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CODIMENSION-ONE FOLIATIONS AND ORIENTED GRAPHS

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Abstract. In this paper, an oriented graph $G(M, F)$ is assigned to each codimension-one foliation (M, F), and topological relations between (M, F) and $G(M, F)$ are studied. A strong relation between admissible functions of (M, F) and $G(M, F)$ is given.

1. Introduction. Let (M, F) be a transversely oriented codimension-one foliation F of a closed oriented manifold M. On the set of all leaves of F, Novikov [6] introduced a partial order to define a so-called Novikov component. On the other hand, it is well-known that a partially ordered set is described as an oriented graph. In this paper, we assign to each (M, F), in a unique way, an oriented graph $G(M, F)$ by a similar way to Novikov's method, and show that for any oriented graph G, there is a codimension-one foliation (M, F) with $G=G(M, F)$. These are done in §3. We also show that there is a 'nice' embedding $\phi : G(M, F) \to M$, and in §4 we prove that the induced homomorphism $\phi_* : \pi_1(G(M, F)) \to \pi_1(M)$ is injective. Walczak [15] introduced the notion of admissible functions of (M, F) and the present author defined the notion of admissible functions of oriented graphs in [10]. As an application of the viewpoint obtained above, we show that these two notions of admissible functions are essentially same. This is done in §5. Finally, in §6, we give a brief discussion on Riemannian labels of oriented graphs, whose definition comes naturally from our viewpoint, and on the Laplacians on graphs.

2. Preliminaries. We begin this section with some definitions on graphs. For the definition of cellular complexes, see Spanier [13], and for generalities on graph theory, see Bollobas [2].

G is called a graph if G is a finite one-dimensional cellular complex. We set $V=V(G)=\{v_i\}=\{\text{all 0-cells of } G\}$ and $E=E(G)=\{e_a\}=\{\text{all 1-cells of } G\}$. We call each $v \in V(G)$ a vertex, and $e \in E(G)$ an edge. For $e \in E(G)$, we also set $V(e)=\mathrm{Cl}(e)-e=\{\text{endpoints of } e\}$, where the closure $\mathrm{Cl}(e)$ of e is taken in G.

REMARK. (a) $V(e)$ may consist of only one point $\{v\}$. In this case, we call e a loop at v.

(b) $V(e_a)=V(e_b)$ may occur even if $e_a \neq e_b$. In this case, G is called a multigraph (see Bollobas [2]).

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A path $P=(V(P), E(P))$ is a pair of ordered elements of $V(G)$ and $E(G)$ of the form

$$V(P)=\langle v_0, v_1, \dots, v_l \rangle\,, \quad E(P)=\langle e_1, e_2, \dots, e_l \rangle \quad \text{with} \quad V(e_i)=\{v_{i-1}, v_i\}\,.$$

The length $l(P)$ of P is defined to be the cardinality of the set $E(P)$, that is, l. In case $v_l=v_0$, we call P a closed path.

Let $G=(V, E)$ be a graph. Since each edge e is homeomorphic to $(0, 1)$, e has a natural orientation induced, via a fixed homeomorphism, from that of $(0, 1)$. In this case, we say that e is oriented. With this in mind, we give the following definition.

G is called an oriented graph if G is a graph and each edge is oriented. For each edge $e \in E(G)$, we call $I(e)=H(0)$ the initial vertex, and $T(e)=H(1)$ the terminal vertex. Here $H:[0, 1]\to G$ is the extended map of the given homeomorphism $h:(0, 1)\to e\subset G$. In case $I(e)=v$ and $T(e)=w$, we occasionally denote e by $[v, w]$. Note that if e is a loop at the vertex v, then $v=I(e)=T(e)$.

Now, let (M, F) be a transversely oriented codimension-one foliation F of a closed oriented manifold M. For generalities on foliations, see Hector and Hirsh [3]. In the following, we shall work in the C^∞-category.

