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CODES OVER RINGS OF SIZE FOUR, HERMITIAN LATTICES, AND CORRESPONDING THETA FUNCTIONS

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Abstract. Let $K = Q(\sqrt{-\ell})$ be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, where ℓ is a square free integer such that $\ell \equiv 3 \mod 4$, and let $C = [n, k]$ is a linear code defined over $\mathcal{O}_K/2\mathcal{O}_K$. The level ℓ theta function $\Theta_{\Lambda_\ell(C)}$ of C is defined on the lattice $\Lambda_\ell(C) := \{x \in \mathcal{O}_K^n : \rho_\ell(x) \in C\}$, where $\rho_\ell : \mathcal{O}_K \to \mathcal{O}_K/2\mathcal{O}_K$ is the natural projection. In this paper, we prove that:

i) for any ℓ, ℓ' such that $\ell \leq \ell'$, $\Theta_{\Lambda_\ell}(q)$ and $\Theta_{\Lambda_{\ell'}}(q)$ have the same coefficients up to $q^{\frac{\ell+1}{4}}$,

ii) for $\ell \geq \frac{2(n+1)(n+2)}{n} - 1$, $\Theta_{\Lambda_\ell}(C)$ determines the code C uniquely,

iii) for $\ell < \frac{2(n+1)(n+2)}{n} - 1$, there is a positive dimensional family of symmetrized weight enumerator polynomials corresponding to $\Theta_{\Lambda_\ell}(C)$.

1. Introduction

Let $K = Q(\sqrt{-\ell})$ be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, where ℓ is a square free integer such that $\ell \equiv 3 \mod 4$. Then the image $\mathcal{O}_K/2\mathcal{O}_K$ of the projection $\rho_\ell : \mathcal{O}_K \to \mathcal{O}_K/2\mathcal{O}_K$ is $\mathbb{F}_4$ (resp., $\mathbb{F}_2 \times \mathbb{F}_2$) if $\ell \equiv 3 \mod 8$ (resp., $\ell \equiv 7 \mod 8$).

Let $\mathcal{R}$ be a ring isomorphic to $\mathbb{F}_4$ or $\mathbb{F}_2 \times \mathbb{F}_2$ and $C = [n, k]$ be a linear code over $\mathcal{R}$ of length n and dimension k. An admissible level ℓ is an ℓ such that $\ell \equiv 3 \mod 8$ if $\mathcal{R}$ is isomorphic to $\mathbb{F}_4$ or $\ell \equiv 7 \mod 8$ if $\mathcal{R}$ is isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$. Fix an admissible ℓ and define $\Lambda_\ell(C) := \{x \in \mathcal{O}_K^n : \rho_\ell(x) \in C\}$.

Then, the **level ℓ theta function** $\Theta_{\Lambda_\ell(C)}(\tau)$ of the lattice $\Lambda_\ell(C)$ is given as the symmetric weight enumerator swe_C of C, evaluated on the theta functions defined on cosets of $\mathcal{O}_K/2\mathcal{O}_K$. In this paper we study the following two questions:

i) How do the theta functions $\Theta_{\Lambda_\ell(C)}(\tau)$ of the same code C differ for different levels ℓ?

ii) Can nonequivalent codes give the same theta functions for all levels ℓ?

In an attempt to study the second question, Chua in [1] gives an example of two nonequivalent codes that give the same theta function for level $\ell = 7$ but not for higher level thetas. We will show in this paper how such an example is not a coincidence. Our main results are as follows:

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Theorem 1. *Let C be a code defined over R. For all admissible ℓ, ℓ' such that $\ell > \ell'$, the following holds:*

$$\Theta_{\Lambda_\ell}(C) = \Theta_{\Lambda_{\ell'}}(C) + \mathcal{O}(q^{\frac{\ell'+1}{4}}).$$

Theorem 2. *Let C be a code of size n defined over $\mathcal{R}$ and $\Theta_{\Lambda_\ell}(C)$ be its corresponding theta function for level ℓ. Then the following hold:*

- **i)** *For $\ell < \frac{2(n+1)(n+2)}{n} - 1$, there is a δ-dimensional family of symmetrized weight enumerator polynomials corresponding to $\Theta_{\Lambda_\ell}(C)$, where $\delta \geq \frac{(n+1)(n+2)}{2} - \frac{n(\ell+1)}{4} - 1$.*
- **ii)** *For $\ell \geq \frac{2(n+1)(n+2)}{n} - 1$ and $n < \frac{\ell+1}{4}$, there is a unique symmetrized weight enumerator polynomial which corresponds to $\Theta_{\Lambda_\ell}(C)$.*

