



COEFFICIENT INEQUALITY FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

T. AL-HAWARY, B. A. FRASIN, AND M. DARUS

Received 26 January, 2014

Abstract. Let $\mathcal{T}(s, \lambda, \mu, \beta)$ be the class of normalized functions, f defined in the open unit disk \mathbb{U} by

$$\operatorname{Re} \left(\frac{\Theta_{\mu}^{\lambda, s+1} f(z)}{\Theta_{\mu}^{\lambda, s} g(z)} \right) > 0 \quad (\mu \in \mathbb{N}, \lambda, s \in \mathbb{N}_0, z \in \mathbb{U})$$

for some $g \in \mathcal{R}_{\beta}(s, \lambda, \mu, \beta)$. The authors in [15] introduced the operator $\Theta_{\mu}^{\lambda, s}$ defined by

$$\Theta_{\mu}^{\lambda, s} f(z) = z + \sum_{k=2}^{\infty} \frac{(k + \lambda - 1)!(\mu - 1)!}{\lambda!(k + \mu - 2)!} k^s a_k z^k,$$

where $\mathcal{R}_{\beta}(s, \lambda, \mu, \beta)$ denotes the class of normalized functions g in the open unit disk \mathbb{U} defined by

$$\left| \arg \left(\frac{\Theta_{\mu}^{\lambda, s+1} g(z)}{\Theta_{\mu}^{\lambda, s} g(z)} \right) \right| < \frac{\pi}{2} \beta \quad (0 < \beta \leq 1).$$

For $f \in \mathcal{T}(s, \lambda, \mu, \beta)$ and given by $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$, a sharp upper bound is obtained for $|a_3 - \tau a_2^2|$ when $\tau \geq 1$.

2010 *Mathematics Subject Classification:* 30C45

Keywords: univalent and analytic functions, starlike and convex functions, Fekete-Szegő problem

1. INTRODUCTION AND DEFINITIONS

Let \mathcal{A} denote the family of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Further, let \mathcal{S} denote the class of functions which are univalent in \mathbb{U} . A function $f(z)$ belonging to \mathcal{A} is said

The work was fully supported by AP-2013-009.

to be strongly starlike of order β in \mathbb{U} , and denoted by $\mathcal{S}\mathcal{S}^*(\mathcal{L})$ if it satisfies

$$\left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2}\beta \quad (0 < \beta \leq 1, z \in \mathbb{U}). \quad (1.2)$$

If $f(z) \in \mathcal{A}$ satisfies

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2}\beta \quad (0 < \beta \leq 1, z \in \mathbb{U}), \quad (1.3)$$

then we say that $f(z)$ is strongly convex of order β in \mathbb{U} , and we denote by $\mathcal{S}\mathcal{C}(\mathcal{L})$ the class of all such functions.

With the help of the differential operator $\Theta_{\mu}^{\lambda,s}$, we say that a function $f(z)$ belonging to \mathcal{A} is said to be in the class $\mathcal{R}_{\rho}(s, \lambda, \mu, \beta)$ if it satisfies

$$\left| \arg \left(\frac{\Theta_{\mu}^{\lambda,s+1} f(z)}{\Theta_{\mu}^{\lambda,s} f(z)} \right) \right| < \frac{\pi}{2}\beta \quad (\mu \in \mathbb{N}, \lambda, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \quad (1.4)$$

for some $\beta(0 < \beta \leq 1)$ and for all $z \in \mathbb{U}$. Note that $\mathcal{R}_{\rho}(0, 0, 1, \beta) = \mathcal{S}\mathcal{S}^*(\mathcal{L})$ and $\mathcal{R}_{\rho}(1, 0, 1, \beta) = \mathcal{S}\mathcal{C}(\mathcal{L})$.

For the class \mathcal{S} of analytic univalent functions, [6] obtained the maximum value of $|a_3 - \tau a_2^2|$ when τ is real. For various functions of \mathcal{S} , the upper bound for $|a_3 - \tau a_2^2|$ is investigated by many different authors (see [1–5, 7, 8, 11–14, 16, 17] and [18]).

