

CODIMENSION ONE SYMPLECTIC FOLIATIONS AND REGULAR POISSON STRUCTURES

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ABSTRACT. In this short note we give a complete characterization of a certain class of compact corank one Poisson manifolds, those equipped with a closed one-form defining the symplectic foliation and a closed two-form extending the symplectic form on each leaf. If such a manifold has a compact leaf, then all the leaves are compact, and furthermore the manifold is a mapping torus of a compact leaf.

These manifolds and their regular Poisson structures admit an extension as the critical hypersurface of a b -Poisson manifold as we will see in [GMP].

1. INTRODUCTION

Given a regular Poisson structure we have an associated symplectic foliation \mathcal{F} given by the distribution of Hamiltonian vector fields. In this short paper we study some properties of codimension one symplectic foliations for regular Poisson manifolds and define some invariants associated to them.

In Section 2.1 we introduce the first invariant, associated to the defining one-form of the foliation. We will see in Section 3.1 that this invariant measures how far a Poisson manifold is from being unimodular. When this invariant vanishes we can choose a defining one-form for the symplectic foliation which is closed. In particular when this invariant vanishes, the Godbillon-Vey class of the foliation vanishes too (as had been previously observed by Weinstein in [We]).

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In Section 2.2, we introduce another invariant also related to the global geometry of these manifolds. The second invariant measures the obstruction to the existence of a closed 2-form on the manifold that restricts to the symplectic structure on each leaf. This invariant had been previously studied by Gotay in [Go] in the setting of coisotropic embeddings.

In section 3.2, we explore some of the global implications of the vanishing of these two invariants. In particular we show that if they vanish and the foliation \mathcal{F} has at least one compact leaf L , then all leaves are compact and M itself is the mapping torus associated with the holonomy map of L onto itself. (In particular, the leaves of \mathcal{F} are the fibers of a fibration of M over \mathbb{S}^1 .) In Section 3.3, we give the Poisson version of this mapping torus theorem. We also briefly describe another global consequence of the vanishing of these invariants: A $2n$ -dimensional Poisson manifold (X, Π) , is a b -Poisson manifold if the section Π^n of $\Lambda^n(X)$ intersects the zero section of this bundle transversally. For such manifolds it is easy to see that this intersection is a regular Poisson manifold with codimension one symplectic leaves and that its first and second invariants vanish. A much harder result (which will be the topic of a sequel to this paper) is that the converse is also true.

As an application of theorem 19 we give an explicit description of the Weinstein's groupoid integrating the Poisson structure of these regular manifolds in the case there is a compact leaf. We do it in section 3.4.

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2. INTRODUCING TWO INVARIANTS OF A FOLIATION

2.1. The defining one-form of a foliation and the first obstruction class. Let M be a manifold, \mathcal{L} a codimension one foliation of M , and write $\Omega(M)$ simply as Ω .

Definition 1. A form $\alpha \in \Omega^1$ is a **defining one-form** of the foliation \mathcal{L} if it is nowhere vanishing and $i_L^* \alpha = 0$ for all leaves $L \xrightarrow{i_L} M$.

A basic property of this defining one-form is:

Lemma 2. *For $\mu \in \Omega^k$, we have $\mu \in \alpha \wedge \Omega^{k-1}$ if and only if $i_L^* \mu = 0$ for all $L \in \mathcal{L}$.*

If u and v are vector fields tangent to the foliation, then $[u, v]$ is as well, and thus $d\alpha(u, v) = u(\alpha(v)) - v(\alpha(u)) - \alpha([u, v]) = 0$. Lemma 2 then implies

$$(1) \quad d\alpha = \beta \wedge \alpha \text{ for some } \beta \in \Omega^1.$$

Thus, $\Omega_1 = \alpha \wedge \Omega$ is a subcomplex of Ω and $\Omega_3 = \Omega / \alpha \wedge \Omega$ a quotient complex. Lemma 2 gives us a necessary and sufficient condition for a k -form $\mu \in \Omega^k$ to be in Ω_1^k .

