# Codifference as a practical tool to measure interdependence

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#### Abstract

Correlation and spectral analysis represent the standard tools to study interdependence in statistical data. However, for the stochastic processes with heavy-tailed distributions such that the variance diverges, these tools are inadequate. The heavy-tailed processes are ubiquitous in nature and finance. We here discuss codifference as a convenient measure to study statistical interdependence, and we aim to give a short introductory review of its properties. By taking different known stochastic processes as generic examples, we present explicit formulas for their codifferences. We show that for the Gaussian processes codifference is equivalent to covariance. For processes with finite variance these two measures behave similarly with time. For the processes with infinite variance the covariance does not exist, however, the codifference is relevant. We demonstrate the practical importance of the codifference by extracting this function from simulated as well as from real experimental data. We conclude that the codifference serves as a convenient practical tool to study interdependence for stochastic processes with both infinite and finite variances as well.

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# 1. Introduction

Stochastic processes with diverging variance are ubiquitous in nature and finance. A remarkable example is an alpha-stable Lévy motion, or Lévy flights, that is the class of non-Gaussian Markovian random processes whose stationary independent increments are distributed according to Lévy stable distributions [1]. Lévy stable laws are important for three fundamental properties: (i) according to generalized central limit theorem, they form the basin of attraction for sums of random variables with diverging variance [2]; (ii) the probability density functions of Lévy stable laws decay in asymptotic power-law form and thus appear naturally in the description of many fluctuation processes with largely scattering statistics characterized by bursts or large outliers; (iii) Lévy flights are statistically self-affine, a property used for the description of random fractal processes. Examples of Lévy flights range from light propagation in fractal medium called Lévy glass [3] and plasma fluctuations in fusion devices [4, 5, 6, 7] to circulation of dollar bills [8] and behavior of the marine verterbrates in response to patchy distribution of food resources [9]; more examples can be found, e.g., in [10, 11]. The Lévy flight dynamics can also stem from a simple Brownian random walk in systems whose operational time typically grows superlinearly with physical time t [12, 13].

Other prominent examples of the processes with heavy-tailed distributions are Lévy Ornstein-Uhlenbeck (OU) process describing the overdamped harmonic oscillator driven by alpha-stable Lévy noise, and fractional Lévy stable motion. The Lévy OU process is a natural generalization of the Gaussian OU process; such generalization has gained popularity, e.g., in finance [14, 15, 16]. The weakly damped harmonic oscillator driven by Lévy noise was discussed in [17, 18]. The Lévy OU process accounts for interdependence (association) of exponential type. On the contrary, fractional motions and fractional noises have an infinite span of interdependence [19, 20]. Fractional processes are also

widely spread in applications [21, 22, 23]. Indeed, in a large class of many-particle systems whose overall dynamics is Markovian, the probe particle coupled with the rest of the system through space correlations exhibit fractional motion with long-ranged non-Markovian memory effects [24, 25, 26]. Fractional Lévy stable motion with long-range dependence was detected in beat-to-beat heart rate fluctuations [27], in solar flare time series [28], and was shown to be a model qualitatively mimicking self-organized criticality signatures in data [29].

What is the measure of interdependence for the processes with infinite variance? Apparently, correlation or spectral power analysis, strictly speaking, can not be used. The alternative measures of dependence are rarely discussed in application-oriented literature.

The notion of covariance (CV) used in correlation analysis, can be generalized for the alpha-stable Lévy process, leading to the notion of covariation [20, 30]. Its definition is based on the Lévy measure, and its practical usefulness is limited. Some results related to the covariation of autoregressive process with Lévy stable distribution are presented in [31]. The other measure is the Lévy correlation cascade [32]. It is defined for infinitely divisible processes, and the properties of that measure are discussed in [33] in detail, see also [34]. However, similar to covariation, the Lévy correlation cascade is of limited practical value because of complicated definition based on the Lévy measure of the underlying process.

Our paper deals with another measure of interdependence called codifference (CD). It is based on the characteristic function of a given process, therefore it can be used not only for alpha-stable processes. Moreover, the codifference in the Gaussian case reduces to the classical covariance, so it can be treated as the natural extension of the well-known measure. On the other hand, according to the definition, it is easy to evaluate the empirical codifference which is based on the empirical characteristic function of the analyzed data. It is worth to mention that the codifference is closely related to the so-called dynamical functional used to study ergodic properties of stochastic processes [35, 36, 20]. In our paper we aim to give an introduction to the concept of codifference and to show its usefulness for analyzing interdependence not only for the processes with infinite variance but for those with finite variance, as well.

The rest of the paper is structured as follows: In Section 2 we give definitions of codifference. In Section 3 we compare autocovariance and autocodifference for Gaussian processes and for non-Gaussian processes with finite variance. In Section 4 we present the autocodifference for processes with infinite variance. The method how to estimate autocodifference from experimental data is described in Section 5 together with the examples taken from simulations. In Section 6 we present the results of real data analysis for two processes with infinite variance. The conclusions are presented in Section 7, and several technical details of calculating autocodifference are collected in the Appendix.

#### 2. Definitions

We start from the definition of codifference for the symmetric alpha-stable (S $\alpha$ S) random variables. Let us remind that the random variable *X* which is S $\alpha$ S with parameter 0 <  $\alpha \le 2$  and scale parameter  $\sigma_X > 0$  has the following characteristic function  $\Phi_X(k)$  [2]:

$$\Phi_X(k) = \langle \exp(ikX) \rangle = \exp\{-\sigma_X^{\alpha}|k|^{\alpha}\},\tag{1}$$

where < ... > denote ensemble average or average over realizations. The codifference of two jointly S $\alpha$ S, 0 <  $\alpha \le 2$ , random variables *X* and *Y* is defined as follows [20]:

$$CD(X,Y) = \sigma_{X-Y}^{\alpha} - \sigma_{X}^{\alpha} - \sigma_{Y}^{\alpha}, \tag{2}$$

where  $\sigma_X, \sigma_Y$  and  $\sigma_{X-Y}$  denote, respectively the scale parameters of *X*, *Y* and *X* – *Y*. The codifference can be also defined in the language of the characteristic function [30, 31, 33]

$$CD(X, Y) = \ln(\langle exp\{iX\} \rangle) + \ln(\langle exp\{-iY\} \rangle) - \ln(\langle exp\{i(X - Y)\} \rangle).$$
(3)

Thus, the definition given in Eq.(2) can be extended to a more general class of random variables, and in the further analysis we use the representation given in (3). We note, that one of the reasons to take the minus sign, i.e., -iY in the second term of the right-hand side of Eq.(3) is to ensure that CD is reduced to covariance in the Gaussian case (with the minus sign); see below.

