CODING INTO HOD VIA NORMAL MEASURES WITH SOME APPLICATIONS

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Abstract. We develop a new method for coding sets while preserving GCH in the presence of large cardinals, particularly supercompact cardinals. We will use the number of normal measures carried by a measurable cardinal as an oracle, and therefore, in order to code a subset A of κ, we require that our model contain κ many measurable cardinals above κ. Additionally we will describe some of the applications of this result.

1. Introduction

The model HOD has long been of interest to set theorists. Philosophically, large cardinal consistency with an inner model such as HOD can be seen as further verification of the existence of large cardinals. One approach to achieve this compatibility has been to construct a canonical inner model with a particular large cardinal property. Another approach to this problem has been to start with a model that exhibits the desired large cardinal property, then force over it to get a model with some inner model like properties, particularly V=HOD. To that end, if one wishes to obtain a model of V=HOD via forcing or to code a generic subset into HOD while preserving large cardinals, the standard method available is to use the continuum function as an oracle, which requires significant failures of GCH. Recently, a method for forcing V=HOD while preserving certain large cardinals and GCH was developed by Brooke-Taylor [BT09]. His method involves using whether $\diamondsuit^*$ holds as an oracle, and his proofs do not indicate how to preserve supercompactness in general.

In this paper, we present an alternative method of coding a subset A of κ while preserving GCH and supercompact cardinals. We will use the number of normal measures carried by a measurable cardinal as an oracle, and therefore, in order to code a subset A of κ, we require that our model contain κ many measurable cardinals above κ. Additionally we will describe some of the applications of this result, where the original coding required violating GCH.

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2. The coding oracle

Before describing this coding, we present some relevant definitions and lemmas. A poset $\mathbb{P}$ is *κ-closed* if for every $\delta \leq \kappa$, given every sequence $\langle p_\alpha : \alpha < \delta \rangle$ of elements of $\mathbb{P}$ such that $\beta < \gamma < \delta$ implies $p_\gamma \leq p_\beta$ (a decreasing chain of length less than or equal to δ), there is some $p \in \mathbb{P}$ (a lower bound to this chain) such that $p \leq p_\alpha$ for all $\alpha < \delta$. $\mathbb{P}$ is *κ-strategically closed* if in the two person game in which the players construct a decreasing sequence $\langle p_\alpha : \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even stages, player II has a strategy which ensures the game can always be continued. For κ a regular cardinal and λ an ordinal, $\mathrm{Add}(\kappa, \lambda)$ is the standard poset for adding λ many Cohen subsets of κ.

We will take this opportunity to discuss a generalization of Hamkins' Gap Forcing Theorem [Ham99], [Ham01a] (as it is stated in [ACH07]), as its results are used extensively throughout this paper. A forcing notion $\mathbb{P}$ (and the forcing extension to which it gives rise) *admits a closure point at δ* if it factors as $\mathbb{Q} * \dot{\mathbb{R}}$, where $\mathbb{Q}$ is nontrivial, $|\mathbb{Q}| \leq \delta$, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ-strategically closed." Our arguments will rely on the following consequence of the main result of [Ham03].

Theorem 1. *([Ham03]) If $V \subseteq V[G]$ admits a closure point at δ and $j : V[G] \longrightarrow M[j(G)]$ is an ultrapower embedding in $V[G]$ with $\delta < \mathrm{cp}(j)$, then $j \restriction V : V \longrightarrow M$ is a definable class in V.*

This theorem follows from [Ham03, Theorem 3, Corollary 14]. If $j : V[G] \longrightarrow M[j(G)]$ witnesses the λ-supercompactness of κ in $V[G]$, then by [Ham03, Corollary 4], the restriction $j \restriction V : V \longrightarrow M$ witnesses the λ-supercompactness of κ in V. This theorem clearly can be applied to measurability embeddings as well, which gives us the result that if our forcing exhibits the closure point property at a sufficiently small cardinal, we can infer that the measurable cardinals and supercompact cardinals of the forcing extension already existed in the ground model.

Definition 2. *Let σ be a cardinal. Then $\mathrm{Coll}(\sigma^+, \sigma^{++})$ is the standard Lévy collapse of σ^{++} to σ^+ using partial functions from σ^+ to σ^{++} of cardinality less than σ^+ .*

Definition 3. *Let $\alpha < \gamma$. Let $M_{\alpha,\gamma}$ be the reverse Easton support iteration using $\mathrm{Add}(\delta^+, 1)$ at every inaccessible cardinal $\delta \in (\alpha, \gamma)$ and trivial forcing at all other stages.*

Standard arguments show that this forcing preserves GCH. In addition, standard arguments (see the first paragraph of the proof of the main theorem of [ACH07]) together with Theorem 1 show that no new measurable cardinals are created, all ground model measurable cardinals are preserved, and every measurable cardinal δ in (α, γ) carries the maximum number of normal measures, that is, $2^{2^\delta} = \delta^{++}$.