Writing $\Omega_2 = \Omega$ we have a short exact sequence of complexes

$$0 \longrightarrow \Omega_1 \xrightarrow{i} \Omega_2 \xrightarrow{j} \Omega_3 \longrightarrow 0.$$

From (1) we get $0 = d(d\alpha) = d\beta \wedge \alpha - \beta \wedge d\alpha = d\beta \wedge \alpha$, that is, α and $d\beta$ are dependent, so we must have $d\beta \in \Omega_1$. Then, $d(j\beta) = 0$ and we can define the **first obstruction class** $c_{\mathcal{L}} \in H^1(\Omega_3)$ to be

$$c_{\mathcal{L}} = [j\beta].$$

Proposition 3. *The first obstruction class $c_{\mathcal{F}}$ does not depend on the choice of the defining one form α .*

Proof. Let α and α' be distinct defining one forms for the foliation \mathcal{F} . We must have $\alpha' = f\alpha$ for some nonvanishing $f \in C^\infty(M)$. Then

$$d\alpha' = df \wedge \alpha + f d\alpha = df \wedge \alpha + f\beta \wedge \alpha = \left(\frac{df}{f} + \beta\right) \wedge \alpha',$$

so $\beta' = d(\log f) + \beta$. Thus $[j\beta'] = [j\beta]$. □

Theorem 4. *The first obstruction class $c_{\mathcal{F}}$ vanishes identically if and only if we can choose α the defining one-form of the foliation \mathcal{F} to be closed.*

Proof. The first obstruction class $c_{\mathcal{F}} = [j\beta]$ vanishes identically if and only if $\beta = df + g\alpha$ for some $f, g \in C^\infty(M)$. Replacing α by $\alpha' = e^{-f}\alpha$

we get

$$\begin{aligned}
d(e^{-f}\alpha) &= -e^{-f}df \wedge \alpha + e^{-f}d\alpha \\
&= -e^{-f}df \wedge \alpha + \beta \wedge \alpha \\
&= -e^{-f}df \wedge \alpha + e^{-f}df \wedge \alpha + e^{-f}g \alpha \wedge \alpha \\
&= 0.
\end{aligned}$$

□

2.2. The defining two-form of a foliation and the second obstruction class. Assume now that M is endowed with a regular corank one Poisson structure Π and that \mathcal{F} is the corresponding foliation of M by symplectic leaves.

Furthermore, assume that the first obstruction class $c_{\mathcal{F}}$ vanishes and therefore the foliation is defined by a closed one form α .

Definition 5. A form $\omega \in \Omega^2$ is a **defining two-form** of the foliation \mathcal{F} induced by the Poisson structure Π if $i_L^*\omega = \omega_L$ is the symplectic form on each leaf $L \xrightarrow{i_L} M$.

Using the formula

$$\begin{aligned}
d\omega(u, v, w) &= u(\omega(v, w)) + v(\omega(w, u)) + w(\omega(u, v)) - \\
&\quad - \omega([u, v], w) - \omega([v, w], u) - \omega([w, u], v)
\end{aligned}$$

with u, v, w vector fields tangent to the foliation we conclude that $i_L^*d\omega = 0$ for all leaves $L \in \mathcal{F}$, so by Lemma 2 and since α is closed,

$$(2) \quad d\omega = \mu \wedge \alpha \text{ for some } \mu \in \Omega^2.$$

From (2) we conclude that $d\mu \wedge \alpha = 0$, that is, α and $d\mu$ are dependent, so $d\mu \in \Omega_1$. Then, $d(j\mu) = 0$ and we can define the **second obstruction class** $\sigma_{\mathcal{F}} \in H^2(\Omega_3)$ to be

$$\sigma_{\mathcal{F}} = [j\mu].$$

Proposition 6. *The second obstruction class $\sigma_{\mathcal{F}}$ does not depend on the choice of the defining two-form ω .*

Proof. Let ω and ω' be distinct defining two-forms for the foliation \mathcal{F} . We must have $\omega' = \omega + \nu$ for some $\nu \in \Omega^2$ such that $i_L^*\nu = 0$ for every

leaf $L \in \mathcal{F}$. Then, by Lemma 2, $\nu = \alpha \wedge \xi$ for some $\xi \in \Omega^1$ and we have:

$$\begin{aligned} d\omega' &= d\omega + d\nu \\ &= \mu \wedge \alpha - \alpha \wedge d\xi \\ &= (\mu + d\xi) \wedge \alpha. \end{aligned}$$

