KYUNGPOOK Math. J. 49(2009), 349-353

Coefficient Estimates in a Class of Strongly Starlike Functions

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Abstract. In this paper we consider some coefficient estimates in the subclass $\mathcal{SL}^*$ of strongly starlike functions defined by a certain geometric condition.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. Everywhere in this paper $z \in \mathcal{U}$ unless we make a note. We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in $\mathcal{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$ and $f(z) = g(\omega(z))$. In particular, if g is univalent in $\mathcal{U}$, we have the following equivalence

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathcal{U}) \subseteq g(\mathcal{U}).$$

Let us denote $Q(f,z) = \dfrac{zf'(z)}{f(z)}$. The class $\mathcal{SS}^*(\beta)$ of strongly starlike functions of order β

$$\mathcal{SS}^*(\beta) := \{f \in \mathcal{A} : |\text{Arg } Q(f,z)| < \beta\pi/2\}, \quad 0 < \beta \le 1$$

was introduced in [5] and [1]. For $\beta = 1$ this class becomes the well known class S^* of starlike functions. In this paper we consider the class $\mathcal{SL}^*$:

$$\mathcal{SL}^* := \{f \in \mathcal{A} : |Q^2(f,z) - 1| < 1\}. \tag{1}$$

It is easy to see that $f \in \mathcal{SL}^*$ if and only if $Q(f,z) \prec q_0(z) = \sqrt{1+z}$, $q_0(0) = 1$. We observe that $\mathcal{L} := \{w \in \mathbb{C} : \text{Re } w > 0, \ |w^2 - 1| < 1\}$ is the interior of the right half of the lemniscate of Bernoulli $\gamma_1 : (x^2+y^2)^2 - 2(x^2-y^2) = 0$, see Figure 1. Moreover $\mathcal{L} \subset \{w : |\text{Arg } w| < \pi/4\}$, thus $\mathcal{SL}^* \subset \mathcal{SS}^*(1/2) \subset \mathcal{S}^*$. The class $\mathcal{SL}^*$ was introduced in [4] and there the authors give also the following representation formula.

Received July 14, 2008; accepted September 12, 2008.
2000 Mathematics Subject Classification: 30C50, 30C45.
Key words and phrases: analytic functions; starlike functions; k–starlike functions; strongly starlike functions.

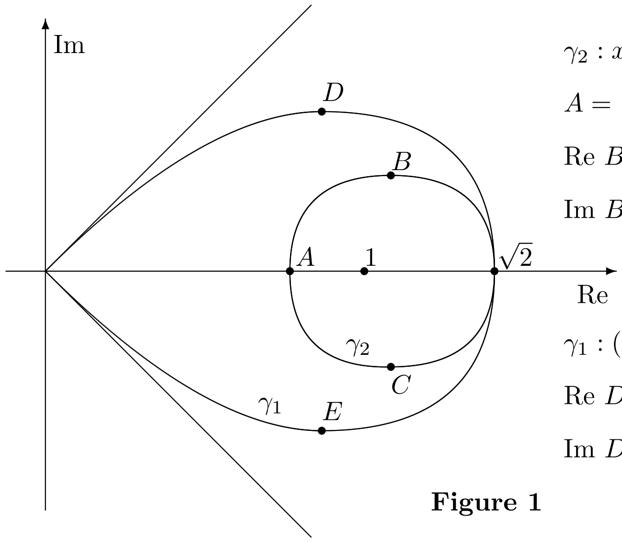

Figure 1

Theorem A([4]). *The function f belongs to the class $\mathcal{SL}^*$ if and only if there exists an analytic function $q \in \mathcal{H}$, $q(0) = 0$, $q(z) \prec q_0(z) = \sqrt{1+z}$, $q_0(0) = 1$ such that*