The basic strategy of our coding is if we would like to code a subset A of κ, we take κ many measurable cardinals above κ, blow up the number of normal measures on those κ measurable cardinals to the maximum number, then force to reduce the number of normal measures on the α^{th} measurable cardinal above κ to fewer than the maximum number, according to whether or not α is in A.

Now let us define our coding and the notation σ_α and γ_σ, which will be used throughout the rest of the paper.

Definition 4. *Let σ be a cardinal. Let γ_σ be the supremum of the first σ many measurable cardinals beyond σ. Suppose that $A \subseteq \sigma$. For every $\alpha < \sigma$, let σ_α be* *the $(\alpha+1)^{\rm st}$ measurable cardinal beyond σ and then let $\mathbb{C}\mathrm{oll}_{\sigma,\gamma_\sigma}(\mathrm{A})$ be the reverse Easton support iteration which forces with $\mathrm{Coll}(\sigma_\alpha^+,\sigma_\alpha^{++})$ for every $\alpha\in A$, and does trivial forcing otherwise. Now we define $\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)=\mathrm{Add}(\sigma^+,1)*\dot{\mathbb{C}}\mathrm{oll}_{\sigma,\gamma_\sigma}(A)$.*

Let $\bar{V}=V^{M_{\sigma,\gamma_\sigma}}$ be our ground model.

Lemma 4.1. *Suppose $A\subseteq\sigma, A\in\bar{V}$. Forcing over $\bar{V}$ with $\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)$ will give us the required number of normal measures in our forcing extension, that is, in $\bar{V}^{\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)}$*

- **(i):** *if $\alpha\in A$, σ_α carries fewer than the maximum number of normal measures, specifically, σ_α^+ many normal measures.*
- **(ii):** *if $\alpha\notin A$, σ_α carries the maximum number of normal measures, that is, $2^{2^{\sigma_\alpha}}=\sigma_\alpha^{++}$ many normal measures.*

Proof. We begin by observing that standard arguments show that GCH is preserved to $\bar{V}^{\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)}$. Also, by Theorem 1, no new measurable cardinals in the open interval (σ,γ_σ) are created by forcing with $\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)$. Since our proof will show that all $\bar{V}$-measurable cardinals in (σ,γ_σ) are preserved when forcing with $\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)$, we will write σ_α without fear of ambiguity.

- **(i):** If $\alpha\in A$, then we force with $\mathrm{Coll}(\sigma_\alpha^+,\sigma_\alpha^{++})$ at stage σ_α. Let $\mathbb{P}_0$ be the forcing up until stage σ_α, let $\mathbb{P}_1=\mathrm{Coll}(\sigma_\alpha^+,\sigma_\alpha^{++})$ be the forcing at stage σ_α, and let $\mathbb{P}_2$ be the forcing beyond stage σ_α. Since $|\mathbb{P}_0|<\sigma_\alpha$, and the next forcing is $\mathrm{Coll}(\sigma_\alpha^+,\sigma_\alpha^{++})$, the arguments of [ACH07] hold (Main Theorem, paragraph 2). Namely, $\bar{V}^{\mathbb{P}_0*\dot{\mathbb{P}}_1}\models$ "σ_α carries σ_α^+ many normal measures." Additionally, any nontrivial forcing beyond σ_α will be sufficiently closed so as not to add any new normal measures to σ_α. So in $\bar{V}^{\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)}=\bar{V}^{\mathbb{P}_0*\dot{\mathbb{P}}_1*\dot{\mathbb{P}}_2}$, there are exactly σ_α^+ many normal measures on σ_α, as desired.
- **(ii):** If $\alpha\notin A$, then no nontrivial forcing occurs at σ_α. Any nontrivial forcing which occurs below σ_α will be small with respect to it, and so by [LS67] will not affect the number of normal measures on σ_α. In addition, any nontrivial forcing beyond σ_α will be sufficiently closed so as not to add any new normal measures to σ_α. So in $\bar{V}^{\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)}=\bar{V}^{\mathbb{P}_0*\dot{\mathbb{P}}_1*\dot{\mathbb{P}}_2}$, there are exactly $\sigma_\alpha^{++}=2^{2^{\sigma_\alpha}}$ many normal measures on σ_α, as desired.

□

In the subsequent sections, we will describe various results that arise from this method.

3. Extending a property concerning HOD-supercompactness with GCH

As mentioned in [Sar08], at a set theory seminar at Berkeley in 2005, Woodin asked if it were possible to construct a model of set theory in which κ is supercompact, but not HOD-supercompact. We will extend the following recent result of Sargsyan [Sar08], which answers Woodin's question with the following theorem:

Theorem 5. ([Sar08]) *Suppose $V\models$ ZFC + GCH + "κ is a supercompact cardinal." Then there is a forcing extension of V in which κ is supercompact, but not HOD-supercompact.*

Note that the cardinal κ is HOD-*supercompact* iff κ is supercompact and for all strong limit cardinals λ, there exists an embedding $j:V\longrightarrow M$ such that $\mathrm{cp}(j)=\kappa$, $j(\kappa)>\lambda$, $M^\lambda\subseteq M$, and $j(\mathrm{HOD})\cap V_\lambda=\mathrm{HOD}\cap V_\lambda$. We follow standard convention and abuse notation by using $j(N)$, where N is a proper class, to mean $j(N) = \bigcup_{\alpha<\text{ORD}} j(V_\alpha^N)$. Since $N = \text{HOD}$ is a definable class, $j \restriction \text{HOD} : \text{HOD} \longrightarrow j(\text{HOD})$ is fully elementary.