In this paper we obtain a sharp upper bounds for $|a_3 - \tau a_2^2|$ when f belongs to the class of functions defined as follows:

Definition 1. Let $\beta(0 < \beta \leq 1)$ and let $f \in \mathcal{A}$. Then $f \in \mathcal{T}(s, \lambda, \mu, \beta)$ if and only if there exist $g \in \mathcal{R}_{\rho}(s, \lambda, \mu, \beta)$ such that

$$\operatorname{Re} \left(\frac{\Theta_{\mu}^{\lambda,s+1} f(z)}{\Theta_{\mu}^{\lambda,s} g(z)} \right) > 0 \quad (\mu \in \mathbb{N}, \lambda, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U}), \quad (1.5)$$

where $g(z) = z + b_2 z^2 + b_3 z^3 + \dots$.

Note that $\mathcal{T}(0, 0, 1, \beta) = \mathcal{K}(\beta)$ the class of close-to-convex functions defined by [3], $\mathcal{T}(0, 0, 1, 1) = \mathcal{K}(1)$ is the class of normalized close-to-convex functions defined by [10].

2. MAIN RESULTS

In order to derive our main results, we have to recall here the following lemma.

Lemma 1 ([17]). *Let $h \in \mathcal{P}$ i.e. h be analytic in \mathbb{U} and be given by $h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots$, and $\operatorname{Re} h(z) > 0$ for $z \in \mathbb{U}$. Then*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}. \quad (2.1)$$

Theorem 1. Let $f(z) \in \mathcal{T}(s, \lambda, \mu, \beta)$ and be given by (1.1). Then for $\beta \geq 1$ and $\tau \geq 1$ we have the sharp inequality

$$|a_3 - \tau a_2^2| \leq \frac{\beta^2 [(3^s \tau \mu^2 (\lambda + 2) - 2^{2s} \mu (\mu + 1) (\lambda + 1))]}{2^{2s} 3^s (\lambda + 2) (\lambda + 1)^2} + \frac{[(3^{s+1} \tau \mu^2 (\lambda + 2) - 2^{2s+1} \mu (\mu + 1) (\lambda + 1)) (\beta + 1)]}{2^{2s} 3^{s+1} (\lambda + 2) (\lambda + 1)^2}. \quad (2.2)$$

Proof. Let $f(z) \in \mathcal{T}(s, \lambda, \mu, \beta)$. It follows from (1.5) that

$$\Theta_{\mu}^{\lambda, s+1} f(z) = \Theta_{\mu}^{\lambda, s} g(z) q(z), \quad (2.3)$$

for $z \in \mathbb{U}$, with $q \in \mathcal{P}$ given by $q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \dots$. Equating coefficients, we obtain

$$\frac{(\lambda + 1)}{\mu} 2^{s+1} a_2 = q_1 + \frac{(\lambda + 1)}{\mu} 2^s b_2, \quad (2.4)$$

and

$$\frac{(\lambda + 2)(\lambda + 1)}{\mu(\mu + 1)} 3^{s+1} a_3 = q_2 + \frac{(\lambda + 1)}{\mu} 2^s b_2 q_1 + \frac{(\lambda + 2)(\lambda + 1)}{\mu(\mu + 1)} 3^s b_3. \quad (2.5)$$

Also, it follows from (1.4) that

$$\Theta_{\mu}^{\lambda, s+1} g(z) = \Theta_{\mu}^{\lambda, s} g(z) (p(z))^{\beta}.$$

where for $z \in \mathbb{U}$, $p \in \mathcal{P}$ and $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$. Thus equating coefficients, we obtain

$$\frac{(\lambda + 1)}{\mu} 2^s b_2 = \beta p_1, \quad (2.6)$$

$$\frac{(\lambda + 2)(\lambda + 1)}{\mu(\mu + 1)} (2) 3^s b_3 = \beta (p_2 + \frac{3\beta - 1}{2} p_1^2). \quad (2.7)$$

