# Coefficient Estimates for a General Subclass of Analytic and Bi-Univalent Functions 

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#### Abstract

In this paper, we introduce and investigate an interesting subclass $\mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu)$, we obtain estimates on the first two Taylor-Maclaurin coefficients $\left|a_{2}\right|$ and $\left|a_{3}\right|$. The results presented in this paper would generalize and improve some recent works of Çağlar et al. [3], Xu et al. [10], and other authors.


## 1. Introduction

Let $\mathbb{R}=(-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$
\mathbb{N}:=\{1,2,3, \ldots\}=\mathbb{N}_{0} \backslash\{0\}
$$

be the set of positive integers.
Let $\mathcal{A}$ denote the class of all functions of the form:

$$
\begin{equation*}
f(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n} \tag{1}
\end{equation*}
$$

which are analytic in the open unit disk

$$
\mathbb{U}=\{z: z \in \mathbb{C} \quad \text { and } \quad|z|<1\} .
$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, which is defined by

$$
f^{-1}(f(z))=z \quad(z \in \mathbb{U})
$$

[^0]and
$$
f\left(f^{-1}(w)\right)=w \quad\left(|w|<r_{0}(f) ; r_{0}(f) \geqq \frac{1}{4}\right) .
$$

In fact, the inverse function $f^{-1}$ is given by

$$
f^{-1}(w)=w-a_{2} w^{2}+\left(2 a_{2}^{2}-a_{3}\right) w^{3}-\left(5 a_{2}^{3}-5 a_{2} a_{3}+a_{4}\right) w^{4}+\cdots
$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). For a brief history and interesting examples of functions in the class $\Sigma$, see [8] (see also [1]). In fact, the aforecited work of Srivastava et al. [8] essentially revived the investigation of various subclasses of the bi-univalent function class $\Sigma$ in recent years; it was followed by such works as those by Frasin and Aouf [4], Xu et al. [9, 10], Hayami and Owa [6], and others (see, for example, [5], [7] and [11]).

Recently, Çağlar et al. [3] introduced the following two subclasses of the bi-univalent function class $\Sigma$ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients $\left|a_{2}\right|$ and $\left|a_{3}\right|$ of functions in each of these subclasses (see also [4] and [10]). It should be mentioned in passing that the functional expression used in the inequalities in (2) and (7) of Definitions 1 and 2 is precisely the same as that used by Zhu [12] for investigating various extensions, generalizations and improvements of the starlikeness criteria which were proven by earlier authors (see, for details, Remark 1 below).

Definition 1. (see [3]) A function $f(z)$ given by (1) is said to be in the class $\mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda)$ if the following conditions are satisfied:

$$
\begin{equation*}
f \in \Sigma \quad \text { and } \quad\left|\arg \left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f^{\prime}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right|<\frac{\alpha \pi}{2} \tag{2}
\end{equation*}
$$

$$
(0<\alpha \leqq 1 ; \lambda \geqq 1 ; \mu \geqq 0 ; z \in \mathbb{U})
$$

and

$$
\begin{aligned}
& \left|\arg \left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g^{\prime}(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)\right|<\frac{\alpha \pi}{2} \\
& (0<\alpha \leqq 1 ; \lambda \geqq 1 ; \mu \geqq 0 ; w \in \mathbb{U})
\end{aligned}
$$

where the function $g$ is given by

$$
\begin{equation*}
g(w)=w-a_{2} w^{2}+\left(2 a_{2}^{2}-a_{3}\right) w^{3}-\left(5 a_{2}^{3}-5 a_{2} a_{3}+a_{4}\right) w^{4}+\cdots \tag{4}
\end{equation*}
$$

Theorem 1. (see [3]) Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the class

$$
\mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda) \quad(0<\alpha \leqq 1 ; \lambda \geqq 1 ; \mu \geqq 0)
$$