In a previous paper, two of the natural questions which arise as a result of this theorem were answered in the affirmative, namely:

(1) *Can this result be extended to the class of supercompact cardinals, K, assuming K has more than one member?*
(2) *Can this result be obtained by doing set forcing over a model of* ZFC *that does not satisfy* GCH*?*

See [Fri] for more details.

We now can answer another natural question that arises from this theorem, that is, *can this result hold in a forcing extension which still satisfies* GCH?

Using the coding described in the previous section, we can answer the question in the affirmative with the following theorem:

Theorem 6. *Let* $V \models \text{ZFC} + \text{GCH} +$ "*κ is a supercompact cardinal.*" *Then there is a forcing extension* $V^{\mathbb{P}}$ *such that* $V^{\mathbb{P}} \models \text{GCH} +$ "*κ is supercompact, but not* HOD*-supercompact.*"

Proof. Let f be a Laver function for κ [Lav78]. Let $S = \{\delta : \delta$ is a measurable limit of measurable cardinals, $f``\delta \subseteq \delta$ and $f(\delta) > \delta\}$. Our partial ordering $\mathbb{P}$ will be a length $\kappa + 1$ reverse Easton support iteration with $\mathbb{P} = \mathbb{P}_\kappa * \dot{\text{Add}}(\kappa, 1)$. Let $\mathbb{P}_\kappa = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle : \alpha < \kappa\rangle$ with $\mathbb{P}_0 = \text{Add}(\omega, 1)$. Let $\dot{\mathbb{Q}}_\alpha$ be a term for trivial forcing, unless $\alpha \in S$. If $\alpha \in S$, let $\dot{\mathbb{Q}}_\alpha = \dot{\text{Add}}(\alpha, 1) * \dot{M}_{\alpha^*,\gamma_{\alpha^*}} * \dot{\bar{\mathbb{S}}}_{\alpha^*,\gamma_{\alpha^*}}(\dot{X})$, where $\dot{X}$ is the name of the generic subset of α added by $\text{Add}(\alpha, 1)$ and α^* is the least measurable cardinal greater than $f(\alpha)$. Thus, forcing with $\mathbb{Q}_\alpha$ introduces a new subset X of α and codes X beyond $f(\alpha)$ by first forcing to blow up the number of normal measures on the first α^* many consecutive measurable cardinals beyond α^* and then forcing to reduce the number of normal measures according to the subset added to α. So $\eta \in X \leftrightarrow \sigma_\eta$ carries σ_η^+ many normal measures.

Let $G \subseteq \mathbb{P}_\kappa$ be V-generic, and let g be $V[G]$-generic for $(\text{Add}(\kappa, 1))^{V[G]}$. Standard arguments show that $V[G][g] \models \text{GCH}$.

We follow the proof of [Sar08, Lemma 2.1].

Lemma 6.1. $V[G][g] \models$ "*κ is supercompact.*"

Proof. Let $\lambda > \kappa$ be an arbitrary strong limit cardinal, and let $j : V \longrightarrow M$ be a λ-supercompactness embedding with $j(f)(\kappa) = \lambda$. Since $\kappa \in j(S)$, the stage κ forcing in $M^{\mathbb{P}_\kappa}$ is nontrivial, and we have that $j(\mathbb{P}) = \mathbb{P}_\kappa * \dot{\text{Add}}(\kappa, 1) * \dot{M}_{\kappa^*,\gamma_{\kappa^*}} * \dot{\bar{\mathbb{S}}}_{\kappa^*,\gamma_{\kappa^*}}(\dot{X}) * \dot{\mathbb{P}}_{tail}$, with $\dot{X}$ the name for the generic subset added by $\text{Add}(\kappa, 1)$, κ^* the least measurable cardinal greater than λ in M and $\dot{\mathbb{P}}_{tail}$ a term for the forcing defined in the half-open interval $(\gamma_{\kappa^*}, j(\kappa)]$. Since the first stage of nontrivial forcing in $M_{\kappa^*,\gamma_{\kappa^*}} * \dot{\bar{\mathbb{S}}}_{\kappa^*,\gamma_{\kappa^*}}(\dot{X})$ is beyond λ, we may write $j(\mathbb{P})$ as $\mathbb{P}_\kappa * \dot{\text{Add}}(\kappa, 1) * \dot{\mathbb{P}}_{tail}$, where the first stage of nontrivial forcing in $\dot{\mathbb{P}}_{tail}$ is beyond λ. Standard arguments show (see [Fri, Lemma 4.1] and [Lav78]) that for any cardinal $\gamma < \lambda$, $V[G][g] \models$ "κ is γ-supercompact". Since λ was arbitrary, this completes the proof of Lemma 6.1. $\square$