It is worth to mention that the codifference shares useful properties [20]

- it is always well-defined, since the definition of CD(X, Y) is based on the characteristic functions of appropriate random variables;
- if the random variables are symmetric, then CD(X, Y) = CD(Y, X);
- if X and Y are independent and jointly S $\alpha$ S, then CD(X, Y) = 0. Conversely, if CD(X, Y) = 0 and  $0 < \alpha < 1$ , then X and Y are independent. When  $1 < \alpha < 2$ , CD(X, Y) = 0 does not imply that X and Y are independent. We note that in case of two Gaussian random variables (i.e. S $\alpha$ S with  $\alpha = 2$ ) the independence of X and Y is equivalent to zero covariance.

The above properties confirm that the codifference is an appropriate mathematical tool for measuring the dependence between alpha-stable random variables as well as random variables from more general class of distributions (e.g., infinitely divisible). In the literature one can also find the generalized codifference which is defined as [20]:

$$GCD(X, Y; \theta_1, \theta_2) = \ln(\langle exp\{i\theta_1 X\} \rangle) + \ln(\langle exp\{i\theta_2 Y\} \rangle) - \ln(\langle exp\{i(\theta_1 X + \theta_2 Y)\} \rangle), \tag{4}$$

where  $\theta_1, \theta_2 \in R$ . The generalized codifference is useful in cases when the random variables considered have different scales or different units [18]. Thus, the proper choice of  $\theta_1$  and  $\theta_2$  parameters allows one to analyze the interdependence at the same scale. Also, to make the measure of dependence invariant against the change of units, it is reasonable to take  $\theta_1$  equal to  $\sigma_X^{-1}$  and  $\theta_2$  equal to  $\sigma_Y^{-1}$ . Equation (4) will then define the "cosum" of non-dimensional, normalized variables [18]. The usage of "cosum" has an additional advantage when e.g., asymmetric alpha-stable Lévy variables are discussed, since the sum of two such variables has a distribution which differ only by a scaling factor, whereas the difference has a distribution which belongs to a different class. In the present paper we however stick to the definition (3).

For a stochastic process  $\{X(t)\}$ , the measure CD(X(t), X(s)) called autocodifference is defined as <sup>1</sup>

$$CD(X(t), X(s)) = \ln(\langle exp\{iX(t)\} \rangle) + \ln(\langle exp\{-iX(s)\} \rangle) - \ln(\langle exp\{i(X(t) - X(s))\} \rangle).$$
(5)

For the stationary process the autocodifference CD(X(t), X(s)) depends on |t - s|.

#### 3. Autocodifference versus autocovariance for processes with finite second moment

In this Section we consider autocodifference for the several processes with finite variance, which are widely used in applications. We restrict ourselves with stationary processes and processes with stationary increments.

# 3.1. Gaussian processes

For the Gaussian processes the autocodifference is simply reduced to autocovariance with the minus sign. Indeed, let  $\{X(t)\}$  be a Gaussian process with mean f(t) and variance g(t), then the characteristic function reads:

$$\Phi_{X(t)}(k) = \langle e^{ikX(t)} \rangle = e^{ikf(t) - g(t)k^2/2}.$$
(6)

Then, for fixed t and s (s < t) the increments X(t) - X(s) have also Gaussian distribution with the mean f(t) - f(s) and variance equal to g(t) + g(s) - 2cov(Y(t), Y(s)). Therefore, from Eq.(5) we get:

$$CD(X(t), X(s)) = +if(t) - \frac{g(t)}{2} - if(s) - \frac{g(s)}{2} - i(f(t) - f(s)) + \frac{g(t) + g(s) - 2cov(X(t), X(s))}{2} = -cov(X(t), X(s)).$$

Below we give some examples of the autocodifference for Gaussian processes.

<sup>&</sup>lt;sup>1</sup>In this paper we denote a stochastic process as  $\{X(t)\}$  and the value of the process at time t as X(t).

#### 3.1.1. Gaussian white noise process

The process  $\{b(t)\}$  is called white Gaussian noise if for each t the random variable b(t) has Gaussian distribution with zero mean and finite variance  $\sigma^2$ , and for each  $t \neq s$  the values b(t) and b(s) are uncorrelated. Thus, the autocodifference is given by:

$$CD(b(t), b(s)) = -cov(b(t), b(s)) = \begin{cases} -\sigma^2, & \text{if } t = s, \\ 0, & \text{otherwise.} \end{cases}$$
(7)

### 3.1.2. Ordinary Brownian motion

The process is called ordinary Brownian motion if it has stationary independent increments possessing Gaussian distribution. The standard ordinary Brownian motion  $\{B(t)\}$  has zero mean and variance equal to t. In this case we have:

$$CD(B(t), B(s)) = -cov(B(t), B(s)) = -\min\{t, s\}.$$
 (8)

### 3.1.3. Gaussian Ornstein-Uhlenbeck process

The Gaussian Ornstein-Uhlenbeck process is defined as a stationary solution of the Langevin equation for the overdamped oscillator:

$$\frac{dY(t)}{dt} + \lambda Y(t) = b(t), \ \lambda > 0 \tag{9}$$

where  $\{b(t)\}\$  is the Gaussian white noise (heuristically b(t) = dB(t)/dt). The Gaussian Ornstein-Uhlenbeck process has the following moving average representation:

$$Y(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} d\tilde{B}(s), \tag{10}$$

where  $\tilde{B}(t)$  is an extension of the Brownian motion for the negative axis, that is

$$\tilde{B}(t) = \begin{cases} B(t), & \text{when } t \ge 0, \\ B(-t), & \text{otherwise.} \end{cases}$$
(11)

We remind that  $\{Y(t)\}$  is the only stationary and Markovian - Gaussian stochastic process, according to the Doob's theorem [38]. The autocodifference is given by:

$$CD(Y(t), Y(s)) = -cov(Y(t), Y(s)) = -\frac{e^{-\lambda|t-s|}}{2\lambda}.$$
 (12)