Then

$$
\begin{equation*}
\left|a_{2}\right| \leqq \frac{2 \alpha}{\sqrt{(\lambda+\mu)^{2}+\alpha\left(\mu+2 \lambda-\lambda^{2}\right)}} \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|a_{3}\right| \leqq \frac{4 \alpha^{2}}{(\lambda+\mu)^{2}}+\frac{2 \alpha}{2 \lambda+\mu} \tag{6}
\end{equation*}
$$

Definition 2. (see [3]) A function $f(z)$ given by (1) is said to be in the class $\mathcal{N}_{\Sigma}^{\mu}(\beta, \lambda)$ if the following conditions are satisfied:

$$
\begin{aligned}
& f \in \Sigma \quad \text { and } \quad \mathfrak{R}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f^{\prime}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)>\beta \\
& (0 \leqq \beta<1 ; \lambda \geqq 1 ; \mu \geqq 0 ; z \in \mathbb{U})
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathfrak{R}\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g^{\prime}(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)>\beta \\
& (0 \leqq \beta<1 ; \lambda \geqq 1 ; \mu \geqq 0 ; w \in \mathbb{U})
\end{aligned}
$$

where the function $g$ is defined by (4).
Remark 1. For functions $f(z)$, which are analytic in $\mathbb{U}$ and normalized by

$$
f(z)=z+\sum_{k=n+1}^{\infty} a_{k} z^{k} \quad(n \in \mathbb{N})
$$

Zhu [12] determined the conditions on the parameters $M, \alpha, \lambda$ and $\mu$ such that the following inequality:

$$
\left|(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f^{\prime}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-1\right|<M
$$

implies that the so-normalized function $f(z)$ is in the corresponding class of starlike functions of order $\alpha(0 \leqq \alpha<1)$. Interestingly, the functional expression used by Zhu [12] is precisely the same as that used in the inequalities in (2) and (7) above. The work of Zhu [12] provided extensions, generalizations and improvements of the various starlikeness criteria which were proven by a number of earlier authors (see, for details, [12]).

Theorem 2. (see [3]) Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the class

$$
\mathcal{N}_{\Sigma}^{\mu}(\beta, \lambda) \quad(0 \leqq \beta<1 ; \lambda \geqq 1 ; \mu \geqq 0)
$$

Then

$$
\begin{equation*}
\left|a_{2}\right| \leqq \min \left\{\sqrt{\frac{4(1-\beta)}{(\mu+1)(2 \lambda+\mu)}}, \frac{2(1-\beta)}{\lambda+\mu}\right\} \tag{9}
\end{equation*}
$$

and

$$
\left|a_{3}\right| \leqq \begin{cases}\min \left\{\frac{4(1-\beta)}{(\mu+1)(2 \lambda+\mu)}, \frac{4(1-\beta)^{2}}{(\lambda+\mu)^{2}}+\frac{2(1-\beta)}{2 \lambda+\mu}\right\} & (0 \leqq \mu<1)  \tag{10}\\ \frac{2(1-\beta)}{2 \lambda+\mu} & (\mu \leqq 1) .\end{cases}
$$

Remark 2. The following special cases of Definitions 1 and 2 are worthy of note:
(i) For $\mu=1$, we obtain the bi-univalent function classes

$$
\mathcal{N}_{\Sigma}^{1}(\alpha, \lambda)=\mathcal{B}_{\Sigma}(\alpha, \lambda) \quad \text { and } \quad \mathcal{N}_{\Sigma}^{1}(\beta, \lambda)=\mathcal{B}_{\Sigma}(\beta, \lambda)
$$

introduced by Frasin and Aouf [4].
(ii) For $\mu=1$ and $\lambda=1$, we have the bi-univalent function classes

$$
\mathcal{N}_{\Sigma}^{1}(\alpha, 1)=\mathcal{H}_{\Sigma}^{\alpha} \quad \text { and } \quad \mathcal{N}_{\Sigma}^{1}(\beta, 1)=\mathcal{H}_{\Sigma}(\beta)
$$

introduced by Srivastava et al. [8].
(iii) For $\mu=0$ and $\lambda=1$, we get the well-known classes

$$
\mathcal{N}_{\Sigma}^{0}(\alpha, 1)=\mathcal{S}_{\Sigma}^{*}[\alpha] \quad \text { and } \quad \mathcal{N}_{\Sigma}^{0}(\beta, 1)=\mathcal{S}_{\Sigma}^{*}(\beta)
$$

of strongly bi-starlike functions of order $\alpha$ and of bi-starlike functions of order $\beta$, respectively.
This paper is essentially a sequel to some of the aforecited works (especially see [3] and [10]). Here we introduce and investigate the general subclass $\mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu) \quad(\lambda \geqq 1 ; \mu \geqq 0)$ of the analytic function class $\mathcal{A}$, which is given by Definition 3 below.