Lemma 6.2. $V[G][g] \models$ "*κ is not* HOD*-supercompact.*"

Proof. Since $\mathbb{P}$ admits a closure point at ω, Sargsyan's argument of [Sar08, Lemma 2.2] shows that κ is not HOD-supercompact in $V[G][g]$. Namely, let $G \subseteq \mathbb{P}$ be V-generic. Factor $\mathbb{P}$ as $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{P}}_{tail}$, where $\mathbb{P}_\kappa$ is the forcing up to stage κ, $\dot{\mathbb{Q}}_\kappa = \dot{\mathrm{A}}\mathrm{dd}(\kappa, 1)$ and $\dot{\mathbb{P}}_{tail}$ is a term for the forcing beyond κ. Let $G = G_\kappa * g * G_{tail}$ be the corresponding factorization of the generic. Assume κ is HOD-supercompact in $V[G] = W$ and let $\mathrm{HOD} = \mathrm{HOD}^W$. Fix a strong limit cardinal λ such that $\mathrm{HOD}^{W_\lambda} = \mathrm{HOD} \cap W_\lambda$. Let $j : W \longrightarrow M$ be a λ-supercompactness embedding such that $j(\mathrm{HOD}) \cap W_\lambda = \mathrm{HOD} \cap W_\lambda = \mathrm{HOD}^{W_\lambda}$. By Theorem 1, $i = j \restriction V$ is definable in V and j is the lift of i. Let $N = j(V) = \bigcup_{\alpha < \mathrm{ORD}} i(V_\alpha)$. If H is the N-generic for $i(\mathbb{P})$, then $M = N[H] = N[j(G)]$. We also have that $H \cap \mathbb{P}_\kappa = G_\kappa$. Let g' be the generic for $\mathrm{Add}(\kappa, 1)$ given by H. Then in $N[H] = M$, g' is ordinal definable. But because κ is HOD-supercompact, $j(\mathrm{HOD}) \cap W_\lambda = \mathrm{HOD} \cap W_\lambda = \mathrm{HOD}^{W_\lambda}$. This implies $g' \in \mathrm{HOD}^{W_\lambda}$. Thus g' is ordinal definable in $W_\lambda = V_\lambda^{V[G]} = V_\lambda^{V[G_\kappa][g][G_{tail}]}$. Since $\mathbb{P}_{tail}$ is κ-closed, g' could not have been added by $\mathbb{P}_{tail}$. So g' had to have been added over $V_\lambda[G_\kappa]$, and more particularly, g' is added over $V_\lambda[G_\kappa]$ by homogeneous forcing. This fact, along with g' being ordinal definable in W_λ, implies that g' is in $V_\lambda[G_\kappa]$. This is impossible, as g' is a $V[G_\kappa]$-generic object for $\mathrm{Add}(\kappa, 1)$. Therefore κ is not HOD-supercompact.

□

Lemmas 6.1-6.2 prove Theorem 6. □

4. MCA AND A PROPER CLASS OF SUPERCOMPACT CARDINALS

Another application of this result is forcing V=HOD in the presence of a proper class of supercompact cardinals while preserving GCH. Note that in order to carry out the coding it is only necessary to have a proper class of measurable cardinals.

Definition 7. *The* Measurable Cardinals Coding Axiom (MCA) *is the assertion that for every cardinal δ, and for every $A \subseteq \delta$, $\exists \sigma > \delta$ such that for every $\alpha < \delta$, $\alpha \in A \leftrightarrow \sigma_\alpha$ carries σ_α^+ many normal measures.*

Note that MCA implies the existence of a proper class of measurable cardinals.

Following the strategy found in Reitz [Rei06], [Rei07], who forces a similar axiom he calls the *Continuum Coding Axiom* (CCA), we will force the MCA, which we note is a strong form of V=HOD.

We define a building block of our forcing, which we call the lottery sum after Hamkins [Ham00]. Specifically, the *lottery sum* of a collection A of posets is defined as $\oplus A = \{\langle \mathbb{Q}, p\rangle : \mathbb{Q} \in A \text{ and } p \in \mathbb{Q}\} \cup \{\mathbf{1}\}$, ordered with $\mathbf{1}$ above everything and $\langle \mathbb{Q}, p\rangle \leq \langle \mathbb{Q}', q\rangle$ when $\mathbb{Q} = \mathbb{Q}'$ and $p \leq_{\mathbb{Q}} q$. Since all compatible conditions must be in the same $\mathbb{Q}$, the forcing effectively holds a lottery of all the posets in A, and the generic chooses the "winning" poset $\mathbb{Q}$ and then forces with it.