This paper is organized as follows. In the second section, we give a basic introduction of lattices and theta functions. We define a lattice Λ over a number field K in general, the theta series of a lattice, and the one-dimensional theta series and its shadow. Then we discuss the lattices over imaginary quadratic fields $K = Q(\sqrt{-\ell})$ with a ring of integers $\mathcal{O}_K$, where ℓ is a square free integer such that $\ell \equiv 3 \mod 4$. The ring $\mathcal{O}_K/(2\mathcal{O}_K)$ is equivalent to either the field of order 4 or a ring of order 4 depending on whether $\ell \equiv 3 \mod 8$ or $\ell \equiv 7 \mod 8$. We define bi-dimensional theta functions for the four cosets of $\mathcal{O}_K/(2\mathcal{O}_K)$.

In the third section, we define codes over $\mathbb{F}_4$ and $\mathbb{F}_2 \times \mathbb{F}_2$, the weight enumerators of a code, and recall the main result of [1]. We simplify the expressions for bi-dimensional theta series and prove Theorem 1.

In the fourth section, we study families of codes corresponding to the same theta function. We call an **acceptable theta series** $\Theta(q)$ a series for which there exists a code C such that $\Theta(q) = \Theta_{\Lambda_\ell}(C)(q)$. For any given ℓ and an acceptable theta series $\Theta(q)$ we can determine a family of symmetrized weight enumerators that correspond to $\Theta(q)$. For small ℓ this is a positive dimensional family, where the dimension is given by Theorem 2i). Hence, the example given in [1] is no surprise. For large ℓ (see Theorem 2ii)) this is a 0-dimensional family of symmetrized weight enumerators that correspond to $\Theta(q)$. Therefore, the example that Chua provides cannot occur for larger ℓ.

2. Introduction to lattices and theta functions

Let K be a number field and $\mathcal{O}_K$ be its ring of integers. A lattice Λ over K is an $\mathcal{O}_K$-submodule of K^n of full rank. The Hermitian dual is defined by

$$\Lambda^* = \{x \in K^n \mid x \cdot \bar{y} \in \mathcal{O}_K, \text{ for all } y \in \Lambda\}, \tag{2.1}$$

where $x \cdot y := \sum_{i=1}^n x_i y_i$. In the case that Λ is a free $\mathcal{O}_K$-module, for every $\mathcal{O}_K$ basis $\{v_1, v_2, ..., v_n\}$ we can associate a Gram matrix G(Λ) given by $G(\Lambda) = (v_i . v_j)_{i,j=1}^n$ and the determinant $\det \Lambda := \det(G)$ defined up to squares of units in $\mathcal{O}_K$. If $\Lambda = \Lambda^*$, then Λ is Hermitian self-dual (or unimodular) and integral if and only if $\Lambda \subset \Lambda^*$. An integral lattice has the property $\Lambda \subset \Lambda^* \subset \frac{1}{\det \Lambda}\Lambda$. An integral lattice is called even if $x \cdot x \equiv 0 \mod 2$ for all $x \in \Lambda$, and otherwise it is odd. An odd unimodular lattice is called a Type 1 lattice, and an even unimodular lattice is called a Type 2 lattice.

The theta series of a lattice Λ in K^n is given by $\Theta_\Lambda(\tau) = \sum_{z \in \Lambda} e^{\pi i \tau z \bar{z}}$, where $\tau \in H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Usually we let $q = e^{\pi i \tau}$. Then, $\Theta_\Lambda(q) = \sum_{z \in \Lambda} q^{z\bar{z}}$.

The 1-dimensional theta series and its **shadow** are given by

$$\theta_3(q) := \sum_{m\in\mathbb{Z}} q^{m^2}, \quad \theta_2(q) := \sum_{m\in\mathbb{Z}+1/2} q^{m^2}. \tag{2.2}$$

Let $\ell > 0$ be a square free integer and $K = Q(\sqrt{-\ell})$ be the imaginary quadratic field with discriminant d_K. Recall that $d_K = -\ell$ if $\ell \equiv 3 \mod 4$ and $d_K = -4\ell$ otherwise.

