

Research Article

Coefficient Bounds for Certain Subclasses of Bi-Univalent Function

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We introduce two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disc. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Also consequences of the results are pointed out.

1. Introduction

Denote by \mathcal{A} the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, denote by \mathcal{S} the class of all functions in \mathcal{A} which are univalent and normalized by $f(0) = 0 = f'(0)$ in \mathbb{U} . The well-investigated subclasses of the univalent function class \mathcal{S} are the class of starlike functions of order α ($0 \leq \alpha < 1$), denoted by $\mathcal{S}^*(\alpha)$ and the class of convex functions of order α denoted by $\mathcal{K}(\alpha)$ in \mathbb{U} . It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} , defined by

$$\begin{aligned} f^{-1}(f(z)) &= z, \quad z \in \mathbb{U}, \\ f(f^{-1}(w)) &= w, \quad |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}, \end{aligned} \quad (2)$$

where

$$\begin{aligned} f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 - a_3) w^3 \\ &\quad - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \end{aligned} \quad (3)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{U} if both $f(z)$ and $f^{-1}(z)$ are univalent in \mathbb{U} . Let Σ denote the class of bi-univalent functions in \mathbb{U} given by (1).

Analogous to the function class \mathcal{S} , the bi-univalent function class Σ include, for example, the class $\mathcal{S}_{\Sigma}^*(\alpha)$ of bi-starlike functions of order α ($0 \leq \alpha < 1$), the class $\mathcal{K}_{\Sigma}(\alpha)$ of biconvex functions of order α ($0 \leq \alpha < 1$), and the class $\mathcal{S}_{\Sigma}^{\alpha}$ of strongly bi-starlike functions of order α ($0 < \alpha \leq 1$). For some intriguing examples of functions and characterization of the class Σ , one could refer to Srivastava et al. [1] and the references stated therein (see also [2]). Recently there has been triggering, interest to study the bi-univalent functions class Σ (see [2–5]) and obtain nonsharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}$, $\mathbb{N} := \{1, 2, 3, \dots\}$ is presumably still an open problem.

Motivated by the earlier works of Srivastava et al. [1] and Frasin and Aouf [3] in the present paper we introduce the following two new subclasses of the function class Σ .

Definition 1. A function $f(z)$ given by (1) is said to be in the class $\mathcal{G}_{\Sigma}(\alpha, \lambda)$ if the following conditions are satisfied:

$$\begin{aligned} f \in \Sigma, \quad & \left| \arg \left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} \right) \right| < \frac{\alpha\pi}{2}, \\ & 0 < \alpha \leq 1, \quad 0 \leq \lambda < 1, \quad z \in \mathbb{U}, \end{aligned}$$

$$\left| \arg \left(\frac{wg'(w)}{(1-\lambda)g(w) + \lambda wg'(w)} \right) \right| < \frac{\alpha\pi}{2},$$

$$0 < \alpha \leq 1, \quad 0 \leq \lambda < 1, \quad w \in \mathbb{U}, \tag{4}$$

where the function g is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \tag{5}$$

Definition 2. A function $f(z)$ given by (1) is said to be in the class $\mathcal{M}_\Sigma(\beta, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \Re \left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} \right) > \beta,$$

$$0 \leq \beta < 1, \quad 0 \leq \lambda < 1, \quad z \in \mathbb{U}, \tag{6}$$

$$\Re \left(\frac{wg'(w)}{(1-\lambda)g(w) + \lambda wg'(w)} \right) > \beta,$$

$$0 \leq \beta < 1, \quad 0 \leq \lambda < 1, \quad w \in \mathbb{U},$$

where the function g is given by (5).

It is of interest to note that, for $\lambda = 0$, the class $\mathcal{G}_\Sigma(\alpha, \lambda)$ reduces to $\mathcal{S}_\Sigma^\alpha$ of strongly bi-starlike functions of order α and the class $\mathcal{M}_\Sigma(\beta, \lambda)$ leads to $\mathcal{S}_\Sigma^*(\beta)$ bi-starlike functions of order β .

The object of the present paper is to find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclasses $\mathcal{G}_\Sigma(\alpha, \lambda)$ and $\mathcal{M}_\Sigma(\beta, \lambda)$ of the function class Σ by employing the techniques used earlier by Srivastava et al. [1].

In order to derive our main results, we recall the following lemma.

Lemma 3 (see [6]). *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each k , where \mathcal{P} , is the family of all functions h analytic in \mathbb{U} for which $\Re\{h(z)\} > 0$, where $h(z) = 1 + c_1z + c_2z^2 + \dots$ for $z \in \mathbb{U}$.*

2. Coefficient Bounds for the Function

Class $\mathcal{G}_\Sigma(\alpha, \lambda)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $f \in \mathcal{G}_\Sigma(\alpha, \lambda)$.