Thus, $\mu' = \mu + d\xi$ and $[j\mu'] = [j\mu]$. \square

Theorem 7. *The second obstruction class $\sigma_{\mathcal{F}}$ vanishes identically if and only if we can choose ω the defining two-form of the foliation \mathcal{F} to be closed.*

Proof. The second obstruction class $\sigma_{\mathcal{F}} = [j\mu]$ vanishes identically if and only if $\mu = d\nu + \gamma \wedge \alpha$ for some $\nu, \gamma \in \Omega^1$. Then, (2) implies that $d\omega = d\nu \wedge \alpha$, and replacing ω by $\omega' = \omega - \nu \wedge \alpha$ yields still $i_L^* \omega' = i_L^* \omega = \omega_L$ for every $L \in \mathcal{F}$ but now with $d\omega' = 0$. \square

2.3. The modular vector field and modular class of a Poisson manifold. We follow Weinstein [We] for the description of modular vector field and the modular class of a Poisson manifold. A complete presentation of these can also be found in [Ko].

The modular vector field of a Poisson manifold measures how far Hamiltonian fields are from preserving a given volume form. A simple example is that of a symplectic manifold: the top power of the symplectic 2-form is a volume form invariant under the flow of any Hamiltonian vector field, so the corresponding modular vector field is zero.

Definition 8. Let (M, Π) be a Poisson manifold and Θ a volume form on it, and denote by u_f the Hamiltonian vector field associated to a smooth function f on M . The **modular vector field** X_{Π}^{Θ} is the derivation given by the mapping

$$f \mapsto \frac{\mathcal{L}_{u_f} \Theta}{\Theta}.$$

When both the Poisson structure and the volume form on M are implicit, we denote the modular vector field by v_{mod} .

The modular vector field has the following properties [We]:

- (1) The flow of the modular vector field preserves the volume and the Poisson structure:

$$\mathcal{L}_{X_{\Pi}^{\Theta}}(\Pi) = 0, \quad \mathcal{L}_{X_{\Pi}^{\Theta}}(\Theta) = 0;$$

- (2) When we change the volume form, the modular vector field changes by:

$$X_{\Pi}^{H\Theta} = X_{\Pi}^{\Theta} - u_{\log(H)}, \text{ where } H \in C^{\infty}(M).$$

This implies that the class of the modular vector field in the first Poisson cohomology group is independent of the volume form chosen. This class is called the **modular class** of the Poisson manifold. A Poisson manifold is called **unimodular** if this cohomology class is zero. As remarked above, symplectic manifolds are unimodular.

- (3) If M is 2-dimensional with Poisson structure given by the bracket $\{x, y\} = f(x, y)$ and volume form $\Theta = dx \wedge dy$, then the modular vector field is u_f , the Hamiltonian vector field associated to f . In particular, the modular vector field is tangent to the zero level set of f .

3. THE CASE WITH VANISHING INVARIANTS

3.1. Vanishing first invariant: Unimodular Poisson manifolds.

Assume again that (M^{2n+1}, Π) is a corank one Poisson manifold and \mathcal{F} its foliation by symplectic leaves. Fix $\alpha \in \Omega^1(M)$ and $\omega \in \Omega^2(M)$ defining one- and two-forms of the foliation, respectively. Let us compute the modular vector field associated to the volume form $\Theta = \alpha \wedge \omega^n$:

$$\begin{aligned} \mathcal{L}_{u_f} \Theta &= d(\iota(u_f)\Theta) \\ &= d(\iota(u_f)\alpha \wedge \omega^n) \\ &= -n d(\alpha \wedge df \wedge \omega^{n-1}) \\ &= n \alpha \wedge df \wedge \beta \wedge \omega^{n-1}, \end{aligned}$$

where $\beta \in \Omega^1(M)$ is such that $d\alpha = \alpha \wedge \beta$. On the last step we use the fact that $d\omega \wedge \alpha = 0$.