# 3.1.4. Fractional Brownian motion

The fractional Brownian motion is a zero mean Gaussian process  $\{B_H(t)\}$  defined as follows [19]:

$$B_{H}(t) = \int_{-\infty}^{0} \left\{ (t-u)^{H-1/2} - (-u)^{H-1/2} \right\} dB(u) + \int_{0}^{t} (t-u)^{H-1/2} dB(u),$$
(13)

where  $\{B(t)\}\$  is the classical Brownian motion. The *H* parameter called Hurst exponent is a real number from the interval (0, 1). The process is self-similar and reduces to ordinary Brownian motion, if H = 1/2. The parameter *H* controls the type of diffusion, namely the process is superdiffusive for H > 1/2 while subdiffusive for H < 1/2. The autocodifference takes the following form:

$$CD(B_H(t), B_H(s)) = -cov(B_H(t), B_H(s)) = -\frac{k(H)}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$
(14)

where  $k(H) = Var(B_H(1)) = \int_0^\infty \left| (1+x)^{H-1/2} - x^{H-1/2} \right|^2 dx + \frac{1}{2H}.$ 

# 3.1.5. Fractional Gaussian noise

The fractional Gaussian noise  $\{b_H(t)\}$  is defined heuristically as the derivative of the fractional Brownian motion, i.e.  $b_H(t) = dB_H(t)/dt$ . Therefore, the autocodifference of the process takes the following form [39]:

$$CD(b_H(t), b_H(s)) = -cov(b_H(t), b_H(s)) = \frac{k(H)}{2}(|t - s + 1|^{2H} - 2|t - s|^{2H} + |t - s - 1|^{2H}),$$
(15)

where  $k(H) = Var(b_H(t)) = \int_0^\infty |(1+x)^{H-1/2} - x^{H-1/2}|^2 dx + \frac{1}{2H}$ . Detailed derivation of covariance structure for both fractional Brownian motion and fractional Gaussian noise is presented in [39, 40].

# 3.2. Non-Gaussian processes

# 3.2.1. Poisson process

The Poisson process with intensity  $\lambda$  is a continuous-time counting process  $\{P(t)\}$  which has stationary, independent increments [41, 42]. The increment P(t) - P(s) (for t > s) has Poisson distribution with parameter  $\lambda(t - s)$ . The covariance function of the Poisson process takes the following form:

$$cov(P(t), P(s)) = \min\{t, s\}\lambda.$$
(16)

The characteristic function of the Poisson process is given by  $\Phi_{P(t)}(k) = e^{t\lambda(e^{ik}-1)}$ , therefore the autocodifference takes the following form:

$$CD(P(t), P(s)) = -2\lambda \min\{t, s\}(1 - \cos 1).$$
 (17)

Some interesting extensions of Poisson process one can find in [43].

# 3.2.2. Tempered stable Lévy process

The tempered stable distributions were suggested in [44] and studied in [45] in more detail. The general mathematical description of this class of processes was established in [46]. In our paper we consider the tempered stable distribution with the Lévy triplet ( $\kappa^2$ ,  $\nu$ ,  $\gamma$ ), which is defined as follows [47]:

$$\begin{split} \kappa &= 0 \\ \nu(dx) &= (\tilde{C}_+ e^{-\lambda_+ x} \mathbf{1}_{x > 0} + \tilde{C}_- e^{-\lambda_- |x|} \mathbf{1}_{x < 0}) \frac{dx}{|x|^{\alpha + 1}} \\ \gamma &= m - \int_{|x| > 1} x \nu(dx), \end{split}$$

where  $\tilde{C}_+, \tilde{C}_-, \lambda_+, \lambda_- > 0, \alpha \in (0, 2)$  and  $m \in R$ .

A Lévy process (i.e. process with independent stationary increments) having tempered stable distribution is called a tempered stable process with parameters  $\alpha$ ,  $\tilde{C}_+$ ,  $\tilde{C}_-$ ,  $\lambda_+$ ,  $\lambda_-$ , m.

For simplicity we consider the special case, namely we assume here m = 0 and  $\lambda_+ = \lambda_- = \lambda$ ,  $\alpha > 1$ . Moreover if we substitute

$$C_{+} = \frac{1}{\Gamma(-\alpha)}\tilde{C}_{+}, \quad C_{-} = \frac{1}{\Gamma(-\alpha)}\tilde{C}_{-},$$

then it can be shown that in this case the characteristic function  $\phi_{T(t)}(k) = Ee^{ikT(t)}$  of the tempered stable process  $\{T(t)\}$  takes the following form:

$$\phi_{T(t)}(k) = \exp\left\{t\left[i\alpha k\lambda^{\alpha-1}(C_{+} - C_{-}) + C_{+}(\lambda - ik)^{\alpha} + C_{-}(\lambda + ik)^{\alpha} - \lambda^{\alpha}(C_{+} + C_{-})\right]\right\}.$$
(18)

The tempered stable Lévy process (called also truncated Lévy flight) is a process  $\{T(t)\}$  with independent stationary increments possessing tempered stable distribution described above. As an example, we analyze the symmetric tempered stable process  $\{T(t)\}$ , i.e. such that the characteristic function of T(t) is given by:

$$\Phi_{T(t)}(k) = \exp\left\{t\left[(\lambda + ik)^{\alpha} + (\lambda - ik)^{\alpha} - 2\lambda^{\alpha}\right]\right\}.$$
(19)

The tempered stable process is an extension of the alpha-stable Lévy process, and for  $\lambda = 0$  the random variable  $T_{\alpha,\lambda}(t)$  reduces to the appropriate alpha-stable random variable, see [48].

The autocovariance of the analyzed tempered stable process has the following form:

$$cov(T(t), T(s)) = 2\alpha(\alpha - 1)\lambda^{\alpha - 2} \min\{s, t\}.$$
(20)

The autocodifference is given by

$$CD(T(t), T(s)) = 2[(\lambda - i)^{\alpha} + (\lambda + i)^{\alpha} - 2\lambda^{\alpha}]\min\{t, s\}.$$
(21)

For more details see Appendix, Section A1.