We now present the following theorem:

Theorem 8. *Let $V \models$ ZFC + GCH + "There is a (proper) class of supercompact cardinals K." Then there is a class forcing $\mathbb{Q} \subseteq V$ such that $V^{\mathbb{Q}} \models$ ZFC + GCH + V=HOD + "K is the class of supercompact cardinals."*

Proof. Our proof will combine the methods we have introduced with a technique due to Brooke-Taylor, which was given by him in [BT09]. Let $\mathbb{M}$ be the reverse Easton support class iteration using $\mathrm{Add}(\delta^+, 1)$ at every inaccessible cardinal δ and trivial forcing at all other stages. Let $\bar{V} = V^{\mathbb{M}}$. In analogy to what we discussed in the paragraph immediately following Definition 3, $\bar{V} \models$ GCH+ "Every measurable cardinal carries the maximum number of normal measures." In addition, standard arguments show that for every $\kappa \in K$, $\bar{V} \models$ "κ is supercompact." Therefore, by our discussion in the paragraph immediately following the statement of Theorem 1, $\bar{V} \models$ "K is the class of supercompact cardinals."

We now work in $\bar{V}$. Let $\mathbb{C}_\sigma = \mathrm{Coll}(\sigma^+, \sigma^{++})$ and let $\mathbb{B}_\sigma = \{\emptyset\}$. Let $\mathbb{P}$ be the reverse Easton support class iteration defined as $\mathbb{P} = \mathrm{Add}(\omega, 1) * \langle \dot{\mathbb{Q}}_\sigma : \sigma \in \text{ORD} \rangle$. Then $\dot{\mathbb{Q}}_\sigma$ will be taken as a term for trivial forcing unless $\sigma \in \bar{V}$ is a "successor" measurable cardinal, that is, σ is not a measurable limit of measurable cardinals in $\bar{V}$. At a nontrivial stage of forcing σ, we take $\dot{\mathbb{Q}}_\sigma$ as a term for the lottery sum between the collapse forcing $\mathbb{C}_\sigma$ and trivial forcing, that is, $\dot{\mathbb{Q}}_\sigma = \oplus\{\dot{\mathbb{C}}_\sigma, \dot{\mathbb{B}}_\sigma\} = \oplus\{\dot{\mathrm{Coll}}(\sigma^+, \sigma^{++}), \{\emptyset\}\}$. We let the "generic decide" which "bit" of information will be coded.

Lemma 8.1. $\bar{V}^{\mathbb{P}} \models$ MCA. *In particular, in $\bar{V}^{\mathbb{P}}$, every set of ordinals is ordinal definable, that is, $\bar{V}^{\mathbb{P}} \models$ V=HOD.*

Proof. First of all, we need a lemma to ensure that our coding is the same in $\bar{V}^{\mathbb{P}}$ as in $\bar{V}$.

Lemma 8.2. $\bar{V}^{\mathbb{P}} \models$ "*δ is a successor measurable cardinal*" $\leftrightarrow \bar{V} \models$ "*δ is a successor measurable cardinal.*"

Proof. Since $\mathbb{P}$ admits a closure point at ω, by Theorem 1, no new measurable cardinals were created by $\mathbb{P}$.

- **(i):** $\Longleftarrow$ Let δ be a successor measurable cardinal in $\bar{V}$. Let $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}^\delta$, where $\mathbb{P}_\delta$ is the forcing up to stage δ. Since δ is a successor measurable cardinal in $\bar{V}$, $|\mathbb{P}_\delta| < \delta$. By the results of [LS67], forcing with $\mathbb{P}_\delta$ will therefore not affect the measurability of δ, and the remnant of the forcing is sufficiently closed so as not to affect δ's measurability. So δ is measurable in $\bar{V}^{\mathbb{P}}$.

  Suppose that δ is now a measurable limit of measurable cardinals in $\bar{V}^{\mathbb{P}}$. If that were the case, then many new measurable cardinals would have to have been created by $\mathbb{P}$. As before, by Theorem 1, no new measurable cardinals were created. So successor measurable cardinals are preserved to $\bar{V}^{\mathbb{P}}$. In particular, δ is a successor measurable cardinal in $\bar{V}^{\mathbb{P}}$.
- **(ii):** $\Longrightarrow$ Let δ be a successor measurable cardinal in $\bar{V}^{\mathbb{P}}$. Since no new measurable cardinals were created by $\mathbb{P}$, we know that δ was measurable in $\bar{V}$. Suppose that δ were a measurable limit of measurable cardinals in $\bar{V}$. In particular, it was a measurable limit of successor measurable cardinals. But the successor measurable cardinals are preserved by $\mathbb{P}$ (see (i)), so δ would remain a measurable limit of measurable cardinals in $\bar{V}^{\mathbb{P}}$. This is a contradiction, so δ was a successor measurable cardinal in $\bar{V}$.