A compact saturated domain D of M is said to be a foliated trivial I-bundle if D is the total space of a trivial I-bundle over a compact leaf L of F and if the induced foliation on D from F is everywhere transverse to the fibers I. Note that the boundary ∂D consists of two copies of the compact leaf L. A compact saturated domain D of M is said to be a $(+)$-fcd (resp. $(-)$-fcd) if N is outward (resp. inward) everywhere on the boundary ∂D of D, where N is a non-vanishing vector field on M transverse to F so that the direction of N coincides with the transverse orientation of F.

It is well-known that if F has an infinite number of compact leaves, then all but a finite number of them are contained in some foliated trivial I-bundles (cf. Hector and Hirsh [3]).

Let $L\subset \operatorname{Int} M$ be a compact leaf of a foliated manifold (M, F) with a boundary which is a union of compact leaves of F. Construct a new foliated manifold (M_0, F_0) as follows: Delete the subset L from M and add two copies of L to $M-L$ by the natural identification so that the resulting manifold M_0 to be compact with $\partial M_0=\partial M\cup\{$two copies of $L\}$ and $F_0=(F-L)\cup\{$two copies of $L\}$. We say that (M_0, F_0) is obtained from (M, F) by cutting M along L.

Let (M, F) be as above and (G, V, E) be an oriented graph. We say that a mapping $\phi: G\to M$ is a transverse embedding if ϕ is a continuous injection and the restriction $\phi|_{\mathrm{Cl}(e)}$ of ϕ to each $\mathrm{Cl}(e)$, $e\in E$, can be extended to a smooth transverse embedding of some open interval containing $[0, 1]$, the domain of the extended map $H:[0, 1]\to G$ of e. Furthermore, if the image of ϕ intersects all leaves of F and the induced orientation on e from the transverse orientation of F coincides with the original one of e, we call ϕ to be nice.

3. Construction of graphs and foliations. Let (M, F) be a transversely oriented codimension-one foliation F of a closed oriented manifold M with $\dim M \geq 3$. In this section, we shall construct, in a unique way, an oriented graph $G(M, F)$ from (M, F). Furthermore, from an arbitrarily given oriented graph G, we shall construct a transversely oriented codimension-one foliation of a closed oriented manifold (M, F) so that $G = G(M, F)$.

First, assume that F has no compact leaves. In this case, it is well-known that there is a closed transversal intersecting all leaves of F, where a closed transversal means an embedding $\phi : S^1 \to M$ which is transverse to the leaves of F. Then take a point v on S^1 and regard S^1 as an oriented graph $G(M, F)$ with one vertex $\{v\}$ and one loop $\{S^1\}$ at v with the orientation induced from the transverse orientation of F.

Second, assume that F has at least one compact leaf, say L, and by cutting along which the foliated manifold obtained from M is a foliated trivial I-bundle. In this case, it is also well-known that there is a closed transversal S^1 intersecting all leaves of F. By the same way as in the first case, take a point v on S^1 and regard S^1 as an oriented graph $G(M, F)$ with one vertex $\{v\}$ and one loop $\{S^1\}$ at v with the orientation induced from the transverse orientation of F.

Finally, we assume that F has compact leaves, but none of them have the property in the second case. In this case, take all (set-theoretical) maximal foliated trivial I-bundles $D_1, D_2, \ldots, D_s$, and set $M_1 = M - \bigcup_{i=1}^{s} \mathrm{Int}(D_i)$. By assumption, M_1 is not empty. Then take minimal $(\pm)$-fcd's $D_{s+1}, D_{s+2}, \ldots, D_t$, and set $M_2 = M_1 - \bigcup_{i=s+1}^{t} \mathrm{Int}(D_i)$. If M_2 is not empty, cut M_2 along all compact leaves in the interior of M_2, and list all connected components as $D_{t+1}, D_{t+2}, \ldots, D_u$. Note that the number of compact leaves in M_2 is finite from the fact stated in Section 2.