Let $\mathcal{O}_K$ be the ring of integers of K. The Hermitian lattice Λ over K is an $\mathcal{O}_K$-submodule of K^n of full rank. Let $\ell \equiv 3 \mod 4$ and let d be a positive number such that $\ell = 4d - 1$. Then, $-\ell \equiv 1 \mod 4$. This implies that the ring of integers is $\mathcal{O}_K = \mathbb{Z}[\omega_\ell]$, where $\omega_\ell = \frac{-1+\sqrt{-\ell}}{2}$ and $\omega_\ell^2 + \omega_\ell + d = 0$. The principal norm form of K is given by $Q_d(x, y) = |x - y\omega_\ell|^2 = x^2 + xy + dy^2$. Since $\ell \equiv 3 \mod 4$, we can consider two cases:

(1) If $\ell \equiv 3 \mod 8$, then $-\ell \equiv 5 \mod 8$. Thus, the prime ideal $\langle 2\rangle \subset \mathbb{Z}$ lifts to a prime $2\mathcal{O}_K \subset \mathcal{O}_K$. Since the ring of integers $\mathcal{O}_K$ is a Dedekind domain, $2\mathcal{O}_K$ is a maximal ideal. Therefore $\mathcal{O}_K/(2\mathcal{O}_K) \simeq \mathbb{F}_4$.

(2) If $\ell \equiv 7 \mod 8$, then $-\ell \equiv 1 \mod 8$. Then the prime ideal $\langle 2\rangle \in \mathbb{Z}$ splits in K. Therefore $2\mathcal{O}_K$ splits in $\mathcal{O}_K$. Hence, $\mathcal{O}_K/(2\mathcal{O}_K) \simeq \mathbb{F}_2 \times \mathbb{F}_2$. In either case, a complete set of coset representatives is $\{0, 1, \omega_\ell, 1 + \omega_\ell\}$.

Let the following be the bi-dimensional theta series for the four cosets:

$$\begin{aligned}
A_d(q) &:= \Theta_{2\mathcal{O}_K}(\tau) = \sum_{m,n\in\mathbb{Z}} q^{4Q_d(m,n)},\\
C_d(q) &:= \Theta_{1+2\mathcal{O}_K}(\tau) = \sum_{m,n\in\mathbb{Z}} q^{4Q_d(m+\frac{1}{2},n)},\\
G_d(q) &:= \Theta_{\omega_\ell+2\mathcal{O}_K}(\tau) = \sum_{m,n\in\mathbb{Z}} q^{4Q_d(m,n+\frac{1}{2})},\\
H_d(q) &:= \Theta_{1+\omega_\ell+2\mathcal{O}_K}(\tau) = \sum_{m,n\in\mathbb{Z}} q^{4Q_d(m+\frac{1}{2},n+\frac{1}{2})}.
\end{aligned} \tag{2.3}$$

Then we have the following lemma.

Lemma 1. *Bi-dimensional theta series can be further expressed in terms of the standard one-dimensional theta series and its shadow:*

$$\begin{aligned}
A_d(q) &= \theta_3(q^4)\theta_3(q^{4\ell}) + \theta_2(q^4)\theta_2(q^{4\ell}),\\
C_d(q) &= \theta_2(q^4)\theta_3(q^{4\ell}) + \theta_3(q^4)\theta_2(q^{4\ell}),\\
G_d(q) &= H_d(q) = \frac{\theta_2(q)\theta_2(q^\ell)}{2}.
\end{aligned} \tag{2.4}$$

Moreover,

$$2G_d(q) = A_d(q^{1/4}) - A_d(q) - C_d(q). \tag{2.5}$$

Proof. See [3] for details. □

3. Codes over $\mathbb{F}_4$ and $\mathbb{F}_2 \times \mathbb{F}_2$

Let $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, where $\omega^2 + \omega + 1 = 0$, be the finite field of four elements. The conjugation is given by $\bar{x} = x^2$, $x \in \mathbb{F}_4$. In particular $\bar{\omega} = \omega^2 = \omega + 1$. Let $R_4 = \mathbb{F}_2 + \omega\mathbb{F}_2$ where the new equation for ω being $\omega^2 + \omega = 0$. Notice that R_4 has two maximal ideals, namely $\langle\omega\rangle$ and $\langle\omega + 1\rangle$. Furthermore, one can show that

$R_4/\langle\omega\rangle$ and $R_4/\langle\omega+1\rangle$ are both isomorphic to $\mathbb{F}_2$. The Chinese remainder theorem tells us that $R_4 = \langle\omega\rangle \oplus \langle\omega+1\rangle$. Therefore, $R_4 \simeq \mathbb{F}_2 \times \mathbb{F}_2$. The conjugate of ω is $\omega+1$. Let $\mathcal{R}$ be the field $\mathbb{F}_4$ if $\ell \equiv 3 \mod 8$ or the ring $R_4 \simeq \mathbb{F}_2 \times \mathbb{F}_2$ when $\ell \equiv 7 \mod 8$. A linear code C of length n over $\mathcal{R}$ is an $\mathcal{R}$-submodule of $\mathcal{R}^n$. The dual is defined as $C^\perp = \{u \in \mathcal{R} : u \cdot \bar{v} = 0 \text{ for all } v \in C\}$. If $C = C^\perp$, then C is self-dual.