Theorem 4. *Let $f(z)$ given by (1) be in the class $\mathcal{G}_\Sigma(\alpha, \lambda)$, $0 < \alpha \leq 1$, and $0 \leq \lambda < 1$. Then*

$$|a_2| \leq \frac{2\alpha}{(1-\lambda)\sqrt{1+\alpha}}, \tag{7}$$

$$|a_3| \leq \frac{4\alpha^2}{(1-\lambda)^2} + \frac{\alpha}{1-\lambda}. \tag{8}$$

Proof. It follows from (4) that

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} = [p(z)]^\alpha, \tag{9}$$

$$\frac{wg'(w)}{(1-\lambda)g(w) + \lambda wg'(w)} = [q(w)]^\alpha,$$

where $p(z)$ and $q(w)$ in \mathcal{P} have the forms

$$p(z) = 1 + p_1z + p_2z^2 + \dots, \tag{10}$$

$$q(z) = 1 + q_1w + q_2w^2 + \dots. \tag{11}$$

Now, equating the coefficients in (9), we get

$$(1-\lambda)a_2 = \alpha p_1, \tag{12}$$

$$(\lambda^2 - 1)a_2^2 + 2(1-\lambda)a_3 = \frac{1}{2}[\alpha(\alpha-1)p_1^2 + 2\alpha p_2], \tag{13}$$

$$-(1-\lambda)a_2 = \alpha q_1, \tag{14}$$

$$(\lambda^2 - 4\lambda + 3)a_2^2 - 2(1-\lambda)a_3 = \frac{1}{2}[\alpha(\alpha-1)q_1^2 + 2\alpha q_2]. \tag{15}$$

From (12) and (14), we get

$$p_1 = -q_1, \tag{16}$$

$$2(1-\lambda)^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2). \tag{17}$$

From (13), (15), and (17), we obtain

$$a_2^2 = \frac{\alpha^2 (p_2 + q_2)}{(\alpha + 1)(1-\lambda)^2}. \tag{18}$$

Applying Lemma 3 for the coefficients p_2 and q_2 , we immediately have

$$|a_2| \leq \frac{2\alpha}{(1-\lambda)\sqrt{1+\alpha}}. \tag{19}$$

This gives the bound on $|a_2|$ as asserted in (7).

Next, in order to find the bound on $|a_3|$, by subtracting (15) from (13), we get

$$4(1-\lambda)a_3 - 4(1-\lambda)a_2^2 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2 - q_1^2). \tag{20}$$

It follows from (16), (17), and (20) that

$$a_3 = \frac{\alpha(p_2 - q_2)}{4(1-\lambda)} + \frac{\alpha^2(p_1^2 + q_1^2)}{2(1-\lambda)^2}. \tag{21}$$

Applying Lemma 3 once again for the coefficients $p_1, p_2, q_1,$ and $q_2,$ we readily get

$$|a_3| \leq \frac{4\alpha^2}{(1-\lambda)^2} + \frac{\alpha}{1-\lambda}. \tag{22}$$

This completes the proof of Theorem 4. \square

In the following section we find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{M}_\Sigma(\beta, \lambda).$

3. Coefficient Bounds for the Function Class $\mathcal{M}_\Sigma(\beta, \lambda)$

Theorem 5. *Let $f(z)$ given by (1) be in the class $\mathcal{M}_\Sigma(\beta, \lambda), 0 \leq \beta < 1,$ and $0 \leq \lambda < 1.$ Then*

$$\begin{aligned} |a_2| &\leq \frac{\sqrt{2(1-\beta)}}{1-\lambda}, \\ |a_3| &\leq \frac{4(1-\beta)^2}{(1-\lambda)^2} + \frac{(1-\beta)}{1-\lambda}. \end{aligned} \tag{23}$$

Proof. It follows from (6) that there exists $p, q \in \mathcal{P}$ such that

$$\begin{aligned} \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} &= \beta + (1-\beta)p(z), \\ \frac{wg'(w)}{(1-\lambda)g(w) + \lambda wg'(w)} &= \beta + (1-\beta)q(w), \end{aligned} \tag{24}$$

where $p(z)$ and $q(w)$ have the forms of (10) and (11), respectively. Equating coefficients in (24) we get

$$\begin{aligned} (1-\lambda)a_2 &= (1-\beta)p_1, \\ (\lambda^2-1)a_2^2 + 2(1-\lambda)a_3 &= (1-\beta)p_2, \\ -(1-\lambda)a_2 &= (1-\beta)q_1, \\ (\lambda^2-4\lambda+3)a_2^2 - 2(1-\lambda)a_3 &= (1-\beta)q_2. \end{aligned} \tag{25}$$

The proof follows, by employing the techniques used in the proof of Theorem 4. \square

Taking $\lambda = 0$ in Theorems 4 and 5 one can get the following corollaries.

Corollary 6. *Let $f(z)$ given by (1) be in the class $\mathcal{S}_\Sigma^\alpha$ and $0 < \alpha \leq 1.$ Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha+1}}, \quad |a_3| \leq 4\alpha^2 + \alpha. \tag{26}$$

Corollary 7. *Let $f(z)$ given by (1) be in the class $\mathcal{S}_\Sigma^*(\beta)$ and $0 \leq \beta < 1.$ Then*

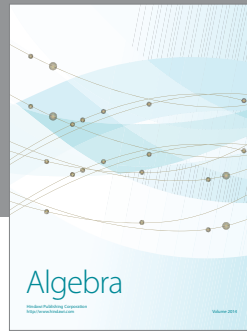
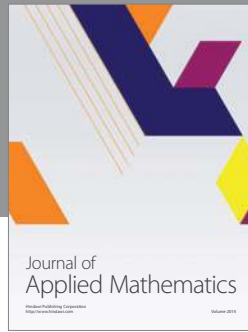
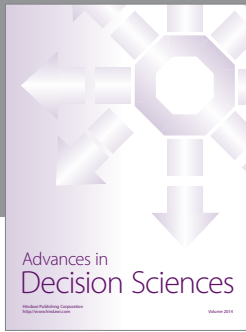
$$|a_2| \leq \sqrt{2-2\beta}, \quad |a_3| \leq 4(1-\beta)^2 + (1-\beta). \tag{27}$$

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