Now we construct an oriented graph $G(M, F)$ from (M, F). Take $v_i \in \mathrm{Int}(D_i)$ $(i = 1, 2, \ldots, u)$ and set $V(G) = \{v_1, v_2, \ldots, v_u\}$. In case $M_2 = \varnothing$, the argument below is valid by simply replacing u with t. For each compact leaf $L_{i_j} \subset \partial D_i$, take a point $p_{i_j} \in L_{i_j}$. If $L_{i_j} = L_{k_l} \subset \partial D_i \cap \partial D_k$, then choose p_{i_j}'s so that $p_{i_j} = p_{k_l}$. On each D_i it is easy to construct smooth arcs $\{c_{i_j}\}$ satisfying the following conditions: c_{i_j} is a smooth arc between v_i and p_{i_j}, $c_{i_j} \cap c_{i_l} = \{v_i\}$ if $j \neq l$, each c_{i_j} is properly contained in a smooth transverse curve, and the set $\bigcup_j c_{i_j}$ intersects all leaves of $F|D_i$. For each compact leaf $L = L_{i_j} = L_{k_l} \subset \partial D_i \cap \partial D_k$, take a union $c_{i_j} \cup c_{k_l}$ and deform it slightly near L so that the resulting curve is again a smooth transverse curve between v_i and v_k. We denote this curve by e_L and give e_L an orientation induced from the transverse orientation of F. Note that this definition makes sense even in the case $i = k$. In this case, e_L is a loop. Set $E(G) = \{e_L\}$. It is easy to see that $G = (V(G), E(G))$ is an oriented graph. We define $G(M, F) = (V(G), E(G))$.

By the construction above, we get the following result.

THEOREM 1. *Let (M, F) be as above. For each (M, F) there exist an oriented graph $G(M, F)$ and a nice transverse embedding $\phi : G(M, F) \to (M, F)$. Furthermore, for each* *edge* $e \in E(G)$, $\phi(\mathrm{Int}(e))$ *intersects each compact leaf of* F *at most once.*

Conversely, we have the following

THEOREM 2. *Let* G *be an oriented graph. Then there is a foliated manifold* (M, F) *so that* $G = G(M, F)$.

PROOF. Let $G = (V, E)$ be an arbitrarily given oriented graph. We shall construct a codimension-one foliation (M^3, F) on a 3-dimensional manifold M^3 so that $G(M^3, F) = G$.

The idea is the following: For each vertex $v \in V$ adjacent k edges, construct a 3-dimensional manifold M_v^3 with k tori T^2's as boundary components and a codimension-one foliation F_v with $\partial M_v^3 \subset F_v$. If v is adjacent to w, then glue suitable $T^2 \subset \partial M_v$ and $T^2 \subset \partial M_w$. After glue all T^2's, we get the desired (M^3, F).

Let $v \in V$ be a vertex adjacent k_v outward edges and l_v inward edges, that is, v is a initial point of k_v edges and is a terminal point of l_v edges. Take a 2-dimensional sphere S^2, delete $k_v + l_v$ small open discs from S^2, and denote by D_v the resulting disc with $k_v + l_v$ boundary components. Set $M_v = D_v \times S^1$, and list all boundary components of ∂M_v as $C_1^+, \dots, C_{k_v}^+, C_1^-, \dots, C_{l_v}^-$ so that each C_i^+ corresponds to a vertex adjacent to v with an oriented edge outward at v and that each C_j^- corresponds to a vertex adjacent to v with an oriented edge inward at v. Now construct a transversely oriented codimension-one foliation F_v with $\partial M_v = \partial D_v \times S^1 \subset F_v$ as follows: Give $S^1 = \boldsymbol{R}/\boldsymbol{Z}$ the canonical orientation induced from the one of $\boldsymbol{R}$. Wind $(\mathrm{Int}\, D_v) \times \{t\}$, $t \in S^1$, along S^1 in the negative direction near C_i^+'s and in the positive direction near C_j^-'s (cf. turbulization in [3] or [7]). Then we get a foliation F_v consisting of these leaves and compact leaves $(\partial D_v) \times S^1$. Note that the transverse orientation along $C_i^+ \times S^1$ is outward and is inward along $C_l^- \times S^1$. The resulting foliated manifold (M_v, F_v) is the desired one. In case $k_v = l_v = 1$, this construction simply gives a foliated trivial I-bundle over T^2 and we need to deform it. To do this, the simplest way is to use the $*$-operation defined by Lawson [5]. Let (M_v', F_v') be the foliated manifold obtained by the above construction. Define $(M_v, f_v) = (M_v', F_v') * (T^3, F_a)$, where (T^3, F_a) is the codimension-one foliation of T^3 with irrational a 'slant', that is, F_a is defined by a closed 1-form and all leaves are dense in T^3. $*$-operation is an identification of foliations along closed transversals, and produces no new compact leaves.