Definition 3. Let the functions $h, p: \mathbb{U} \rightarrow \mathbb{C}$ be so constrained that

$$
\min \{\mathfrak{R}(h(z)), \mathfrak{R}(p(z))\}>0 \quad(z \in \mathbb{U}) \quad \text { and } \quad h(0)=p(0)=1
$$

Also let the function $f$, defined by (1), be in the analytic function class $\mathcal{A}$. We say that

$$
f \in \mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu) \quad(\lambda \geqq 1 ; \mu \geqq 0)
$$

if the following conditions are satisfied:

$$
\begin{equation*}
f \in \Sigma \quad \text { and } \quad(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f^{\prime}(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \in h(\mathbb{U}) \quad(z \in \mathbb{U}) \tag{11}
\end{equation*}
$$

and

$$
\begin{equation*}
(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g^{\prime}(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \in p(\mathbb{U}) \quad(w \in \mathbb{U}) \tag{12}
\end{equation*}
$$

where the function $g$ is defined by (4).
We note that the class $\mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu)$ reduces to the function classes $\mathcal{B}_{\Sigma}^{h, p}(\lambda)$ and $\mathcal{H}_{\Sigma}^{h, p}$ given by

$$
\begin{aligned}
& \mathcal{B}_{\Sigma}^{h, p}(\lambda)=\mathcal{N}_{\Sigma}^{h, p}(\lambda, 1), \\
& \mathcal{B}_{\Sigma}^{h, p}=\mathcal{N}_{\Sigma}^{h, p}(1,0)
\end{aligned}
$$

and

$$
\mathcal{H}_{\Sigma}^{h, p}=\mathcal{N}_{\Sigma}^{h, p}(1,1)
$$

respectively, each of which was introduced and studied recently by Xu et al. [10], Bulut [2] and Xu et al. [9], respectively.
Remark 3. There are many choices of the functions $h(z)$ and $p(z)$ which would provide interesting subclasses of the analytic function class $\mathcal{A}$. For example, if we let

$$
\begin{equation*}
h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\alpha} \quad(0<\alpha \leqq 1 ; z \in \mathbb{U}) \tag{13}
\end{equation*}
$$

or

$$
\begin{equation*}
h(z)=p(z)=\frac{1+(1-2 \beta) z}{1-z} \quad(0 \leqq \beta<1 ; z \in \mathbb{U}), \tag{14}
\end{equation*}
$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 3. If $f \in \mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu)$, then

$$
\begin{equation*}
f \in \Sigma \text { and }\left|\arg \left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f^{\prime}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right|<\frac{\alpha \pi}{2} \tag{15}
\end{equation*}
$$

$$
(0<\alpha \leqq 1 ; \lambda \geqq 1 ; \mu \geqq 0 ; z \in \mathbb{U})
$$

and

$$
\begin{aligned}
& \left|\arg \left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g^{\prime}(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)\right|<\frac{\alpha \pi}{2} \\
& (0<\alpha \leqq 1 ; \lambda \geqq 1 ; \mu \geqq 0 ; w \in \mathbb{U})
\end{aligned}
$$

or

$$
\begin{aligned}
& f \in \Sigma \quad \text { and } \quad \mathfrak{R}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f^{\prime}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)>\beta \\
& (0 \leqq \beta<1 ; \lambda \geqq 1 ; \mu \geqq 0 ; z \in \mathbb{U})
\end{aligned}
$$

and

$$
\mathfrak{R}\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g^{\prime}(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)>\beta
$$

$$
(0 \leqq \beta<1 ; \lambda \geqq 1 ; \mu \geqq 0 ; w \in \mathbb{U}),
$$

where the function $g$ is defined by (4). This means that

$$
f \in \mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda) \quad(0<\alpha \leqq 1 ; \lambda \geqq 1 ; \mu \geqq 0)
$$

or

$$
f \in \mathcal{N}_{\Sigma}^{\mu}(\beta, \lambda) \quad(0 \leqq \beta<1 ; \lambda \geqq 1 ; \mu \geqq 0) .
$$