We define $\Lambda_\ell(C) := \{x \in \mathcal{O}_K^n : \rho_\ell(x) \in C\}$ where $\rho_\ell : \mathcal{O}_K \to \mathcal{O}_K/2\mathcal{O}_k \to \mathcal{R}$. In other words, $\Lambda_\ell(C)$ consists of all vectors in $\mathcal{O}_K^n$ which when taken mod $2\mathcal{O}_K$ componentwise are in $\rho_\ell^{-1}(C)$. The following is immediate.

Lemma 2. (1) *$\Lambda_\ell(C)$ is an $\mathcal{O}_K$-lattice.*

(2) $\Lambda_\ell(C^\perp) = 2\Lambda_\ell(C)^*$.

(3) *C is self-dual if and only if $\frac{\Lambda_\ell(C)}{\sqrt{2}}$ is self-dual.*

Let $u = (u_1, u_2, \ldots, u_n) \in \mathcal{R}^n$ be a codeword and $\alpha \in \mathcal{R}$. Then the counting function $n_\alpha(u)$ is defined as the number of elements in the set $\{j : u_j = \alpha\}$. For a code C we define the complete weight enumerator (*cwe*), symmetrized weight enumerator (*swe*) and Hamming weight enumerator (W) to be

$$
\begin{aligned}
cwe_C(X,Y,Z,W) &:= \sum_{u\in C} X^{n_0(u)} Y^{n_1(u)} Z^{n_\omega(u)} W^{n_{1+\omega}(u)},\\
swe_C(X,Y,Z) &:= cwe_C(X,Y,Z,Z),\\
W_C(X,Y) &:= swe_C(X,Y,Y).
\end{aligned}
\tag{3.1}
$$

Then we have the following.

Proposition 1. *Let $\ell \equiv 3 \mod 4$, C be a linear code over $\mathcal{R}$, and $\frac{\Lambda_\ell(C)}{\sqrt{2}}$ be a Hermitian lattice constructed via the construction A. Then*

$$
\theta_{\Lambda_\ell(C)}(\tau) = swe_C(A_d(q), C_d(q), G_d(q))
\tag{3.2}
$$

where $A_d(q)$, $C_d(q)$, and $G_d(q)$ are given as in (2.4).

For a proof of the above statement the reader can see [1]. From the definition of a one-dimensional theta series we have

$$
\theta_2(q) = 2q^{1/4}\sum_{i\in S} q^i, \quad \theta_2(q^4) = 2q\sum_{i:\text{odd}} q^{i^2-1}, \quad \theta_3(q^4) = 1 + 2q^4 \sum_{i\in\mathbb{Z}^+} q^{4(i^2-1)},
$$

where $S = \left\{\frac{j^2-1}{4} : j \equiv 1 \mod 2\right\}$. From (2.4) we can write

$$
G_d(q) = \frac{\theta_2(q)\theta_2(q^\ell)}{2} = q^{\frac{(\ell+1)}{4}}\alpha_1,
$$

where $\alpha_1 = \sum_{i\in S} q^i \sum_{j\in S} q^{\ell j}$. Then,

$$
\begin{aligned}
A_d(q) &= \theta_3(q^4)\theta_3(q^{4\ell}) + \theta_2(q^4)\theta_2(q^{4\ell})\\
&= (1 + 2q^4 \sum_{i\in\mathbb{Z}^+} q^{4(i^2-1)})(1 + 2q^{4\ell}\sum_{j\in\mathbb{Z}^+} q^{4\ell(j^2-1)})\\
&+ 4q^{\ell+1}\sum_{i:\text{odd}} q^{i^2-1}\sum_{j:\text{odd}} q^{(j^2-1)\ell}\\
&= \alpha_2 + q^{\ell+1}\alpha_3 + q^{4\ell}\alpha_4,
\end{aligned}
$$

where α_2, α_3 and α_4 have the following forms:

$$\alpha_2 = 1 + 2q^4 \sum_{i\in\mathbb{Z}^+} q^{4(i^2-1)},$$
$$\alpha_3 = 4 \sum_{i:\text{odd}} q^{i^2-1} \sum_{j:\text{odd}} q^{(j^2-1)\ell},$$
$$\alpha_4 = 2 \sum_{j\in\mathbb{Z}^+} q^{4\ell(i^2-1)}(1 + 2q^4 \sum_{i\in\mathbb{Z}^+} q^{4(i^2-1)}).$$

