



## Coefficient Bounds for a Certain Class of Analytic and Bi-Univalent Functions

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**Abstract.** In this paper, we introduce and investigate a subclass of analytic and bi-univalent functions in the open unit disk  $\mathbb{U}$ . By using the Faber polynomial expansions, we obtain upper bounds for the coefficients of functions belonging to this analytic and bi-univalent function class. Some interesting recent developments involving other subclasses of analytic and bi-univalent functions are also briefly mentioned.

### 1. Introduction

Let  $\mathcal{A}$  denote the class of functions  $f(z)$  which are *analytic* in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and normalized by the following Taylor-Maclaurin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Also let  $\mathcal{S}$  denote the subclass of functions in  $\mathcal{A}$  which are univalent in  $\mathbb{U}$  (see, for details, [8]).

It is well known that every function  $f \in \mathcal{S}$  has an inverse  $f^{-1}$ , which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right), \quad (1.2)$$

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according to the Koebe One-Quarter Theorem (see, for example, [8]). In fact, the inverse function  $f^{-1}$  is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \tag{1.3}$$

A function  $f \in \mathcal{A}$  is said to be *bi-univalent* in  $\mathbb{U}$  if both  $f(z)$  and  $f^{-1}(z)$  are univalent in  $\mathbb{U}$ . Let  $\Sigma$  denote the class of analytic and bi-univalent functions in  $\mathbb{U}$  given by the Taylor-Maclaurin series expansion (1.1). Some examples of functions in the class  $\Sigma$  are presented below:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2} \log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of the class  $\Sigma$ . Other common examples of functions in  $\mathcal{S}$  such as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of the class  $\Sigma$ .

For a brief history of functions in the class  $\Sigma$ , see [22] (see also [4], [14], [18] and [25]). In fact, judging by the remarkable flood of papers on the subject (see, for example, [5–7, 9–12, 15–17, 19–21, 23, 26, 27, 29, 30]), the recent pioneering work of Srivastava *et al.*[22] appears to have revived the study of analytic and bi-univalent functions in recent years (see also [3], [13] and [24]).

The object of the present paper is to introduce a new subclass of the function class  $\Sigma$  and use the Faber polynomial expansion techniques to derive bounds for the general Taylor-Maclaurin coefficients  $|a_n|$  for the functions in this class. We also obtain estimates for the first two coefficients  $|a_2|$  and  $|a_3|$  of these functions.

## 2. Bounds Derivable by the Faber Polynomial Expansion Techniques

We begin by introducing the function class  $\mathcal{N}_\Sigma^{(\alpha, \lambda)}$  by means of the following definition.

**Definition.** A function  $f(z)$  given by (1.1) is said to be in the class  $\mathcal{N}_\Sigma^{(\alpha, \lambda)}$  ( $0 \leq \alpha < 1$ ;  $\lambda \geq 0$ ) if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \Re \{f'(z) + \lambda z f''(z)\} > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1; \lambda \geq 0). \tag{2.1}$$

By using the Faber polynomial expansions of functions  $f \in \mathcal{A}$  of the form (1.1), the coefficients of its inverse map  $g = f^{-1}$  may be expressed as follows (see [1] and [2]; see also [12]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots, a_n) w^n. \tag{2.2}$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} [a_5 + (-n+2)a_3^2] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5)a_3 a_4] + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

where such expressions as (for example)  $(-n)!$  are to be interpreted *symbolically* by

$$(-n)! \equiv \Gamma(1 - n) := (-n)(-n - 1)(-n - 2) \cdots \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \ (\mathbb{N} := \{1, 2, 3, \dots\})) \quad (2.3)$$

and  $V_j$  ( $7 \leq j \leq n$ ) is a homogeneous polynomial in the variables  $a_2, a_3, \dots, a_n$  (see, for details, [2]). In particular, the first three terms of  $K_{n-1}^{-n}$  are given below:

$$K_1^{-2} = -2a_2,$$

$$K_2^{-3} = 3(2a_2^2 - a_3)$$

and

$$K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

In general, an expansion of  $K_n^p$  is given by (see, for details, [1])

$$K_n^p = pa_n + \frac{p(p-1)}{2}D_n^2 + \frac{p!}{(p-3)!3!}D_n^3 + \cdots + \frac{p!}{(p-n)!n!}D_n^n \quad (p \in \mathbb{Z})$$

