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# Coefficient Estimates for Certain Subclasses of Analytic and Bi-univalent Functions

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**Abstract.** For $\lambda \geq 0$ and $0 \leq \alpha < 1 < \beta$, we denote by $\mathcal{K}(\lambda;\alpha,\beta)$ the class of normalized analytic functions satisfying the two sided-inequality

$$\alpha < \Re\left(\frac{zf'(z)}{f(z)} + \lambda\frac{z^2 f''(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U}),$$

where $\mathbb{U}$ is the open unit disk. Let $\mathcal{K}_\Sigma(\lambda;\alpha,\beta)$ be the class of bi-univalent functions such that $f$ and its inverse $f^{-1}$ both belong to the class $\mathcal{K}(\lambda;\alpha,\beta)$. In this paper, we establish bounds for the coefficients, and solve the Fekete-Szegő problem, for the class $\mathcal{K}(\lambda;\alpha,\beta)$. Furthermore, we obtain upper bounds for the first two Taylor-Maclaurin coefficients of the functions in the class $\mathcal{K}_\Sigma(\lambda;\alpha,\beta)$.

## 1. Introduction

Let $\mathcal{A}$ denote the class of the functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{S}$ be the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

It is well known that every function $f \in \mathcal{S}$ of the form (1.1) has an inverse $f^{-1}$, defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U}),$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r;\ r \geq \frac{1}{4}\right),$$

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where

$$f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^2 - 5a_2a_3 + a_4\right)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions defined in the open unit disk $\mathbb{U}$. Recently, the bounds of coefficients of analytic and bi-univalent functions have been studied by many authors. We refer the reader to [2, 3, 5, 12, 14–17, 19, 20] for recent investigations in this topic.

For two analytic functions $f$ and $g$ in $\mathbb{U}$, we say that $f$ is subordinate to $g$ in $\mathbb{U}$, and write $f \prec g$ $(z \in \mathbb{U})$, if

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U})$$

for some analytic function $\omega(z)$ such that

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

If $g$ is univalent in $\mathbb{U}$, then the subordination $f \prec g$ is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha$ $(0 \le \alpha < 1)$, if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}).$$

We denote $\mathcal{S}^*(\alpha)$ by the class of starlike functions of order $\alpha$. Also, we denote $\mathcal{M}(\beta)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U}),$$

for some $\beta > 1$. Moreover, the subclass $\mathcal{S}^*(\alpha, \beta) \subset \mathcal{A}$ consists of functions, which satisfy the following inequality

$$\alpha < \Re\left(\frac{zf'(z)}{f(z)}\right) < \beta \qquad \big(0 \le \alpha < 1 < \beta;\ z \in \mathbb{U}\big).$$

We remark that the functions classes $\mathcal{M}(\beta)$ and $\mathcal{S}^*(\alpha, \beta)$ were first investigated by Uralegaddi *et al.* [18] and Kuroki and Owa [11], respectively.

Next we consider the following two new subclasses of $\mathcal{A}$.

**Definition 1.1.** Let $\lambda$, $\alpha$ and $\beta$ be real numbers such that $\lambda \ge 0$ and $0 \le \alpha < 1 < \beta$. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}(\lambda; \alpha, \beta)$ if $f$ satisfies the inequality:

$$\alpha < \Re\left(\frac{zf'(z)}{f(z)} + \lambda\frac{z^2f''(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U}).$$

**Remark 1.2.** If we set $\lambda = 0$ in Definition 1.1, then it reduces to the class $\mathcal{S}^*(\alpha, \beta)$. It is clear that $\mathcal{S}^*(\alpha, \beta) \subset \mathcal{S}^*(\alpha)$ and $\mathcal{S}^*(\alpha, \beta) \subset \mathcal{M}(\beta)$.

**Definition 1.3.** Let $\lambda \ge 0$ and $0 \le \alpha < 1 < \beta$, we denote by $\mathcal{K}_\Sigma(\lambda; \alpha, \beta)$ the class of bi-univalent functions consisting of the functions in $\mathcal{A}$ such that

$$f \in \mathcal{K}(\lambda; \alpha, \beta) \qquad \text{and} \qquad f^{-1} \in \mathcal{K}(\lambda; \alpha, \beta),$$

where $f^{-1}$ is the inverse function of $f$.