Furthermore,

$$\begin{aligned} C_d(q) &= \theta_2(q^4)\theta_3(q^{4\ell}) + \theta_3(q^4)\theta_2(q^{4\ell}) \\ &= 2q \sum_{i:\text{odd}} q^{i^2-1}(1 + 2q^{4\ell} \sum_{i\in\mathbb{Z}^+} q^{4\ell(i^2-1)}) \\ &+ (1 + 2q^4 \sum_{i\in\mathbb{Z}^+} q^{4(i^2-1)})2q^\ell \sum_{i:\text{odd}} q^{(i^2-1)\ell} \\ &= \alpha_5 + q^\ell\alpha_6 + q^{4\ell+1}\alpha_7, \end{aligned}$$

where α_5, α_6 and α_7 have the following forms:

$$\alpha_5 = 2 \sum_{i:\text{odd}} q^{i^2-1},$$
$$\alpha_6 = 2 \sum_{j:\text{odd}} q^{(j^2-1)\ell}(1 + 2q^4 \sum_{i\in\mathbb{Z}^+} q^{4(i^2-1)}),$$
$$\alpha_7 = 4 \sum_{i:\text{odd}} q^{i^2-1} \sum_{j\in\mathbb{Z}^+} q^{4\ell(j^2-1)}.$$

The next result shows that for large enough admissible ℓ and ℓ' the theta functions $\Theta_{\Lambda_\ell}(C)$ and $\Theta_{\Lambda_{\ell'}}(C)$ are virtually the same.

Theorem 3. *Let C be a code defined over R. For all admissible ℓ, ℓ' such that $\ell > \ell'$, the following holds:*

$$\Theta_{\Lambda_\ell}(C) = \Theta_{\Lambda_{\ell'}}(C) + \mathcal{O}(q^{\frac{\ell'+1}{4}}). \tag{3.3}$$

Proof. Let

$$swe_C(X, Y, Z) = \sum_{i+j+k=n} a_{i,j,k} \cdot X^i Y^j Z^k$$

be a degree n polynomial. Write this as a polynomial in Z. Then

$$swe_C(Z) = \sum_{k=0}^{n} L_k Z^k = L_0 + Z(\sum_{k=1}^{n} L_k Z^{k-1}).$$

Terms in L_0 are of the form of $a_{i,j}X^iY^j$, where $i + j = n$. From the above we have

$$\begin{aligned} A_d(q)^i \cdot C_d(q)^j &= (\alpha_2 + q^{\ell+1}\alpha_3 + q^{4\ell}\alpha_4)^i \cdot (\alpha_5 + q^\ell\alpha_6 + q^{4\ell+1}\alpha_7)^j \\ &= (\text{terms independent from } \ell) + q^\ell(\cdots). \end{aligned}$$

Also we have seen that $G_d(q) = q^{(\ell+1)/4}\alpha_1$. This gives

$$\begin{aligned} \Theta_{\Lambda_\ell}(C) &= swe_C(A_d(q), C_d(q), G_d(q)) \\ &= (\text{terms independent from } \ell) + \mathcal{O}(q^{\frac{\ell+1}{4}})^{\cdot} \end{aligned}$$

Then the result follows. ☐

Example 1. Let C be a code defined over R_4 that has symmetrized weight enumerator

$$swe_C(X,Y,Z) = X^3 + X^2Z + XY^2 + 2XZ^2 + Y^2Z + 2Z^3.$$

Then we have the following:

$$\begin{aligned} &\Theta_{\Lambda_{63}}(C) = 1 + 6q^4 + 12q^8 + 8q^{12} + 12q^{16} + 6q^{18} + 48q^{20} + 30q^{22} + \cdots, \\ &\Theta_{\Lambda_{79}}(C) = 1 + 6q^4 + 12q^8 + 8q^{12} + 6q^{16} + 30q^{20} + 6q^{22} + 48q^{24} + \cdots, \\ &\Theta_{\Lambda_{79}}(C) = \theta_{\Lambda_{63}}(C) + \mathcal{O}(q^{16}). \end{aligned} \tag{3.4}$$

4. A family of codes corresponding to the same theta function

If we are given the code over $\mathcal{R}$ and its symmetrized weight enumerator polynomial, then by (3.2) we can find the theta function of the lattice constructed from the code by using the construction A. Now, we would like to give a way to construct families of codes corresponding to the same theta function.