If v and w are adjacent, then w corresponds to one of $C_i^{\pm} \times S^1$'s $\subset M_v$, say, $C_i^+ \times S^1$, and v to one of $C_j^{\mp} \times S^1$'s $\subset M_w$, say, $C_j^- \times S^1$. Identify M_v and M_w along $C_i^+ \times S^1$ and $C_j^- \times S^1$ naturally. In this way, identifying all $C_i^{\pm} \times S^1$'s in M_v for $v \in V(G)$, we get the desired codimension-one foliation

$$(M^3, F) = \bigcup_{v \in V(G)} (M_v, F_v) / \{\text{identification given above}\} .$$

It is easy to see that $G(M, F) = G$. This completes the proof of Theorem 2.

4. A topological relation. In this section, we show the following topological relation between (M, F) and $G(M, F)$ constructed in Section 3.

THEOREM 3. *Let (M, F), $G(M, F)$ and ϕ be as in Theorem 1. If F has a compact leaf, then the induced map $\phi_* : \pi_1(G) \to \pi_1(M)$ is injective.*

PROOF. We shall identify the oriented graph $G = G(M, F)$ and $\phi(G(M, F)) \subset M$. We assume that $\operatorname{Ker} \phi_* \neq \{1\}$ and derive a contradiction. Let α be a non-trivial element in $\operatorname{Ker} \phi_*$. Represent α by a closed path of the smallest length, say, $\alpha = \langle e_1 e_2 \cdots e_k \rangle$. Note that, as F has a compact leaf, each edge e_i intersects at least one compact leaf and distinct edges do not intersect the same compact leaf except at their vertices. Let $f: D^2 \to M$ be a continuous map with $f(\partial D) = e_1 e_2 \cdots e_k$. We deform f so that except near vertices of e_i's f is smooth and is in general position with respect to F. If L_1 is a compact leaf intersecting $\operatorname{Int}(e_1)$, then $f^{-1}(L_1 \cap f(D^2))$ is a set of circles and arcs on D^2, and one of the arcs connects a point in $\operatorname{Int}(e_1)$ to a point in $\operatorname{Int}(e_j)$ for some j. If $e_1 = [v, w]$, then, by considering the orientations, it is easy to see that $e_j = [w, v]$. If L_2 is a compact leaf intersecting $\operatorname{Int}(e_2)$, then $f^{-1}(L_2 \cap f(D^2))$ is a set of circles and arcs on D^2, and one of the arcs connects a point in $\operatorname{Int}(e_2)$ to a point in $\operatorname{Int}(e_l)$ for some l. As the compact leaves L_1 and L_2 does not intersect, the arc between $\operatorname{Int}(e_2)$ and $\operatorname{Int}(e_l)$ does not intersect the arc between $\operatorname{Int}(e_1)$ and $\operatorname{Int}(e_j)$. This implies $l < j$, and if $e_2 = [x, y]$, then $e_l = [y, x]$. We can repeat this process until we find i so that $e_i = [u, z]$ and $e_{i+1} = [z, u]$. Therefore, $\alpha = \langle e_1 \cdots e_{i-1} [u, z][z, u] e_{i+2} \cdots e_k \rangle = \langle e_1 \cdots e_{i-1} e_{i+2} \cdots e_k \rangle$, which contradicts the minimality of the length of $e_1 e_2 \cdots e_k$ representing α. This completes the proof of Theorem 3.

REMARK. It is still an open problem whether any smooth codimension-one foliation on a simply connected closed manifold always admit compact leaves or not (cf. Langevin [4]).