Our paper is motivated and stimulated especially by the works of Çağlar et al. [3] and Xu et al. [10]. Here we propose to investigate the bi-univalent function class $\mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu)$ introduced in Definition 3 and derive coefficient estimates on the first two Taylor-Maclaurin coefficients $\left|a_{2}\right|$ and $\left|a_{3}\right|$ for a function $f \in \mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu)$ given by (1). Our results for the bi-univalent function class $\mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu)$ would generalize and improve the related works of Çağlar et al. [3] and Xu et al. [10] (see also [4] and [8]).

## 2. A Set of General Coefficient Estimates

In this section, we state and prove our general results involving the bi-univalent function class $\mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu)$ given by Definition 3 .

Theorem 3. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the function class $\mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu)$. Then

$$
\begin{equation*}
\left|a_{2}\right| \leqq \min \left\{\sqrt{\frac{\left|h^{\prime}(0)\right|^{2}+\left|p^{\prime}(0)\right|^{2}}{2(\lambda+\mu)^{2}}}, \sqrt{\frac{\left|h^{\prime \prime}(0)\right|+\left|p^{\prime \prime}(0)\right|}{2(\mu+1)(2 \lambda+\mu)}}\right\} \tag{19}
\end{equation*}
$$

and

$$
\begin{align*}
\left|a_{3}\right| \leqq \min \left\{\frac{\left|h^{\prime}(0)\right|^{2}+\left|p^{\prime}(0)\right|^{2}}{2(\lambda+\mu)^{2}}+\frac{\left|h^{\prime \prime}(0)\right|+\left|p^{\prime \prime}(0)\right|}{4(2 \lambda+\mu)}\right. \\
\left.\frac{(3+\mu)\left|h^{\prime \prime}(0)\right|+|1-\mu|\left|p^{\prime \prime}(0)\right|}{4(\mu+1)(2 \lambda+\mu)}\right\} . \tag{20}
\end{align*}
$$

Proof. First of all, we write the argument inequalities in (11) and (12) in their equivalent forms as follows:

$$
(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f^{\prime}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}=h(z) \quad(z \in \mathbb{U})
$$

and

$$
(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g^{\prime}(w)\left(\frac{g(w)}{w}\right)^{\mu-1}=p(w) \quad(w \in \mathbb{U})
$$

respectively, where $h(z)$ and $p(w)$ satisfy the conditions of Definition 3. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expensions:

$$
h(z)=1+h_{1} z+h_{2} z^{2}+\cdots
$$

and

$$
p(w)=1+p_{1} w+p_{2} w^{2}+\cdots
$$

respectively. Now, upon equating the coefficients of

$$
(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f^{\prime}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}
$$

with those of $h(z)$ and the coefficients of

$$
(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g^{\prime}(w)\left(\frac{g(w)}{w}\right)^{\mu-1}
$$

with those of $p(w)$, we get

$$
\begin{align*}
& (\lambda+\mu) a_{2}=h_{1}  \tag{21}\\
& (2 \lambda+\mu) a_{3}+(\mu-1)\left(\lambda+\frac{\mu}{2}\right) a_{2}^{2}=h_{2}  \tag{22}\\
& -(\lambda+\mu) a_{2}=p_{1} \tag{23}
\end{align*}
$$

and

$$
\begin{equation*}
-(2 \lambda+\mu) a_{3}+(\mu+3)\left(\lambda+\frac{\mu}{2}\right) a_{2}^{2}=p_{2} . \tag{24}
\end{equation*}
$$