By definition of modular vector field, we have

$$(3) \quad n \alpha \wedge df \wedge \beta \wedge \omega^{n-1} = (\iota(v_{\text{mod}})df) \alpha \wedge \omega^n.$$

Furthermore, v_{mod} is tangent to the leaves L of \mathcal{F} , so (3) implies that

$$(4) \quad n df_L \wedge \beta_L \wedge \omega_L^{n-1} = (\iota(v_{\text{mod}})df_L) \omega_L^n,$$

where as before $\omega_L = i_L^* \omega$, and similarly $f_L = i_L^* f$ and $\beta_L = i_L^* \beta$.

On the other hand, because $(df_L \wedge \omega_L^n)$ is a $(2n+1)$ -form on a $(2n)$ -dimensional manifold L , we have

$$\iota(v_{\text{mod}})(df_L \wedge \omega_L^n) = 0$$

and hence

$$(5) \quad (\iota(v_{\text{mod}})df_L)\omega_L^n = df_L \wedge n \iota(v_{\text{mod}})\omega_L \wedge \omega_L^{n-1}.$$

From (4) and (5) we conclude that for all $f_L \in C^\infty(L)$,

$$df_L \wedge \iota(v_{\text{mod}})\omega_L \wedge \omega_L^{n-1} = df_L \wedge \beta_L \wedge \omega_L^{n-1},$$

which implies the following:

Theorem 9. *Consider the Poisson manifold (M^{2n+1}, Π) with volume form $\Theta = \alpha \wedge \omega^n$, where α and ω are defining one- and two-forms of the induced foliation.*

Then, the modular vector field is the vector field which on every symplectic leaf $L \in \mathcal{F}$ satisfies

$$\iota(v_{\text{mod}})\omega_L = \beta_L.$$

A corollary of Theorem 9 is the following criterion for unimodularity¹ of corank one Poisson manifolds:

Theorem 10. *A corank one Poisson manifold (M, Π) with induced symplectic foliation \mathcal{F} is unimodular if and only if the first obstruction class of the foliation $c_{\mathcal{F}}$ vanishes identically.*

Proof. Recall that (M, Π) is unimodular if and only if we can choose a volume form on it such that the corresponding modular vector field is zero. Also, recall from Theorem 4 that $c_{\mathcal{F}}$ vanishes identically if and only if we can choose a closed defining one-form α .

But by Theorem 9, the modular vector field $v_{\text{mod}} = X_{\Pi}^{\alpha \wedge \omega^n}$ is zero if and only if $\beta_L = i_L^* \beta = 0$ for every leaf $L \in \mathcal{F}$, which by Lemma 2 is equivalent to $\beta \in \alpha \wedge \Omega$, thus making α closed:

$$d\alpha = \beta \wedge \alpha = 0.$$

□

Remark 11. Given a transversally orientable foliation with defining one-form α , the Godbillon-Vey class is defined as the class of the 3-form $\beta \wedge d\beta$. From Theorem 10 we deduce in particular that for unimodular Poisson manifolds this 3-form vanishes. The converse is not true as observed by Weinstein in [We].

¹We thank David Martínez-Torres for pointing out that an alternative proof of this can be obtained via formula 4.6 in [KLW]. However, this alternative proof requires a use of the interpretation of the modular class as one-dimensional representations and this seems an interesting but longer path.

3.2. Vanishing (first and) second invariant(s): a topological result. We begin by recalling Reeb's global stability theorem about codimension one foliations:

Theorem 12. [Re] *Let \mathcal{F} be a transversely orientable codimension one foliation of a compact connected manifold M . If \mathcal{F} contains a compact leaf L with finite fundamental group, then every leaf of \mathcal{F} is diffeomorphic to L .*

Furthermore, M is the total space of a fibration $f : M \rightarrow \mathbb{S}^1$ with fiber L , and \mathcal{F} is the fiber foliation $\{f^{-1}(\theta) | \theta \in \mathbb{S}^1\}$.