By the well-known Novikov's compact leaf theorem (see Novikov [6]), any smooth codimension-one foliation on S^3 has a compact leaf. Thus, combining this with Theorem 3, we have the following.

COROLLARY 1. *For any (S^3, F), the graph $G(S^3, F)$ is a tree. Here, the orientation of edges of $G(S^3, F)$ are negrected.*

5. Admissible functions. First, we give further definitions on graphs. Let $G = (V(G), E(G))$ be a graph. A graph K is called a full subgraph of G if

(i) K is a non-empty subcomplex of G, and

(ii) any $e \in E(G)$ with $V(e) \subset K$ implies $e \in E(K)$.

A proper full subgraph K of an oriented graph G is called a $(+)$-subgraph (resp. $(-)$-subgraph) if $e \in E(G)$ with $V(e) \cap V(K) \neq \emptyset$ and $V(e) \cap (V(G) - V(K)) \neq \emptyset$ implies $I(e) \in V(K)$ (resp. $T(e) \in V(K)$).

Recall the definition of admissible functions of an oriented graph G (see Oshikiri [10]). We call a function $f\colon V(G)\to \boldsymbol{R}$ admissible if every minimal $(+)$-subgraph contains a vertex v with $f(v)>0$, and every minimal $(-)$-subgraph contains a vertex w with $f(w)<0$. Here "minimal" means the usual set theoretical sense, that is, being non-empty and containing no non-empty proper $(+)$-subgraphs (resp. $(-)$-subgraphs). In case G has no $(+)$-subgraphs, any function f with $f(v)>0$ and $f(w)<0$ for some $v, w\in V(G)$ or $f\equiv 0$ is called admissible.

Next we recall some definitions on foliations. Let F be a transversely oriented codimension-one foliation of a closed connected oriented manifold M. Let g be a Riemannian metric on M. Then there is a unique unit vector field N orthogonal to F whose direction coincides with the given transverse orientation of F. We give an orientation to F as follows: Let $\{E_1, \dots, E_n\}$ be an oriented local orthonormal frame for the tangent bundle TF of F. The orientation of M given by $\{N, E_1, \dots, E_n\}$ then coincides with the given one of M. We denote the mean curvature of a leaf L at x with respect to N by $H(x)$, that is,

$$H=\sum_{i=1}^{n}\langle\nabla_{E_i}E_i, N\rangle\,,$$

where $\langle\ ,\ \rangle$ means $g(\ ,\ )$, ∇ is the Riemannian connection of (M, g), and $\{E_i\}$ is a local orthonormal frame for TF with $\dim F=n$. We call $H(x)$ the mean curvature function of F with respect to g. We also define an n-form χ_F on M by

$$\chi_F(V_1, \dots, V_n)=\det(\langle E_i, V_j\rangle)_{i,j=1,\cdots,n} \qquad \text{for} \quad V_j\in TM\,,$$

where $\{E_1, \dots, E_n\}$ is an oriented local orthonormal frame for TF. Note that the restriction $\chi_F|_L$ is the volume element of $(L, g|_L)$ for $L\in F$. Then we have the following formula.

Proposition R (Rummler [12]). *$d\chi_F=-H\,dV(M, g)=\mathrm{div}_g(N)\,dV(M, g)$, where $dV(M, g)$ is the volume element of (M, g) and $\mathrm{div}_g(N)$ is the divergence of N with respect to g, that is,*

$$\mathrm{div}_g(N)=\sum_{i=1}^{n}\langle\nabla_{E_i}N, E_i\rangle\,.$$

Let f be a smooth function on M. We call f admissible if there is a Riemannian metric g on M so that $-f$ coincides with the mean curvature function of F with respect to g (see Walczak [15] or Oshikiri [8], [9]). A characterization of admissible functions, which is conjectured by Walczak (see Langevin [4]) and proved affirmatively by the author (see Oshikiri [11]), is the following

Theorem O. *Let F be a transversely oriented codimension-one foliation of a closed connected oriented manifold M. Assume that F contains at least one $(+)$-fcd. Then f is admissible if and only if $f(x)>0$ somewhere in any minimal $(+)$-fcd and $f(y)<0$*

somewhere in any minimal $(-)$*-fcd. In case* F *contains no* $(+)$*-fcd's, any smooth function* f *with* $f(x)>0$ *and* $f(y)<0$ *for some* $x, y \in M$ *or* $f \equiv 0$ *is admissible.*