Let $\Theta(q) = \sum_{i=0}^{\infty} \lambda_i q^i$ be an acceptable theta series for level ℓ and

$$f(x,y,z) = \sum_{i+j+k=n} c_{i,j,k} x^i y^j z^k$$

be a degree n generic ternary homogeneous polynomial. We want to find out how many polynomials $f(x,y,z)$ correspond to $\Theta(q)$ for a fixed ℓ.

We have the following lemma.

Lemma 3. *Let C be a code of size n defined over $\mathcal{R}$ and $\Theta(q)$ be its theta function for level ℓ. Then, $\Theta(q)$ is uniquely determined by its first $\frac{n(\ell+1)}{4}$ coefficients.*

Proof. Let C be a code over $\mathcal{R}$, $\Theta(q) = \sum_{i=0}^{\infty} \lambda_i q^i$ be its theta series, $s = \frac{n(\ell+1)}{4}$ and

$$f(x,y,z) = \sum_{i+j+k=n} c_{i,j,k} x^i y^j z^k$$

be a degree n generic ternary homogeneous polynomial. Find $A_d(q), C_d(q), G_d(q)$ for the given ℓ and substitute it in $f(x,y,z)$. Hence $f(x,y,z)$ is now written as a series in q. Recall that a generic degree n ternary polynomial has $r = \frac{(n+1)(n+2)}{2}$ coefficients. So, the corresponding coefficients of the two sides of the equation are equal:

$$f(A_d(q), C_d(q), G_d(q)) = \sum_{i=0}^{\infty} \lambda_i q^i.$$

Consider the term

$$c_{i,j,k}(\alpha_2 + q^{\ell+1}\alpha_3 + q^{4\ell}\alpha_4)^i (\alpha_5 + q^{\ell}\alpha_6 + q^{4\ell+1}\alpha_7)^j (q^{\frac{(\ell+1)}{4}}\alpha_1)^k.$$

Then $c_{i,j,k}$ appears first as a coefficient of $q^{j+\frac{k(\ell+1)}{4}}$. For all such j,k we have $j + \frac{k(\ell+1)}{4} \leq \frac{n(\ell+1)}{4}$. Consider the equations where $c_{i,j,k}$ appears first. This is a system of equations with $\leq \frac{(n+1)(n+2)}{2}$ equations. Let us denote this system of equations as Ξ. Solve this system for $c_{i,j,k}$. Hence, $c_{i,j,k}$ is a function of $\lambda_0, \ldots, \lambda_s$. For each $\mu > s$, λ_μ is a function of $c_{i,j,k}$ for $i,j,k = 0,\ldots,n$, and therefore a rational function on $\lambda_0, \ldots, \lambda_s$. This completes the proof. ☐

Next we have the following theorem:

Theorem 2. *Let C be a code of size n defined over $\mathcal{R}$ and $\Theta_{\Lambda_\ell}(C)$ be its corresponding theta function for level ℓ. Then the following hold:*

- **i)** *For $\ell < \frac{2(n+1)(n+2)}{n} - 1$, there is a δ-dimensional family of symmetrized weight enumerator polynomials corresponding to $\Theta_{\Lambda_\ell}(C)$, where $\delta \geq \frac{(n+1)(n+2)}{2} - \frac{n(\ell+1)}{4} - 1$.*
- **ii)** *For $\ell \geq \frac{2(n+1)(n+2)}{n} - 1$ and $n < \frac{\ell+1}{4}$, there is a unique symmetrized weight enumerator polynomial that corresponds to $\Theta_{\Lambda_\ell}(C)$.*

Proof. We want to find out how many polynomials $f(x, y, z)$ correspond to $\Theta_{\Lambda_\ell}(C)$ for a fixed ℓ. $\Theta_{\Lambda_\ell}(C)$ and $f(x, y, z)$ are defined as above. Consider the system of equations Ξ.

If $\frac{n(\ell+1)}{4} < r$, then our system has more variables than equations. Since the system is linear, the solution space is a family of positive dimension.