The key point in the proof of Theorem 12 is that such a foliation with a compact leaf has trivial holonomy (see for example [CN]). Since a foliation defined by a closed one-form has trivial holonomy as well (see again [CN], p.80), the proof and conclusions of Reeb's theorem hold in the following case:

Theorem 13. *Let \mathcal{F} be a transversely orientable codimension one foliation of a compact connected manifold M with $c_{\mathcal{F}} = 0$. If \mathcal{F} contains a compact leaf L , then every leaf of \mathcal{F} is diffeomorphic to L .*

Furthermore, M is the total space of a fibration $f : M \rightarrow \mathbb{S}^1$ with fiber L , and \mathcal{F} is the fiber foliation $\{f^{-1}(\theta) | \theta \in \mathbb{S}^1\}$.

Remark 14. A theorem of Tischler [Ti] says that a compact manifold endowed with a non-vanishing closed one form must be a fibration over a circle. However, this fibration need not coincide with the codimension one foliation defined by the closed one form. Theorem 13 asserts that when the foliation contains a compact leaf, then it is itself a fibration over a circle.

We outline an alternative proof of Theorem 13 that does not use Reeb's theorem (or rather, its proof):

Proposition 15. *Let \mathcal{F} be a transversely orientable codimension one foliation of a compact connected manifold M with $c_{\mathcal{F}} = 0$. Then:*

- (1) *there exists a family of diffeomorphisms $\Phi_t : M \rightarrow M$, defined for $t \in (-\varepsilon, \varepsilon)$, that takes leaves to leaves;*
- (2) *if \mathcal{F} contains a compact leaf L , then all leaves are compact;*
- (3) *and furthermore there exists a saturated neighbourhood \mathcal{U} of L and a projection $f : \mathcal{U} \rightarrow I$ such that the foliation is diffeomorphic to the foliation given by the fibers of f .*

Proof. (1) Let α be a closed defining one form of the foliation \mathcal{F} , and v a vector field on M such that $\alpha(v) = 1$; this vector field is transversal to the foliation \mathcal{F} . The flow Φ_t of the vector field v , defined for $t \in (-\varepsilon, \varepsilon)$ for small enough ε , takes leaves to leaves diffeomorphically, since

$$\mathcal{L}_v \alpha = \iota_v d\alpha + d\iota_v \alpha = 0.$$

(2) Let N be the union of all compact leaves in M . The set N is open: Given a leaf L contained in N , the set

$$\{\Phi_t(L) | t \in (-\varepsilon, \varepsilon)\}$$

is an open neighborhood of L contained in N . The same argument can be used to show that $M \setminus N$ is open. Since N is nonempty and M is connected, we must have $N = M$.

(3) Because $i_L^* \alpha = 0$ with L compact, Poincaré lemma guarantees the existence of a tubular neighbourhood \mathcal{U} of L and function f on it such that the leaf L is the zero level set of f and $\alpha = df$ on \mathcal{U} . Shrinking \mathcal{U} as necessary, we can assume it is a saturated neighbourhood and that the leaves are level sets of f .

□

So far we have proved that foliations with vanishing first invariant $c_{\mathcal{F}}$ are locally trivial fibrations in the neighbourhood of a given compact leaf. Using the transverse vector field v of the proof of (1) in Proposition 15, we could now drag this vector bundle structure and by a Čech-type construction obtain a global fiber bundle over \mathbb{S}^1 . Furthermore, the choice of a transverse vector field v whose flow takes leaves to leaves gives us an Ehresmann connection, which we use to lift the closed loop on the base of the fibration and thus obtain a holonomy map $\phi : L \rightarrow L$, thus obtaining the following:

Corollary 16. *Let \mathcal{F} be a transversely orientable codimension one foliation of a compact connected manifold M with $c_{\mathcal{F}} = 0$, and assume that the foliation contains a compact leaf L . Then, the manifold M is the mapping torus of the diffeomorphism $\phi : L \rightarrow L$ given by the holonomy map of the \mathbb{S}^1 fibration.²*

Remark 17. If the foliation \mathcal{F} satisfies $c_{\mathcal{F}} = 0$ but has no compact leaves, and instead there exists a leaf L such that $H^1(L, \mathbb{R}) = 0$ (hypothesis of Thurston's theorem [Th]), then it can be proved that the foliation is a fibration over \mathbb{S}^1 .

²The mapping torus of $\phi : L \rightarrow L$ is the space $\frac{L \times [0,1]}{(x,0) \sim (\phi(x),1)}$.