3.2.3. Laplace motion

The Laplace motion (called also variance gamma process) is a process { $\Lambda(t)$ },  $t \ge 0$  defined as follows [49]:

$$\Lambda(t) = B(G(t)),\tag{22}$$

where  $\{B(\cdot)\}$  is an ordinary Brownian motion, and  $\{G(t)\}$  is a Gamma process with parameters  $\gamma$ ,  $\lambda$  defined as a pure jump Lévy increasing process with increments having Gamma distribution. It has the following characteristic function:

$$\Phi_{G(t)}(k) = \langle \exp(ikG(t)) \rangle = (1 - ik/\lambda)^{-\gamma t}, k \in \mathbb{R}.$$
(23)

The Laplace motion has been successfully applied in the modeling of credit risk in structural models [50, 51, 52]. Some extensions of this process one can find, e.g., in [53].

The process  $\{\Lambda(t)\}\$  has a zero mean and stationary independent increments. The covariance function reads

$$cov(\Lambda(t), \Lambda(s)) = \frac{\Gamma(\gamma s + 1)}{\lambda \Gamma(\gamma s)}, s < t,$$
(24)

where  $\Gamma(a)$  is a gamma function, while the autocodifference reads

$$CD(\Lambda(t), \Lambda(s)) = -2\gamma s \ln(1 + 1/(2\lambda)), s < t.$$
<sup>(25)</sup>

See Appendix, Section A2, for more details.

### 4. Autocodifference for processes with infinite variance

In this section we analyze the processes for which the autocovariance function is not defined, and then the autocodifference is the main measure of dependence.

### 4.1. White Lévy noise

The white Lévy noise  $\{l_{\alpha}(t)\}, t \ge 0$ , is a process such that for each  $s \ne t$  the random variables  $l_{\alpha}(s)$  and  $l_{\alpha}(t)$  are independent, and for each *t* the random variable  $l_{\alpha}(t)$  has the following characteristic function:

$$\Phi_{l_{\alpha}(t)}(k) = \begin{cases} \exp\{-\sigma^{\alpha}|k|^{\alpha}(1-i\beta sgn(k)\tan\frac{\pi\alpha}{2}) + ik\mu\} & \text{if } \alpha \neq 1, \\ \exp\{-\sigma|k|(1-i\beta\frac{2}{\pi}sgn(k)\ln|k|) + ik\mu)\} & \text{if } \alpha = 1, \end{cases}$$
(26)

where  $0 < \alpha \le 2$  is the stability index,  $\sigma > 0$  the scale parameter,  $-1 \le \beta \le 1$  the skewness parameter, and  $\mu \in R$  is the shift parameter.

The autocodifference reads as

$$CD(l_{\alpha}(t), l_{\alpha}(s)) = \begin{cases} -2\sigma^{\alpha}, & \text{if } t = s, \\ 0, & \text{otherwise.} \end{cases}$$
(27)

It is worth to mention that the above formula is valid not only for symmetric Lévy noise.

# 4.2. Lévy flights

The Lévy flight (called also alpha-stable Lévy motion) is the process  $\{L_{\alpha}(t)\}, t \ge 0$ , with independent stationary increments possessing alpha-stable distribution, i.e. in general case for each t the random variable  $L_{\alpha}(t)$  has the stable distribution with index of stability  $\alpha$ , scale parameter  $t^{1/\alpha}\sigma$ , skewness  $\beta$  and shift parameter  $\mu = 0$ . The autocodifference for the analyzed process takes the form

$$CD(L_{\alpha}(t), L_{\alpha}(s)) = -2\sigma^{\alpha} \min\{t, s\}.$$
(28)

Let us note, for  $\alpha = 2$  and  $\sigma = 1/\sqrt{2}$  (parameters of symmetric alpha-stable distribution corresponding to the standard Gaussian one) the above formula reduces to (8).

#### 4.3. Lévy Ornstein-Uhlenbeck process

The Lévy Ornstein-Uhlenbeck process  $\{Y_{\alpha}(t)\}$  is defined via the following Langevin equation:

$$\frac{dY_{\alpha}(t)}{dt} + \lambda Y_{\alpha}(t) = l_{\alpha}(t), \quad \lambda > 0,$$
(29)

where  $\{l_{\alpha}(t)\}\$  is white Lévy noise. The process was examined, for example, in [54]. The stationary solution of equation (29) reads

$$Y_{\alpha}(t) = \int_{-\infty}^{t} e^{-\lambda(t-u)} d\tilde{L}_{\alpha}(u),$$
(30)

where

$$\tilde{L}_{\alpha}(t) = \begin{cases} L_{\alpha}(t), & \text{when } t \ge 0, \\ L_{\alpha}(-t), & \text{otherwise.} \end{cases}$$
(31)

For simplicity, we assume that in the considered case  $\{L_{\alpha}(t)\}\$  is symmetric alpha-stable Lévy process i.e. process, for which the increments have alpha-stable distribution with the  $\mu = 0$ ,  $\beta = 0$  and scale parameter  $\sigma$ . Then the autocodifference takes the form

$$CD(Y_{\alpha}(t), Y_{\alpha}(s)) = -\sigma^{\alpha} \frac{1 + e^{-\lambda \alpha |t-s|} - |1 - e^{-\lambda |t-s|}|^{\alpha}}{\lambda \alpha}.$$
(32)

For more details see Appendix, Section A3. We note that for  $\alpha = 2$  and  $\sigma = 1/\sqrt{2}$  (parameters of symmetric alphastable distribution corresponding to the standard Gaussian one) the above formula reduces to (12).