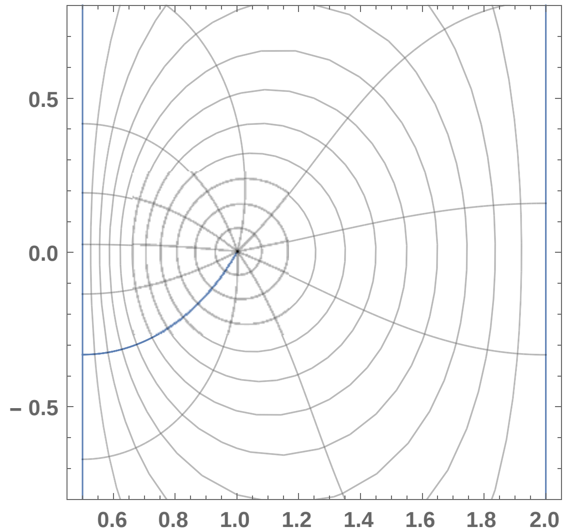


Figure 1: The image of $\mathbb{D}$ under the function $p(z)$ for $\alpha = 1/2$ and $\beta = 2$.

**Remark 1.4.** If $\lambda = 0$ in Definition 1.3, for simplicity, we write $\mathcal{S}^*_\Sigma(\alpha, \beta)$ instead of $\mathcal{K}_\Sigma(0; \alpha, \beta)$.

A classical theorem of Fekete and Szegő [7] states that for $f \in \mathcal{S}$ of the form (1.1), the functional $\left|a_3 - \lambda a_2^2\right|$ satisfies the inequality

$$\left|a_3 - \lambda a_2^2\right| \le \begin{cases} 3 - 4\lambda, & \lambda \le 0, \\ 1 + 2e^{-(2\lambda)/(1-\lambda)}, & 0 \le \lambda \le 1, \\ 4\lambda - 3, & \lambda \ge 1. \end{cases}$$

This inequality is sharp in the sense that for each real $\lambda$ there exists a function in $\mathcal{S}$ such that equality holds (see [1, 9]). Thus the determination of sharp upper bounds for the nonlinear functional $\left|a_3 - \lambda a_2^2\right|$ for any compact family $\mathcal{F}$ of functions in $\mathcal{A}$ is often called the Fekete-Szegő problem for $\mathcal{F}$.

This paper is organized as follows. We start with coefficient estimates for functions of the classes $\mathcal{K}(\lambda; \alpha, \beta)$ and $\mathcal{K}_\Sigma(\lambda; \alpha, \beta)$. The first of our main results, Theorem 3.1, gives bounds of coefficients for the the functions of the class $\mathcal{K}(\lambda; \alpha, \beta)$. The second of our main results, Theorem 3.4, solves the Fekete-Szegő problem for the class $\mathcal{K}(\lambda; \alpha, \beta)$. Finally, in Theorem 3.6, we estimate the upper bounds of initial coefficients of inverse functions and bi-univalent functions of the class $\mathcal{K}_\Sigma(\lambda; \alpha, \beta)$.

## 2. Preliminary Results

In [11], Kuroki and Owa defined an analytic function $p\colon \mathbb{U} \to \mathbb{C}$ by

$$p(z) = 1 + \frac{(\beta - \alpha)i}{\pi} \log\left(\frac{1 - ze^{2\pi(1-\alpha)i/(\beta-\alpha)}}{1 - z}\right) \qquad (0 \le \alpha < 1 < \beta;\ z \in \mathbb{U}), \tag{2.1}$$

and they proved that $p$ maps $\mathbb{U}$ onto the convex domain (see Figure 1)

$$\Omega = \left\{\omega :\ \alpha < \Re(\omega) < \beta\right\}.$$

We observe that the function $p$, defined by (2.1), has the representation

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n z^n \qquad (z \in \mathbb{U}), \tag{2.2}$$

where

$$B_n = \frac{(\beta-\alpha)i}{n\pi}\left(1 - e^{2n\pi(1-\alpha)i/(\beta-\alpha)}\right) \qquad (n \in \mathbb{N}). \tag{2.3}$$

In order to prove our main results, we need the following lemmas.

**Lemma 2.1.** ([8]) *Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be a function with positive real part in $\mathbb{U}$. Then, for any complex number $\nu$,*

$$\left|c_2 - \nu c_1^2\right| \leq 2\max\{1, |1-2\nu|\}.$$

The proof of the next lemma is similar to that of Lemma 1.3 in [11], and we omit the details.