If $\frac{n(\ell+1)}{4} \geq r$, then for each equation in Ξ (see the proof of the previous lemma) we have only one $c_{i,j,k}$ appearing for the first time. Otherwise suppose $c_{i,j,k}$ and $c_{i',j',k'}$ appear for the first time in an equation of Ξ. Then $j + \frac{k(\ell+1)}{4} = j' + \frac{k'(\ell+1)}{4}$. This implies

$$4(j - j') = (k' - k)(\ell + 1). \tag{4.1}$$

Without loss of generality, assume $k' \geq k$. We can consider three cases.

Case 1: If $k' - k \geq 2$, then from (4.1) we have $4n(j - j') = n(k' - k)(\ell + 1) \geq 4r(k' - k)$. Then we have $n(j - j') \geq (n + 1)(n + 2)$. Since $n \geq (j - j')$, we have a contradiction.

Case 2: If $k' - k = 1$, then by (4.1), $j - j' = \frac{\ell+1}{4}$. Since $j - j' \leq n$ and $\frac{\ell+1}{4} > n$, we get a contradiction.

Case 3: If $k' - k = 0$, then by (4.1) we have $j = j'$. Hence $i = i'$.

Notice that $c_{n,0,0}$ is uniquely determined by the equation corresponding to the equation of the coefficient of q^0. Solve the system Ξ in the order of the equation that corresponds to the power of q. We have a unique solution for $c_{i,j,k}$. □

4.1. **Families of codes of length 3.** In this section we discuss the codes of length 3 for different levels ℓ. Our main goal is to investigate the example provided in [1] and provide some computational evidence for the above two cases. We assume that the symmetrized weight enumerator polynomial is a generic homogenous polynomial of degree three.

Let $P(x, y, z)$ be a generic ternary cubic homogeneous polynomial given as below:

$$\begin{aligned} P(x, y, z) = {} & c_1x^3 + c_2y^3 + c_3z^3 + c_4x^2y + c_5x^2z + c_6y^2x + c_7y^2z \\ & + c_8z^2x + c_9z^2y + c_{10}xyz. \end{aligned} \tag{4.2}$$

Assume that there is a code C, of length 3, defined over $\mathcal{R}$ such that $swe_C(x, y, z) = P(x, y, z)$. First we have to fix the level ℓ. When we fix the level, we can find $A_d(q), C_d(q), G_d(q)$. By equating both sides of

$$p(A_d(q), C_d(q), G_d(q)) = \sum_{i=0}^{\infty} \lambda_i q^i,$$

we can get a system of equations. When $\ell = 7$, we are in the first case of the previous theorem. The system of equations is given by the following:

$$\begin{cases} c_1 - \lambda_0 = 0, \\ 2c_4 - \lambda_1 = 0, \\ 4c_6 + 2c_5 - \lambda_2 = 0, \\ 8c_2 + 4c_{10} - \lambda_3 = 0 \end{cases} \qquad \begin{cases} 6c_1 + 4c_8 + 2c_5 + 8c_7 - \lambda_4 = 0, \\ 8c_4 + 8c_9 + 4c_{10} - \lambda_5 = 0, \\ 8c_5 + 8c_3 + 8c_7 + 8c_8 + 8c_6 - \lambda_6 = 0. \end{cases} \tag{4.3}$$

The solution for the above system is given by $c_1 = \lambda_0, c_4 = \frac{1}{2}\lambda_1$, and

$$\begin{aligned} & c_2 = \frac{1}{2}\lambda_1 + \frac{1}{8}\lambda_3 - \frac{1}{8}\lambda_5 + c_9, \quad c_3 = \frac{3}{2}\lambda_0 - \frac{1}{4}\lambda_2 - \frac{1}{4}\lambda_4 + \frac{1}{8}\lambda_6 + c_7, \\ & c_5 = -3\lambda_0 + \frac{1}{2}\lambda_4 - 4c_7 - 2c_8, \quad c_6 = \frac{3}{2}\lambda_0 + \frac{1}{4}\lambda_2 - \frac{1}{4}\lambda_4 + 2c_7 + c_8, \\ & c_{10} = -\lambda_1 + \frac{1}{4}\lambda_5 - 2c_9 \end{aligned} \tag{4.4}$$

where c_7, c_8, c_9 are free variables. By giving different triples (c_7, c_8, c_9), we can construct different polynomials $P(x, y, z)$ for the same $\sum_{i=0}^{\infty} \lambda_i q^i$.