$$f(z) = z \exp \int_0^z \frac{q(t)-1}{t} dt. \tag{2}$$

Let $q_1(z) = \dfrac{3+2z}{3+z}$, $q_2(z) = \dfrac{5+3z}{5+z}$, $q_3(z) = \dfrac{8+4z}{8+z}$. Because $q_i(z) \prec q_0(z)$, $i = 1, 2, 3$, then by (2) we obtain that the functions $f_1(z) = z + \dfrac{z^2}{3}$, $f_2(z) = z(1+\dfrac{z}{5})^2$, $f_3(z) = z(1+\dfrac{z}{8})^3$ are in $\mathcal{SL}^*$. If we take $q_0(z) = \sqrt{1+z}$, $q_0(0) = 1$ then we obtain from (2) the function f_0

$$f_0(z) := \frac{4z \exp(2\sqrt{1+z}-2)}{(1+\sqrt{1+z})^2} = z + \frac{1}{2}z^2 + \frac{1}{16}z^3 + \frac{1}{96}z^4 - \frac{1}{128}z^5 + \cdots. \tag{3}$$

Rønning considered in [3] an analogously defined class connected with a parabolic region:

$$\mathcal{S}_p^* := \{f \in \mathcal{A} : \mathrm{Re}[Q(f,z)] > |Q(f,z) - 1|\}.$$

Kanas and Wiśniowska introduced in [2] the concept of a k–starlike functions

$$k - \mathcal{ST} := \{f \in \mathcal{A} : \mathrm{Re}[Q(f,z)] > k|Q(f,z) - 1|\}, \quad k \geq 0.$$

In this way they obtained a continuous passage from starlike functions ($k = 0$) to the class $\mathcal{S}_p^*$ ($k = 1$). Moreover for $0 < k < 1$ the quantity $Q(f,z)$ takes its values in a convex domain on the right of a hyperbola while for $k > 1$ inside an ellipse. Let us consider the conic region $P(k) = \{w \in \mathbb{C} : \text{Re } w > k|w-1|\}$ connected with the class $k - \mathcal{ST}$ described above. For $k > 1$ the curve $\partial P(k)$ is the ellipse $\gamma_2 : x^2 = k^2(x-1)^2 + k^2y^2$. For $k \geq 2 + \sqrt{2}$ this ellipse lies entirely inside $\overline{\mathcal{L}}$. Therefore $k - \mathcal{ST} \subset \mathcal{SL}^*$, for $k \geq 2 + \sqrt{2}$.

2. Main results

Theorem 1. *If the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to the class $\mathcal{SL}^*$, then*

$$\sum_{k=2}^{\infty}(k^2-2)|a_k|^2 \leq 1. \tag{4}$$

Proof. If $f \in \mathcal{SL}^*$, then $Q(f,z) \prec q_0(z) = \sqrt{1+z}$. Hence $Q(f,z) = \sqrt{1+\omega(z)}$, where ω satisfies $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$. Therefore $f^2(z) = (zf'(z))^2 - f^2(z)\omega(z)$ and using this we can obtain

$$\begin{aligned}
2\pi\sum_{k=1}^{\infty}|a_k|^2 r^{2k} &= \int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta \\
&\geq \int_0^{2\pi} \left|\omega(re^{i\theta})\right| \left|f^2(re^{i\theta})\right| d\theta \\
&= \int_0^{2\pi} \left|(re^{i\theta}f'(re^{i\theta}))^2 - f^2(re^{i\theta})\right| d\theta \\
&\geq \int_0^{2\pi} \left|re^{i\theta}f'(re^{i\theta})\right|^2 - \left|f(re^{i\theta})\right|^2 d\theta \\
&= 2\pi\sum_{k=1}^{\infty}k^2|a_k|^2 r^{2k} - 2\pi\sum_{k=1}^{\infty}|a_k|^2 r^{2k}
\end{aligned}$$

for $0 < r < 1$. The extremes in this sequence of inequalities give

$$2\sum_{k=1}^{\infty}|a_k|^2r^{2k} \geq \sum_{k=1}^{\infty}k^2|a_k|^2r^{2k}, \quad 0 < r < 1.$$