□

We return to the proof of Lemma 8.1. Now it will suffice to show that $\forall A \subseteq$ ORD, $A \in \bar{V}^{\mathbb{P}}$, $D_A = \{p \in \mathbb{P} : p \Vdash$ "$\dot{A}$ is coded, as in the statement of MCA"$\}$ is dense in $\mathbb{P}$. Fix some $A \in \bar{V}^{\mathbb{P}}$. There exists a δ such that $A \in \bar{V}^{\mathbb{P}_\delta}$. Fix any $p \in \mathbb{P}$ and an ordinal ϱ such that $p \Vdash$ "$\dot{A} \subseteq \varrho$". Since $\mathbb{P}$ uses Easton support, support$(p) \subseteq \gamma$ for some γ. Let σ be the least measurable cardinal $> \max(\gamma, \delta, \varrho)$. Since there exists a proper class of supercompact cardinals, in particular σ exists, and there are ϱ many measurable cardinals above σ. This allows us to use Brooke-Taylor's coding technique of [BT09]. We now extend p by going to σ and then if $\alpha \in A$, at stage σ_α, extend p to a condition which forces that $\mathbb{C}_{\sigma_\alpha}$ will be chosen. If $\alpha \notin A$, we extend p to choose $\mathbb{B}_{\sigma_\alpha}$. Therefore p can be extended to code A as in the statement of MCA for any $A \in \bar{V}^{\mathbb{P}}$, $A \subseteq$ ORD. Therefore $\bar{V}^{\mathbb{P}} \models$ MCA, so $\bar{V}^{\mathbb{P}} \models$ V=HOD. □

Lemma 8.3. $\bar{V}^{\mathbb{P}} \models$ "*K is the class of supercompact cardinals.*"

Proof. Again, since $\mathbb{P}$ admits a closure point at ω, by an application of Theorem 1, no new supercompact cardinals were created by $\mathbb{P}$. So it suffices to show that all supercompact cardinals in $\bar{V}$ are preserved to $\bar{V}^{\mathbb{P}}$.

Let $\kappa \in K$. Now it remains to show that κ is supercompact in $\bar{V}^{\mathbb{P}}$.

Let $G \subseteq \mathbb{P}$ be $\bar{V}$-generic. Let $\lambda > \kappa$ be such that λ has cofinality κ and is a limit of measurable cardinals. Note that $\lambda^{<\kappa} = \lambda$. Let $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{P}}_{\kappa,\lambda} * \dot{\mathbb{P}}_{tail}$ and let $G = G_\kappa * G_{\kappa,\lambda} * G_{tail}$, where $\mathbb{P}_\kappa$ is the forcing up to stage κ, $\dot{\mathbb{P}}_{\kappa,\lambda}$ is a term for the forcing from κ to λ, and $\dot{\mathbb{P}}_{tail}$ is a term for the forcing beyond λ. $\mathbb{P}_{tail}$ will be sufficiently closed so that by the choice of λ, $\bar{V}^{\mathbb{P}_\kappa * \dot{\mathbb{P}}_{\kappa,\lambda}} \models$ "κ is λ-supercompact" $\leftrightarrow \bar{V}^{\mathbb{P}} \models$ "κ is λ-supercompact." Since λ may be chosen to be arbitrarily large, it will suffice to show that $\bar{V}^{\mathbb{P}_\kappa * \dot{\mathbb{P}}_{\kappa,\lambda}} \models$ "κ is λ-supercompact."

To this end, let $j : \bar{V} \longrightarrow M$ be a 2^λ-supercompactness embedding for κ. Since M and $\bar{V}$ agree up to 2^λ, it follows that up to and including stage λ, this forcing is the same in M as it is in $\bar{V}$ and that we may factor $j(\mathbb{P}_\kappa * \dot{\mathbb{P}}_{\kappa,\lambda})$ as $\mathbb{P}_\kappa * \dot{\mathbb{P}}_{\kappa,\lambda} * \dot{\mathbb{P}}_{\lambda,j(\kappa)} * j(\dot{\mathbb{P}}_{\kappa,\lambda})$, where $\dot{\mathbb{P}}_{\lambda,j(\kappa)}$ is a term for the forcing defined in the open interval $(\lambda, j(\kappa))$ and $j(\dot{\mathbb{P}}_{\kappa,\lambda})$ is a term for the forcing defined in the closed interval $[j(\kappa), j(\lambda)]$. As in the proof of Lemma 6.1, standard arguments (see [Fri, Lemma 4.1] and [Lav78]) once again show that $\bar{V}^{\mathbb{P}_\kappa * \dot{\mathbb{P}}_{\kappa,\lambda}} \models$ "κ is λ-supercompact." This proves Lemma 8.3. □

Standard arguments show that $\bar{V}^{\mathbb{P}} \models$ GCH. With $\mathbb{Q} = \mathbb{M} * \dot{\mathbb{P}}$, Lemmas 8.1-8.3 prove Theorem 8. □

We note that the results of this section produce a model of set theory that satisfies the Ground Axiom introduced by Hamkins [Ham05] and Reitz [Rei06], [Rei07].