From (2.4), (2.5), (2.6) and (2.7) we have

$$\begin{aligned} a_3 - \tau a_2^2 &= \frac{\mu(\mu + 1)}{3^{s+1} (\lambda + 2) (\lambda + 1)} (q_2 - \frac{1}{2} q_1^2) \\ &+ \frac{2^{2s+1} \mu(\mu + 1) (\lambda + 1) - 3^{s+1} \tau \mu^2 (\lambda + 2)}{3^{s+1} \times 2^{2s+2} (\lambda + 2) (\lambda + 1)^2} q_1^2 \\ &+ \frac{\beta \mu(\mu + 1)}{2 \times 3^{s+1} (\lambda + 2) (\lambda + 1)} (p_2 - \frac{1}{2} p_1^2) \\ &+ \frac{[2^{2s+1} \mu(\mu + 1) (\lambda + 1) - 3^{s+1} \tau \mu^2 (\lambda + 2)] \beta}{3^{s+1} \times 2^{2s+1} (\lambda + 2) (\lambda + 1)^2} p_1 q_1 \\ &+ \frac{3 \times 2^{2s} \beta^2 \mu(\mu + 1) (\lambda + 1) - 3^{s+1} \tau \beta^2 \mu^2 (\lambda + 2)}{3^{s+1} 2^{2s+2} (\lambda + 2) (\lambda + 1)^2} p_1^2. \end{aligned} \quad (2.8)$$

Assume that $a_3 - \tau a_2^2$ is positive. Thus we now estimate $\operatorname{Re}(a_3 - \tau a_2^2)$, so from (2.8) and by using the Lemma 1 and letting $p_1 = 2re^{i\theta}$, $q_1 = 2Re^{i\phi}$, $0 \leq r \leq 1$, $0 \leq R \leq 1$, $0 \leq \theta < 2\pi$ and $0 \leq \phi < 2\pi$, we obtain

$$\begin{aligned}
& 3^{s+1} \operatorname{Re}(a_3 - \tau a_2^2) \\
&= \frac{\mu(\mu+1)}{(\lambda+2)(\lambda+1)} \operatorname{Re}(q_2 - \frac{1}{2}q_1^2) + \frac{[2^{2s+1}\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\mu^2(\lambda+2)]}{2^{2s+2}(\lambda+2)(\lambda+1)^2} \operatorname{Re}q_1^2 \\
&\quad + \frac{\beta\mu(\mu+1)}{2(\lambda+2)(\lambda+1)} \operatorname{Re}(p_2 - \frac{1}{2}p_1^2) \\
&\quad + \frac{\beta[2^{2s+1}\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\mu^2(\lambda+2)]}{2^{2s+1}(\lambda+2)(\lambda+1)^2} \operatorname{Re}p_1q_1 \\
&\quad + \frac{3 \times 2^{2s}\beta^2\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\beta^2\mu^2(\lambda+2)}{2^{2s+2}(\lambda+2)(\lambda+1)^2} \operatorname{Re}p_1^2 \tag{2.9} \\
&\leq \frac{2\mu(\mu+1)}{(\lambda+2)(\lambda+1)}(1-R^2) + \frac{[2^{2s+1}\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\mu^2(\lambda+2)]}{2^{2s}(\lambda+2)(\lambda+1)^2} R^2 \cos 2\phi \\
&\quad + \frac{\beta\mu(\mu+1)}{(\lambda+2)(\lambda+1)}(1-r^2) \\
&\quad + \frac{2\beta[2^{2s+1}\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\mu^2(\lambda+2)]}{2^{2s}(\lambda+2)(\lambda+1)^2} rR \cos(\theta + \phi) \\
&\quad + \frac{3\beta^2[2^{2s}\mu(\mu+1)(\lambda+1) - 3^s\tau\mu^2(\lambda+2)]}{2^{2s}(\lambda+2)(\lambda+1)^2} r^2 \cos 2\theta \\
&\leq \frac{3^{s+1}\tau\mu^2(\lambda+2) - 2^{2s+2}\mu(\mu+1)(\lambda+1)}{2^{2s}(\lambda+2)(\lambda+1)^2} R^2 \\
&\quad + \frac{2\beta\mu[3^{s+1}\tau\mu(\lambda+2) - 2^{s+1}(\mu+1)(\lambda+1)]}{2^{2s}(\lambda+2)(\lambda+1)^2} rR \\
&\quad + \beta \left(\frac{3\beta[3^s\tau\mu^2(\lambda+2) - 2^{2s}(\mu+1)(\lambda+1)]}{2^{2s}(\lambda+2)(\lambda+1)^2} - \frac{\mu(\mu+1)}{(\lambda+2)(\lambda+1)} \right) r^2 \\
&\quad + \frac{\mu(\mu+1)}{(\lambda+2)(\lambda+1)}(\beta+2) \\
&= \Psi(r, R). \tag{2.10}
\end{aligned}$$