Eventually, if we let $r \to 1^-$ then we obtain (4). □

Corollary 1. *If the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to the class $\mathcal{SL}^*$, then $|a_k| \leq \sqrt{\dfrac{1}{k^2-2}}$ for $k \geq 2$.*

Theorem 2. *If the function* $f(z)=\sum_{k=1}^{\infty}a_kz^k$ *belongs to the class* $\mathcal{SL}^*$*, then*

$$|a_2|\le 1/2,\quad |a_3|\le 1/4,\quad |a_4|\le 1/6. \tag{5}$$

Those estimations are sharp.

Proof. If $f(z)=\sum_{k=1}^{\infty}a_kz^k$ belongs to the class $\mathcal{SL}^*$ then $[zf'(z)]^2=f^2(z)[\omega(z)-1]$, where ω satisfies $\omega(0)=0,\ |\omega(z)|<1$ for $|z|<1$. Let us denote

$$[zf'(z)]^2=\sum_{k=2}^{\infty}A_kz^k,\quad f^2(z)=\sum_{k=2}^{\infty}B_kz^k,\quad \omega(z)=\sum_{k=1}^{\infty}C_kz^k. \tag{6}$$

Then we have

$$A_k=\sum_{l=1}^{k-1}l(k-l)a_la_{k-l}\ ,\quad B_k=\sum_{l=1}^{k-1}a_la_{k-l} \tag{7}$$

and

$$\sum_{k=2}^{\infty}(A_k-B_k)z^k=\left[\sum_{k=1}^{\infty}C_kz^k\right]\left[\sum_{k=2}^{\infty}B_kz^k\right]. \tag{8}$$

Thus

$$A_2=a_1=1,\quad A_3=4a_1a_2=4a_2,\quad A_4=6a_3+4a_2^2,\quad A_5=8a_1a_4+12a_2a_3 \tag{9}$$

and

$$B_2=a_1=1,\quad B_3=2a_2,\quad B_4=2a_3+a_2^2,\quad B_5=2a_1a_4+2a_2a_3. \tag{10}$$

Equating the second, third and fourth coefficients of both sides of (8) we obtain

(i) $A_3-B_3=C_1B_2$,

(ii) $A_4-B_4=C_1B_3+C_2B_2$,

(iii) $A_5-B_5=C_1B_4+C_2B_3+C_3B_2$.

So by (9), (10) we have

(j) $a_2=\frac{1}{2}C_1$,

(jj) $a_3=\frac{1}{16}C_1^2+\frac{1}{4}C_2$,

(jjj) $a_4=\frac{1}{96}C_1^3+\frac{1}{24}C_1C_2+\frac{1}{6}C_3$.

It is well known that $|C_k| \leq 1$, $\sum_{k=1}^{\infty} |C_k|^2 \leq 1$ therefore we obtain (5).
For the proof of sharpness let us consider $q(z) = \sqrt{1+z^n}$. Using the representation formula (2) we obtain the function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ such that $[zf'(z)]^2 = f^2(z)[z^n - 1]$ and with the notation (6) we have

$$\sum_{k=2}^{\infty}(A_k - B_k)z^k = \sum_{k=2}^{\infty} B_k z^{k+n}.$$

So $A_k = B_k$ for $k \leq n+1$. This gives $a_1 = 1$, $a_2 = \cdots = a_n = 0$. While $A_{n+2} - B_{n+2} = B_2$ gives

$$\sum_{l=1}^{n+1}[l(n+2-l) - 1]a_l a_{n+2-l} = 1$$

thus $2na_{n+1} = 1$. Therefore there exists a function f in the class $\mathcal{SL}^*$ such that $f(z) = z + \frac{1}{2n} z^{n+1} + \cdots$. □

Conjecture. *Let $f \in \mathcal{SL}^*$ and $f(z) = \sum_{k=1}^{\infty} a_k z^k$. Then $|a_{n+1}| \leq \frac{1}{2n}$.*

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