From (21) and (23), we obtain

$$
\begin{equation*}
h_{1}=-p_{1} \tag{25}
\end{equation*}
$$

and

$$
\begin{equation*}
2(\lambda+\mu)^{2} a_{2}^{2}=h_{1}^{2}+p_{1}^{2} . \tag{26}
\end{equation*}
$$

Also, from (22) and (24), we find that

$$
\begin{equation*}
(\mu+1)(2 \lambda+\mu) a_{2}^{2}=h_{2}+p_{2} . \tag{27}
\end{equation*}
$$

Therefore, we find from the equations (26) and (27) that

$$
\left|a_{2}\right|^{2} \leqq \frac{\left|h^{\prime}(0)\right|^{2}+\left|p^{\prime}(0)\right|^{2}}{2(\lambda+\mu)^{2}}
$$

and

$$
\left|a_{2}\right|^{2} \leqq \frac{\left|h^{\prime \prime}(0)\right|+\left|p^{\prime \prime}(0)\right|}{2(\mu+1)(2 \lambda+\mu)^{\prime}},
$$

respectively. So we get the desired estimate on the coefficient $\left|a_{2}\right|$ as asserted in (19).
Next, in order to find the bound on the coefficient $\left|a_{3}\right|$, we subtract (24) from (22). We thus get

$$
\begin{equation*}
2(2 \lambda+\mu) a_{3}-2(2 \lambda+\mu) a_{2}^{2}=h_{2}-p_{2} . \tag{28}
\end{equation*}
$$

Upon substituting the value of $a_{2}^{2}$ from (26) into (28), it follows that

$$
a_{3}=\frac{h_{1}^{2}+p_{1}^{2}}{2(\lambda+\mu)^{2}}+\frac{h_{2}-p_{2}}{2(2 \lambda+\mu)} .
$$

We thus find that

$$
\begin{equation*}
\left|a_{3}\right| \leqq \frac{\left|h^{\prime}(0)\right|^{2}+\left|p^{\prime}(0)\right|^{2}}{2(\lambda+\mu)^{2}}+\frac{\left|h^{\prime \prime}(0)\right|+\left|p^{\prime \prime}(0)\right|}{4(2 \lambda+\mu)} . \tag{29}
\end{equation*}
$$

On the other hand, upon substituting the value of $a_{2}^{2}$ from (27) into (28), it follows that

$$
a_{3}=\frac{(3+\mu) h_{2}+(1-\mu) p_{2}}{2(\mu+1)(2 \lambda+\mu)}
$$

Consequently, we have

$$
\begin{equation*}
\left|a_{3}\right| \leqq \frac{(3+\mu)\left|h^{\prime \prime}(0)\right|+|1-\mu|\left|p^{\prime \prime}(0)\right|}{4(\mu+1)(2 \lambda+\mu)} . \tag{30}
\end{equation*}
$$

This evidently completes the proof of Theorem 3.

## 3. Corollaries and Consequences

By setting $\mu=1$ in Theorem 3, we get Corollary 1 below.
Corollary 1. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the bi-univalent function class $\mathcal{B}_{\Sigma}^{h, p}(\lambda)(\lambda \geqq 1)$. Then

$$
\begin{equation*}
\left|a_{2}\right| \leqq \min \left\{\sqrt{\frac{\left|h^{\prime}(0)\right|^{2}+\left|p^{\prime}(0)\right|^{2}}{2(1+\lambda)^{2}}}, \sqrt{\frac{\left|h^{\prime \prime}(0)\right|+\left|p^{\prime \prime}(0)\right|}{4(1+2 \lambda)}}\right\} \tag{31}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|a_{3}\right| \leqq \min \left\{\frac{\left|h^{\prime}(0)\right|^{2}+\left|p^{\prime}(0)\right|^{2}}{2(1+\lambda)^{2}}+\frac{\left|h^{\prime \prime}(0)\right|+\left|p^{\prime \prime}(0)\right|}{4(1+2 \lambda)}, \frac{\left|h^{\prime \prime}(0)\right|}{2(1+2 \lambda)}\right\} . \tag{32}
\end{equation*}
$$