Letting β, λ, μ and τ fixed and differentiating $\Psi(r, R)$ partially when $\mu \geq 1, \lambda \geq 0, \beta \geq 1$ and $\tau \geq 1$ we observe that

$$\begin{aligned}
& \Psi_{rr}\Psi_{RR} - (\Psi_{rR})^2 \\
&= \frac{2^{2s+4}\beta\mu^2(\mu+1)^2}{(\lambda+2)}[2\beta+1] - \frac{4 \times 3^{s+1}\tau\beta\mu^3(\mu+1)}{(\lambda+1)}[5\beta-1] < 0.
\end{aligned}$$

Therefore, the maximum of $\Psi(r, R)$ occurs on the boundaries. Thus the desired inequality follows by observing that

$$\Psi(r, R) \leq \Psi(1, 1) = \frac{3\beta^2 [3^s \tau \mu^2 (\lambda + 2) - 2^{2s} \mu (\mu + 1) (\lambda + 1)]}{2^{2s} (\lambda + 2) (\lambda + 1)^2} + \frac{[(3^{s+1} \tau \mu^2 (\lambda + 2) - 2^{2s+1} \mu (\mu + 1) (\lambda + 1)) (2\beta + 1)]}{2^{2s} (\lambda + 2) (\lambda + 1)^2}.$$

The equality for (2.2) is attained when $p_1 = q_1 = 2i$ and $p_2 = q_2 = -2$. \square

Letting $s = \lambda = 0$ and $\mu = 1$ in the above Theorem, we have the result given by Jahangiri [9]:

Corollary 1. *Let $f(z) \in \mathcal{K}(\beta)$ and be given by (1.1). Then for $\beta \geq 1$ and $\tau \geq 1$ we have the sharp inequality*

$$|a_3 - \tau a_2^2| \leq \beta^2 (\tau - 1) + \frac{(2\beta + 1)(3\tau - 2)}{3}. \quad (2.11)$$

Letting $s = \mu = 1$ and $\lambda = 0$ in the above Theorem, we have the following result:

Corollary 2. *Let $f(z) \in \mathcal{T}(1, 0, 1, \beta)$ and be given by (1.1). Then for $\beta \geq 1$ and $\tau \geq 1$ we have the sharp inequality*

$$|a_3 - \tau a_2^2| \leq \frac{1}{36} [3\beta^2 (3\tau - 4) + (9\tau - 8)(2\beta + 1)].$$

ACKNOWLEDGEMENT

The authors would like to thank the referee for the informative comments and suggestions in order to improve the manuscript.