3.3. Vanishing (first and) second invariant(s): a Poisson result. We wanted to look in this section at the case of vanishing first and second invariants, and for that we must now assume that M is endowed with a regular corank one Poisson structure and that \mathcal{F} is the corresponding symplectic foliation.

Proposition 18. *The two invariants $c_{\mathcal{F}}$ and $\sigma_{\mathcal{F}}$ vanish if and only if there exists a Poisson vector field transversal to the foliation.*

Proof. Given α and ω defining one- and two-forms respectively, let v be the vector field uniquely defined by

$$(6) \quad \alpha(v) = 1 \text{ and } \iota(v)\omega = 0.$$

Conversely, given a vector field v on M transversal to the foliation \mathcal{F} , these equalities uniquely give us defining one- and two-forms α and ω .

The vector field v is Poisson if and only if $\mathcal{L}_v\alpha = \mathcal{L}_v\omega = 0$ which by (6) becomes $\iota(v)d\alpha = \iota(v)d\omega = 0$. But then necessarily $d\alpha = d\omega = 0$, or equivalently, $c_{\mathcal{F}} = \sigma_{\mathcal{F}} = 0$. \square

As before, the choice of a transverse vector field v gives an Ehresmann connection on the fibration that we can use to lift the closed loop on the base of the fibration to obtain a holonomy map $\phi : L \rightarrow L$.

Because v is a Poisson vector field, its flow drags the symplectic structure of one leaf to define the Poisson structure on M (because M is a symplectic fibration over a one dimensional base, the Guillemin-Lerman-Sternberg condition [GLS] guarantees that the minimal coupling structure yields a unique Poisson structure on M). Furthermore, the parallel transport of the connection preserves the symplectic structure on the leaves, and in particular the holonomy map ϕ is a symplectomorphism:

Theorem 19. *Let M be an oriented compact connected regular Poisson manifold of corank one and \mathcal{F} its symplectic foliation. If $c_{\mathcal{F}} = \sigma_{\mathcal{F}} = 0$ and \mathcal{F} contains a compact leaf L , then every leaf of \mathcal{F} is symplectomorphic to L .*

Furthermore, M is the total space of a fibration over \mathbb{S}^1 and it is the mapping torus of the symplectomorphism $\phi : L \rightarrow L$ given by the holonomy map of the \mathbb{S}^1 fibration.

Regular corank one Poisson manifolds with $c_{\mathcal{F}} = \sigma_{\mathcal{F}} = 0$ are interesting in Poisson geometry because they can be characterized as manifolds which are the critical hypersurfaces of Poisson b -manifolds:

Definition 20. An oriented Poisson manifold (X^{2n}, Π) is a **Poisson b -manifold** if the map

$$x \in X \mapsto (\Pi(x))^n \in \Lambda^{2n}(TX)$$

is transverse to the zero section.³

Theorem 21. [GMP] *Let (M, Π) be an oriented compact regular Poisson manifold of corank one, \mathcal{F} its symplectic foliation and v the Poisson vector field transversal to \mathcal{F} .*

If $c_{\mathcal{F}} = \sigma_{\mathcal{F}} = 0$, then there exists an extension of (M, Π) to a b -Poisson manifold $(U, \tilde{\Pi})$. This extension is unique up to isomorphism among the extensions for which $[v]$ is the image of the modular class under the map

$$H^1_{\text{Poisson}}(U) \longrightarrow H^1_{\text{Poisson}}(M).$$

Example 22. Let $M = \mathbb{T}^3$ with coordinates $\theta_1, \theta_2, \theta_3$ and \mathcal{F} the codimension one foliation with leaves given by

$$\theta_3 = a\theta_1 + b\theta_2 + k, \quad k \in \mathbb{R},$$

where $a, b \in \mathbb{R}$ are fixed and independent over \mathbb{Q} . This implies that each leaf is diffeomorphic to \mathbb{R}^2 [Ma]. Then,

$$\alpha = \frac{a}{a^2 + b^2 + 1} d\theta_1 + \frac{b}{a^2 + b^2 + 1} d\theta_2 - \frac{1}{a^2 + b^2 + 1} d\theta_3$$

is a defining one-form for \mathcal{F} and there is a Poisson structure Π on M of which

$$\omega = d\theta_1 \wedge d\theta_2 + b d\theta_1 \wedge d\theta_3 - a d\theta_2 \wedge d\theta_3$$

is the defining two-form. Note that α and ω are closed, and so the invariants $c_{\mathcal{F}}$ and $\sigma_{\mathcal{F}}$ vanish.