Remark 4. Corollary 1 is an improvement of the following estimates obtained by Xu et al. [10].
Corollary 2. (see [10]) Suppose that $f(z)$ given by its Taylor-Maclaurin series expansion (1) is in the function class $\mathcal{B}_{\Sigma}^{h, p}(\lambda)(\lambda \geqq 1)$. Then

$$
\begin{equation*}
\left|a_{2}\right| \leqq \sqrt{\frac{\left|h^{\prime \prime}(0)\right|+\left|p^{\prime \prime}(0)\right|}{4(1+2 \lambda)}} \tag{33}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|a_{3}\right| \leqq \frac{\left|h^{\prime \prime}(0)\right|}{2(1+2 \lambda)} \tag{34}
\end{equation*}
$$

By setting $\mu=1$ and $\lambda=1$ in Theorem 3, we get the following consequence.
Corollary 3. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the function class $\mathcal{H}_{\Sigma}^{h, p}$. Then

$$
\begin{equation*}
\left|a_{2}\right| \leqq \min \left\{\sqrt{\frac{\left|h^{\prime}(0)\right|^{2}+\left|p^{\prime}(0)\right|^{2}}{8}}, \sqrt{\frac{\left|h^{\prime \prime}(0)\right|+\left|p^{\prime \prime}(0)\right|}{12}}\right\} \tag{35}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|a_{3}\right| \leqq \min \left\{\frac{\left|h^{\prime}(0)\right|^{2}+\left|p^{\prime}(0)\right|^{2}}{8}+\frac{\left|h^{\prime \prime}(0)\right|+\left|p^{\prime \prime}(0)\right|}{12}, \frac{\left|h^{\prime \prime}(0)\right|}{6}\right\} \tag{36}
\end{equation*}
$$

Remark 5. Corollary 3 is an improvement of the following estimates obtained by Xu et al. [9].
Corollary 4. (see [9]) Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the function class $\mathcal{H}_{\Sigma}^{h, p}$. Then

$$
\begin{equation*}
\left|a_{2}\right| \leqq \sqrt{\frac{\left|h^{\prime \prime}(0)\right|+\left|p^{\prime \prime}(0)\right|}{12}} \tag{37}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|a_{3}\right| \leqq \frac{\left|h^{\prime \prime}(0)\right|}{6} \tag{38}
\end{equation*}
$$

Remark 6. By setting $\mu=0$ and $\lambda=1$ in Theorem 3, we get [2, Theorem 2.1].
If we set

$$
h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\alpha} \quad(0<\alpha \leqq 1 ; z \in \mathbb{U})
$$

in Theorem 3, we can readily deduce Corollary 5.
Corollary 5. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the bi-univalent function class

$$
\mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda)(0<\alpha \leqq 1 ; \lambda \geqq 1 ; \mu \geqq 0)
$$

Then

$$
\left|a_{2}\right| \leqq \begin{cases}\frac{2 \alpha}{\lambda+\mu} & (\lambda \geqq 1+\sqrt{1+\mu})  \tag{39}\\ \frac{2 \alpha}{\sqrt{(\mu+1)(2 \lambda+\mu)}} & (1 \leqq \lambda<1+\sqrt{1+\mu})\end{cases}
$$

and

$$
\left|a_{3}\right| \leqq \begin{cases}\min \left\{\frac{4 \alpha^{2}}{(\lambda+\mu)^{2}}+\frac{2 \alpha^{2}}{2 \lambda+\mu}, \frac{4 \alpha^{2}}{(\mu+1)(2 \lambda+\mu)}\right\} & (0 \leqq \mu<1)  \tag{40}\\ \frac{2 \alpha^{2}}{2 \lambda+\mu} & (\mu \geqq 1)\end{cases}
$$