**Lemma 2.2.** *Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1 < \beta$. Then $f \in \mathcal{K}(\lambda;\alpha,\beta)$ if and only if*

$$\frac{zf'(z)}{f(z)} + \lambda\frac{z^2 f''(z)}{f(z)} \prec p(z) \qquad (z \in \mathbb{U}), \tag{2.4}$$

*where $p(z)$ is given by* (2.1).

**Lemma 2.3.** ([13]) *Let $p(z) = \sum_{n=1}^{\infty} C_n z^n$ be analytic and univalent in $\mathbb{U}$ and suppose that $p(z)$ maps $\mathbb{U}$ onto a convex domain. If $q(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in $\mathbb{U}$ and satisfies the subordination:*

$$q(z) \prec p(z) \qquad (z \in \mathbb{U}),$$

*then*

$$|A_n| \leq |C_1| \qquad (n = 1, 2, \ldots).$$

## 3. Main Results

We begin by presenting some coefficient problems involving functions of the class $\mathcal{K}(\lambda;\alpha,\beta)$.

**Theorem 3.1.** *If $f \in \mathcal{K}(\lambda;\alpha,\beta)$, then*

$$|a_2| \leq \frac{|B_1|}{2\lambda+1} \qquad \text{and} \qquad |a_n| \leq \frac{|B_1|}{(n-1)(n\lambda+1)}\prod_{k=2}^{n-1}\left(1 + \frac{|B_1|}{(k-1)(k\lambda+1)}\right) \qquad (n = 3, 4, 5, \ldots), \tag{3.1}$$

*where $|B_1|$ is given by*

$$|B_1| = \frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}. \tag{3.2}$$

*Proof.* Let us define

$$q(z) = \frac{zf'(z)}{f(z)} + \lambda\frac{z^2 f''(z)}{f(z)} \qquad (z \in \mathbb{U}), \tag{3.3}$$

and let the function $p$ be given by (2.1). Then, the subordination (2.4) can be written as follows:

$$q(z) \prec p(z) \qquad (z \in \mathbb{U}). \tag{3.4}$$

Note that the function $p$ defined by (2.1) is convex in $\mathbb{U}$ and has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n z^n \qquad (z \in \mathbb{U}),$$

where $B_n$ is given by (2.3). If we let

$$q(z) = 1 + \sum_{n=1}^{\infty} A_n z^n \qquad (z \in \mathbb{U}),$$

then from Lemma 2.3 we see that the subordination (3.4) implies

$$|A_n| \le |B_1| \qquad (n = 1, 2, \ldots), \tag{3.5}$$

where $|B_1|$ is given by (3.2).

Now, (3.3) implies that

$$zf'(z) + \lambda z^2 f''(z) = q(z) f(z) \qquad (z \in \mathbb{U}).$$

Then, by comparing the coefficients of $z^n$ on the both sides, we see that

$$a_n = \frac{1}{(n-1)(n\lambda+1)} \times \left(A_{n-1} + a_2 A_{n-2} + a_3 A_{n-3} + \cdots + a_{n-1} A_1\right).$$

A simple calculation together with the inequality (3.5) yields that

$$\begin{aligned} |a_n| &= \frac{1}{(n-1)(n\lambda+1)} \times \left|A_{n-1} + a_2 A_{n-2} + a_3 A_{n-3} + \cdots + a_{n-1} A_1\right| \\ &\le \frac{1}{(n-1)(n\lambda+1)} \times \left(|A_{n-1}| + |a_2||A_{n-2}| + |a_3||A_{n-3}| + \cdots + |a_{n-1}||A_1|\right) \\ &\le \frac{|B_1|}{(n-1)(n\lambda+1)} \sum_{k=1}^{n-1} |a_k|, \end{aligned}$$

where $|B_1|$ is given by (3.2) and $|a_1| = 1$. Hence, we have $|a_2| \le |B_1| / (2\lambda + 1)$. To prove the remaining part of the theorem, we need to show that

$$\frac{|B_1|}{(n-1)(n\lambda+1)} \sum_{k=1}^{n-1} |a_k| \le \frac{|B_1|}{(n-1)(n\lambda+1)} \prod_{k=2}^{n-1} \left(1 + \frac{|B_1|}{(k-1)(k\lambda+1)}\right), \tag{3.6}$$

for $n = 3, 4, 5, \ldots$. We use induction to prove (3.6). The case $n = 3$ is clear. Next, assume that the inequality (3.6) holds for $n = m$. Then, a straightforward calculation gives