#### 4.4. Fractional Lévy motion

The fractional Lévy motion is a process  $\{L_{\alpha,H}(t)\}$  defined for any 0 < H < 1 as follows [20, 55, 56, 57]:

$$L_{\alpha,H}(t) = \int_{-\infty}^{0} \left\{ (t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha} \right\} dL_{\alpha}(u) + \int_{0}^{t} (t-u)^{H-1/\alpha} dL_{\alpha}(u),$$
(33)

where  $\{L_{\alpha}(t)\}$  is the symmetric alpha-stable Lévy process with index of stability  $\alpha \in (0, 2)$ . For simplicity we assume  $\sigma = 1$ . The process  $\{L_{\alpha,H}(t)\}$  is self-similar, stationary-increment process with infinite second moment. Similar to fractional Brownian motion, the parameter *H* controls the diffusion law, namely for  $H < 1/\alpha$  and  $H > 1/\alpha$  the fractional Lévy motion exhibits sub- and superdiffusive behavior, respectively [58]. The autocodifference for fractional Lévy motion has the following form:

$$CD(L_{\alpha,H}(t), L_{\alpha,H}(s)) = k(H, \alpha) \left( |t - s|^{\alpha H} - |t|^{\alpha H} - |s|^{\alpha H} \right), \tag{34}$$

where  $k(H, \alpha) = \int_0^\infty \left| (1+u)^{H-1/\alpha} - u^{H-1/\alpha} \right|^\alpha du + \frac{1}{H\alpha}$ . See Appendix, Section A4, for more details.

Let us remind that for  $H = 1/\alpha$  the fractional Lévy motion reduces to Lévy flights. In this case the autocodifference (34) takes the form  $-2 \min\{s, t\}$  which is consistent with formula (28) under the assumption  $\sigma = 1$ .

# 4.5. Fractional Lévy noise

The fractional Lévy noise  $\{l_{\alpha,H}(t)\}$  is defined heuristically as the derivative of the fractional Lévy motion  $\{L_{\alpha,H}(t)\}$ . For large *t* the autocodifference has a power law form [59]:

For large *t* the autocodifference has a power law form [59]: If either  $0 < \alpha \le 1$ , 0 < H < 1 or  $1 < \alpha < 2$ ,  $1 - \frac{1}{\alpha(\alpha-1)} < H < 1$ ,  $H \ne 1/\alpha$ , then for  $t \to \infty$ 

$$CD(l_{\alpha,H}(t), l_{\alpha,H}(0)) \sim Ct^{\alpha H - \alpha}.$$
(35)

If  $1 < \alpha < 2$  and  $0 < H < 1 - \frac{1}{\alpha(\alpha-1)}$ , then for  $t \to \infty$ 

$$CD(l_{\alpha,H}(t), l_{\alpha,H}(0)) \sim Dt^{H-1/\alpha-1},$$
(36)

where the constants C and D are time-independent.

Note that, for  $\alpha = 2$  the fractional Lévy noise reduces to fractional Gaussian noise. In this case, the autocodifference given in (15) behaves as  $t^{2H-2}$  for  $t \to \infty$  which is consistent with (35) [59].

# 4.6. Superdiffusion continuous time random walk-like process

We consider superdiffusive continuous time random walk-like process which is defined as follows [12]:

$$Z(t) = B(L_{\alpha}(t)), \tag{37}$$

where  $\{B(t)\}\$  is the ordinary Brownian motion and  $\{L_{\alpha}(t)\}\$  the Lévy flight process described in Section 4.2, with  $0 < \alpha < 1, \beta = 1, \mu = 0$  and scale parameter  $\sigma > 0$ , i.e. the process for which the Laplace transform is given by:

$$\mathcal{L}_{L_{\alpha}(t)}(k) = \langle \exp\{-kL_{\alpha}(t)\} \rangle = \exp\{-t(\sigma|k|)^{\alpha}\}.$$
(38)

The processes  $\{B(t)\}$  and  $\{L_{\alpha}(t)\}$  are independent.

The process  $\{L_{\alpha}(t)\}\$  in definition (37) plays the role of time (i.e. is should be non-negative increasing process), therefore the above restrictions on the parameters have to be assumed.

Since Brownian motion and Lévy flight processes have independent stationary increments, then the process  $\{Z(t)\}$  also possesses this property [60]. The possible applications of superdiffusion continuous time random walk-like process are transport in heterogeneous catalysis, micelle systems, reactions and transport in polymer systems under conformational motion and dynamical systems [13].

Since  $\langle B^2(L_\alpha(t)) \rangle = \langle L_\alpha(t) \rangle$  and the mean of  $L_\alpha(t)$  does not exists, then the process  $\{Z(t)\}$  has infinite second moment. The autocodifference for the process  $\{Z(t)\}$  is given by:

$$CD(Z(t), Z(s)) = -2^{-\alpha+1}\sigma^{\alpha}\min\{t, s\}.$$
 (39)

See Appendix, Section A5, for more details.

#### 5. How to estimate codifference from data

We define an estimator of autocodifference in the form:

$$\widehat{CD}(X(t), X(s)) = \ln(\hat{\phi}(1, 0, X(t), X(s))) + \ln(\hat{\phi}(0, -1, X(t), X(s))) - \ln(\hat{\phi}(1, -1, X(t), X(s))),$$
(40)

where  $\hat{\phi}(u, v, X(t), X(s))$  is an estimator of the characteristic function:

$$\hat{\phi}(u, v, X(t), X(s)) = <\exp\{i(uX(t) + vX(s))\} > .$$
(41)

In [61] an efficient methodology is introduced for estimating the codifference from a single trajectory of stationary process. Namely, if  $\{x_k, k = 1, ..., N\}$  is realization of a stationary process  $\{X(t)\}$ , then the estimator of the characteristic function takes the form:

$$\hat{\phi}(u, v, X(t), X(s)) = \frac{1}{N} \sum_{k=1}^{N-|t-s|} \exp(i(ux_{k+|t-s|} + vx_k)).$$
(42)

For a nonstationary process we are not able to estimate empirical autocodifference by using only a single trajectory, therefore the above estimator requires modification. Suppose, we have *M* trajectories of a nonstationary process  $\{X(t)\}$ . Let us take a sample  $\{x_k^t, k = 1, ..., M\}$  being a realization of a random variable X(t), that is the values of the process  $\{X(t)\}$  taken at a fixed time *t*, and a sample  $\{x_k^s : k = 1, ..., M\}$  composed from the values of the process  $\{X(t)\}$  taken at a fixed time *s*. By construction, both samples consist of independent identically distributed random variables. Thus, in the nonstationary case the estimator of characteristic function  $\phi(u, v, X(t), X(s))$  is defined as:

$$\hat{\phi}(u, v, X(t), X(s)) = \frac{1}{M} \sum_{k=1}^{M} \exp(i(ux_k^t + vx_k^s)).$$
(43)

In Fig. 1 we show a single trajectory of the tempered stable Lévy motion and compare estimator of the autocodifference obtained from Eqs.(40) and (43) with theoretical value given by Eq.(21). One observes perfect agreement between theoretical and empirical results. In Fig. 2 we present a trajectory of Lévy stable Ornstein-Uhlenbeck process together with the theoretical autocodifference given by Eq.(32) and its estimator obtained from Eqs.(40) and (42). In Fig. 3 we plot the path of fractional Lévy noise and also compare the theoretical autocodifference, Eq.(36), and its estimator. In all three cases one observes almost perfect agreement between the empirical and theoretical results.