Consider the following theta function. From [1] there are two nonisomorphic codes that give this theta function for level $\ell = 7$:

$$\theta_{\sqrt{2}\mathcal{O}_K{}^3_7} = 1 + 6q^2 + 24q^4 + 56q^6 + 114q^8 + 168q^{10} + 280q^{12} + 294q^{14} + \cdots . \tag{4.5}$$

For this particular theta function, we can rewrite the solution (Eq. (4.4)) as follows: $c_1 = 1, c_2 = c_9, c_3 = 1 + c_7, c_4 = 0, c_5 = 9 - 4c_7 - 2c_8, c_6 = -3 - 2c_7 + c_8, c_{10} = -2c_9$.

For the triple $(1, 2, 0)$ (resp., $(0, 3, 0)$) we get the symmetrized weight enumerator polynomial for the code $C_{3,2}$ (resp. $C_{3,3}$). That is, $swe_{C_{3,2}}(X, Y, Z) = X^3 + X^2 Z + XY^2 + 2XZ^2 + Y^2 Z + 2Z^3$ (resp., $swe_{C_{3,3}}(X, Y, Z) = X^3 + 3X^2 Z + 3XZ^2 + Z^3$), where $C_{3,2}$ and $C_{3,3}$ are given by

$$\begin{aligned} C_{3,2} &= \omega\langle[0, 1, 1]\rangle + (\omega + 1)\langle[0, 1, 1]\rangle^{\perp}, \\ C_{3,3} &= \omega\langle[0, 0, 1]\rangle + (\omega + 1)\langle[0, 0, 1]\rangle^{\perp}. \end{aligned} \tag{4.6}$$

When $\ell = 15$, we are in the second case of the above theorem. The system of equations is as follows:

$$\begin{cases} c_1 - \lambda_0 = 0, \\ 2c_4 - \lambda_1 = 0, \\ 4c_6 - \lambda_2 = 0, \\ 8c_2 - \lambda_3 = 0, \\ 6c_1 + 2c_5 - \lambda_4 = 0, \end{cases} \qquad \begin{cases} 8c_4 + 4c_{10} - \lambda_5 = 0, \\ 2c_5 + 8c_7 + 8c_6 - \lambda_6 = 0, \\ 4c_8 + 8c_7 + 12c_1 + 8c_5 - \lambda_8 = 0, \\ 10c_4 + 8c_9 + 8c_{10} - \lambda_9 = 0, \\ 8c_7 + 8c_5 + 12c_8 + 8c_3 + 8c_1 - \lambda_{12} = 0. \end{cases} \tag{4.7}$$

Each c_i appears first in exactly one equation. For example, consider the seventh equation. c_7 is the only variable that appears first in the seventh equation. Solve the system in the given order. The solution for the above system is given by: $c_1 = \lambda_0$, $c_2 = \frac{1}{8}\lambda_3$, $c_4 = \frac{1}{2}\lambda_1$, $c_6 = \frac{1}{4}\lambda_2$, and

$$\begin{aligned} & c_3 = -\lambda_0 - \frac{1}{2}\lambda_2 + \frac{3}{4}\lambda_4 + \frac{1}{4}\lambda_6 - \frac{3}{8}\lambda_8 + \frac{1}{8}\lambda_{12}, && c_5 = -3\lambda_0 + \frac{1}{2}\lambda_4, \\ & c_7 = \frac{3}{4}\lambda_0 - \frac{1}{4}\lambda_2 - \frac{1}{8}\lambda_4 + \frac{1}{8}\lambda_6, && c_9 = \frac{3}{8}\lambda_1 - \frac{1}{4}\lambda_5 + \frac{1}{8}\lambda_9, \\ & c_8 = \frac{3}{2}\lambda_0 + \frac{1}{2}\lambda_2 - \frac{3}{4}\lambda_4 - \frac{1}{4}\lambda_6 + \frac{1}{4}\lambda_8, && c_{10} = -\lambda_1 + \frac{1}{4}\lambda_5. \end{aligned} \tag{4.8}$$

We have a unique solution. This implies that two nonequivalent codes cannot give the same theta function for $\ell = 15$ and $n = 3$.

5. Concluding remarks

The main goal of this paper was to find out how theta functions determine the codes over a ring of size 4. First we have shown how the theta functions of the same code C differ for different levels ℓ. The first $\frac{\ell+1}{4}$ terms of the theta functions for levels ℓ and ℓ' are the same, where $\ell' \geq \ell$.

In [1], two nonisomorphic codes that give the same theta function for level $\ell = 7$ but not under higher level constructions are given. We justified the reason why we don't have a similar situation for higher level constructions. In this note we have addressed a method that we can use for finding a family of polynomials that correspond to a given acceptable theta series for some fixed level ℓ. We have studied two cases depending upon ℓ that give either a positive dimensional family of polynomials or a unique polynomial.

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