where

$$\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\} \quad \text{and} \quad D_n^p = D_n^p(a_2, a_3, \dots)$$

and, alternatively, by (see, for details, [28])

$$D_n^m(a_1, a_2, \dots, a_n) = \sum \left( \frac{m!}{\mu_1! \cdots \mu_n!} \right) a_1^{\mu_1} \cdots a_n^{\mu_n},$$

where  $a_1 = 1$  and the sum is taken over all nonnegative integers  $\mu_1, \dots, \mu_n$  satisfying the following conditions:

$$\begin{cases} \mu_1 + \mu_2 + \cdots + \mu_n = m \\ \mu_1 + 2\mu_2 + \cdots + n\mu_n = n. \end{cases}$$

It is clear that

$$D_n^n(a_1, a_2, \dots, a_n) = a_1^n.$$

Our first main result is given by Theorem 1 below.

**Theorem 1.** Let  $f$  given by (1.1) be in the class  $\mathcal{N}_{\Sigma}^{\alpha, \lambda}$  ( $0 \leq \alpha < 1$  and  $\lambda \geq 0$ ). If  $a_k = 0$  for  $2 \leq k \leq n - 1$ , then

$$|a_n| \leq \frac{2(1 - \alpha)}{n[1 + \lambda(n - 1)]} \quad (n \in \mathbb{N} \setminus \{1, 2\}). \quad (2.4)$$

*Proof.* For analytic functions  $f$  of the form (1.1), we have

$$f'(z) + \lambda z f''(z) = 1 + \sum_{n=2}^{\infty} [1 + \lambda(n - 1)] n a_n z^{n-1} \quad (2.5)$$

and, for its inverse map  $g = f^{-1}$ , it is seen that

$$\begin{aligned}
 g'(w) + \lambda w g''(w) &= 1 + \sum_{n=2}^{\infty} [1 + \lambda(n - 1)] n b_n w^{n-1} \\
 &= 1 + \sum_{n=2}^{\infty} [1 + \lambda(n - 1)] K_{n-1}^{-n}(a_2, a_3, \dots, a_n) w^{n-1}.
 \end{aligned}
 \tag{2.6}$$

On the other hand, since  $f \in \mathcal{N}_{\Sigma}^{\alpha, \lambda}$  and  $g = f^{-1} \in \mathcal{N}_{\Sigma}^{\alpha, \lambda}$ , by definition, there exist two positive real-part functions

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

and

$$q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n,$$

where

$$\Re(p(z)) > 0 \quad \text{and} \quad \Re(q(w)) > 0 \quad (z, w \in \mathbb{U}),$$

so that

$$\begin{aligned}
 f'(z) + \lambda z f''(z) &= \alpha + (1 - \alpha)p(z) \\
 &= 1 + (1 - \alpha) \sum_{n=1}^{\infty} K_n^1(c_1, c_2, \dots, c_n) z^n
 \end{aligned}
 \tag{2.7}$$

and

$$\begin{aligned}
 g'(w) + \lambda w g''(w) &= \alpha + (1 - \alpha)q(w) \\
 &= 1 + (1 - \alpha) \sum_{n=1}^{\infty} K_n^1(d_1, d_2, \dots, d_n) w^n.
 \end{aligned}
 \tag{2.8}$$

Thus, upon comparing the corresponding coefficients in (2.5) and (2.7), we get

$$[1 + \lambda(n - 1)] n a_n = (1 - \alpha) K_{n-1}^1(c_1, c_2, \dots, c_{n-1}). \tag{2.9}$$

Similarly, by using (2.6) and (2.8), we find that

$$[1 + \lambda(n - 1)] K_{n-1}^{-n}(a_1, a_2, \dots, a_n) = (1 - \alpha) K_{n-1}^1(d_1, d_2, \dots, d_{n-1}). \tag{2.10}$$