$$\begin{aligned} |a_{m+1}| &\le \frac{|B_1|}{m[(m+1)\lambda+1]} \sum_{k=1}^{m} |a_k| = \frac{|B_1|}{m[(m+1)\lambda+1]} \left(\sum_{k=1}^{m-1} |a_k| + |a_m|\right) \\ &\le \frac{|B_1|}{m[(m+1)\lambda+1]} \prod_{k=2}^{m-1} \left(1 + \frac{|B_1|}{(k-1)(k\lambda+1)}\right) \\ &\quad + \frac{|B_1|}{m[(m+1)\lambda+1]} \times \frac{|B_1|}{(m-1)(m\lambda+1)} \prod_{k=2}^{m-1} \left(1 + \frac{|B_1|}{(k-1)(k\lambda+1)}\right) \\ &= \frac{|B_1|}{m[(m+1)\lambda+1]} \prod_{k=2}^{m} \left(1 + \frac{|B_1|}{(k-1)(k\lambda+1)}\right), \end{aligned}$$

which implies that the inequality (3.6) holds for $n = m+1$. Hence, the desired estimate for $|a_n|$ $(n = 3, 4, 5, \ldots)$ follows, as asserted in (3.1). This completes the proof of Theorem 3.1. □

Taking $\lambda = 0$ in Theorem 3.1, and using the identity

$$\frac{|B_1|}{n-1}\prod_{k=2}^{n-1}\left(1+\frac{|B_1|}{k-1}\right)=\prod_{k=2}^{n}\left(\frac{k-2+|B_1|}{k-1}\right) \qquad (n=3,4,5,\ldots),$$

we obtain the following corollary.

**Corollary 3.2.** *If $f \in \mathcal{S}^*(\alpha,\beta)$, then*

$$|a_n| \leq \prod_{k=2}^{n}\left(\frac{k-2+|B_1|}{k-1}\right) \qquad (n=2,3,4,\ldots),$$

*where $|B_1|$ is given by* (3.2).

**Remark 3.3.** For $0 \leq \alpha < 1 < \beta$, we have

$$|B_1| = \frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha} \leq \frac{2(\beta-\alpha)}{\pi}\times\frac{\pi(1-\alpha)}{\beta-\alpha} = 2(1-\alpha) \leq 2,$$

thus, we obtain

$$|a_n| \leq \prod_{k=2}^{n}\left(\frac{k-2+|B_1|}{k-1}\right) \leq \prod_{k=2}^{n}\left(\frac{k}{k-1}\right) = n \qquad (n=2,3,4,\ldots),$$

shows how that the coefficient bounds in Corollary 3.2 are related to the well-known Bieberbach conjecture [4] proved by de Branges in 1985 [6] (cf. [10]).

Next, we will solve the Fekete-Szegő problem for functions $f \in \mathcal{K}(\lambda;\alpha,\beta)$.

**Theorem 3.4.** *Let $f \in \mathcal{K}(\lambda;\alpha,\beta)$. Then, for a complex number $\mu$,*

$$\left|a_3-\mu a_2^2\right| \leq \frac{|B_1|}{2(3\lambda+1)}\max\left\{1, \left|\frac{B_2}{B_1}-\frac{2(3\lambda+1)\mu-(2\lambda+1)}{(2\lambda+1)^2}B_1\right|\right\}, \tag{3.7}$$

*where $B_1$ and $B_2$ are given by* (2.3). *The result is sharp.*

*Proof.* Let us consider the functions $p$ and $q$ were given by (2.1) and (3.3), respectively. Then, since $f \in \mathcal{K}(\lambda;\alpha,\beta)$, in view of Lemma 2.2, we have

$$q(z) \prec p(z) = 1+\sum_{n=1}^{\infty}B_n z^n \qquad (z \in \mathbb{U}),$$

where $B_n$ is given by (2.3). Let

$$h(z) = \frac{1+p^{-1}\big(q(z)\big)}{1-p^{-1}\big(q(z)\big)} = 1+h_1 z+h_2 z^2+\cdots \qquad (z \in \mathbb{U}). \tag{3.8}$$

Then $h$ is analytic, and it has positive real part in $\mathbb{U}$. We obtain

$$q(z) = p\left(\frac{h(z)-1}{h(z)+1}\right) \qquad (z \in \mathbb{U}). \tag{3.9}$$