Indeed, (M, Π) can be extended to $(U, \tilde{\Pi})$ where $U = M \times (-\varepsilon, \varepsilon)$ and $\tilde{\Pi}$ is the bivector field dual to the 2-form

$$\tilde{\omega} = d(\log t) \wedge \alpha + \omega.$$

3.4. Vanishing (first and) second invariants: Explicit integration of the Poisson structure. A Lie algebroid structure over a manifold M consists of a vector bundle E together with a bundle map $\rho : E \rightarrow TM$ and a Lie bracket $[\cdot, \cdot]_E$ on the space of sections satisfying, for all $\alpha, \beta \in \Gamma(E)$ and all $f \in C^\infty(M)$, the Leibniz identity:

$$[\alpha, f\beta]_E = f[\alpha, \beta]_E + L_{\rho(\alpha)}(f)\beta.$$

³In particular, this implies that the critical set $\{x \in X | (\Pi(x))^n = 0\}$ is a hypersurface.

Given a Lie groupoid one can naturally associate to it a Lie algebroid structure, but the converse is not true: given a Lie algebroid it is not always possible to produce a smooth Lie groupoid with the prescribed Lie algebroid structure. However, when the Lie algebroid is in fact a Lie algebra (case of Lie algebroid over a point), the integration to a Lie group is guaranteed by Lie's third theorem. In [CF], Marius Crainic and Rui Loja Fernandes solve the problem of integrability of a Lie algebroid to a Lie groupoid.

Given a Poisson manifold (M, π) , there exists a natural Lie algebroid structure on T^*M : the Poisson cotangent Lie algebroid has anchor map $\pi^\#$ and Lie bracket defined by

$$[\alpha, \beta] = L_{\pi^\#(\alpha)}(\beta) - L_{\pi^\#(\beta)}(\alpha) - d(\pi(\alpha, \beta)).$$

In this case the integrability problem consists of associating a symplectic groupoid to this Lie algebroid structure, as studied by Marius Crainic and Rui Loja Fernandes in [CF04]. The canonical integration, or Weinstein's groupoid, is a symplectic groupoid integrating the algebroid which is source simply connected.

Recall the following theorem [CF04, Corollary 14],

Theorem 23. *[Crainic-Fernandes] Let M be a regular Poisson manifold. Then:*

- (1) *If M admits a leafwise symplectic embedding then every leaf of M is a Lie-Dirac submanifold.*
- (2) *If every leaf of M is a Lie-Dirac submanifold then M is integrable.*

The second obstruction class $\sigma_{\mathcal{F}}$ can be interpreted via Gotay's embedding theorem [Go] and measures the obstruction for a closed 2-form to exist on Z which restricts to ω_L on each leaf L . If $c_{\mathcal{F}} = 0$, we get a leafwise symplectic embedding and Theorem 23 guarantees integrability. Furthermore, using Theorem 19 we can obtain an explicit characterization of the Weinstein's integrating groupoid:

Corollary 24. *The Poisson structure on manifolds with vanishing invariants $c_{\mathcal{F}}$ and $\sigma_{\mathcal{F}}$ is integrable (in the Crainic-Fernandes sense) and the Weinstein's symplectic groupoid is a mapping torus.*

Proof. Consider the Weinstein's groupoid of a symplectic leaf $(\Pi_1(L), \Omega)$ and the product $(\Pi_1(L) \times T^*(\mathbb{R}), \Omega + d\lambda_{\text{liouville}})$. Let f be the time-1 flow of the Poisson vector field transverse to the symplectic foliation v .

On this product define $\phi : (\tilde{x}, (t, \eta)) \mapsto (\tilde{f}, (t + 1, \eta))$. This map preserves the symplectic groupoid structure and thus by identification it induces a symplectic groupoid structure on the mapping torus which has ϕ as gluing diffeomorphism. \square

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