We note that, for  $a_k = 0$  ( $2 \leq k \leq n - 1$ ), we have

$$b_n = -a_n$$

and so

$$[1 + \lambda(n - 1)] n a_n = (1 - \alpha) c_{n-1} \tag{2.11}$$

and

$$-[1 + \lambda(n - 1)] n a_n = (1 - \alpha) d_{n-1}. \tag{2.12}$$

Thus, according to the Carathéodory Lemma (see [8]), we also observe that

$$|c_n| \leq 2 \quad \text{and} \quad |d_n| \leq 2 \quad (n \in \mathbb{N}).$$

Now, taking the moduli in (2.11) and (2.12) and applying the Carathéodory Lemma, we obtain

$$|a_n| \leq \frac{(1 - \alpha) |c_{n-1}|}{n[1 + \lambda(n - 1)]} = \frac{(1 - \alpha) |d_{n-1}|}{n[1 + \lambda(n - 1)]} \leq \frac{2(1 - \alpha)}{n[1 + \lambda(n - 1)]}, \tag{2.13}$$

which evidently completes the proof of Theorem 1.  $\square$

### 3. Estimates for the Initial Coefficients $a_2$ and $a_3$

In this section, we choose to relax the coefficient restrictions imposed in Theorem 1 and derive the resulting estimates for the initial coefficients  $a_2$  and  $a_3$  of functions  $f \in \mathcal{N}_{\Sigma}^{\alpha, \lambda}$  given by the Taylor-Maclaurin series expansion (1.1). We first state the following theorem.

**Theorem 2.** *Let  $f$  given by (1.1) be in the class  $\mathcal{N}_{\Sigma}^{\alpha, \lambda}$  ( $0 \leq \alpha < 1$  and  $\lambda \geq 0$ ). Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1 - \alpha)}{3(1 + 2\lambda)}} & 0 \leq \alpha < \frac{1 + 2\lambda - 2\lambda^2}{3(1 + 2\lambda)} \\ \frac{1 - \alpha}{1 + \lambda} & \frac{1 + 2\lambda - 2\lambda^2}{3(1 + 2\lambda)} \leq \alpha < 1 \end{cases} \tag{3.1}$$

and

$$|a_3| \leq \frac{2(1 - \alpha)}{3(1 + 2\lambda)}. \tag{3.2}$$

*Proof.* If we set  $n = 2$  by and  $n = 3$  in (2.9) and (2.10), respectively, we obtain

$$2(1 + \lambda)a_2 = (1 - \alpha)c_1, \tag{3.3}$$

$$3(1 + 2\lambda)a_3 = (1 - \alpha)c_2, \tag{3.4}$$

$$-2(1 + \lambda)a_2 = (1 - \alpha)d_1 \tag{3.5}$$

and

$$3(1 + 2\lambda)(2a_2^2 - a_3) = (1 - \alpha)d_2. \tag{3.6}$$

Upon dividing both sides of (3.3) or (3.5) by  $2(1 + \lambda)$ , if we take their moduli and apply the Carathéodory Lemma, we find that

$$|a_2| \leq \frac{(1 - \alpha) |c_1|}{2(1 + \lambda)} = \frac{(1 - \alpha) |d_1|}{2(1 + \lambda)} \leq \frac{1 - \alpha}{1 + \lambda}. \tag{3.7}$$

Now, by adding (3.4) to (3.6), we have

$$6(1 + 2\lambda)a_2^2 = (1 - \alpha)(c_2 + d_2), \tag{3.8}$$

that is,

$$a_2^2 = \frac{(1-\alpha)(c_2+d_2)}{6(1+2\lambda)}. \quad (3.9)$$

Another application of the Carathéodory Lemma followed by taking the square roots in this last equation (3.9) yields

$$|a_2| \leq \sqrt{\frac{2(1-\alpha)}{3(1+2\lambda)}}, \quad (3.10)$$

which proves the first assertion (3.1) of Theorem 2.

Next, for

$$\frac{1+2\lambda-2\lambda^2}{3(1+2\lambda)} \leq \alpha < 1,$$

we note that

$$\frac{1-\alpha}{1+\lambda} \leq \sqrt{\frac{2(1-\alpha)}{3(1+2\lambda)}}. \quad (3.11)$$

Thus, upon dividing both sides of (3.4) by  $3(1+2\lambda)$ , if we take the modulus of each side and apply the Carathéodory Lemma once again, we get

$$|a_3| \leq \frac{2(1-\alpha)}{3(1+2\lambda)}, \quad (3.12)$$

which completes the proof of the second assertion (3.2) of Theorem 2.  $\square$

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