Definition 9. *The* Ground Axiom *(*GA*) is the assertion that the universe of sets V is not a forcing extension of any inner model $W \subseteq V$ by nontrivial set forcing $\mathbb{P} \in W$.*

In particular, we can produce a model of set theory which satisfies V=HOD + GA + GCH + "There exists a proper class of supercompact cardinals" with the following additional theorem.

Theorem 10. *The* MCA *implies the* GA*.*

Proof. We follow Reitz's proof of [Rei06, Theorem 9] and [Rei07, Theorem 10]. Suppose $V \models$ MCA. Suppose further that V is a set forcing extension of an inner model $V = W[h]$, where h is W-generic for some poset $\mathbb{Q} \in W$. For $\sigma > |\mathbb{Q}|$, by the results of [LS67], the models W and V will agree on the properties "σ is a measurable cardinal" and "σ carries σ^+ many normal measures." Every set of ordinals A in V is coded into the "number of normal measures" function of V. The claim is that one such code for A must appear above $|\mathbb{Q}|$. If $A \subseteq \delta$ and $|\mathbb{Q}| = \gamma$, let $\gamma^* = \max(\gamma, \delta)$. Then by the MCA, there exists a $\sigma > \gamma^*$ such that A is coded using the first δ many measurable cardinals beyond σ. Thus A is coded into the "number of normal measures" function of V above $|\mathbb{Q}|$, and so the code appears also in W. Thus $A \in W$, and so every set of ordinals of V is also in W. This shows that $V = W$, and so the forcing $\mathbb{Q}$ was trivial. Thus $V \models$ GA. □

Hence, the model constructed in Theorem 8 is the witnessing model.

5. The Wholeness Axiom, WA_0, with V=HOD, GA and GCH

Using the methods of this paper, we can extend a result of Hamkins [Ham01b] involving the hierarchy of Wholeness Axioms and V=HOD, which are derived from the Wholeness Axiom, proposed by Paul Corazza [Cor00]. The Wholeness Axiom is intended as a slight weakening of Kunen's famous inconsistency result [Kun71] concerning the nonexistence of a nontrivial elementary embedding from the universe to itself.

The hierarchy of Wholeness Axioms is formalized in the language $\{\in, \mathbf{j}\}$, augmenting the usual language of set theory $\{\in\}$ with an additional unary function symbol $\mathbf{j}$ to represent the embedding. We begin with the following definitions.

Definition 11. *The Wholeness Axiom* WA_n*, where* n *is among* $0, 1, ..., \infty$*, consists of the following:*

(1) *(Elementarity) All instances of* $\varphi(x) \leftrightarrow \varphi(\boldsymbol{j}(\mathrm{x}))$ *for* φ *in the language* $\{\in\}$.
(2) *(Separation) All instances of the* $\sum_n$*-Separation Axioms for formulae in the full language* $\{\in, \boldsymbol{j}\}$.
(3) *(Nontriviality) The axiom* $\exists x(\boldsymbol{j}(\mathrm{x}) \neq \mathrm{x})$.

The following theorem generalizes the Main Theorem of [Ham01b].

Theorem 12. *If the Wholeness Axiom* WA_0 *is itself consistent, then it is consistent with* V=HOD + GA+ GCH.

Proof. Suppose that $V \models \mathrm{WA}_0$. As in [Ham01b], every model of one of the Wholeness Axioms has the form $\langle V, \in, j\rangle$, where $\langle V, \in\rangle$ satisfies ZFC and $j : V \longrightarrow V$ is a nontrivial amenable elementary embedding.

We now have the following easy lemma.

Lemma 12.1. *If* $V \models \mathrm{WA}_0$*, then* $V \models$*"There exists a proper class of measurable cardinals."*

Proof. Let $j : V \longrightarrow V$ be the witnessing embedding, with critical sequence $\{\kappa_n : n \in \omega\}$, defined by $\kappa_0 = \kappa = \mathrm{cp(j)}$ and $\kappa_{n+1} = j(\kappa_n)$. Since $j \upharpoonright V_\kappa$ is the identity function, it follows that $V_\kappa \prec V_{\kappa_1}$. As in [Ham01b], by iteratively applying j to this fact, one easily concludes that

$$V_{\kappa_0} \prec V_{\kappa_1} \prec V_{\kappa_2} \prec \cdots \prec V.$$

By [Ham01b], $V \models$ "κ is supercompact." Hence, $V_\kappa \models$ "There exists a proper class of measurable cardinals." But since $V_\kappa \prec V$, $V \models$ "There exists a proper class of measurable cardinals." □

We take this opportunity to make several remarks concerning Lemma 12.1. First, as pointed out to us by Paul Corazza, [Cor06] shows that ZFC+ WA_0 implies the existence of a proper class of cardinals that are super-n-huge for any particular $n \in \omega$ (and more). Thus, V_κ has an incredibly rich large cardinal structure. In addition, as Corazza has pointed out, our proof of Lemma 12.1 skirts an ambiguity (which poses no difficulties in our case) that is addressed more precisely in [Cor06, Section 8].

