\delta_n}$ be the normalized empirical measures of a Poisson point process with uniform (in w) intensity measure

$$\frac{Z_2}{Z_1} n e^{-V(x)+V(x_0)} \mathbb{1}_{\tilde{B}_{\delta_n}}(w) dw = n(1 + O(\delta_n)) \mathbb{1}_{\tilde{B}_{\delta_n}}(w) dw$$

The choice of the constant is substantiated by the following. Note that by Notation 3.7,

$$d\tilde{\mu}_x^{\delta_n}(w) = \frac{1}{Z_1} \mathbb{1}_{\tilde{B}_{\delta_n}(x)}(w) dw, \quad d\nu_x^{\delta_n}(z) = \frac{e^{-V(z)+V(x)}}{Z_2} \mathrm{dvol}(z).$$

Then by Lemma 4.2, $(\exp_x)_* \psi_* \tilde{\eta}_x^{\delta_n}$ are the normalized empirical measures of a Poisson point process with intensity

$$
\begin{aligned}
(\exp_x)_* \psi_* (\frac{Z_2}{Z_1} n e^{-V(x)+V(x_0)} \mathbb{1}_{\tilde{B}_{\delta_n}}(w) dw) &= Z_2 n e^{-V(x)+V(x_0)} d((\exp_x)_* \psi_* \tilde{\mu}_x^{\delta_n})(z) \\
&= Z_2 n e^{-V(x)+V(x_0)} d\nu_x^{\delta_n}(z) \\
&= Z_2 n e^{-V(x)+V(x_0)} \frac{e^{-V(z)+V(x)}}{Z_2} \mathrm{dvol}(z) \\
&= n e^{-V(z)+V(x_0)} \mathrm{dvol}(z)
\end{aligned}
$$

and hence $(\exp_x)_* \psi_* \tilde{\eta}_x^{\delta_n} = \eta_x^{\delta_n}$ in distribution.

The distance upon applying ψ satisfies the bound

$$
\begin{aligned}
d(\psi(w_1), \psi(w_2)) &= \|w_1 + F(w_1) - w_2 - F(w_2)\|(1 + O(\delta_n^2)) \\
&\leqslant \left(\|w_1 - w_2\| + \left\| \int_0^1 \nabla F(w_1 + t(w_2 - w_1)(w_2 - w_1) dt \right\| \right) (1 + O(\delta_n^2)) \\
&\leqslant \|w_1 - w_2\|(1 + O(\sup_{t \in [0,1]} \|\nabla F(w_1 + t(w_2 - w_1)\|))(1 + O(\delta_n^2)) \\
&\leqslant \|w_1 - w_2\|(1 + C\delta_n)
\end{aligned}
$$

using Lemma 2.7 on the first line and (4.13) on the last line. Therefore Lemma 4.18 applies and hence, together with Lemma 4.17,

$$
\begin{aligned}
\mathbb{E}[W_1(\eta_x^{\delta_n}, \nu_x^{\delta_n})] &= \mathbb{E}[W_1((\exp_x)_* \psi_* \tilde{\eta}_x^{\delta_n}, (\exp_x)_* \psi_* \tilde{\mu}_x^{\delta_n})] \\
&\leqslant \mathbb{E}[W_1(\tilde{\eta}_x^{\delta_n}, \tilde{\mu}_x^{\delta_n})](1 + C\delta_n) \\
&\leqslant c_n' \delta_n^3 (1 + C\delta_n) \leqslant c_n \delta_n^3
\end{aligned}
$$

where $(c_n)_{n \in \mathbb{N}}$ is such that $c_n \to 0$ as $n \to \infty$. □

This proved the last missing piece, namely that (4.9) converges to 0 fast enough, thereby concluding the proof of Theorem 4.7.

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