Figure 1: Top panel: single trajectory of the tempered stable Lévy motion  $\{T_{\alpha,\lambda}(t)\}$  with parameters  $\alpha = 0.9, \lambda = 0.01$ . Bottom panel: Estimator of autocodifference  $CD(T_{\alpha,\lambda}(t), T_{\alpha,\lambda}(20))$  calculated with 1000 trajectories, and the theoretical value given by Eq.(21).

# 6. Real data analysis

### 6.1. Plasma data

In this section we investigate the data obtained in experiment on the controlled fusion device. The important characteristics of edge plasma turbulence, such as fluctuation amplitudes, spectra, and turbulence-induced transport are investigated in the Uragan-3M (U-3M) stellarator torsatron by the use of high resolution measurements of density (ion saturation current) and potential (floating potential) fluctuations with the help of movable Langmuir probe arrays. We address the reader to [62] for the details of experimental set-up and description of the data base. Here we present the analysis of the ion saturation current fluctuations (in mA) measured at the small torus radial position r = 9.9cm. Similar data were analyzed in [7]. In Fig. 4 we present the examined time series (top panel) and corresponding



Figure 2: The trajectory of stable Lévy Ornstein-Uhlenbeck process  $\{Y_{\alpha}(t)\}$  with parameters  $\alpha = 1.6382$ ,  $\lambda = 0.0045$  (top panel), together with theoretical autocodifference  $CD(Y_{\alpha}(t), Y_{\alpha}(0))$  and its estimator (bottom panel).



Figure 3: The trajectory of fractional Lévy noise  $l_{\alpha,H}(t)$  with parameters  $\alpha = 1.95$ , H = 0.8 (top panel), together with the estimator of autocodifference  $CD(l_{\alpha,H}(t), l_{\alpha,H}(0))$  and fitted power function  $t^{\alpha(H-1)}$  (bottom panel).



Figure 4: The examined time series of the ion saturation current fluctuations (in mA) registered in the U-3M torsatron at the small torus radial positions r = 9.9cm (top panel), and the estimator of autocodifference (bottom panel).

estimator of the autocodifference (bottom panel).

The estimator of autocodifference has non-zero value at t = 0 and then drops to zero. Therefore, we infer that the data can be considered as a white noise. Moreover, we also observe non-Gaussian behavior of the underlying series. Thus, we propose to model the process by using white Lévy noise. We confirm our assumption by using the Anderson-Darling goodness-of-fit test for stable distribution [63], which indicates that the data come from Lévy stable distribution (p-value is equal to 0.88). Then, by using the regression method [63] we estimate the index of stability. As a result we get  $\hat{\alpha} = 1.95$ .

On the basis of autocodifference we can also estimate the scale parameter  $\sigma$  of the analyzed series. In order to do this we compare the value of the estimator of autocodifference for t = 0 with the theoretical value  $-2\sigma^{\alpha}$ . As the result we obtain  $\hat{\sigma} = 1.05$ . We then compare the value obtained with the estimates of  $\sigma$  parameter made by other methods, namely, regression [63] and McCulloch [64] methods. The obtained values are  $\hat{\sigma} = 1.06$  and  $\hat{\sigma} = 1.03$ , respectively, which is in good agreement with the value obtained from the autocodifference.

### 6.2. Financial data

As a second example we analyze time series that describes closing prices of the investment holding company Cosco Pacific Ltd. The data are quoted daily in the period 04.01.2000-14.01.2013, [65]. In Fig. 5 (top panel) we present the examined time series.

We recall that the Gaussian Ornstein-Uhlenbeck process originally used by Vasicek for the analysis of interest rates [14], was successfully applied to many other data from financial markets [15]. In line with these findings, we intend to describe the data analyzed by using Ornstein-Uhlenbeck process. However, in contrast to the classical model, significant jumps in the examined data are clearly observed, which may indicate non-Gaussian behavior of the process. We thus suggest the Lévy Ornstein-Uhlenbeck process defined in (29) as a candidate for fitting the data after removing the mean. At the first step of our analysis we estimate the relaxation parameter  $\lambda$  that enters Eq.(29). Here we employ the Whittle estimation method described in [66] (also used in [67]), which is based on the sample periodogram of the analyzed time series. Once the  $\hat{\lambda}$  estimate is found, the residuals of the process can be derived (recall that in our case the residual of the Lévy Ornstein-Uhlenbeck model is a process  $l_{\alpha}(t)$  in (29)). On the basis of



Figure 5: The examined time series that describes closing prices (in USD) of Cosco Pacific Ltd. in the period 04.01.2000-14.01.2013 (top panel), and the estimator of autocodifference and theoretical function given by Eq.(32) (bottom panel).

the residual series, by using the regression method, we can estimate the index of stability  $\alpha$ . We also confirmed the stable distribution of the residual series by using the Anderson-Darling goodness of fit test. As the result, we obtain  $\hat{\lambda} = 0.0045$  and  $\hat{\alpha} = 1.64$ . In Fig. 5 we present the estimator of autocodifference along with the theoretical value given by Eq.(32).

Similar to the plasma data analysis, we estimate the scale parameter  $\sigma$  from the autocodifference function by using estimated values of  $\lambda$  and  $\alpha$  in Eq.(32). As the result we obtain  $\hat{\sigma} = 0.17$ , which is exactly the same value that the regression and McCulloch methods give for these data.