Now we shall discuss a relation between these two definitions of admissible functions. Let (M, F) and $G=G(M, F)$ be as in Section 3. Let $v \in V(G)$ and $e \in E(G)$ correspond to a foliated compact domain $D_v \subset M$ and to a compact leaf $L_e \in F$, respectively. For a saturated compact domain D of (M, F) we denote by $G(D)$ the subgraph of G consisting of all vertices $v \in V(G)$ with $\operatorname{Int} D \cap D_v \neq \emptyset$ and all edges $e \in E(G)$ with $L_e \subset \operatorname{Int} D$. It is easy to see the following.

LEMMA. *If* D *is a* $(+)$*-fcd* (*resp.* $(-)$*-fcd*), *then* $G(D)$ *is a* $(+)$*-subgraph* (*resp.* $(-)$*-subgraph*). *Furthermore,* D *is a minimal* $(+)$*-fcd* (*resp.* $(-)$*-fcd*) *if and only if* $G(D)$ *is a minimal* $(+)$*-subgraph* (*resp.* $(-)$*-subgraph*).

Let f be a smooth function on M and dV a volume element on M. Define a function $G_{dV}(f)\colon V(G) \to \boldsymbol{R}$ by

$$G_{dV}(f)(v)=\int_{D_v} f\,dV \qquad \text{for} \quad v \in V(G)\,.$$

The main result of this section is the following

THEOREM 4. *For a smooth function* f *on* M, *the following two conditions are equivalent.*

(1) f *is a admissible on* (M, F).

(2) *There is a volume element* dV *on* M *so that* $G_{dV}(f)$ *is admissible on* $G(M, F)$.

PROOF. First assume that f is admissible. By definition, there is a Riemannian metric g on M so that $f=-H$, where H is the mean curvature function of F with respect to g. Set $dV=dV(M, g)$, that is, the volume element of (M, g). We show that $G_{dV}(f)$ is admissible. Let K be a minimal $(+)$-subgraph of G. Set $D_K=\bigcup_{v \in V(K)} D_v$. By the above lemma, D_K is a minimal $(+)$-fcd. Using Rummler's formula (Proposition R) we have

$$\sum_{v \in V(K)} G_{dV}(f)(v)=\int_{D_K} f\,dV=\int_{\partial D_K} \chi_F>0\,,$$

since D_K is a $(+)$-fcd. Thus $G_{dV}(f)(v)>0$ for some $v \in V(K)$. Similary, we also have $G_{dV}(f)(v)<0$ for some $v \in V(K)$ when K is a $(-)$-subgraph.

We prove the converse. By Theorem O, it is sufficient to show that $f(x)>0$ somewhere in any minimal $(+)$-fcd and $f(y)<0$ somewhere in any minimal $(-)$-fcd. Let D be a minimal $(+)$-fcd. By the above lemma, $G(D)$ is a minimal $(+)$-subgraph. Thus, there is a vertex $v \in V(G(D))$ so that $G_{dV}(f)(v)>0$, as $G_{dV}(f)$ is admissible on $G(M, F)$. By definition, this means that $\int_{D_v} f\,dV>0$. Therefore there must be a point

$x \in D_v \subset D$ with $f(x) > 0$. Similary, there must be a point $y \in D_v \subset D$ with $f(y) < 0$ for any minimal $(-)$-fcd D. This completes the proof.

COROLLARY 2. *For any admissible function h on $G(M, F)$, there are an admissible function f and a volume element dV on M so that $h = G_{dV}(f)$.*

PROOF. Let h be an admissible function on $G(M, F)$. If there are a smooth function f and a volume element dV on M so that $h = G_{dV}(f)$, then, by the above theorem, f is automatically admissible. Thus, we have only to show the existence of a smooth function f and a volume element dV on M so that $h = G_{dV}(f)$. Choose an arbitrary volume element dV on M and fix it. Set $f_1 \equiv 0$. For each $v \in V(G)$, deform f_1 smoothly on $\mathrm{Int}\, D_v \subset M$ so that $h(v) = \int_{D_v} f_1 dV$, and set the resulting smooth function f. It is easy to see $h = G_{dV}(f)$.