We find from (3.8) and (3.9) that

$$(2\lambda+1)a_2 = \frac{1}{2}B_1h_1 \qquad \text{and} \qquad 2(3\lambda+1)a_3 - (2\lambda+1)a_2^2 = \frac{1}{2}B_1h_2 + \frac{1}{4}(B_2 - B_1)h_1^2.$$

Therefore, we have

$$a_2 = \frac{B_1h_1}{2(2\lambda+1)} \qquad \text{and} \qquad a_3 = \frac{2(2\lambda+1)B_1h_2 + \left[B_1^2 + (2\lambda+1)(B_2 - B_1)\right]h_1^2}{8(2\lambda+1)(3\lambda+1)},$$

which imply that

$$a_3 - \mu a_2^2 = \frac{B_1}{4(3\lambda+1)}(h_2 - \nu h_1^2),$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{2(3\lambda+1)\mu - (2\lambda+1)}{(2\lambda+1)^2}B_1\right).$$

By applying Lemma 2.1, we obtain

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &= \frac{|B_1|}{4(3\lambda+1)}\left|h_2 - \nu h_1^2\right| \leq \frac{|B_1|}{2(3\lambda+1)}\max\{1; |1 - 2\nu|\} \\ &= \frac{|B_1|}{2(3\lambda+1)}\max\left\{1; \left|\frac{B_2}{B_1} - \frac{2(3\lambda+1)\mu - (2\lambda+1)}{(2\lambda+1)^2}B_1\right|\right\},\end{aligned}$$

where $B_1$ and $B_2$ are given by (2.3). This implies the desired estimate of (3.7).

The estimate is sharp for the function $f\colon \mathbb{U} \to \mathbb{C}$ defined by

$$f(z) = \int_0^z \left\{\exp\left(\int_0^\zeta \frac{p(\xi) - 1}{\xi}d\xi\right)\right\} d\zeta, \tag{3.10}$$

where the function $p$ is given by (2.1) (see Figure 2). Hence the proof of Theorem 3.4 is completed. □

Using Theorem 3.4, we can easily get the following result.

**Corollary 3.5.** *Let $f \in \mathcal{K}(\lambda; \alpha, \beta)$, and let $f^{-1}$ be the inverse function of $f$. If*

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n \qquad \left(|w| < r;\ r \geq \frac{1}{4}\right), \tag{3.11}$$

*then*

$$|b_2| \leq \frac{|B_1|}{2\lambda+1} \qquad \text{and} \qquad |b_3| \leq \frac{|B_1|}{2(3\lambda+1)}\max\left\{1, \left|\frac{B_2}{B_1} - \frac{10\lambda+3}{(2\lambda+1)^2}B_1\right|\right\},$$

*where $B_1$ and $B_2$ are given by* (2.3).

*Proof.* The relations (1.2) and (3.11) yield

$$b_2 = -a_2 \qquad \text{and} \qquad b_3 = 2a_2^2 - a_3.$$

Thus, in view of (3.1) and the identity $|b_2| = |a_2|$, the estimate for $|b_2|$ follows immediately. Furthermore, applying Theorem 3.4 with $\mu = 2$ gives the estimate for $|b_3|$. □

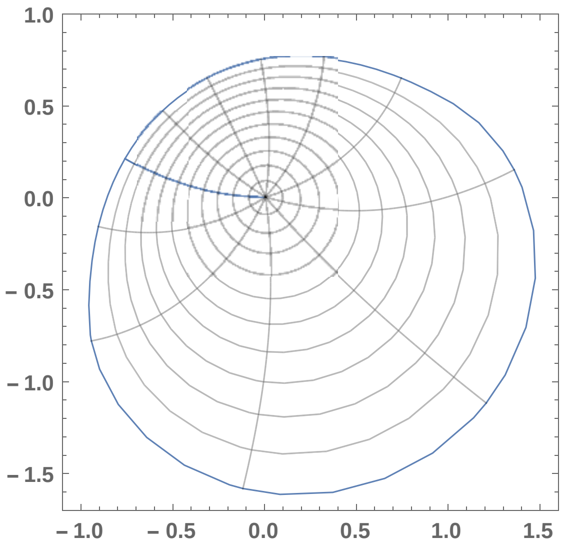


Figure 2: The image of $\mathbb{D}$ under the function $f(z)$, defined by (3.10), for $\alpha = 1/2$ and $\beta = 2$.