Remark 7. It is easy to see, for the coefficient $\left|a_{2}\right|$, that

$$
\begin{aligned}
& \frac{2 \alpha}{\lambda+\mu} \leqq \frac{2 \alpha}{\sqrt{(\lambda+\mu)^{2}+\alpha\left(\mu+2 \lambda-\lambda^{2}\right)}} \\
& (0<\alpha \leqq 1 ; \lambda \geqq 1+\sqrt{1+\mu} ; \mu \geqq 0)
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{2 \alpha}{\sqrt{(\mu+1)(2 \lambda+\mu)}} \leqq \frac{2 \alpha}{\sqrt{(\lambda+\mu)^{2}+\alpha\left(\mu+2 \lambda-\lambda^{2}\right)}} \\
& (0<\alpha \leqq 1 ; 1 \leqq \lambda<1+\sqrt{1+\mu} ; \mu \geqq 0) .
\end{aligned}
$$

On the other hand, for the coefficient $\left|a_{3}\right|$, we make the following observations:
(i) If $0 \leqq \mu<1$ and

$$
\min \left\{\frac{4 \alpha^{2}}{(\lambda+\mu)^{2}}+\frac{2 \alpha^{2}}{2 \lambda+\mu^{\prime}}, \frac{4 \alpha^{2}}{(\mu+1)(2 \lambda+\mu)}\right\}=\frac{4 \alpha^{2}}{(\lambda+\mu)^{2}}+\frac{2 \alpha^{2}}{2 \lambda+\mu^{\prime}}
$$

then

$$
\frac{4 \alpha^{2}}{(\lambda+\mu)^{2}}+\frac{2 \alpha^{2}}{2 \lambda+\mu} \leqq \frac{4 \alpha^{2}}{(\lambda+\mu)^{2}}+\frac{2 \alpha}{2 \lambda+\mu} \quad(0<\alpha \leqq 1 ; \lambda \geqq 1) ;
$$

(ii) If $0 \leqq \mu<1$ and

$$
\min \left\{\frac{4 \alpha^{2}}{(\lambda+\mu)^{2}}+\frac{2 \alpha^{2}}{2 \lambda+\mu^{\prime}} \frac{4 \alpha^{2}}{(\mu+1)(2 \lambda+\mu)}\right\}=\frac{4 \alpha^{2}}{(\mu+1)(2 \lambda+\mu)^{\prime}}
$$

then

$$
\begin{aligned}
\frac{4 \alpha^{2}}{(\mu+1)(2 \lambda+\mu)} & \leqq \frac{4 \alpha^{2}}{(\lambda+\mu)^{2}}+\frac{2 \alpha^{2}}{2 \lambda+\mu} \\
& \leqq \frac{4 \alpha^{2}}{(\lambda+\mu)^{2}}+\frac{2 \alpha}{2 \lambda+\mu} \quad(0<\alpha \leqq 1 ; \lambda \leqq 1) ;
\end{aligned}
$$

(iii) If $\mu \geqq 1$, then

$$
\begin{aligned}
\frac{2 \alpha^{2}}{2 \lambda+\mu} & \leqq \frac{2 \alpha}{2 \lambda+\mu} \\
& \leqq \frac{4 \alpha^{2}}{(\lambda+\mu)^{2}}+\frac{2 \alpha}{2 \lambda+\mu} \quad(0<\alpha \leqq 1 ; \lambda \leqq 1) .
\end{aligned}
$$

Thus, clearly, Corollary 5 is an improvement of Theorem 1.
By setting $\mu=1$ in Corollary 5 , we obtain the following consequence.
Corollary 6. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the bi-univalent function class

$$
\mathcal{B}_{\Sigma}(\alpha, \lambda)(0<\alpha \leqq 1 ; \lambda \geqq 1) .
$$

Then

$$
\left|a_{2}\right| \leqq \begin{cases}\frac{2 \alpha}{\lambda+1} & (\lambda \leqq 1+\sqrt{2})  \tag{41}\\ \sqrt{\frac{2}{2 \lambda+1}} \alpha & (1 \leqq \lambda<1+\sqrt{2})\end{cases}
$$

and

$$
\begin{equation*}
\left|a_{3}\right| \leqq \frac{2 \alpha^{2}}{2 \lambda+1} . \tag{42}
\end{equation*}
$$