### 7. Conclusions

In this paper we have examined the codifference, a general measure of interdependence, that can be considered as an alternative to covariance function. We have indicated the importance of codifference especially for processes with non-Gaussian distribution, for which the correlation function is not defined. For a class of Gaussian processes the autocodifference is simply reduces to autocovariance with negative sign. We then analyzed this measure for several well-known processes with finite variance and show close similarity between autocodifference and autocovariance. We furthermore present the generic examples of the processes, for which the autocovariance does not exist, therefore the codifference becomes of particular importance. After giving a simple practical recipe how to estimate the autocodifference from experimental data for both stationary and nonstationary processes, we estimated codifference from the surrogate data and analyze two real data sets representing random fluctuations observed in laboratory plasma and prices on financial market. For both examples we demonstrated how the estimated autocodifference it is possible to estimate parameters of the fitted model. Summarizing, we conclude that the codifference serves as a convenient practical tool to study interdependence for stochastic processes with infinite and finite variances as well.

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# Appendix

# A1. Tempered stable Lévy process

Since the process  $\{T(t)\}$  has stationary independent increments, zero mean, and  $\langle T^2(t) \rangle = 2t\alpha(\alpha - 1)\lambda^{\alpha-2}$ , then for s < t we obtain

$$cov(T(t), T(s)) = \langle T(t)(T(s) \rangle = \langle T(s)(T(t) - T(s)) \rangle + \langle T^{2}(s) \rangle = 2s\alpha(\alpha - 1)\lambda^{\alpha - 2}$$

Using the same reasoning for t < s we get  $cov(T(t), T(s)) = 2\alpha(\alpha - 1)\lambda^{\alpha-2} \min\{s, t\}$ . Now let us calculate characteristic function of T(t) - T(s) for s < t. We have

$$<\exp\{i(T(t) - T(s))\}> = <\exp\{iT(t - s)\}> =\exp\{(t - s)[(\lambda + i)^{\alpha} + (\lambda - i)^{\alpha} - 2\lambda^{\alpha}]\}.$$

Moreover,

$$<\exp\{iT(t)\}>=\exp\{t\left[(\lambda+i)^{\alpha}+(\lambda-i)^{\alpha}-2\lambda^{\alpha}\right]\},\\<\exp\{-iT(s)\}>=\exp\{s\left[(\lambda+i)^{\alpha}+(\lambda-i)^{\alpha}-2\lambda^{\alpha}\right]\}$$

As a final result, we obtain:

$$CD((T(t), T(s)) = -(t-s) [(\lambda+i)^{\alpha} + (\lambda-i)^{\alpha} - 2\lambda^{\alpha}] + t [(\lambda+i)^{\alpha} + (\lambda-i)^{\alpha} - 2\lambda^{\alpha}] + s [(\lambda+i)^{\alpha} + (\lambda-i)^{\alpha} - 2\lambda^{\alpha}]$$
$$= 2s [(\lambda-i)^{\alpha} + (\lambda+i)^{\alpha} - 2\lambda^{\alpha}].$$

Therefore, we have

$$CD((T(t), T(s)) = 2\min\{t, s\}[(\lambda - i)^{\alpha} + (\lambda + i)^{\alpha} - 2\lambda^{\alpha}]$$

# A2. Laplace motion

The gamma process  $\{G(t)\}\$  with parameters  $\gamma$ ,  $\lambda$  is a pure jump Lévy increasing process with the moments

$$E(G^{n}(t)) = \lambda^{-n} \Gamma(\gamma t + n) / \Gamma(\gamma t), n \ge 0,$$
(44)

and  $\Gamma(\cdot)$  is a Gamma function. If  $\Lambda(t) = B(G(t))$ , then  $\langle \Lambda(t) \rangle = 0$ , and we have

$$\begin{aligned} \cos(\Lambda(t), \Lambda(s)) &= < \Lambda(t)\Lambda(s) > = < (\Lambda(t) - \Lambda(s) + \Lambda(s))\Lambda(s) > = < (\Lambda(t) - \Lambda(s))\Lambda(s) + \Lambda^{2}(s) > \\ &= < \Lambda^{2}(s) > = < B^{2}(u)|G(s) = u > = < G(s) > = \frac{\Gamma(\gamma s + 1)}{\lambda\Gamma(\gamma s)}. \end{aligned}$$

Let us calculate the autocodifference of the process { $\Lambda(t)$ }. For s < t we have:

$$\begin{aligned} \ln < \exp(i(\Lambda(t) - \Lambda(s)) > &= \ln < \exp(i(\Lambda(t - s)) > = \ln < \exp(iB(u))|G(t - s) = u > \\ &= \ln < \exp(-1/2\Gamma(t - s)) > = (-\gamma(t - s))\ln(1 + 1/(2\lambda)). \end{aligned}$$

Thus the autocodifference for s < t is equal to

$$CD(\Lambda(t), \Lambda(s)) = -\gamma t \ln(1 + 1/(2\lambda)) - \gamma s \ln(1 + 1/(2\lambda)) + \gamma(t - s) \ln(1 + 1/(2\lambda))$$
  
= -2\gamma s \ln(1 + 1/(2\lambda)).

# A3. Lévy Ornstein-Uhlenbeck process

We note that the random variable  $Y_{\alpha}(t)$  given by (30) for  $\lambda > 0$  is S $\alpha$ S with  $\sigma^{\alpha} = \frac{\sigma^{\alpha}}{\alpha \lambda}$ . This fact follows directly from the Propositions 3.4.1 and 3.5.2 in [20]. To obtain the formula for the autocodifference, we use the relation between the scale parameters and codifference given in (2). Then we obtain:

$$\sigma^{\alpha}_{Y_{\alpha}(t),Y_{\alpha}(t)} = \sigma^{\alpha}_{Y_{\alpha}(s),Y_{\alpha}(s)} = \frac{\sigma^{\alpha}}{\lambda\alpha}.$$

From Eq.(30) for s < t we get:

$$Y_{\alpha}(t) - Y_{\alpha}(s) = \int_{-\infty}^{\infty} f(t-x) - f(s-x) dL_{\alpha}(x),$$

where  $f(t-x) = e^{-\lambda(t-x)} \mathbf{1}_{\{x < t\}}$ . Using Proposition 3.5.2 in [20] we obtain the scale parameter of random variable  $Y_{\alpha}(t) - Y_{\alpha}(s)$  for s < t:

$$\sigma_{Y_{\alpha}(t)-Y_{\alpha}(s)}^{\alpha} = \sigma^{\alpha} \int_{-\infty}^{\infty} |f(t-x) - f(s-x)|^{\alpha} dx.$$