Finally, we will estimate some initial coefficients for the bi-univalent functions $f \in \mathcal{K}_\Sigma(\lambda;\alpha,\beta)$.

**Theorem 3.6.** *Let $f \in \mathcal{K}_\Sigma(\lambda;\alpha,\beta)$. Then*

$$|a_2| \le \frac{|B_1|\sqrt{|B_1|}}{\sqrt{\left|(4\lambda+1)B_1^2 + (2\lambda+1)^2(B_1 - B_2)\right|}} \qquad \text{and} \qquad |a_3| \le \frac{|B_1| + |B_1 - B_2|}{4\lambda+1}, \tag{3.12}$$

*where $B_1$ and $B_2$ are given by* (2.3).

*Proof.* If $f \in \mathcal{K}_\Sigma(\lambda;\alpha,\beta)$, then $f \in \mathcal{K}(\lambda;\alpha,\beta)$ and $g = f^{-1} \in \mathcal{K}(\lambda;\alpha,\beta)$. Hence

$$M(z) := \frac{zf'(z)}{f(z)} + \lambda\frac{z^2 f''(z)}{f(z)} \prec p(z) \qquad (z \in \mathbb{U}),$$

$$L(z) := \frac{zg'(z)}{g(z)} + \lambda\frac{z^2 g''(z)}{g(z)} \prec p(z) \qquad (z \in \mathbb{U}),$$

where the function $p$ is given by (2.1). Let

$$t(z) = \frac{1 + p^{-1}\big(M(z)\big)}{1 - p^{-1}\big(M(z)\big)} = 1 + t_1 z + t_2 z^2 + \cdots \qquad (z \in \mathbb{U}),$$

and

$$k(z) = \frac{1 + p^{-1}\big(L(z)\big)}{1 - p^{-1}\big(L(z)\big)} = 1 + k_1 z + k_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

Then $t$ and $k$ are analytic and have positive real part in $\mathbb{U}$, and satisfy the well-known estimates

$$|t_n| \le 2 \quad \text{and} \quad |k_n| \le 2 \qquad (n \in \mathbb{N}). \tag{3.13}$$

Therefore, we have

$$M(z) = p\left(\frac{t(z) - 1}{t(z) + 1}\right) \qquad \text{and} \qquad L(z) = p\left(\frac{k(z) - 1}{k(z) + 1}\right) \qquad (z \in \mathbb{U}).$$

By comparing the coefficients, we get

$$(2\lambda+1)a_2 = \frac{1}{2}B_1t_1, \tag{3.14}$$

$$2(3\lambda+1)a_3-(2\lambda+1)a_2^2 = \frac{1}{2}B_1t_2+\frac{1}{4}(B_2-B_1)t_1^2, \tag{3.15}$$

$$-(2\lambda+1)a_2 = \frac{1}{2}B_1k_1, \tag{3.16}$$

and

$$-2(3\lambda+1)a_3+(10\lambda+3)a_2^2 = \frac{1}{2}B_1k_2+\frac{1}{4}(B_2-B_1)k_1^2, \tag{3.17}$$

where $B_1$ and $B_2$ are given by (2.3). From (3.14) and (3.16), we obtain

$$t_1 = -k_1. \tag{3.18}$$

Also, from (3.15), (3.16), (3.17) and (3.18), we see that

$$a_2^2 = \frac{B_1^3(t_2+k_2)}{4[(4\lambda+1)B_1^2+(2\lambda+1)^2(B_1-B_2)]}$$

and

$$a_3 = \frac{B_1[(10\lambda+3)t_2+(2\lambda+1)k_2]+2(3\lambda+1)(B_2-B_1)t_1^2}{8(3\lambda+1)(4\lambda+1)}.$$

These equations, together with (3.13), give the bounds on $|a_2|$ and $|a_3|$ as asserted in (3.12). This completes the proof of Theorem 3.6. □

By setting $\lambda = 0$ in Theorem 3.6, we obtain the following corollary.

**Corollary 3.7.** *Let $f \in \mathcal{S}_\Sigma^*(\alpha,\beta)$. Then*

$$|a_2| \le \frac{|B_1|\sqrt{|B_1|}}{\sqrt{\left|B_1^2+B_1-B_2\right|}} \qquad \text{and} \qquad |a_3| \le |B_1|+|B_1-B_2|,$$

*where $B_1$ and $B_2$ are given by* (2.3).

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