REFERENCES

- [1] H. Abdel-Gawad and D. Thomas, "A subclass of close-to-convex functions," *Publ. de L'Inst. Math. Nouvelle série tome*, vol. 16, pp. 279–290, 1985.
- [2] H. Abdel-Gawad and D. Thomas, "The Fekete-Szegő problem for strongly close-to-convex functions," *Proc. Amer. Math. Soc.*, vol. 114, no. 2, pp. 345–349, 1992, doi: [10.2307/2159653](https://doi.org/10.2307/2159653).
- [3] A. Chonweerayoot, D. Thomas, and W. Upakarnitakaset, "On the coefficients of close-to-convex functions," *Math. Japon.*, vol. 36, no. 5, pp. 819–826, 1991.
- [4] M. Darus and D. Thomas, "On the Fekete-Szegő problem for close-to-convex functions," *Math. Japon.*, vol. 44, no. 3, pp. 507–511, 1996.
- [5] M. Darus and D. Thomas, "On the Fekete-Szegő problem for close-to-convex functions," *Math. Japon.*, vol. 47, no. 1, pp. 125–132, 1998.
- [6] M. Fekete and G. Szegő, "Eine Bemerkung über ungrade schlicht Funktionen," *J. London Math. Soc.*, vol. 8, pp. 85–89, 1933.
- [7] B. Frasin and M. Darus, "On the Fekete-Szegő problem," *Inter. J. Math. and Math. Sci. (IJMMS)*, vol. 24, no. 9, pp. 577–581, 2000, doi: [10.1155/S0161171200005111](https://doi.org/10.1155/S0161171200005111).
- [8] R. Goel and B. Mehrotra, "A coefficient inequality for certain classes of analytic functions," *Tamkang J. Math.*, vol. 22, no. 2, pp. 153–163, 1991.

- [9] M. Jahangiri, "A coefficient inequality for a class of close-to-convex functions," *Math. Japon.*, vol. 41, no. 3, pp. 557–559, 1995.
- [10] W. Kaplan, "Close-to-convex schlicht functions," *Mich. Math. J.*, vol. 1, pp. 169–185, 1952.
- [11] F. Keogh and E. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proc. Amer. Math. Soc.*, vol. 20, pp. 8–12, 1969, doi: [10.2307/2035949](https://doi.org/10.2307/2035949).
- [12] W. Koepf, "On the Fekete-Szegő for close-to-convex functions II," *Arch. Math.*, vol. 49, pp. 420–433, 1987, doi: [10.1007/BF01194100](https://doi.org/10.1007/BF01194100).
- [13] W. Koepf, "On the Fekete-Szegő problem for close-to-convex functions," *Proc. Amer. Math. Soc.*, vol. 101, pp. 89–95, 1987.
- [14] R. London, "Fekete-Szegő inequalities for close-to-convex functions," *Proc. Amer. Math. Soc.*, vol. 117, pp. 947–950, 1993, doi: [10.1090/S0002-9939-1993-1150652-2](https://doi.org/10.1090/S0002-9939-1993-1150652-2).
- [15] A. Mohammed and M. Darus, "An operator defined by convolution involving the generalized Hurwitz-Lerch zeta function," *Sains Malaysiana*, vol. 41, no. 12, pp. 1657–1661, 2012.
- [16] M. Nasr and H. Abdel-Gawad, "On the Fekete-Szegő problem for close-to-convex functions of order p ," *New Trends in Geometric Functions Theory and Applications (Madaras 1990)*, World Sci. Publ. Co., River Edge, N.J., pp. 66–74, 1991.
- [17] C. Pommerenke, *Univalent functions*, 1st ed. Göttingen: Vandenhoeck and Ruprecht, 1975.
- [18] S. Trimple, "A coefficient inequality for convex univalent functions," *Proc. Amer. Math. Soc.*, vol. 48, no. 1, pp. 266–267, 1975.

Authors' addresses

T. Al-Hawary

Irbid National University, Faculty of Science and Technology, Department of Mathematics, P.O. Box: 2600, Irbid, Jordan

E-mail address: tariq.amh@yahoo.com

B. A. Frasin

Faculty of Science, Department of Mathematics, Al al-Bayt University, P.O. Box: 130095, Mafraq, Jordan

E-mail address: bafrasin@yahoo.com

M. Darus

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor D. Ehsan, Malaysia

E-mail address: maslina@ukm.edu.my