Therefore we have:

$$\sigma_{Y_{\alpha}(t)-Y_{\sigma}(s)}^{\alpha} = \sigma^{\alpha} \int_{-\infty}^{s} |f(t-x) - f(s-x)|^{\alpha} dx + \sigma^{\alpha} \int_{s}^{t} |f(t-x)|^{\alpha} dx$$
$$= \sigma^{\alpha} \int_{-\infty}^{s} |e^{-\lambda(t-x)} - e^{-\lambda(s-x)}|^{\alpha} dx + \sigma^{\alpha} \int_{s}^{t} e^{-\lambda\alpha(t-x)} dx$$
$$= \sigma^{\alpha} |e^{-\lambda t} - e^{-\lambda s}|^{\alpha} \int_{-\infty}^{s} e^{\lambda\alpha x} dx + \sigma^{\alpha} \frac{1 - e^{-\lambda\alpha(t-s)}}{\lambda\alpha}$$
$$= \sigma^{\alpha} \frac{|1 - e^{-\lambda(t-s)}|^{\alpha} + 1 - e^{-\lambda\alpha(t-s)}}{\lambda\alpha}.$$

Then we obtain:

$$CD(Y_{\alpha}(t), Y_{\alpha}(s)) = -\sigma^{\alpha} \frac{1 + e^{-\lambda \alpha(t-s)} - |1 - e^{-\lambda(t-s)}|^{\alpha}}{\lambda \alpha}.$$

# A4. Fractional Lévy motion

From Proposition 3.4 in [68] we infer that for the stable integrals  $\int_S f(x)dL_\alpha(x)$ , where  $\{L_\alpha(t)\}$  is a symmetric Lévy motion with  $0 < \alpha < 2$  and  $S \subset R$ , the following holds:

$$<\exp\left\{i\theta\int_{S}f(x)dL_{\alpha}(x)\right\}>=\exp\left\{-|\theta|^{\alpha}\int_{S}|f(x)|^{\alpha}dx\right\}$$
(45)

if  $\int_{S} |f(x)|^{\alpha} dx < \infty$ .

Using formula (45) and the fact that fractional Lévy motion  $\{L_{\alpha,H}(t)\}$  is self similar and has stationary increments, for t < s we obtain:

$$CD(L_{\alpha,H}(t), L_{\alpha,H}(s)) = \ln(\langle exp\{it^{H}L_{\alpha,H}(1)\} \rangle) + \ln(\langle exp\{-is^{H}L_{\alpha,H}(1)\} \rangle) - \ln(\langle exp\{i(t-s)^{H}L_{\alpha,H}(1)\} \rangle).$$

Therefore, we have:

$$CD(L_{\alpha,H}(t), L_{\alpha,H}(s)) = \left(|t-s|^{\alpha H} - |t|^{\alpha H} - |s|^{\alpha H}\right)k(H, \alpha),$$

where

$$k(H,\alpha) = \int_{-\infty}^{\infty} \left| \max(1-u,0)^{H-1/\alpha} - \max(-u,0)^{H-1/\alpha} \right|^{\alpha} du.$$

Now we show that the above integral converges. Indeed, we have

$$\begin{split} k(H,\alpha) &= \int_{-\infty}^{0} \left| (1-u)^{H-1/\alpha} - (-u)^{H-1/\alpha} \right|^{\alpha} du + \int_{0}^{1} |1-u|^{H\alpha-1} du = \int_{0}^{\infty} \left| (1+u)^{H-1/\alpha} - u^{H-1/\alpha} \right|^{\alpha} du + \frac{1}{H\alpha} \\ &= \int_{0}^{1} \left| (1+u)^{H-1/\alpha} - u^{H-1/\alpha} \right|^{\alpha} du + \int_{1}^{\infty} \left| (1+u)^{H-1/\alpha} - u^{H-1/\alpha} \right|^{\alpha} du + \frac{1}{H\alpha}. \end{split}$$

The first integral converges, because

$$\int_0^1 \left| (1+u)^{H-1/\alpha} - u^{H-1/\alpha} \right|^\alpha du \le \int_0^1 (1+u)^{H\alpha-1} du = \frac{2^{H\alpha}}{H\alpha}.$$
14

Note that

$$(1+u)^{H-1/\alpha} - u^{H-1/\alpha} = (H-1/\alpha) \int_0^1 (x+u)^{H-1/\alpha-1} du$$

Therefore, for  $H \neq 1/\alpha$  we have

$$\left| (1+u)^{H-1/\alpha} - u^{H-1/\alpha} \right|^{\alpha} = \left| (H-1/\alpha) \right|^{\alpha} \left| \int_{0}^{1} (x+u)^{H-1/\alpha-1} dx \right|^{\alpha} \le \left| (H-1/\alpha) \right|^{\alpha} |1+u|^{\alpha(H-1)-1},$$

which gives for H < 1

$$\int_{1}^{\infty} \left| (1+u)^{H-1/\alpha} - u^{H-1/\alpha} \right|^{\alpha} du \le \int_{1}^{\infty} \left| (H-1/\alpha) \right|^{\alpha} |1+u|^{\alpha(H-1)-1} du = \frac{|(H-1/\alpha)|^{\alpha}}{\alpha(1-H)}.$$

A5. Supediffusion continuous time random walk-like process

Let us first calculate the characteristic function of Z(t). We have:

$$\Phi_{Z(t)}(k) = <\exp\{-\frac{L_{\alpha}(t)k^2}{2}\}>.$$

Now taking the formula of the Laplace transform of the process  $\{Z(t)\}$  given in (38), we obtain:

$$\Phi_{Z(t)}(k) = \exp\{-tk^2\sigma^{\alpha}2^{-\alpha}\}$$

Since  $Z(t) - Z(s) \sim Z(t - s)$  for t > s (" ~ " means equality in distribution), then we obtain:

$$CD(Z(t), Z(s)) = -t\sigma^{\alpha} 2^{-\alpha} - s\sigma^{\alpha} 2^{-\alpha} + (t-s)\sigma^{\alpha} 2^{-\alpha} = -\sigma^{\alpha} 2^{-\alpha+1} s = -\sigma^{\alpha} 2^{-\alpha+1} \min\{t, s\}.$$

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