Also, as the referee has observed, it is possible to prove Lemma 12.1 as follows. Since $\kappa_0 = \text{cp}(j)$, κ_0 is measurable, so by elementarity, each member of the critical sequence $\{\kappa_n : n \in \omega\}$ is also measurable. By Kunen's original proof of [Kun71], the critical sequence $\{\kappa_n : n \in \omega\}$ is cofinal in V. Therefore $V \models$ "$\forall\alpha\exists\beta > \alpha(\beta$ is a measurable cardinal)", so $V \models$ "There exists a proper class of measurable cardinals."

Returning to the proof of Theorem 12, as in [Ham01b], we force with the usual reverse Easton support class iteration that forces GCH. Thus, at any cardinal γ, we force with $\text{Add}(\gamma^+, 1)$. By [Ham01b], the resulting forcing extension $\bar{V}$ satisfies WA_0 and GCH. Furthermore, as in [ACH07], this forcing ensures that every measurable cardinal σ carries the maximum number of normal measures.

We define now in $\bar{V}$ the poset $\mathbb{P}_\kappa$ used in the construction of the witnessing model for Theorem 12. $\mathbb{P}_\kappa$ is a reverse Easton support κ-iteration, defined as $\mathbb{P}_\kappa = \text{Add}(\omega, 1) * \langle \dot{\mathbb{Q}}_\sigma : \sigma < \kappa \rangle$. Here, $\dot{\mathbb{Q}}_\sigma$ is a term for trivial forcing unless σ is a "successor" measurable cardinal. When this is true, $\dot{\mathbb{Q}}_\sigma$ is a term for the poset defined as in Theorem 8. As we have observed previously, this iteration preserves GCH. Our earlier arguments also show that this iteration forces V=HOD in $\bar{V}_\kappa^{\mathbb{P}_\kappa}$. In addition, our earlier arguments also show that $\bar{V}_\kappa^{\mathbb{P}_\kappa} \models$ GA. The same lifting arguments as given in [Ham01b] (literally presented unchanged) now show that $\bar{V}_\kappa^{\mathbb{P}_\kappa} \models \text{WA}_0$. □

References

[ACH07] Arthur W. Apter, James Cummings, and Joel David Hamkins. Large cardinals with few measures. *Proceedings of the American Mathematical Society*, 135(7):2291–2300, 2007.

[BT09] Andrew D. Brooke-Taylor. Large cardinals and definable well-orders on the universe. *Journal of Symbolic Logic*, 74(2):641–654, 2009.

[Cor00] Paul Corazza. Consistency of V=HOD with the Wholeness Axiom. *Archive for Mathematical Logic*, 39(3):219–226, 2000.

[Cor06] Paul Corazza. The spectrum of elementary embeddings $j : V \to V$. *Annals of Pure and Applied Logic*, 139(1-3):327–399, 2006.

[Fri] Shoshana Friedman. Extending a property concerning HOD-supercompactness. Submitted for publication.

[Ham99] Joel David Hamkins. Gap forcing: Generalizing the Lévy-Solovay theorem. *Bulletin of Symbolic Logic*, 5(2):264–272, 1999.

[Ham00] Joel David Hamkins. The lottery preparation. *Annals of Pure and Applied Logic*, 101(2-3):103–146, 2000.

[Ham01a] Joel David Hamkins. Gap forcing. *Israel Journal of Mathematics*, 125:237–252, 2001.

[Ham01b] Joel David Hamkins. The Wholeness Axioms and V=HOD. *Archive for Mathematical Logic*, 40(1):1–8, 2001.

[Ham03] Joel David Hamkins. Extensions with the approximation and cover properties have no new large cardinals. *Fundamenta Mathematicae*, 180(3):257–277, 2003.

[Ham05] Joel David Hamkins. The Ground Axiom. *Oberwolfach Report*, 55:3160–3162, 2005.

[Kun71] Kenneth Kunen. Elementary embeddings and infinitary combinatorics. *Journal of Symbolic Logic*, 36(3):407–413, 1971.

[Lav78] Richard Laver. Making the supercompactness of κ indestructible under κ-directed closed forcing. *Israel Journal of Mathematics*, 29(4):385–388, 1978.

[LS67] Azriel Lévy and Robert M. Solovay. Measurable cardinals and the continuum hypothesis. *Israel Journal of Mathematics*, 5:234–248, 1967.

[Rei06] Jonas Reitz. *The Ground Axiom*. PhD thesis, The CUNY Graduate Center, 2006.

[Rei07] Jonas Reitz. The ground axiom. *Journal of Symbolic Logic*, 72(4):1299–1317, 2007.

[Sar08] Grigor Sargsyan. On HOD-supercompactness. *Archive for Mathematical Logic*, 47(7-8):765–768, 2008.

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