Remark 8. Corollary 6 provides an improvement of the following estimates obtained by Frasin and Aouf [4].
Corollary 7. (see [4]) Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the bi-univalent function class

$$
\mathcal{B}_{\Sigma}(\alpha, \lambda) \quad(0<\alpha \leqq 1 ; \lambda \geqq 1) .
$$

Then

$$
\begin{equation*}
\left|a_{2}\right| \leqq \frac{2 \alpha}{\sqrt{(\lambda+1)^{2}+\alpha\left(1+2 \lambda-\lambda^{2}\right)}} \tag{43}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|a_{3}\right| \leqq \frac{4 \alpha^{2}}{(\lambda+1)^{2}}+\frac{2 \alpha}{2 \lambda+1} \tag{44}
\end{equation*}
$$

By setting $\mu=1$ and $\lambda=1$ in Corollary 5, we get the following consequence.
Corollary 8. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the bi-univalent function class $\mathcal{H}_{\Sigma}^{\alpha}(0<\alpha \leqq 1)$. Then

$$
\begin{equation*}
\left|a_{2}\right| \leqq \sqrt{\frac{2}{3}} \alpha \tag{45}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|a_{3}\right| \leqq \frac{2 \alpha^{2}}{3} \tag{46}
\end{equation*}
$$

Remark 9. Corollary 8 is an improvement of the following estimates which were given by Srivastava et al. [8].
Corollary 9. (see [8]) Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the bi-univalent function class $\mathcal{H}_{\Sigma}^{\alpha}(0<\alpha \leqq 1)$. Then

$$
\begin{equation*}
\left|a_{2}\right| \leqq \sqrt{\frac{2}{\alpha+2}} \alpha \tag{47}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|a_{3}\right| \leqq \frac{\alpha(3 \alpha+2)}{3} \tag{48}
\end{equation*}
$$

By setting $\mu=0$ and $\lambda=1$ in Corollary 5, we get the following consequence.
Corollary 10. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the bi-univalent function class $\mathcal{S}_{\Sigma}^{*}[\alpha](0<\alpha \leqq 1)$. Then

$$
\begin{equation*}
\left|a_{2}\right| \leqq \sqrt{2} \alpha \tag{49}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|a_{3}\right| \leqq 2 \alpha^{2} \tag{50}
\end{equation*}
$$

Remark 10. If we set

$$
\begin{equation*}
h(z)=p(z)=\frac{1+(1-2 \beta) z}{1-z} \quad(0 \leqq \beta<1 ; \lambda \geqq 1 ; z \in \mathbb{U}) \tag{51}
\end{equation*}
$$

in Theorem 3, we can readily deduce Theorem 2.
Remark 11. The aforecited work by Çağlar et al. [3] contains several interesting further special cases and consequences of Theorem 2, which we have generalized here by means of Theorem 3 (see Remark 3). The reader will find each of these further special cases and consequences of Theorem 2, too, to be motivatingly interesting.

## 4. Concluding Remarks and Observations

In our present investigation, we have considered an interesting subclass $\mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu)$ of analytic and biunivalent functions in the open unit disk $\mathbb{U}$. We have derived estimates on the first two Taylor-Maclaurin coefficients $\left|a_{2}\right|$ and $\left|a_{3}\right|$ for functions belonging to the class $\mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu)$. By means of corollaries and consequences which we discussed in the preceding section by suitably specializing the functions $h(z)$ and $p(z)$ (and also the parameters $\lambda$ and $\mu$ ), we have also shown already that the results presented in this paper would generalize and improve some recent works of Çağlar et al. [3], Xu et al. [10], and other authors.

Finally, our motivation for introducing the subclass $\mathcal{N}_{\Sigma}^{h, p}(\lambda, \mu)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$ in Definition 3 is motivated at least partially by the work of Zhu [12] who provided extensions, generalizations and improvements of the various starlikeness criteria which were proven by a number of earlier authors (see, for details, Remark 1).

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