6. Concluding remarks. The viewpoint given above enables us to translate many notions on foliated manifolds into the ones on graphs. We shall discuss on this point briefly.

Let $G = (V, E)$ be an oriented graph. Set

$$C^0(G) = \{f : V \to \boldsymbol{R}\} \quad \text{and} \quad C^1(G) = \{\phi : E \to \boldsymbol{R}\}.$$

We call $g_G = (g_V, g_E)$ a Riemannian label, where $g_V : V \to \boldsymbol{R}_+$ and $g_E : E \to \boldsymbol{R}_+$ are functions with positive real values. Define inner products on $C^0(G)$ and $C^1(G)$ by

$$\langle f_1, f_2 \rangle = \sum_{v \in V} g_V(v) f_1(v) f_2(v) \qquad \text{for} \quad f_1, f_2 \in C^0(G)$$

and

$$\langle \phi_1, \phi_2 \rangle = \sum_{e \in E} g_E(e) \phi_1(e) \phi_2(e) \qquad \text{for} \quad \phi_1, \phi_2 \in C^1(G).$$

Recall the boundary operator $d : C^0(G) \to C^1(G)$ defined by

$$df([x, y]) = f(y) - f(x) \qquad \text{for} \quad f \in C^0(G) \text{ and an oriented edge } [x, y] \in E.$$

Define the coboundary operator $\delta : C^1(G) \to C^0(G)$ by

$$\delta\phi(v) = \frac{1}{g_V(v)} \sum_{e_v} \mathrm{sgn}(e_v) g_E(e_v) \phi(e_v),$$

where the summation is taken over all edges $e_v \in E$ adjacent to v, $\mathrm{sgn}(e_v) = +1$ if v is the terminal point of e_v, and $\mathrm{sgn}(e_v) = -1$ if v is the initial point of e_v. It is easy to see that

$$\langle df, \phi \rangle = \langle f, \delta\phi \rangle \qquad \text{for} \quad f \in C^0(G) \text{ and } \phi \in C^1(G).$$

This enables us to define the so-called Laplacians $\Delta_V^g : C^0(G) \to G^0(G)$ and $\Delta_E^g : C^1(G) \to C^1(G)$ by

$$\Delta_V^g(f) = \delta df \quad \text{and} \quad \Delta_E^g(\phi) = d\delta\phi.$$

If we choose $g_V = g_E = 1$, then Δ_V^g is the standard Laplacian on graphs (cf. Biggs [1], Urakawa [14]). Note that the definition of δ involves the orientation of edges, however, the definitions of Δ^g's work without orientation of edges.

Finally, we mention the so-called Stokes' Theorem. For an oriented graph G with a Riemannian label $g = (g_V, g_E)$, define the integrations of f and ϕ by

$$\int_K f = \sum_{v \in V(K)} g_V(v) f(v) \qquad \text{for} \quad f \in C^0(G)$$

and

$$\int_K \phi = \sum_{e \in E(K)} g_E(e) \phi(e) \qquad \text{for} \quad \phi \in C^1(G),$$

where $K = V(K) \cup E(K)$ is a set of vertices and edges with orientations in G. Here the following convention is used: If the orientation of $e \in E(K)$ is opposite to the one of the corresponding edge $e' \in E(G)$, then define $\phi(e) = -\phi(e')$. For a full subgraph H, we have the well-known Stokes' Theorem:

$$\int_H \delta\phi = \int_{\partial H} \phi \qquad \text{for} \quad \phi \in C^1(G),$$

where ∂H is the set of oriented edges e such that $\partial e \cap V(H) \neq \emptyset$ and $\partial e \cap (V(G) - V(H)) \neq \emptyset$. $e \in \partial H$ is oriented from H to the complement of H, that is, outward from H.

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